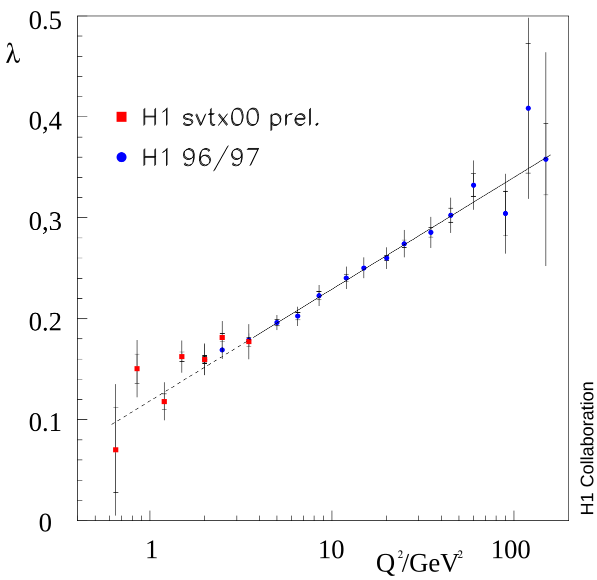
Math 104 Midterm #2

blue=question, black=answer.

1 The H1 collaboration published the data plotted to the right[a].
Which of the following expressions is the best approximation to their data?

[a] *The Rise of the Proton Structure Function F_2 Towards Low x*, Acta Phys.Polon. B33 **(2002)** 2841-2846



a). $\lambda = 0.05\left(\frac{Q^2}{(0.3\ \text{GeV})^2}\right)$

b). $\lambda = 0.05\left(Q^2 \cdot (0.3\ \text{GeV})^2\right)$

c). $\lambda = 0.05\ln\left(\frac{Q^2}{(0.3\ \text{GeV})^2}\right)$

d). $\lambda = 0.05\ln\left(Q^2 \cdot (0.3\ \text{GeV})^2\right)$

e). $\lambda = 0.05\exp\left(\frac{Q^2}{(0.3\ \text{GeV})^2}\right)$

f). $\lambda = 0.05\exp\left(Q^2 \cdot (0.3\ \text{GeV})^2\right)$

Only Q^2/GeV^2 is a number (i.e. without units) and can appear in the argument of ln or exp. According to the plot, there is a linear relation between λ and $\ln(Q^2/GeV^2)$. This already singles out answer c). Plug in simple values for Q^2 if you are unsure.

2 Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1+x^2}$. Using Maple, how would you find the solution satisfying $\lim_{x\to\infty} y(x) = 0$? Hint: A single command suffices.

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> dsolve( { diff(y(x),x)=1/(1+x^2), y(infinity)=0 }, y(x) );
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3 Define the *error function* as

$$\operatorname{erf}(x) \stackrel{\text{def}}{=} \frac{2}{\sqrt{\pi}} \int_{-\infty}^{x} e^{-t^2}\, \mathrm{d}t$$

Find the minimum of the derivative of the inverse function erf^{-1}.

erf has domain $\mathbb{R}$, so the range of erf^{-1} is $\mathbb{R}$.

$$\frac{\mathrm{d}}{\mathrm{d}y}\operatorname{erf}^{-1}(y) = \left.\frac{1}{\frac{2}{\sqrt{\pi}}e^{-x^2}}\right|_{x=\operatorname{erf}^{-1}(y)}$$

The maximum of e^{-x^2} is at $x = 0$. Hence the minimum of the derivative of $\operatorname{erf}^{-1}(y)$ is at $y = \operatorname{erf}(0)$.

$$\frac{\mathrm{d}}{\mathrm{d}y}\operatorname{erf}^{-1}(\operatorname{erf}(0)) = \left.\frac{1}{\frac{2}{\sqrt{\pi}}e^{-x^2}}\right|_{x=0} = \frac{\sqrt{\pi}}{2}$$

4 Compute the derivative of $u(t) = \sqrt{\ln \sqrt[3]{t}}$.

$$\frac{1}{6}\frac{1}{t\sqrt{\ln\left(\sqrt[3]{t}\right)}}$$

5 The function $y(x)$ satisfies $\cot y = e^x - \ln x$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Implicit differentiation:

$$-\left(1+\cot(y)^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} = e^x - \frac{1}{x}$$

Solve for $\mathrm{d}y/\mathrm{d}x$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{e^x - \frac{1}{x}}{-\left(1+\cot(y)^2\right)} = -\frac{e^x - \frac{1}{x}}{1+(e^x - \ln x)^2}$$

6 Integrate $\int_0^{\pi} \frac{2^{\ln x}}{\sqrt{x}}\,\mathrm{d}x$

$$\int_0^{\pi} \frac{2^{\ln x}}{\sqrt{x}}\,\mathrm{d}x = \int_0^{\pi} x^{\ln(2)-\frac{1}{2}}\,\mathrm{d}x = \frac{\pi^{\ln(2)+\frac{1}{2}}}{\ln(2)-\frac{1}{2}}$$

7 Find $\lim_{x\to 0^+} \left(1+\frac{1}{x}\right)^x$.

$$\lim_{x\to 0^+} \left(1+\frac{1}{x}\right)^x = \lim_{x\to 0^+} \exp\left(x\ln\left(1+\frac{1}{x}\right)\right) = $$
$$= \lim_{y\overset{\text{def}}{=}\frac{1}{x}\to\infty} \exp\left(\frac{\ln(1+y)}{y}\right) = 0$$

since $\frac{\ln(1+y)}{y} \longrightarrow 0$ (logarithm grows slower than any nonconstant polynomial).

8 Sort the following functions by their growth rates, from slowest to fastest growing:

$$x, \quad \sqrt{x}, \quad \sqrt{x^2-1}, \quad 1, \quad e^x, \quad (\ln 2)^x$$

Which of those functions are $o(x)$, which are $O(x)$?

x and $\sqrt{x^2-1}$ grow at the same rate.

$$\leftarrow \text{slowest} \quad (\ln 2)^x,\ 1,\ \sqrt{x},\ \left\{ \begin{matrix} x, \\ \sqrt{x^2-1} \end{matrix} \right\},\ e^x \quad \text{fastest} \rightarrow$$

$$\begin{aligned} (\ln 2)^x,\ 1,\ \sqrt{x},\ x,\ \sqrt{x^2-1} &= O(x) \\ (\ln 2)^x,\ 1,\ \sqrt{x} &= o(x) \end{aligned}$$

9 Find $\lim_{x\to 0^+} \frac{\sin^{-1} 2x}{\sqrt{x}}$.

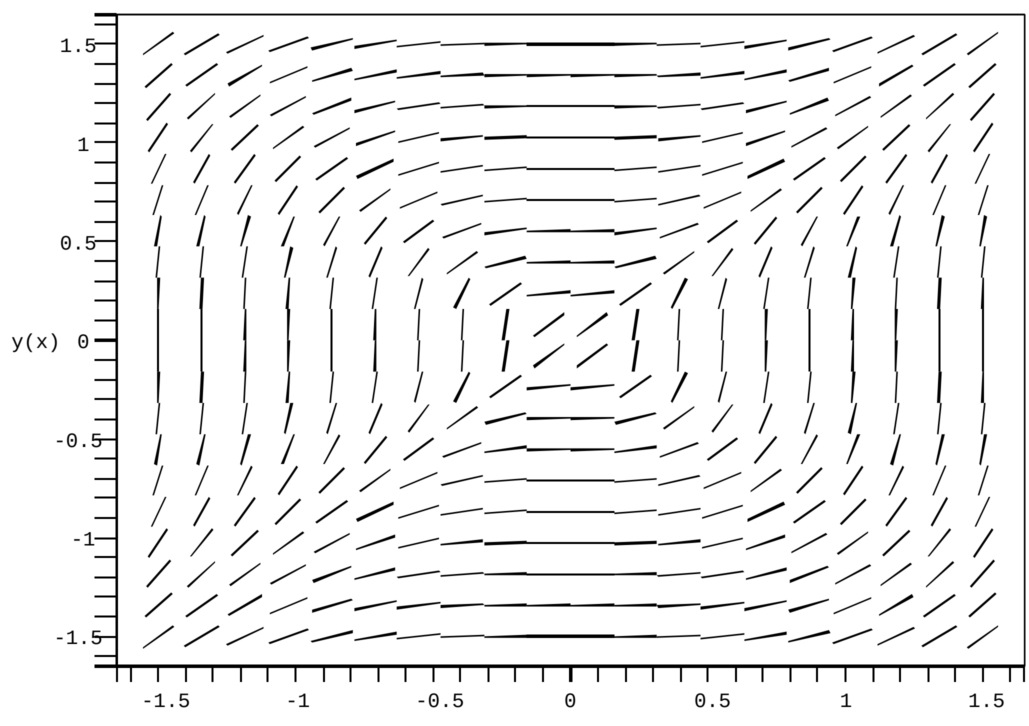
$$\lim_{x\to 0^+} \frac{\sin^{-1} 2x}{\sqrt{x}} = \lim_{x\to 0^+} \frac{2x+\frac{4}{3}x^3+\cdots}{\sqrt{x}} = \lim_{x\to 0^+} \left(2\sqrt{x}+\cdots\right) = 0$$

10 Solve the differential equation $(2+x)y' + y = \sqrt{x}$.

$$(2+x)y' + y = \frac{\mathrm{d}}{\mathrm{d}x}(2+x)y\sqrt{x}$$

$$\Rightarrow (2+x)y = \frac{2}{3}x^{\frac{3}{2}} + C \;\Leftrightarrow\; y = \frac{2}{3}\frac{x^{\frac{3}{2}} + C'}{2+x}$$

11 The following graph is a slope field for the differential equation $y' = \frac{x^2}{y^2}$. Sketch the solution which passes trough the point $(0, 1)$. Make sure you do not continue the solution beyond any points where the differential equation is not defined.



Start at $x = 0$, $y = 1$. Following the slope field to the left you must stop if you reach the x-axis, the differential equation is not defined at $y = 0$.

12 Find $\displaystyle\int_0^\infty \left(x^2+1\right) e^{-2x}\, \mathrm{d}x$, if it exists.

$$\int_0^\infty \left(x^2+1\right) e^{-2x}\, \mathrm{d}x = \frac{3}{4}$$

13 Expand $\dfrac{3\,x^4+7\,x^2+x+1}{\left(1+x^2\right)^2 (x-1)}$ into partial fractions.

$$\frac{3\,x^4+7\,x^2+x+1}{\left(1+x^2\right)^2 (x-1)} = \frac{x+2}{(1+x^2)^2} + \frac{3}{x-1}$$

14 Integrate the quotient of the previous exercise with respect to x.

$$\int \mathrm{d}x \left(\frac{x+2}{(1+x^2)^2} + \frac{3}{x-1}\right) = \frac{4\,x-2}{4(1+x^2)} + \arctan(x) + 3\,\ln|x-1| + C$$

15 Does $\displaystyle\int_2^\infty \frac{e^{-x}}{\left(1+x^2\right)^2 (x-1)}\,\mathrm{d}x$ converge?

$\left(1+x^2\right)^2 (x-1) > 1$ for $x \in [2,\infty[$. So

$$0 \le \frac{e^{-x}}{\left(1+x^2\right)^2 (x-1)} \le e^{x}$$

and since $\int_2^\infty 0\mathrm{d}x$ and $\int_2^\infty e^x \mathrm{d}x$ converge, so does $\displaystyle\int_2^\infty \frac{e^{-x}}{\left(1+x^2\right)^2 (x-1)}\,\mathrm{d}x$.