Math 104 Midterm #3

blue=question, black=answer.

1 The **Fibonacci sequence** is defined recursively by

- $a_1 = a_2 = 1$
- $a_{n+1} = a_n + a_{n-1}$

Find the first 6 terms in the sequence.

$$a_1 = 1,\ a_2 = 1,\ a_3 = 2,\ a_4 = 3,\ a_5 = 5,\ a_6 = 8$$

2 Let $\{a_n\}$ be again the Fibonacci sequence. It can easily be seen that $a_n \geq 1\,\forall n \in \mathbb{Z}$, and hence:

- $\{a_n\}$ is a sequence of positive terms.
- The sum $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

What does that tell you about the limit $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$? Which theorem are you using?

Ratio test: If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$ then the series $\sum a_n$ converges.
Contrapositive: If $\sum a_n$ diverges then $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} \geq 1$.

3 Let $\{a_n\}$ be again the Fibonacci sequence. Assuming that the limit $L \stackrel{\text{def}}{=} \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists, show that it satisfies the equation $L = 1 + \frac{1}{L}$.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{a_n + a_{n-1}}{a_n} = \lim_{n\to\infty} \left(1 + \frac{a_{n-1}}{a_n}\right) =$$
$$= 1 + \lim_{n\to\infty} \frac{a_{n-1}}{a_n} = 1 + \frac{1}{\lim_{n\to\infty} \frac{a_n}{a_{n-1}}}$$

4 Assuming that the limit $L \stackrel{\text{def}}{=} \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ of the previous problem exists, finds its value.

We already know that $L > 1$.

$$L = 1 + \frac{1}{L} \;\Leftrightarrow\; L^2 - L - 1 = 0 \;\Leftrightarrow\; L_{1,2} = \frac{1 \pm \sqrt{5}}{2}$$

The negative root $\frac{1-\sqrt{5}}{2} < 0$ is excluded because $L > 1$. Therefore

$$L = \frac{1+\sqrt{5}}{2}$$

5 Let $\{a_n\}$ be the sequence $a_n = n\left(1 - \cos\frac{1}{n}\right)$. Find $\lim_{n\to\infty} a_n$. Hint: Use the Maclaurin series for cosine.

$$\lim_{n\to\infty} n\left(1 - \cos\frac{1}{n}\right) = \lim_{n\to\infty} n\left[1 - \left(1 - \frac{1}{2}\left(\frac{1}{n}\right)^2 + \cdots\right)\right] =$$
$$= \lim_{n\to\infty} \left(\frac{1}{2}\frac{1}{n} + \cdots\right) = 0$$

6 The following are standard series. Check the appropriate boxes:

Series	**Converges absolutely**	**Converges conditionally**	**Diverges**
$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$	☐	☐	☒
$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$	☐	☒	☐
$\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$	☒	☐	☐
$\displaystyle\sum_{n=1}^{\infty} \left(-\frac{1}{2}\right)^n$	☒	☐	☐
$\displaystyle\sum_{n=1}^{\infty} (-1)^n$	☐	☐	☒

7 Let $p_n =$ (n$^{\text{th}}$ prime number). Show that the alternating series $\sum_{n=1}^{\infty}(-1)^n \frac{1}{p_n}$ converges. Hint: Use Leibniz's Theorem.

- $\frac{1}{p_n} > 0$
- $p_{n+1} > p_n \quad \Rightarrow \quad \frac{1}{p_{n+1}} < \frac{1}{p_n}$
- $\lim_{n\to\infty} \frac{1}{p_n} = 0$

Under these conditions the alternating series converges by Leibniz's Theorem.

8 In your solution to the previous problem, what kind of convergence did you actually prove? Did you show

a). $\sum_{n=1}^{\infty}(-1)^n \frac{1}{p_n}$ converges absolutely.

b). $\sum_{n=1}^{\infty}(-1)^n \frac{1}{p_n}$ converges conditionally.

c). None of the above.

Leibniz's Theorem proves conditional convergence at most.

Here is a way to see that. Leibniz's Theorem implies the convergence of the alternating harmonic series. But the alternating harmonic series converges only conditionally, not absolutely. Hence Leibniz's Theorem proves conditional convergence at best.

Now technically, a series is conditionally convergent if it converges but does not converge absolutely. Since you did not show that $\sum \frac{1}{p_n}$ diverges, it may still converge absolutely. So really c) is the correct answer. I also gave full points if you answered b) though.

9 Find the radius of convergence and identify the function

$$f(x) = \sum_{n=0}^{\infty} nx^n = x + 2x^2 + 3x^3 + \cdots$$

Ratio test for $\sum_{n=0}^{\infty} n|x|^n$:

$$\lim_{n\to\infty} \frac{(n+1)|x|^{n+1}}{n|x|^n} = \lim_{n\to\infty} \frac{n+1}{n}|x| = |x|$$

So the series converges absolutely if $|x| < 1$ and diverges if $|x| > 1$. Hence the radius of convergence is $R = 1$.

$$\sum_{n=0}^{\infty} nx^n = x\sum_{n=0}^{\infty} nx^{n-1} = x\frac{\mathrm{d}}{\mathrm{d}x}\sum_{n=0}^{\infty} x^n = x\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{1-x} = \frac{x}{(1-x)^2}$$

10 Find the first two (nonzero) terms is the Maclaurin series for

$$f(x) = \sqrt{1+x} - e^{\frac{x}{2}}$$

$$\begin{aligned}
\sqrt{1+x} &= \sum \binom{1/2}{n} x^n = 1 + \frac{1}{2}x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2\cdot 1}x^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3\cdot 2\cdot 1}x^3 + \cdots \\
e^{\frac{x}{2}} &= \sum \frac{\left(\frac{x}{2}\right)^n}{n!} = 1 + \frac{x}{2} + \frac{1}{2}\left(\frac{x}{2}\right)^2 + \frac{1}{1\cdot 2\cdot 3}\left(\frac{x}{2}\right)^3 + \cdots \\
&\Rightarrow \sqrt{1+x} - e^{\frac{x}{2}} = -\frac{1}{4}x^2 + \frac{1}{24}x^3 + \cdots
\end{aligned}$$