

Economics 4230: Macro Modeling

Dynamic Fiscal Policy

PARTS III and IV

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Part III

The Life Cycle Model

- Generalization of the two-period model to multiple periods
- Modigliani-Ando life cycle hypothesis focuses on consumption and savings profiles as well as wealth accumulation over a household's lifetime
- Friedman's permanent income hypothesis focuses on impact of timing and characteristics of uncertain income on consumption choices.

- Household lives for T periods. We allow that $T = \infty$
- In each period t household earns after-tax income y_t and consumes c_t .
- May have initial wealth $A \geq 0$

- Period budget constraint

$$c_t + s_t = y_t + (1 + r)s_{t-1}$$

Here r denotes interest rate, s_t denotes financial assets carried over from period t to period $t + 1$ and s_{t-1} denotes assets from period $t - 1$ carried to period t .

- Net Saving in period t is defined as the difference between total income $y_t + rs_{t-1}$ and consumption c_t .

$$s_t - s_{t-1} = y_t + rs_{t-1} - c_t$$

- Period 1 budget constraint

$$c_1 + s_1 = A + y_1.$$

$$U(c_1, c_2, \dots, c_T) = u(c_1) + \beta u(c_2) + \beta^2 u(c_3) + \dots + \beta^{T-1} u(c_T)$$

or

$$U(c) = \sum_{t=1}^T \beta^{t-1} u(c_t)$$

where $c = (c_1, c_2, \dots, c_T)$ denotes the lifetime consumption profile

REWRITE THE PERIOD-BY-PERIOD BUDGET CONSTRAINTS AS A SINGLE INTERTEMPORAL BUDGET CONSTRAINT: NOTE THAT

$$c_1 + s_1 = A + y_1$$

$$c_2 + s_2 = y_2 + (1 + r)s_1$$

- Solve second equation for s_1

$$s_1 = \frac{c_2 + s_2 - y_2}{1 + r}$$

- and plug into first equation, to obtain

$$c_1 + \frac{c_2 + s_2 - y_2}{1 + r} = A + y_1$$

- which can be rewritten as

$$c_1 + \frac{c_2}{1 + r} + \frac{s_2}{1 + r} = A + y_1 + \frac{y_2}{1 + r}$$

- Repeat this procedure: from third period budget constraint

$$c_3 + s_3 = y_3 + (1 + r)s_2$$

we can solve for

$$s_2 = \frac{c_3 + s_3 - y_3}{1 + r}$$

and plug in to obtain

$$c_1 + \frac{c_2}{1 + r} + \frac{c_3}{(1 + r)^2} + \frac{s_3}{(1 + r)^2} = A + y_1 + \frac{y_2}{1 + r} + \frac{y_3}{(1 + r)^2}$$

- Continue the process T times, to arrive at the intertemporal budget constraint

$$\begin{aligned} & c_1 + \frac{c_2}{1 + r} + \frac{c_3}{(1 + r)^2} + \dots + \frac{c_T}{(1 + r)^{T-1}} + \frac{s_T}{(1 + r)^{T-1}} \\ = & A + y_1 + \frac{y_2}{1 + r} + \frac{y_3}{(1 + r)^2} \dots + \frac{y_T}{(1 + r)^{T-1}} \end{aligned}$$

- s_T denotes saving from period T to $T + 1$. Household lives only for T periods, so she has no use for saving in period $T + 1$. We don't allow $s_T < 0$. Thus $s_T = 0$ and

$$c_1 + \frac{c_2}{1+r} + \frac{c_3}{(1+r)^2} + \dots + \frac{c_T}{(1+r)^{T-1}}$$

$$= A + y_1 + \frac{y_2}{1+r} + \frac{y_3}{(1+r)^2} \dots + \frac{y_T}{(1+r)^{T-1}}$$

or

$$\sum_{t=1}^T \frac{c_t}{(1+r)^{t-1}} = A + \sum_{t=1}^T \frac{y_t}{(1+r)^{t-1}}$$

Present discounted value of lifetime consumption (c_1, \dots, c_T) equals the present discounted value of lifetime income (y_1, \dots, y_T) plus initial bequests.

- Household maximizes utility subject to budget constraint

- In order to solve this problem, need to use Lagrangian method.
- ① Rewrite all constraints of the problem in the form

$$\text{stuff} = 0$$

For our problem

$$\begin{aligned} & A + y_1 + \frac{y_2}{1+r} + \frac{y_3}{(1+r)^2} \cdots + \frac{y_T}{(1+r)^{T-1}} \\ & - c_1 - \frac{c_2}{1+r} - \frac{c_3}{(1+r)^2} - \cdots - \frac{c_T}{(1+r)^{T-1}} \\ = & 0 \end{aligned}$$

- Write down the Lagrangian: take the objective function and add all constraints, each pre-multiplied by a so-called Lagrange multiplier. This entity λ can be treated as a constant number. Lagrangian becomes

$$\begin{aligned}
 & \mathcal{L}(c_1, \dots, c_T) \\
 = & u(c_1) + \beta u(c_2) + \beta^2 u(c_3) + \dots + \beta^{T-1} u(c_T) + \\
 & \lambda \left(A + y_1 + \frac{y_2}{1+r} + \frac{y_3}{(1+r)^2} \dots + \frac{y_T}{(1+r)^{T-1}} \right. \\
 & \quad \left. - c_1 - \frac{c_2}{1+r} - \frac{c_3}{(1+r)^2} \dots - \frac{c_T}{(1+r)^{T-1}} \right) \\
 = & \sum_{t=1}^T \beta^{t-1} u(c_t) + \lambda \left(A + \sum_{t=1}^T \frac{y_t}{(1+r)^{t-1}} - \sum_{t=1}^T \frac{c_t}{(1+r)^{t-1}} \right)
 \end{aligned}$$

- Take first order conditions with respect to all choice variables and set them equal to 0. For example chose variables are (c_1, \dots, c_T)

$$u'(c_1) - \lambda = 0 \quad \text{or} \quad u'(c_1) = \lambda.$$

Doing the same for c_2 yields

$$\beta u'(c_2) - \lambda \frac{1}{1+r} = 0 \quad \text{or} \quad (1+r)\beta u'(c_2) = \lambda$$

and for an arbitrary c_t we find $(1+r)^{t-1}\beta^{t-1}u'(c_t) = \lambda$. Combining

$$\begin{aligned} u'(c_1) &= (1+r)\beta u'(c_2) \\ &= \dots = [(1+r)\beta]^{t-1} u'(c_t) = [(1+r)\beta]^t u'(c_{t+1}) \\ &= \dots = [(1+r)\beta]^{T-1} u'(c_T) \end{aligned}$$

These equations determine relative consumption levels across periods, that is, the ratios $\frac{c_2}{c_1}$, $\frac{c_3}{c_2}$ and so forth. For absolute consumption levels need to use the budget constraint.

- For $t = 1$, $u'(c_1) = (1 + r) \beta u'(c_2)$.
- If consume a little less in period 1, and save the amount to consume a bit extra in the second period, then the utility cost is $-u'(c_1)$ and the benefit is $(1 + r) \beta u'(c_2)$. Thus entire utility consequences from saving a little more today and eating it tomorrow are

$$-u'(c_1) + (1 + r) \beta u'(c_2) \leq 0$$

because the household should not be able to improve his lifetime utility from doing so.

- Similar argument for consuming one unit more today and saving one unit less leads to

$$-u'(c_1) + (1 + r) \beta u'(c_2) \geq 0.$$

- Combining the two equations leads to the Euler equation.

- Suppose the market discounts income at the same rate $\frac{1}{1+r}$ as the household discounts utility, β . In this case $\beta = \frac{1}{1+r}$ or $\beta(1+r) = 1$. Euler equation becomes

$$u'(c_1) = u'(c_2) = \dots = u'(c_t) = \dots = u'(c_T)$$

- Since utility function is strictly concave (i.e. $u''(c) < 0$) we have that

$$c_1 = c_2 = \dots = c_t = \dots = c_T = \bar{c}$$

Consumption is constant over a households' lifetime; the timing of income and consumption is completely de-coupled.

- Consumption level: from the intertemporal budget

$$\sum_{t=1}^T \frac{c_t}{(1+r)^{t-1}} = I$$

- Since $c_t = \bar{c}$ for all times t we have:

$$\bar{c} \sum_{t=1}^T \frac{1}{(1+r)^{t-1}} = I$$
$$\bar{c} = \frac{1}{\sum_{t=1}^T \frac{1}{(1+r)^{t-1}}} * I$$

- The term $\sum_{t=1}^T \frac{1}{(1+r)^{t-1}}$ annuitizes a constant stream of consumption.

- Note:

$$\sum_{t=1}^T \frac{1}{(1+r)^{t-1}} = \begin{cases} \frac{1+r - \frac{1}{(1+r)^{T-1}}}{r} & \text{if } T < \infty \\ \frac{1+r}{r} & \text{if } T = \infty \end{cases}$$

- Thus, if households are infinitely lived:

$$\bar{c} = c_1 = c_t = \frac{r}{1+r} I$$

- Household lives 60 years, from age 1 to age 60
- Household inherits nothing, i.e. $A = 0$.
- In the first 45 years of life, household works and makes annual income of \$40,000 per year. For last 15 years of her life household is retired and earns nothing
- We assume that the interest rate is $r = 0$ and $\beta = 1$.

- From previous discussion we know that consumption over the households' lifetime is constant

$$c_1 = c_2 = \dots = c_{60} = c$$

- Level of consumption? Total discounted lifetime value of income.

$$\begin{aligned} & y_1 + \frac{y_2}{1+r} + \frac{y_3}{(1+r)^2} \dots + \frac{y_{60}}{(1+r)^{T-1}} \\ = & y_1 + y_2 + y_3 \dots + y_{60} = y_1 + y_2 + y_3 \dots + y_{45} \\ = & 45 * \$40,000 = \$1,800,000 \end{aligned}$$

- Total discounted lifetime cost of consumption

$$\begin{aligned} & c_1 + \frac{c_2}{1+r} + \frac{c_3}{(1+r)^2} + \dots + \frac{c_{60}}{(1+r)^{59}} \\ = & c_1 + c_2 + \dots + c_{60} = 60 * c \end{aligned}$$

- Equating lifetime income and cost of lifetime consumption yields

$$\begin{aligned}c &= \frac{45}{60} * \$40,000 \\ &= \$30,000\end{aligned}$$

- In all working years the household consumes \$10,000 less than income and puts the money aside for consumption in retirement.

- Savings in all working periods is

$$\begin{aligned} sav_t &= y_t + rs_{t-1} - c_t \\ &= y_t - c_t \\ &= \$40,000 - \$30,000 \\ &= \$10,000 \end{aligned}$$

whereas for all retirement periods

$$\begin{aligned} sav_t &= y_t + rs_{t-1} - c_t \\ &= -c_t \\ &= -\$30,000 \end{aligned}$$

- Asset position of the household. Remember that

$$sav_t = s_t - s_{t-1} \text{ or}$$

$$s_t = s_{t-1} + sav_t$$

Since the household starts with 0 bequests, $s_0 = 0$. Thus

$$s_1 = s_0 + sav_1$$

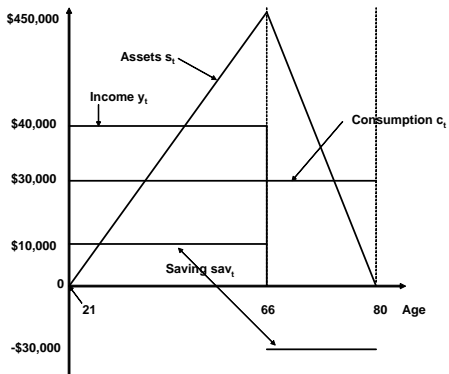
$$= \$0 + \$10,000 = \$10,000$$

$$s_2 = s_1 + sav_2 = \$10,000 + \$10,000 = \$20,000$$

$$s_{45} = s_{44} + sav_{45} = \$440,000 + \$10,000 = \$450,000$$

$$s_{46} = s_{45} + sav_{46} = \$450,000 - \$30,000 = \$420,000$$

$$s_{60} = s_{59} + sav_{60} = \$30,000 - \$30,000 = \$0$$



Life Cycle Profiles, Model

TWO PERIODS AND LOG-UTILITY

- If $\beta \neq \frac{1}{1+r}$, we need stronger assumptions on the utility function to make more progress.
- Two periods and utility function $u(c) = \log(c)$.
- Euler equation becomes

$$\frac{1}{c_1} = \frac{(1+r)\beta}{c_2} \quad \text{or} \quad c_2 = (1+r)\beta c_1$$

- Combining this with the intertemporal budget constraint

$$c_1 + \frac{c_2}{1+r} = A + y_1 + \frac{y_2}{1+r}$$

yields

$$\begin{aligned} c_1 &= \frac{I}{1+\beta} \\ c_2 &= \frac{(1+r)\beta}{1+\beta} I \end{aligned}$$

- If $\beta = \frac{1}{1+r}$, consumption over the life cycle is constant.
- Now suppose $\beta > \frac{1}{1+r}$ or $\beta(1+r) > 1$.
- From Euler equations we have

$$\begin{aligned}u'(c_1) &= (1+r)\beta u'(c_2) \\ &= \dots = [(1+r)\beta]^{t-1} u'(c_t) = [(1+r)\beta]^t u'(c_{t+1}) \\ &= \dots = [(1+r)\beta]^{T-1} u'(c_T)\end{aligned}$$

- This implies $\frac{u'(c_1)}{u'(c_2)} = (1+r)\beta$. and thus

$$\begin{aligned}\frac{u'(c_1)}{u'(c_2)} &> 1 \\ u'(c_1) &> u'(c_2)\end{aligned}$$

- Since $u'(c)$ is a strictly decreasing function we have $c_1 < c_2$.

- Similarly

$$\begin{aligned} [(1+r)\beta]^{t-1} u'(c_t) &= [(1+r)\beta]^t u'(c_{t+1}) \\ \frac{u'(c_t)}{u'(c_{t+1})} &= \frac{[(1+r)\beta]^t}{[(1+r)\beta]^{t-1}} = (1+r)\beta > 1 \end{aligned}$$

so that $c_{t+1} > c_t$.

- Now suppose that $\beta < \frac{1}{1+r}$ or $\beta(1+r) < 1$.
- Identical argument to the one above shows that now

$$c_1 > c_2 > \dots > c_t > \dots > c_T.$$

- Consider the specific CRRA period utility function

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}$$

- Note: for $\sigma = 1$ this utility function becomes

$$u(c) = \log(c)$$

- In this case

$$u'(c) = c^{-\sigma}$$

- Euler equations

$$(c_1)^{-\sigma} = (1+r)\beta(c_2)^{-\sigma} = [(1+r)\beta]^{t-1} (c_t)^{-\sigma} = [(1+r)\beta]^t (c_{t+1})^{-\sigma}$$

- Thus for any period t

$$\begin{aligned} [(1+r)\beta]^{t-1} (c_t)^{-\sigma} &= [(1+r)\beta]^t (c_{t+1})^{-\sigma} \\ (c_t)^{-\sigma} &= [(1+r)\beta] (c_{t+1})^{-\sigma} \\ \left(\frac{c_{t+1}}{c_t}\right)^{\sigma} &= (1+r)\beta \\ \frac{c_{t+1}}{c_t} &= [(1+r)\beta]^{\frac{1}{\sigma}} \end{aligned}$$

- Consumption levels: note that

$$\begin{aligned}c_{t+1} &= [(1+r)\beta]^{\frac{1}{\sigma}} c_t = [(1+r)\beta]^{\frac{2}{\sigma}} c_{t-1} = \dots [(1+r)\beta]^{\frac{t}{\sigma}} c_1 \text{ or} \\c_t &= [(1+r)\beta]^{\frac{t-1}{\sigma}} c_1\end{aligned}$$

- Intertemporal budget constraint

$$\sum_{t=1}^T \frac{c_t}{(1+r)^{t-1}} = I$$

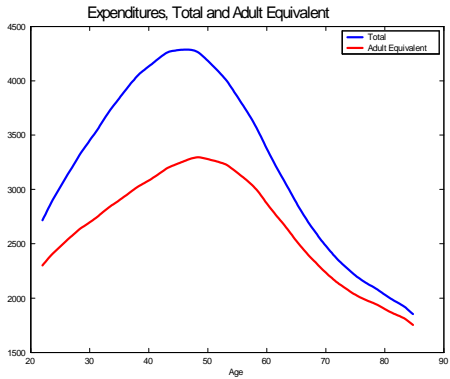
- Plugging in for c_t yields

$$\sum_{t=1}^T \frac{[(1+r)\beta]^{\frac{t-1}{\sigma}} c_1}{(1+r)^{t-1}} = I$$

- Solving this out for c_1 yields

$$c_1 = \frac{1 - \beta^{\frac{1}{\sigma}} (1+r)^{\frac{1}{\sigma}-1}}{1 - \left[\beta^{\frac{1}{\sigma}} (1+r)^{\frac{1}{\sigma}-1} \right]^T} * I$$
$$c_t = [(1+r)\beta]^{\frac{t-1}{\sigma}} * \frac{1 - \beta^{\frac{1}{\sigma}} (1+r)^{\frac{1}{\sigma}-1}}{1 - \left[\beta^{\frac{1}{\sigma}} (1+r)^{\frac{1}{\sigma}-1} \right]^T} * I$$

- Main predictions of the model: consumption should be smooth over the life cycle.
- Assets should display a hump, increasing until retirement and then declining
- In data
 - disposable income follows a hump over the life cycle, with a peak around the age of 45
 - consumption follows a hump over the life cycle



Consumption over the Life Cycle

- Theoretical prediction: consumption is either monotonically upward trending, monotonically downward trending or perfectly flat over the life cycle.

- Data: consumption is hump-shaped over the life cycle (as is income)

- How can we account for the difference?

- Changes in household size and household composition
- Family size also is hump-shaped over the life cycle
- Life cycle model only asserts that marginal utility of consumption should be smooth over the life cycle, not necessarily consumption expenditures themselves.
- But: if adjust data by household equivalence scales, still 50% of the hump persists

- Households spend resources to be able to work:
 - Commuting
 - Working Clothes

- Households can either produce a good at home or buy it. When there is more time available they produce it:
 - Household Chores: cooking, cutting grass, shoveling
 - Tax filing

- Retired Households spend more time shopping hence they have more time:
- As a consequence they purchase consumption goods cheaper
- So expenditures may be lower but consumption is actually not lower

- Same predictions as before if consumption and leisure are separable in the utility function,

$$U(c, l) = \sum_{t=1}^T \beta^{t-1} [u(c_t) + v(l_t)]$$

- But if consumption and leisure are substitutes, then if labor supply is hump-shaped over the live cycle (because labor productivity is), then households may find it optimal to have a hump-shaped labor supply and consumption profile over the life cycle.

- End of Life Can be very Expensive
- Long term Care: Retirement Home.
- In the absence of Annuities Closeness to Death Makes End of Life Scary.

- Declining consumption profile over the life cycle can be explained by $\beta(1+r) < 1$.
- If young households can't borrow against their future labor income, then best thing they can do is to consume whatever income they have when young. Since income is increasing in young ages, so is consumption.
- As households age they want to start saving and the borrowing constraints lose importance. But now the fact that $\beta(1+r) < 1$ kicks in and induces consumption to fall.

- Uncertain life time acts as additional discount factor, make consumption fall when probability of dying increases.
- Uncertain income induces precautionary savings behavior (as long as $u'''(c) > 0$). As more and more uncertainty is resolved, households start to save less for precautionary reasons and save more.
- Combination of changes in household size and income and lifetime uncertainty can generate a hump in consumption over the life cycle of similar magnitude and timing as in the data (see Attanasio et al., 1999).

- Now assume that incomes $\{y_1, y_2, \dots, y_T\}$ are risky.
- Only extra concept required is the conditional expectation E_t of an economic variable that is uncertain.
- Thus, $E_t y_{t+1}$ is expectation in period t of income in period $t + 1$, $E_t y_{t+2}$ is period t expectation of income in period $t + 2$ etc. Timing convention: when expectations are taken in t , y_t is known.

- Consider various possible events tomorrow ω_{t+1} each with probability $p(\omega_{t+1})$:
$$\sum_{\omega_{t+1}} p(\omega_{t+1}) = 1$$
- Associated to each ω_{t+1} we have random variables $c_{t+1}(\omega_{t+1})$, $y_{t+1}(\omega_{t+1})$.
- The Expected value of c_{t+1} is $E_t\{c_{t+1}\} = \sum_{\omega_{t+1}} p(\omega_{t+1}) c_{t+1}$.
- An agent maximizes the utility between today and tomorrow

$$\max_{s_t} \quad u(c_t) + E_t \{u[c_{t+1}(\omega_{t+1})]\} = u(c_t) + \sum_{\omega_{t+1}} p(\omega_{t+1}) u[c_{t+1}(\omega_{t+1})] \quad \text{s.t.}$$

$$c_t = y_t - s_t$$

$$c_{t+1}(\omega_{t+1}) = s_t + y_{t+1}(\omega_{t+1})$$

- With First Order Condition

$$u'(c_t) = \sum_{\omega_{t+1}} p(\omega_{t+1}) u'[c_{t+1}(\omega_{t+1})] = E_t u'[c_{t+1}(\omega_{t+1})]$$

- Assume interest rate r is not random. Also assume lifetime horizon of the household is infinite, $T = \infty$. Generalization of Euler equation

$$u'(c_t) = \beta (1 + r) E_t u'(c_{t+1})$$

- Since income in period $t + 1$ is risky from the perspective of period t , so is consumption c_{t+1} .
- Main problem for analysis: in general cannot pull the expectation into the marginal utility function, since in general

$$E_t u'(c_{t+1}) \neq u'(E_t c_{t+1})$$

- But now assume that the utility function is quadratic:

$$u(c_t) = -\frac{1}{2} (c_t - \bar{c})^2$$

- \bar{c} is the bliss level of consumption, assumed so large that given the household's lifetime income this consumption level cannot be attained.
- For all consumption levels $c_t < \bar{c}$ we have

$$\begin{aligned}u'(c_t) &= -(c_t - \bar{c}) = \bar{c} - c_t > 0 \\u''(c_t) &= -1 < 0\end{aligned}$$

- For all consumption levels $c_t < \bar{c}$ we have

$$u'(c_t) = -(c_t - \bar{c}) = \bar{c} - c_t > 0$$

$$u''(c_t) = -1 < 0$$

- Thus this utility function is strictly increasing and strictly concave for all $c_t < \bar{c}$.
- Recall a household with strictly concave utility function is risk averse

- Euler equation becomes

$$-(c_t - \bar{c}) = -E_t(c_{t+1} - \bar{c})$$

Thus

$$E_t c_{t+1} = c_t$$

- Households arrange consumption such that, in expectation, it stays constant between today and tomorrow.
- But: in presence of income risk realized consumption c_{t+1} in period $t + 1$ might deviate from this plan.

- In order to determine the level of consumption we need the intertemporal budget constraint:

$$E_t \sum_{s=0}^{\infty} \frac{c_{t+s}}{(1+r)^s} = (1+r)s_{t-1} + E_t \sum_{s=0}^{\infty} \frac{y_{t+s}}{(1+r)^s}$$

- Euler equation implies (by law of iterated expectations) that

$$E_t c_{t+1} = c_t$$

$$E_t c_{t+2} = E_t E_{t+1} c_{t+2} = E_t c_{t+1} = c_t$$

$$E_t c_{t+s} = c_t$$

- Left hand side of intertemporal budget constraint:

$$E_t \sum_{s=0}^{\infty} \frac{c_{t+s}}{(1+r)^s} = \sum_{s=0}^{\infty} \frac{E_t c_{t+s}}{(1+r)^s} =$$
$$c_t \sum_{s=0}^{\infty} \frac{1}{(1+r)^s} = \frac{1}{1 - \frac{1}{1+r}} c_t = \frac{1+r}{r} c_t.$$

- Optimal consumption rule:

$$c_t = \frac{r}{1+r} \left((1+r)s_{t-1} + E_t \sum_{s=0}^{\infty} \frac{y_{t+s}}{(1+r)^s} \right)$$

- For period 1, thus consumption becomes

$$c_1 = \frac{r}{1+r} \left(A + E_1 \sum_{s=0}^{\infty} \frac{y_{1+s}}{(1+r)^s} \right) = \frac{r}{1+r} E_1 I$$

- Compare this to the certainty case

$$c_1 = \frac{r}{1+r} I$$

- Both expressions: optimal consumption rules are exactly alike: in both cases the household consumes permanent income!

- Surprising result: despite presence of income risk the household makes the same planned consumption choices as in the absence of risk. Called certainty equivalence behavior

- Households do *not* engage in *precautionary savings behavior* by saving more in the presence than in the absence of future income risk: only expected future income matters for planned consumption, not income risk.

- This is true despite household risk aversion.

- *Realized* consumption in period $t + 1$ will in general deviate from $E_t c_{t+1} = c_t$
- Realized change in consumption between period t and $t + 1$ is given by

$$c_{t+1} - c_t = \frac{r}{1+r} \sum_{s=0}^{\infty} \frac{E_{t+1} y_{t+1+s} - E_t y_{t+1+s}}{(1+r)^s}$$

- Realized change in consumption given by annuity value $\frac{r}{1+r}$ of the sum of discounted *revisions in expectations* about future income in periods $t + 1 + s$, that is, $E_{t+1} y_{t+1+s} - E_t y_{t+1+s}$.

- How large are realized changes in consumption? Depends crucially on type of income shock the household experiences between the two periods.

- Consider two examples: perfectly permanent shock (unexpected but permanent promotion) and fully transitory shock (unexpected one-time bonus).

- Permanent promotion: extra income p for rest of households' life. Since unexpected in period t , for *all* future periods

$$E_{t+1}y_{t+1+s} - E_t y_{t+1+s} = p.$$

- Thus

$$c_{t+1} - c_t = \frac{r}{1+r} \sum_{s=0}^{\infty} \frac{p}{(1+r)^s} = p \frac{r}{1+r} \frac{1}{1 - \frac{1}{1+r}} = p$$

- Consumption goes up by full amount of the unexpected but permanent income increase between period t and $t + 1$.

- Now consider a one time unexpected bonus b in period $t + 1$. Then

$$E_{t+1}y_{t+1} - E_t y_{t+1} = b$$

and for all future periods beyond $t + 1$

$$E_{t+1}y_{t+1+s} - E_t y_{t+1+s} = 0.$$

- Then

$$c_{t+1} - c_t = \frac{r}{1+r} \sum_{s=0}^{\infty} \frac{b}{(1+r)^s} = \frac{r}{1+r} b$$

- Realized consumption change

$$c_{t+1} - c_t = \frac{r}{1+r} b$$

- Increase in consumption is only $\frac{r}{1+r}$ of the bonus (of about 2% if the real interest rate is $r = 2\%$).
- Instead, most of the bonus is saved and used to increase consumption in *all* future periods by a small bit.

WHAT IF PREFERENCES ARE NOT QUADRATIC, BUT LIKE LOGS?

- Example: Two periods. Income in first is 1. Income in second is $1 + \ell$ with probability .5 and $1 - \ell$ with probability .5. Log utility. $1 + r = 1$.

$$\max_{c_1, s, c_{2g}, c_{2b}} \log c_1 + \frac{1}{2} \log c_{2g} + \frac{1}{2} \log c_{2b}$$

$$c_1 + s = 1$$

$$c_{2g} = s + 1 + \ell$$

$$c_{2b} = s + 1 - \ell$$

- Rewriting after substitution

$$\max_{c_1, s, c_{2g}, c_{2b}} \log(1 - s) + \frac{1}{2} \log(s + 1 + \ell) + \frac{1}{2} \log(s + 1 - \ell)$$

WHAT IF PREFERENCES ARE NOT QUADRATIC, BUT LIKE LOGS?

- First order conditions (absent algebra errors)

$$\frac{-1}{1-s} + \frac{1}{2} \frac{1}{s+1+\ell} + \frac{1}{2} \frac{1}{s+1-\ell} = 0$$

- Simplifying

$$\frac{1}{1-s} = \frac{1+s}{s^2 + 2s + 1 - \ell^2}$$

$$(s+1)^2 - \ell^2 = (1+s)(1-s)$$

$$2s^2 + 2s - \ell^2 = 0$$

$$s = \frac{-2 + \sqrt{4 + 8\ell^2}}{4}$$

- Note that if $\ell = 0$ then $s = 0$ while if $\ell = 1$ then $s = .36$.
- The higher the variance (here ℓ) the higher the savings

WHAT IF PREFERENCES ARE NOT QUADRATIC, BUT LIKE LOGS?

- In $u'''(c) > 0$ then agents have precautionary savings, this is they save more the higher the risk that they save.
- When $\beta(1+r) = 1$ then $c_t < E[c_{t+1}]$
- People save extra not so much for a rainy day but for wild weather.
- People are more like this than like quadratic preferences.

Part IV

Positive Theory of Government Activity

- So far: analysis of individual household behavior
- Now: introduction of government activity: taxation, transfers, government spending, issuing and repaying debt
- Question 1: What are the constraints the government faces?
- Question 2: How do government policies affect private household decisions?

2023 (Estimate) Federal Budget (in billion \$)	
Receipts	4,638.2
Individual Income Taxes	2,345.2
Corporate Income Taxes	500.9
Social Insurance Receipts	1,509.9
Excise Taxes	90.7
Other	191.5
Outlays	5,792.0 (1,186.7 off budget)
National Defense	808.6
International Affairs	63.4
Health	782.4
Medicare	854.5
Income Security	688.2
Social Security	1,318.7
Net Interest	395.5
Other	880.7
Surplus	-1,153.8

- Government Expenditures

$$G_t = \text{Defense} + \text{International Affairs} + \text{Health} + \text{Other Outlays}$$

- Net Taxes

$$T_t = \text{Taxes} + \text{Social Insurance Receipts} + \text{Other Receipts} \\ - \text{Medicare} - \text{Social Security} - \text{Income Security}$$

- Interest on government debt: $rB_{t-1} = \text{Net Interest}$

- Denote by $t = 1$ the first period a country exists. Budget constraint of the government reads as

$$G_1 = T_1 + B_1$$

- For an arbitrary period t , the government budget constraint reads as

$$G_t + (1 + r)B_{t-1} = T_t + B_t$$

- For simplicity we assume that all government bonds have a maturity of one period.

- Rewrite budget constraint as

$$G_t - T_t + rB_{t-1} = B_t - B_{t-1}$$

- Primary government deficit: $G_t - T_t$
- Total government deficit: $def_t = G_t - T_t + rB_{t-1}$.
- Note that

$$def_t = B_t - B_{t-1}$$

- For $t = 2$, budget constraint reads as

$$G_2 + (1+r)B_1 = T_2 + B_2 \quad \text{or} \quad B_1 = \frac{T_2 + B_2 - G_2}{1+r}$$

- Plug this into budget constraint for period 1 to get

$$\begin{aligned} G_1 &= T_1 + \frac{T_2 + B_2 - G_2}{1+r} \\ G_1 + \frac{G_2}{1+r} &= T_1 + \frac{T_2}{1+r} + \frac{B_2}{1+r} \end{aligned}$$

- Continue this process to

$$\begin{aligned} &G_1 + \frac{G_2}{1+r} + \frac{G_3}{(1+r)^2} + \dots + \frac{G_T}{(1+r)^{T-1}} \\ &= T_1 + \frac{T_2}{1+r} + \frac{T_3}{(1+r)^2} + \dots + \frac{T_T}{(1+r)^{T-1}} + \frac{B_T}{(1+r)^{T-1}} \end{aligned}$$

CONSOLIDATION OF GOVERNMENT BUDGET CONSTRAINT II

- Assume that even the government cannot die in debt:

$$\begin{aligned} & G_1 + \frac{G_2}{1+r} + \frac{G_3}{(1+r)^2} + \dots + \frac{G_T}{(1+r)^{T-1}} \\ = & T_1 + \frac{T_2}{1+r} + \frac{T_3}{(1+r)^2} + \dots + \frac{T_T}{(1+r)^{T-1}} \end{aligned}$$

or more compactly

$$\sum_{t=1}^T \frac{G_t}{(1+r)^{t-1}} = \sum_{t=1}^T \frac{T_t}{(1+r)^{t-1}}$$

- If country lives forever, government budget constraint becomes

$$\sum_{t=1}^{\infty} \frac{G_t}{(1+r)^{t-1}} = \sum_{t=1}^{\infty} \frac{T_t}{(1+r)^{t-1}}$$

- Present discounted value of total government expenditures equals present discounted value of total taxes.

- Question: How should the government finance a war?
- Two principal ways to levy revenues for a government
 - Tax in the current period
 - Issue government debt, the interest and principal of which has to be paid via taxes in the future.
- What are the macroeconomic consequences of using these different instruments, and which instrument is to be preferred from a normative point of view?

- Ricardian Equivalence: it makes no difference. A switch from taxing today to issuing debt and taxing tomorrow does not change real allocations and prices in the economy.

- Origin: David Ricardo (1772-1823).

- His question: how to finance a war with annual expenditures of \$20 millions. Asked whether it makes difference to finance the \$20 millions via current taxes or to issue government bonds with infinite maturity (so-called consols) and finance the annual interest payments of \$1 million in all future years by future taxes (at an assumed interest rate of 5%).

- His conclusion was (in “Funding System”) that

in the point of the economy, there is no real difference in either of the modes; for twenty millions in one payment [or] one million per annum for ever ... are precisely of the same value

- Ricardo formulates and explains the equivalence hypothesis, but is sceptical about its empirical validity

...but the people who pay the taxes never so estimate them, and therefore do not manage their affairs accordingly. We are too apt to think, that the war is burdensome only in proportion to what we are at the moment called to pay for it in taxes, without reflecting on the probable duration of such taxes. It would be difficult to convince a man possessed of \$20,000, or any other sum, that a perpetual payment of \$50 per annum was equally burdensome with a single tax of \$1,000.

- Ricardo doubts that agents are as rational as they should, according to “in the point of the economy”, or that they rationally believe not to live forever and hence do not have to bear part of the burden of the debt. Since Ricardo didn't believe in the empirical validity of the theorem, he has a strong opinion about which financing instrument ought to be used to finance the war

war-taxes, then, are more economical; for when they are paid, an effort is made to save to the amount of the whole expenditure of the war; in the other case, an effort is only made to save to the amount of the interest of such expenditure.

- Suppose the world only lasts for two periods
- Government has to finance a war in the first period. The war costs G_1 pounds. Assume that government does not do any spending in the second period, so that $G_2 = 0$.
- Question: does it makes a difference whether the government collects taxes for the war in period 1 or issues debt and repays the debt in period 2?

- Budget constraints for the government

$$\begin{aligned}G_1 &= T_1 + B_1 \\(1+r)B_1 &= T_2\end{aligned}$$

where we used the fact that $G_2 = 0$ and $B_2 = 0$

- Policy A: Immediate taxation: $T_1^A = G_1$ and $B_1 = T_2^A = 0$
- Policy B: Debt issue, to be repaid tomorrow: $T_1^B = 0$ and $B_1^B = G_1$, $T_2^B = (1+r)B_1^B = (1+r)G_1$.
- Note that both policies satisfy the intertemporal government budget constraint

$$G_1 = T_1^i + \frac{T_2^i}{1+r} \quad \text{for } i \in \{A, B\}$$

- Individual behavior: Household maximizes utility

$$u(c_1) + \beta u(c_2)$$

- subject to the lifetime budget constraint

$$c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r} + A$$

where y_1 and y_2 are the after-tax incomes in the first and second period of the households' life.

- Let

$$y_1 = e_1 - T_1$$

$$y_2 = e_2 - T_2$$

where e_1, e_2 are the pre-tax earnings of the household and T_1, T_2 are taxes paid by the household.

- Government policies only affect after tax incomes. But

$$c_1 + \frac{c_2}{1+r} = e_1 - T_1 + \frac{e_2 - T_2}{1+r} + A$$

$$c_1 + \frac{c_2}{1+r} + T_1 + \frac{T_2}{1+r} = e_1 + \frac{e_2}{1+r} + A$$

- Household spends present discounted value of pre-tax income $e_1 + \frac{e_2}{1+r} + A$ on present discounted value of consumption $c_1 + \frac{c_2}{1+r}$ and present discounted value of income taxes.
- Two tax-debt policies that imply the same present discounted value of lifetime taxes therefore lead to exactly the same lifetime budget constraint and thus exactly the same individual consumption choices.

- For immediate taxation (policy A) we have $T_1^A = G_1$ and $T_2^A = 0$, and thus
$$T_1^A + \frac{T_2^A}{1+r} = G_1$$
- For debt issue (policy B) we have $T_1^B = 0$ and $T_2^B = (1+r)G_1$, and thus
$$T_1^B + \frac{T_2^B}{1+r} = G_1$$
- Both policies imply the same present discounted value of lifetime taxes for the household. Present discounted value of taxes is not changed.
- Consumption choices do not change, but savings choices do.
- Period by period budget constraints

$$c_1 + s = e_1 - T_1$$

$$c_2 = e_2 - T_2 + (1+r)s$$

- Let (c_1^*, c_2^*) be the optimal consumption choices in the two periods and let s^{*A} denote the optimal saving under policy A.
- Savings choice s^{*B} denote the policy B saving policy. Thus

$$\begin{aligned} c_1^{*A} &= e_1 - T_1^A - s^{*A} \\ &= e_1 - T_1^B - s^{*B} \end{aligned}$$

- Thus

$$\begin{aligned} e_1 - T_1^A - s^{*A} &= e_1 - s^{*B} \\ s^{*B} &= s^{*A} + T_1^A. \end{aligned}$$

- Under policy B the household saves exactly T_1^A more than under the first policy, the full extent of the tax reduction from the second policy. This extra saving T_1^A yields $(1+r)T_1^A$ extra income in the second period, exactly enough to pay the taxes T_2^B levied in the second period by the government to repay its debt.

Theorem

(Ricardian Equivalence) A policy reform that does not change government spending (G_1, \dots, G_T), and only changes the timing of taxes, but leaves the present discounted value of taxes paid by each household in the economy has no effect on aggregate consumption in any time period.

- Key Assumption 1: No Borrowing Constraint
- Key Assumption 2: No Redistribution of the Burden of Taxes
- Key Assumption 3: Lump Sum Taxation

- Binding borrowing constraints can lead a household to change her consumption choices, even if a change in the timing of taxes does not change her discounted lifetime income.
- Proof by example: French British war; costs \$100 per person.
- Utility function

$$\log(c_1) + \log(c_2)$$

and pre-tax income of \$1,000 in both periods of their life.

- For simplicity $r = 0$.
- Policy 1: tax \$100 in the first period
- Policy 2: incur \$100 in government debt, to be repaid in the second period. Since $r = 0$, government has to repay \$100 in the second period

- Without borrowing constraints we know from general theorem that the two policies have identical consequences. Under both policies discounted lifetime income is \$1,900 and

$$c_1 = c_2 = \frac{1,900}{2} = 950$$

- With borrowing constraints: policy 1

$$c_1 = y_1 = 900 \text{ and } c_2 = y_2 = 1000$$

- Second policy

$$c_1 = c_2 = 950$$

- If households are borrowing constrained, current taxes have stronger effects on current consumption than the issuing of debt, since postponing taxes to the future relaxes borrowing constraints.

- Policy *A*: levy the \$100 cost per person by taxing everybody \$100 in period 1
- Policy *B*: issue government debt of \$100 and to repay simply the interest on that debt. Under that households face taxes of $T_2^B = \$5$, $T_3^B = \$5$ and so forth.
- For a person born in period 1: under policy *A*, her present discounted value of lifetime income is

$$I = \$1000 - \$100 + \frac{\$1000}{1.05} = 1852.38$$

and under policy *B* it is

$$I = \$1000 + \frac{\$995}{1.05} = 1947.6$$

- Under policy *A* consumption equals

$$c_1^A = 926.2$$

$$c_2^A = 972.5$$

and under policy *B* it equals

$$c_1^B = 973.8$$

$$c_2^B = 1022.5$$

- Under policy *B*, part of the cost of the war is borne by future generations that inherit the debt from the war, at least the interest on which has to be financed via taxation.

- Ricardian equivalence was thought to be an empirically irrelevant theorem because timing of taxes always shifts tax burden across generations.
- Robert Barro (1974) resurrected debate.
- Step 1: if households live forever, Ricardian equivalence holds.
- Consider two arbitrary government tax policies. Since we keep G_t fixed in every period, the intertemporal budget constraint

$$\sum_{t=1}^{\infty} \frac{G_t}{(1+r)^{t-1}} = \sum_{t=1}^{\infty} \frac{T_t}{(1+r)^{t-1}}$$

requires that the two tax policies have the same present discounted value.

- Without borrowing constraints only the present discounted value of lifetime after-tax income matters for a household's consumption choice. But since the present discounted value of taxes is the same under the two policies it follows that present discounted value of after-tax income is unaffected by the switch from one tax policy to the other. Private decisions thus remain unaffected, therefore all other economic variables in the economy remain unchanged by the tax change. Ricardian equivalence holds.

Do HOUSEHOLDS LIVE FOREVER?

- Step 2: argue that households live forever. Key: bequests.
- Suppose that people live for one period and have utility function

$$U(c_1) + \beta V(b_1)$$

where V is the maximal lifetime utility of children with bequests b .

- Now parameter β measures intergenerational altruism. A value of $\beta > 0$ indicates that you are altruistic, a value of $\beta < 1$ indicates that you love your children not as much as you love yourself.
- Budget constraint

$$c_1 + b_1 = y_1$$

- Bequests are constrained to be non-negative, that is $b_1 \geq 0$.

Do HOUSEHOLDS LIVE FOREVER? II

- Utility function of child is given by

$$U(c_2) + \beta V(b_2)$$

and the budget constraint is

$$c_2 + b_2 = y_2 + (1 + r)b_1$$

- Note that $V(b_1)$ equals the maximized value of $U(c_2) + \beta V(b_2)$
- Economy with one-period lived people that are linked by altruism and bequests is identical to economy with people that live forever and face borrowing constraints (since we have that bequests $b_1 \geq 0$, $b_2 \geq 0$ and so forth).
- But: binding borrowing constraints invalidate Ricardian equivalence.
- Conclusion: in Barro model with one-period lived individuals Ricardian equivalence holds if a) individuals are altruistic ($\beta > 0$) and bequest motives are operative.

- A lump-sum tax is a tax that does not change the relative price between two goods that are chosen by private households.
- Demonstrate that timing of taxes is not irrelevant if the government does not have access to lump-sum taxes by example
- Utility function

$$\log(c_1) + \log(c_2)$$

- Income before taxes of \$1000 in each period and $r = 0$. The war costs \$100.
- Policy A: levy a \$100 tax on first period labor income.
- Policy B: issue \$100 in debt, repaid in the second period with proportional consumption taxes at rate τ .

- Under policy A optimal consumption choice is

$$c_1^A = c_2^A = \$950$$

$$s^A = \$900 - \$950 = -\$50$$

- The two budget constraints under policy B read as

$$c_1^B + s = \$1000$$

$$c_2^B(1 + \tau) = \$1000 + s$$

which can be solved for using $c_1^B + (1 + \tau)c_2 = \2000

- Maximizing utility subject to the lifetime budget constraint yields

$$c_1^B = \$1000$$

$$c_2^B = \frac{\$1000}{1 + \tau}$$

- Under policy B the households consumes strictly more than under the first policy. Reason: tax on second period consumption makes consumption in the second period more expensive, relative to consumption in the first period. Households substitute away from the now more expensive good.

- Fact that the tax changes the effective relative price between the two goods qualifies this tax as a non-lump-sum tax.

- Government must levy \$100 in taxes. Tax revenues are given by

$$\tau c_2^B = \frac{\tau 1000}{1 + \tau} = 100$$

- Thus

$$\begin{aligned}\tau &= \frac{0.1}{0.9} = 0.1111 \\ c_2^B &= 900 \\ s^B &= 0\end{aligned}$$

- Households prefer the lump-sum way of financing the war to the distortionary way:

$$\log(950) + \log(950) > \log(1000) + \log(900).$$

- Report by Jagadeesh Gokhale from 2013, *Spending Beyond our Means: How We are Bankrupting Future Generations*
- The book is
<https://object.cato.org/sites/cato.org/files/pubs/pdf/spending-beyond-our-means.pdf>
- A very short description and required reading is
https://www.cato.org/sites/cato.org/files/articles/gokhale-generational_accounting.pdf

THE FISCAL SITUATION OF THE U.S.

- Fiscal Imbalance:

$$FI_t = PVE_t^{cfp} + B_t - PVR_t^{cfp}$$

where PVE_t^{cfp} is the present discounted value of projected expenditures under current fiscal policy, PVR_t^{cfp} is present discounted value of all projected receipts and B_t is government debt at the end of period t .

- In terms of our previous notation

$$PVE_t = \sum_{\tau=t+1}^{\infty} \frac{G_{\tau}}{(1+r)^{\tau-t}}$$

and

$$PVR_t = \sum_{\tau=t+1}^{\infty} \frac{T_{\tau}}{(1+r)^{\tau-t}}$$

as well as

$$B_t = \sum_{\tau=1}^t \frac{G_{\tau}}{(1+r)^{\tau-t}} - \sum_{\tau=1}^t \frac{T_{\tau}}{(1+r)^{\tau-t}}$$

- Intertemporal budget constraint suggests that feasible fiscal policy must have $FI_t = 0$.
- But

$$PVE_t^{cfp} \neq PVE_t$$

$$PVR_t^{cfp} \neq PVR_t$$

- Which means that either PVE_t^{cfp} or PVR_t^{cfp} will change to adjust to reality.

- In order to assess which generations bear what burden of the total fiscal imbalance, an additional concept is needed.
- Generational imbalance

$$GI_t = PVE_t^{cfpL} + B_t - PVR_t^{cfpL}$$

where PVE_t^{cfpL} is the present discounted value of outlays paid to generations currently alive, with PVR_t^{cfpL} defined correspondingly.

- GI_t is that part of the fiscal imbalance FI_t that results from transactions of the government with past (through B_t) and living generations.
- Difference $FI_t - GI_t$ denotes the projected part of fiscal imbalance due to future generations.

MAIN ASSUMPTIONS

- Real interest rate (discount rate for the present value calculations) of 3.68% per annum (average yield on a 30 year Treasury bond in recent years).
- Annual growth rate of real wages of between 1% and 2%, based on future projections of CBO.
- Growth of health care costs? Account for fact that the expenditure (in per capita terms) growth rate in Medicare is projected to be significantly above the projections from growth rates of wages for immediate future. Beyond 2035 this gap is assumed to gradually shrink to zero.

- 1 Baseline policy scenario corresponds to current fiscal policy

$$PVE_t^{cfp}, PVR_t^{cfp}$$

- 2 Alternative policy scenario factors in likely policy changes.

$$PVE_t^{as}, PVR_t^{as}$$

- In order to compute GI , one needs to break down taxes paid and outlays received by generations.

MAIN RESULTS

Fiscal Imbalance, Baseline (Billion of 2012 Dollars)		Current Fiscal Projections		
Part of the Budget	2012	2017	2022	
<i>FI</i> in Social Insurance	64,853	70,961	82,564	
<i>FI</i> in Rest of Federal Government	-10,502	-10,687	-11,742	
Total <i>FI</i>	54,675	60,274	70,822	

Fiscal Imbalance, Alternative Fiscal Scenario (Billion of 2012 Dollars)			
Part of the Budget	2012	2017	2022
<i>FI</i> in Social Insurance (SS+Med.)	65,934	72,036	83,606
<i>FI</i> in Social Security	20,077	22,272	26,660
<i>FI</i> in Medicare	45,857	49,764	56,946
<i>FI</i> in Rest of Federal Government	25,457	29,826	36,660
Total <i>FI</i>	91,391	101,862	120,266

Fiscal Imbal., Alternative Fiscal Scenario (% of Pres. Val. GDP)			
Part of the Budget	2012	2017	2022
<i>FI</i> in Social Insurance (SS+Med.)	6.5%	6.5%	6.8%
<i>FI</i> in Rest of Federal Government	2.5%	2.7%	3.0%
Total <i>FI</i>	9.0%	9.1%	9.8%

- ① *FI* is huge: requires the confiscation of 9% of GDP in perpetuity to close this imbalance from the perspective of 2012. Required increase in payroll taxes about 20% points
- ② This is before the 2017 TCJA that introduced a further tax reduction and the pandemic induced enormous additional deficits.
- ③ *FI* grows over time at a gross rate of $(1 + r) = 1.0368$ per year.
- ④ Largest part (about 1/2) of *FI* is due to Medicare. Comes from a) fast increases of medical goods prices and b) population aging.
- ⑤ *FI* dwarfs official government debt by a factor of 5.

Generational Imbalance, Alternative (Bill of 2012 Dollars)

Part of the Budget	2012	2017	2022
<i>FI</i> in Social Insurance	65,934	72,036	83,606
<i>FI</i> in Social Security	20,077	22,272	26,660
<i>GI</i> in Social Security (incl. Trust Fund)	19,586	21,726	26,032
<i>FI</i> – <i>GI</i> in Social Security	491	546	628
<i>FI</i> in Medicare	45,857	49,764	56,946
<i>GI</i> in Medicare	34,487	38,311	44,693
<i>FI</i> – <i>GI</i> in Medicare (incl. Trust Fund)	11,370	11,453	12,253

- ① 3/4 of Medicare *FI* is due to generations currently alive. But even future generations have benefits exceeding contributions (mainly because of Medicare prescription drug benefits).
- ② *FI* in social security is due *entirely* to past and current generations.
- ③ Magnitude of numbers depends on: growth rate of wages, discount rate applied to future revenues and outlays, temporary differential between expenditure growth in Medicare and the economy.
- ④ **But** conclusion robust: large spending cuts or tax increases required to restore fiscal balance. Medicare and Social Security key.

- Early U.S. history: few commodity taxes, on alcohol, tobacco and snuff, real estate sold at auctions, corporate bonds and slaves.
- British-American War in 1812: added sales taxes on gold, silverware and other jewelry
- In 1817 all internal taxes were abolished. Government relies exclusively on tariffs on imported goods.
- Civil war from 1861-1865 required increased funds for the federal government.
- In 1862, office of Commissioner of Internal Revenue was established. Right to assess, levy and collect taxes, and to enforce the tax laws through seizure of property and income and through prosecution.
- Individuals with earnings between \$600 – \$10000 had to pay an income tax of 3%; higher rates for people with income above \$10000.

- Additional sales and excise taxes were introduced. For the first time an inheritance tax was introduced.
- Total tax collections reached \$310 million in 1866, highest amount in U.S. history to that point, an amount not reached again until 1911.
- General income tax was scrapped in 1872, with other taxes besides excise taxes on alcohol and tobacco.
- Re-introduced in 1894, but declared unconstitutional in 1895, because it did not levy taxes and distribute funds among states in accordance with the constitution.
- Modern federal income tax was permanently introduced in the U.S. in 1913 through the 16-th Amendment to the Constitution. Gave Congress legal authority to tax income of both individuals and corporations.

- By 1920 IRS collected \$5.4 billion dollars, rising to \$7.3 billion dollars at the eve of WWII. Still, income tax was still largely a tax on corporations and very high income individuals, since exemption levels were high.
- In 1943 the government introduced a withholding tax on wages By 1945 number of income taxpayers increased to 60 million and tax revenues increased to \$43 billion, a six-fold increase from the revenues in 1939.
- Most far-reaching tax reforms in recent history: President Reagan in 1981 and 1986, President Clinton's tax reform of 1993 and the tax reforms of President George W. Bush in 2001-2003. Also the 2017 TCJA. Still too early to assess.
- The Reagan tax reforms reduced income tax rates by individuals drastically (with a total reduction amounting to the order of \$500 – 600 billion), partially offset by an increase in tax rates for corporations and moderate increases of taxes for the very wealthy.

- Mounting budget deficits: President Clinton partially reversed Reagan's tax cuts in 1993.
- Further tax reforms under the Clinton presidency included tax cuts for capital gains, the introduction of a \$500 tax credit per child and tax incentives for education expenses.
- Large tax cuts in 2003 by President Bush temporarily reduced dividend and capital gains taxes as well as increase child tax credits and lower marginal tax rates for most Americans. Were set to expire in 2012. Did partially expire in 2013.
- Further tax cut in 2017 with the TCJA.

- Too early to assess its role properly
- Reduces tax rates for businesses and individuals;
- Seems to be a regressive change that reduces revenue
- Simplifies personal taxes and
 - Increases the standard deduction and family tax credits,
 - Eliminates personal exemptions and making it less beneficial to itemize deductions;
 - Limits deductions for state and local income taxes (SALT) and property taxes;
 - Limits the mortgage interest deduction; reduces the alternative minimum tax for individuals and eliminates it for corporations;
 - Reduces the number of estates impacted by the estate tax;
 - Overall as of July 2022 (see Auerbach, Kotlikoff and Koehler 2022) the effects seem to be quite neutral on progressivity.

- Let y denote taxable income. If we model a deduction d explicitly, then taxable income is $y - d$.
- A tax code is defined by a tax function $T(y)$, which for each possible taxable income y gives the amount of taxes that are due to be paid.
- Example: if $y = \$100,000$ and $T(y) = \$25,000$, then every person with taxable income of $\$100,000$ in 2013 owes the government $\$25,000$ in taxes.

- For a given tax code T we define as

- ① Average tax rate of individual with taxable income y as

$$t(y) = \frac{T(y)}{y}$$

for all $y > 0$.

- ② Marginal tax rate of individual with taxable income y as

$$\tau(y) = T'(y)$$

whenever $T'(y)$ is well-defined (that is, whenever $T'(y)$ is differentiable).

- Interpretation: average tax rate $t(y)$ indicates what *fraction* of her taxable income a person with income y has to deliver to the government as tax. Marginal tax rate $\tau(y)$ measures how high the tax rate is on the last dollar earned, for a total taxable income of y .

- Equivalent definitions of tax code: can define tax code by

- ① Average tax rate schedule, since

$$T(y) = y * t(y)$$

- ② Marginal tax rate schedule (and the tax for $y = 0$), since

$$T(y) = T(0) + \int_0^y T'(y)dy$$

where the equality follows from the fundamental theorem of calculus.

- ③ Current U.S. federal personal income tax code is defined by a collection of marginal tax rates.

- A tax code is progressive if the function $t(y)$ is strictly increasing in y for all income levels y . It is progressive over an income interval (y_l, y_h) if $t(y)$ is strictly increasing for all income levels $y \in (y_l, y_h)$. It is also progressive if it is proportional over all income intervals but the proportion is higher in each successive interval.
- A tax code is regressive if the function $t(y)$ is strictly decreasing in y for all income levels y . It is regressive over an income interval (y_l, y_h) if $t(y)$ is strictly decreasing for all income levels $y \in (y_l, y_h)$.
- A tax code is proportional if the function $t(y)$ is constant y for all income levels y . It is proportional over an income interval (y_l, y_h) if $t(y)$ is constant for all income levels $y \in (y_l, y_h)$.

- Head tax or poll tax

$$T(y) = T$$

where $T > 0$ is a number. This tax is regressive since

$$t(y) = \frac{T}{y}$$

is a strictly decreasing function of y . Also note that the marginal tax $\tau(y) = 0$ for all income levels.

$$T(y) = \tau * y$$

where $\tau \in [0, 1)$ is a parameter.

Note that

$$t(y) = \tau(y) = \tau$$

that is, average and marginal tax rates are constant in income and equal to the tax rate τ . This tax system is proportional.

$$T(y) = \begin{cases} 0 & \text{if } y < d \\ \tau(y - d) & \text{if } y \geq d \end{cases}$$

x where $d, \tau \geq 0$ are parameters. Household pays no taxes if her income does not exceed the exemption level d , and then pays a fraction τ in taxes on every dollar earned above d . Average tax rates

$$t(y) = \begin{cases} 0 & \text{if } y < d \\ \tau \left(1 - \frac{d}{y}\right) & \text{if } y \geq d \end{cases}$$

Marginal tax rates

$$\tau(y) = \begin{cases} 0 & \text{if } y < d \\ \tau & \text{if } y \geq d \end{cases}$$

Tax system is progressive for all income levels above d ; for all income levels below it is proportional.

Such a tax code is defined by its marginal tax rates and the income brackets for which these taxes apply.

Example with three brackets

$$\tau(y) = \begin{cases} \tau_1 & \text{if } 0 \leq y < b_1 \\ \tau_2 & \text{if } b_1 \leq y < b_2 \\ \tau_3 & \text{if } b_2 \leq y < \infty \end{cases}$$

The tax code is characterized by the three marginal rates (τ_1, τ_2, τ_3) and income cutoffs (b_1, b_2) that define the income tax brackets.

- Compute tax schedule

- For $0 \leq y < b_1$

$$T(y) = \int_0^y \tau(y) dy = \int_0^y \tau_1 dy = \tau_1 \int_0^y dy = \tau_1 y,$$

- For $b_1 \leq y < b_2$

$$T(y) = \int_0^y \tau(y) dy = \int_0^{b_1} \tau_1 dy + \int_{b_1}^y \tau_2 dy = \tau_1 b_1 + \tau_2 (y - b_1)$$

- For $y \geq b_2$

$$\begin{aligned} T(y) &= \int_0^{b_1} \tau_1 dy + \int_{b_1}^{b_2} \tau_2 dy + \int_{b_2}^y \tau_3 dy \\ &= \tau_1 b_1 + \tau_2 (b_2 - b_1) + \tau_3 (y - b_2) \end{aligned}$$

- Average tax rates are given by

$$t(y) = \begin{cases} \tau_1 & \text{if } 0 \leq y < b_1 \\ \frac{\tau_1 b_1}{y} + \tau_2 \left(1 - \frac{b_1}{y}\right) & \text{if } b_1 \leq y < b_2 \\ \frac{\tau_1 b_1 + \tau_2 (b_2 - b_1)}{y} + \tau_3 \left(1 - \frac{b_2}{y}\right) & \text{if } b_2 \leq y < \infty \end{cases}$$

- If $\tau_1 < \tau_2 < \tau_3$ then this tax system is proportional for $y \in [0, b_1]$ and progressive for $y > b_1$.
- With just two brackets we get back a flat tax with deduction, if $\tau_1 = 0$.
- Current U.S. tax code resembles the last example closely, but consists of seven marginal tax rates and six income cut-offs that define the income tax brackets. The income cut-offs vary with family structure.

Theorem

A differentiable tax code $T(y)$ is progressive, that is, $t(y)$ is strictly increasing in y (i.e. $t'(y) > 0$ for all y) if and only if the marginal tax rate $T'(y)$ is higher than the average tax rate $t(y)$ for all income levels $y > 0$, that is

$$T'(y) > t(y)$$

Proof: By definition

$$t(y) = \frac{T(y)}{y}$$

Using the definition the rule for differentiating a ratio of two functions we obtain

$$t'(y) = \frac{yT'(y) - T(y)}{y^2}$$

This expression is positive if and only if

$$yT'(y) - T(y) > 0$$

or

$$T'(y) > \frac{T(y)}{y} = t(y)$$

QED.

- Intuition: for average tax rates to increase with income requires that the tax rate you pay on the last dollar earned is higher than the average tax rate you paid on all previous dollars.
- This result provides us with another, equivalent, way to characterize a progressive tax system.
- Differentiability of $T(y)$ not needed for the argument.
- A similar result can be stated and proved for a regressive or proportional tax system.

$$\begin{aligned} \text{Gross Income} &= \text{Wages and Salaries} \\ &+ \text{Interest Income and Dividends} \\ &+ \text{Net Business Income} \\ &+ \text{Net Rental Income} \\ &+ \text{Other Income} \end{aligned}$$

- Other income includes unemployment insurance benefits, alimony, income from gambling, income from illegal activities. Not included: child support, gifts below a certain threshold, interest income from state and local bonds (so-called Muni's), welfare and veterans benefits, employer contributions for health insurance and retirement accounts.

ADJUSTED GROSS INCOME AND TAXABLE INCOME

$$\begin{aligned}\text{Adjusted Gross Income (AGI)} &= \text{Gross Income} \\ &\quad - \text{contributions to IRA's} \\ &\quad - \text{alimony not after 2017 law and new divorces} \\ &\quad - \text{health insurance of self-employed}\end{aligned}$$

$$\begin{aligned}\text{Taxable Income} &= \text{AGI} \\ &\quad - \text{Deductions (Standard or Itemized)} \\ &\quad - \text{Exemptions} \\ &= y\end{aligned}$$

Note

$$\begin{aligned}\text{Taxes due upon filing} &= T(y) \\ &\quad - \text{Tax withholdings} \\ &\quad - \text{Tax credits}\end{aligned}$$

- Who should be taxed?
 - A person
 - A Household (the sum of the income of both).
- Most countries use people. The U.S. taxes the couple. There used to be a marriage tax penalty recently (2017) has changed to eliminate the penalty except for the highest income.
- Still there are large incentives that matter (Kato 2022)
 - It matters who earns the additional dollar.
 - It affects the rewards to experience which may change future outcomes.
 - It also may affect whom to marry
 - Especially, given the existing upper bound on social security contributions

THE MARRIAGE PENALTY: BEFORE 2017

Tax Rates for 2013, Singles

Income	$T'(y)$	$T(y)$
$0 \leq y < \$8,925$	10%	$0.1y$
$\$8,925 \leq y < \$36,250$	15%	$\$892 + 0.15(y - 8,925)$
$\$36,250 \leq y < \$87,850$	25%	$\$4,991 + 0.25(y - 36,250)$
$\$87,850 \leq y < \$183,250$	28%	$\$17,891 + 0.28(y - 87,850)$
$\$183,250 \leq y < \$398,350$	33%	$\$44,603 + 0.33(y - 183,250)$
$\$398,350 \leq y < \$400,000$	35%	$\$115,586 + 0.35(y - 398,350)$
$\$400,000 \leq y < \infty$	39.6%	$\$116,164 + 0.396(y - 400,000)$

Tax Rates for 2013, Married Filing Jointly

Income	$T'(y)$	$T(y)$
$0 \leq y < \$17,850$	10%	$0.1y$
$\$17,850 \leq y < \$72,500$	15%	$\$1,785 + 0.15(y - 17,850)$
$\$72,500 \leq y < \$146,400$	25%	$\$9,982 + 0.25(y - 72,500)$
$\$146,400 \leq y < \$223,050$	28%	$\$28,457 + 0.28(y - 146,400)$
$\$223,050 \leq y < \$398,350$	33%	$\$49,919 + 0.33(y - 223,050)$
$\$398,350 \leq y < \$450,000$	35%	$\$107,768 + 0.35(y - 398,350)$
$\$450,000 \leq y < \infty$	39.6%	$\$125,846 + 0.396(y - 450,000)$

Large Marriage Penalty for incomes above \$146,000 in 2013

DISAPPEARANCE MARRIAGE *PENALTY* IN TAX CUT AND JOBS ACT 2017

Tax Brackets and Tax Rates, 2019 by Family Type

Tax Rate	For Unmarried Individuals	For Married Individuals Filing Joint Returns	For Heads of Households
Taxable Income Over			
10%	\$0	\$0	\$0
12%	\$9,700	\$19,400	\$13,850
22%	\$39,475	\$78,950	\$52,850
24%	\$84,200	\$168,400	\$84,200
32%	\$160,725	\$321,450	\$160,700
35%	\$204,100	\$408,200	\$204,100
37%	\$510,300	\$612,350	\$510,300

2019 Standard Deduction and Personal Exemption

Filing Status	Deduction Amount
Single	\$12,200
Married Filing Jointly	\$24,400
Head of Household	\$18,350

Marriage Penalty Almost Gone

One can show that it is *impossible* to design a tax system that simultaneously is:

- ① Progressive, as defined above
- ② Satisfies across family equity: families with equal household incomes pay equal taxes (independent of how much of that income is earned by different members of each household)
- ③ Marriage-neutral: a given family pays the same taxes independent of whether the partners of the family are married or not.

SO WHY DID PEOPLE GET MARRIED?

- Pension Benefits
- Health Care Benefits
- Swift Legal System
- Tradition
- No privately designed substitute (prenups are not)

Under the new law is a good deal.

ALTERNATIVE MINIMUM TAX (AMT) I

- Created in the 1960s to prevent high-income taxpayers from avoiding the individual income tax: high-income taxpayers calculate their tax bill twice: once under the ordinary income tax system and again under the AMT, then it needs to pay the higher of the two.
- The AMT uses an alternative definition of taxable income, Alternative Minimum Taxable Income). Exemptions are significant but phase out for high-income taxpayers. The AMT is levied at two rates: 26% and 28%.
- The 2019 exemption amount is \$71,700 for singles and \$111,700 for married couples
- The 28 percent AMT rate applies to excess AMTI of \$194,800 for all taxpayers.
- AMT exemptions phase out at 25 cents per dollar earned once taxpayer AMTI hits a certain threshold. In 2019, the exemption will start phasing out at \$510,300 in AMTI for singles and \$1,020,600 for married

- Before the 2017 the individual alternative minimum tax (AMT) primarily affected well-off households, but not those with the very highest incomes.
- It was also more likely to hit taxpayers with large families, those who were married, and those who lived in high-tax states.
- TCJA shields almost all upper-middle and high-income taxpayers from the reach of the AMT.
- The AMT is now most likely to hit those at the top of the income scale who are engaged in certain sheltering activities.
- It has quite less bite than before.

EARNED INCOME TAX CREDIT (EITC)

- It is a refundable tax credit for low- to moderate-income working individuals and couples, particularly those with children.
- The amount of EITC benefit depends on a recipient's income and number of children. For a person or couple to claim one or more persons as their qualifying child, requirements such as relationship, age, and shared residency must be met.
- The maximum Earned Income Tax Credit in 2019 for single and joint filers is \$529, if the filer has no children (Table 5). The maximum credit is \$3,526 for one child, \$5,828 for two children, and \$6,557 for three or more children. All these are relatively small increases from 2018.
- Small Potatoes

- Product of selling an asset at higher price (often a business)
- Long-term capital gains are taxed at 15% up to \$434,550 for single and \$488,850 for married and then at 20%
- There is a Qualified Business Income Deduction. The TCJA includes a 20% deduction for pass-through businesses against up to \$160,700 (singles)
- **Annual Exclusion for Gifts** In 2019, the first \$15,000 of gifts to any person are excluded from tax.

- The Government has the Monopoly of Money Creation
- It uses it as a source of Revenue.
- Excessive Money Creation leads to Inflation so it is used with Caution.
- Modern Monetary Theory claims without support that Government Expenditures can be expanded a lot without problem as it and can ultimately be paid for by printing new money.

- Improvements in Cryptography that secret bilateral transmissions of information without a codebook. Allows a holder to own somethings as specified in the ledger.
- Existence of a Public Ledger that can be written by those that solve certain expensive computer problems.
- Rewards to those that solve the problem and can write on the ledger in terms of bitcoins.
- Allows for the transfer of bitcoins without additional parties.
- Bitcoins privately appropriate the seignorage.

- Simple example: two households in the economy
 - Household 1, taxable income $y = 100,000$ and household 2 with $y = 20,000$.
 - Lifetime utility $\log(c)$ only depends on their current after-tax income $c = y - T(y)$.
 - Their Consumption is Separate
 - Compare social welfare under progressive tax system with a proportional tax system.

- Hypothetical progressive tax system

$$\tau(y) = \begin{cases} 0\% & \text{if } 0 \leq y < 15000 \\ 10\% & \text{if } 15000 \leq y < 50000 \\ 20\% & \text{if } 50000 \leq y < \infty \end{cases}$$

- Under this tax system total tax revenues from the two agents are

$$\begin{aligned} & T(15,000) + T(100,000) \\ = & 0.1 * (20000 - 15000) \\ & + 0.1 * 35000 + 0.2(100000 - 50000) \\ = & \$500 + \$13500 \\ = & \$14000 \end{aligned}$$

and consumption for the households are

$$\begin{aligned} c_1 & = 20000 - 500 = 19500 \\ c_2 & = 100000 - 13500 = 86500 \end{aligned}$$

- Determine proportional tax rate τ such that revenues are same under hypothetical proportional tax system as under progressive system:

$$14000 = \tau * 20,000 + \tau * 100,000 = \tau * 120,000$$
$$\tau = \frac{14,000}{120,000} = 11.67\%$$

- Under proportional tax system consumption of both households equals

$$c_1 = (1 - 0.1167) * 20000 = 17667$$

$$c_2 = (1 - 0.1167) * 100000 = 88333$$

- Which tax system is better? Hard question! Use social welfare function

$$W(u(c_1), \dots, u(c_N))$$

- Household i is a “dictator”

$$W(u(c_1), \dots, u(c_N)) = u(c_i)$$

- If dictator is $i = 1$, prefer U.S. system. If dictator is $i = 2$, prefer proportional tax system.

- Utilitarian social welfare function given by

$$W(u(c_1), \dots, u(c_N)) = u(c_1) + \dots + u(c_N)$$

- Posits that everybody's utility should be counted equally.
- Basis: Jeremy Bentham (1748-1832) & John Stuart Mill's (1806-1873) "Utilitarianism" (published in 1863). Principle of Utility.

*Actions are right in proportion as they tend to promote happiness;
wrong as they tend to produce the reverse of happiness*

- Comparison for example

$$W^{\text{prog}}(u(c_1), u(c_2)) = \log(19500) + \log(86500) = 21.2461$$

$$W^{\text{prop}}(u(c_1), u(c_2)) = \log(17667) + \log(88333) = 21.1683$$

- Interpersonal Comparisons are difficult (need same utility)

- Rawlsian social welfare function

$$W(u(c_1), \dots, u(c_N)) = \min_i \{u(c_1), \dots, u(c_N)\}$$

- Idea: veil of ignorance plus extreme risk aversion

- For example

$$\begin{aligned} W^{\text{prog}}(u(c_1), u(c_2)) &= \min\{\log(c_1), \log(c_2)\} = \log(19500) \\ W^{\text{prop}}(u(c_1), u(c_2)) &= \min\{\log(c_1), \log(c_2)\} = \log(17667) \\ &< W^{\text{prog}}(u(c_1), u(c_2)) \end{aligned}$$

Suppose that taxable incomes are not affected by the tax code and suppose that u is strictly concave and the same for every household. Then under Rawlsian and Utilitarian social welfare function it is optimal to have complete income redistribution:

$$c_1 = c_2 = \dots = c_N = \frac{y_1 + y_2 + \dots + y_N - G}{N} = \frac{Y - G}{N}$$

where G is total required tax revenue and $Y = y_1 + y_2 + \dots + y_N$ Tax code that achieves this is given by

$$T(y_i) = y_i - \frac{Y - G}{N}$$

i.e. tax income at a 100% and then rebate $\frac{Y-G}{N}$ back to everybody.

- Suppose that $N = 2$ and $c_2 > c_1$ as the result of tax code. This cannot be optimal!
- Take away a little from household 2 and give it to household 1
- Under Rawlsian social welfare function this improves societal welfare since the poorest person has been made better off.
- Under Utilitarian social welfare function, loss of agent 2, $u'(c_2)$ is smaller than the gain of agent 1, $u'(c_1)$, since by concavity $c_2 > c_1$ implies

$$u'(c_1) > u'(c_2).$$

- But: assumption that changes in the tax system do not change a households' incentive to work, save and thus generate income is a very strong one. Therefore now want to analyze how income and consumption taxes change the economic incentives of households to work, consume and save.

- Utilitarianism takes utilities more seriously than it should.
 - Monotonic transformations of Utilities are yield the same allocations but not necessarily the same welfare.
- Rawlsian has the issue of only caring about the worst. But this cannot be taking literally: All societies are in terrible shape as long as there is any infant mortality.
- The idea of the veil of ignorance is an excellent one. It separates our circumstances from our assessment.
- It allows us to pick some particular utility function. It ends up yielding very egalitarian results.
- Still it is difficult to use to assess changes as the veil of ignorance does not apply. We know where we were before the policy change

- Household problem

$$\max_{c_1, c_2, s, \ell} \log(c_1) + \theta \log(1 - \ell) + \beta \log(c_2)$$

$$\begin{aligned} s.t. \quad (1 + \tau_{c_1})c_1 + s &= (1 - \tau_\ell)w\ell \\ (1 + \tau_{c_2})c_2 &= (1 + r(1 - \tau_s))s + b \end{aligned}$$

- Parameter θ measures how much households value leisure, relative to consumption.

- Intertemporal budget constraint. Solving second budget constraint yields

$$s = \frac{(1 + \tau_{c2})c_2 - b}{(1 + r(1 - \tau_s))}$$

and thus

$$(1 + \tau_{c1})c_1 + \frac{(1 + \tau_{c2})c_2}{(1 + r(1 - \tau_s))} = (1 - \tau_\ell)w\ell + \frac{b}{(1 + r(1 - \tau_s))}$$

- Rewrite this. Note that $\ell = 1 - (1 - \ell)$. Then

$$\begin{aligned} & (1 + \tau_{c_1})c_1 + \frac{(1 + \tau_{c_2})c_2}{(1 + r(1 - \tau_s))} = \\ = & (1 - \tau_\ell)w * (1 - (1 - \ell)) + \frac{b}{(1 + r(1 - \tau_s))} \end{aligned}$$

$$\begin{aligned} & (1 + \tau_{c_1})c_1 + \frac{(1 + \tau_{c_2})c_2}{(1 + r(1 - \tau_s))} + (1 - \ell)(1 - \tau_\ell)w = \\ = & (1 - \tau_\ell)w + \frac{b}{(1 + r(1 - \tau_s))} \end{aligned}$$

- Interpretation: household has potential income from social security $\frac{b}{(1+r(1-\tau_s))}$ and from supplying *all* her time to the labor market, $(1 - \tau_\ell)w$.
- Buys three goods
 - Consumption c_1 in first period, at effective price $(1 + \tau_{c1})$
 - Consumption c_2 in second period, at effective price $\frac{(1+\tau_{c2})}{(1+r(1-\tau_s))}$
 - Leisure $1 - \ell$ at effective price $(1 - \tau_\ell)w$, equal to the opportunity cost of not working.

- Lagrangian

$$\begin{aligned} L = & \log(c_1) + \theta \log(1 - \ell) + \beta \log(c_2) \\ & + \lambda \left\{ (1 - \tau_\ell)w + \frac{b}{(1 + r(1 - \tau_s))} \right. \\ & \left. - (1 + \tau_{c_1})c_1 - \frac{(1 + \tau_{c_2})c_2}{(1 + r(1 - \tau_s))} - (1 - \ell)(1 - \tau_\ell)w \right\} \end{aligned}$$

- First order conditions:

$$\begin{aligned}\frac{1}{c_1} - \lambda(1 + \tau_{c_1}) &= 0 \\ \frac{\beta}{c_2} - \lambda \frac{(1 + \tau_{c_2})}{(1 + r(1 - \tau_s))} &= 0 \\ \frac{-\theta}{1 - \ell} + \lambda(1 - \tau_\ell)w &= 0\end{aligned}$$

- Rewriting

$$\begin{aligned}\frac{1}{c_1} &= \lambda(1 + \tau_{c_1}) \\ \frac{\beta}{c_2} &= \lambda \frac{(1 + \tau_{c_2})}{(1 + r(1 - \tau_s))} \\ \frac{\theta}{1 - \ell} &= \lambda(1 - \tau_\ell)w\end{aligned}$$

- Intertemporal optimality condition

$$\frac{\beta c_1}{c_2} = \frac{(1 + \tau_{c_2})}{(1 + \tau_{c_1})} * \frac{1}{(1 + r(1 - \tau_s))}$$

Interpretation: marginal rate of substitution

$$\frac{\beta u'(c_2)}{u'(c_1)} = \frac{\beta c_1}{c_2}$$

should equal relative price between consumption in the second to consumption in the first period, $\frac{1}{(1+r(1-\tau_s))}$. With differential consumption taxes, the relative price has to be adjusted by relative taxes $\frac{(1+\tau_{c_2})}{(1+\tau_{c_1})}$.

- Comparative statics

- ① Increase in capital income tax rate τ_s reduces after-tax interest rate $1 + r(1 - \tau_s)$ and induces households to consume more in first period, relative to second period (ratio $\frac{c_1}{c_2}$ increases).
- ② Increase in consumption taxes in first period τ_{c_1} induces households to consume less in first period, relative to second period (ratio $\frac{c_1}{c_2}$ decreases).
- ③ Increase in consumption taxes in second period τ_{c_2} induces households to consume more in first period, relative to second period (ratio $\frac{c_1}{c_2}$ increases).

- Intratemporal optimality condition

$$\frac{\theta_{c_1}}{1 - \ell} = \frac{(1 - \tau_\ell)w}{(1 + \tau_{c_1})}$$

- Interpretation: marginal rate of substitution between current period leisure and current period consumption,

$$\frac{\theta u'(1 - \ell)}{u'(c_1)} = \frac{\theta_{c_1}}{1 - \ell}$$

should equal after-tax wage, adjusted by first period consumption taxes

$$\frac{(1 - \tau_\ell)w}{(1 + \tau_{c_1})}$$

- Comparative statics

① Increase in labor income taxes τ_ℓ reduces after-tax wage and reduces consumption, relative to leisure, that is $\frac{c_1}{1-\ell}$ falls.

② Increase in consumption taxes τ_{c_1} reduces consumption, relative to leisure, that is $\frac{c_1}{1-\ell}$ falls.

Proposition

Proposition: *Suppose we start with tax system with no labor income taxes, $\tau_\ell = 0$ and uniform consumption taxes $\tau_{c_1} = \tau_{c_2} = \tau_c$. Denote by c_1, c_2, ℓ, s the optimal consumption, savings and labor supply decision. Then there exists a labor income tax τ_ℓ and a lump sum tax T such that for $\tau_c = 0$ households find it optimal to make exactly the same consumption choices as before.*

Proof: If consumption tax is uniform, it drops out of the intertemporal optimality condition. Rewrite intratemporal optimality condition as

$$\frac{\theta c_1}{(1 - \ell)w} = \frac{(1 - \tau_\ell)}{(1 + \tau_c)}$$

Right hand side, for $\tau_\ell = 0$, is equal to

$$\frac{1}{(1 + \tau_c)}$$

Set $\hat{\tau}_\ell = \frac{\tau_c}{1 + \tau_c}$ and $\hat{\tau}_c = 0$. Then

$$\frac{(1 - \hat{\tau}_\ell)}{(1 + \hat{\tau}_c)} = 1 - \frac{\tau_c}{1 + \tau_c} = \frac{1}{(1 + \tau_c)},$$

and household faces the same intratemporal optimality condition as before. Appropriate lump-sum tax T guarantees that tax payments remain the same.

- Intratemporal optimality condition yields

$$c_1 = \frac{(1 - \tau_\ell)(1 - \ell)w}{(1 + \tau_{c_1})\theta}$$

- Intertemporal optimality condition yields

$$c_2 = \beta c_1(1 + r(1 - \tau_s)) \frac{(1 + \tau_{c_1})}{(1 + \tau_{c_2})} = \frac{(1 - \tau_\ell)(1 - \ell)w}{\theta} \frac{\beta(1 + r(1 - \tau_s))}{(1 + \tau_{c_2})}$$

- Plugging into budget constraint yields

$$\begin{aligned} \frac{(1 - \tau_\ell)(1 - \ell)w}{\theta} + \beta \frac{(1 - \tau_\ell)(1 - \ell)w}{\theta} &= (1 - \tau_\ell)w\ell + \frac{b}{(1 + r(1 - \tau_s))} \\ (1 + \beta) \frac{(1 - \tau_\ell)(1 - \ell)w}{\theta} &= (1 - \tau_\ell)w\ell + \frac{b}{(1 + r(1 - \tau_s))} \end{aligned}$$

- Solve for ℓ to obtain

$$\ell^* = \frac{1 + \beta}{1 + \beta + \theta} - \frac{b}{(1 + r(1 - \tau_s))\theta w(1 - \tau_\ell)(1 + \beta + \theta)}$$

- If $b = 0$, then

- $\ell^* = \frac{1 + \beta}{1 + \beta + \theta} \in (0, 1)$

- Interpretation: the more the household values leisure (the higher is θ), the less she finds it optimal to work. With $b > 0$, higher social security benefits in retirement reduce labor supply in the working period. If b gets really big, then the optimal $\ell^* = 0$.

- Rest of solution

$$\begin{aligned} c_1 &= \frac{(1 - \tau_\ell)}{(1 + \tau_{c_1})(1 + \beta + \theta)} w \\ c_2 &= \frac{\beta(1 - \tau_\ell)(1 + r(1 - \tau_s))}{(1 + \beta + \theta)(1 + \tau_{c_2})} w \\ s &= \frac{\beta(1 - \tau_\ell)w}{1 + \beta + \theta} \end{aligned}$$

- Let's consider countries i that differ in their
 - tax rates on labor $\tau_{\ell i}$, and consumption $\tau_{c i}$
 - perhaps in their wages w_i
 - and in their use of government revenues: Which fraction ξ_i of $\tau_{\ell i} \ell_i w_i + \tau_{c i} c_i$ is used for consumption:

$$T_i = \xi_i (\tau_{\ell i} \ell_i w_i + \tau_{c i} c_i)$$

- Households live only one period and maximize

$$\log c + \theta \log(1 - \ell_i)$$

subject to:

$$(1 + \tau_{c i})c = \ell w_i(1 - \tau_{\ell i}) + T_i$$

SOLVING THE HOUSEHOLDS' PROBLEM BY SUBSTITUTION

- $\max_{\ell} \quad \log \frac{\ell w_i(1-\tau_{\ell i})+T_i}{(1+\tau_{ci})} + \theta \log(1-\ell)$
- $\max_{\ell} \quad \log [\ell w_i(1-\tau_{\ell i}) + T_i] - \log(1+\tau_{ci}) + \theta \log(1-\ell)$
- The FOC

$$\frac{w_i (1 - \tau_{\ell i})}{\ell_i w_i(1 - \tau_{\ell i}) + T_i} = \frac{\theta}{1 - \ell_i}$$

- Getting rid of the denominators

$$(1 - \ell_i) w_i (1 - \tau_{\ell i}) = \theta[\ell_i w_i(1 - \tau_{\ell i}) + T_i]$$

- Isolating the term with labor

$$w_i (1 - \tau_{\ell i}) - \theta T_i = (1 + \theta)[\ell_i w_i (1 - \tau_{\ell i})]$$

- Which yields

$$\ell_i = \frac{1}{1+\theta} \frac{w_i(1-\tau_{\ell i}) - \theta T_i}{w_i(1-\tau_{\ell i})}$$

- Note that without transfers, labor is independent of wages: $\ell_i = \frac{1}{1+\theta}$
- A very important feature: We work as much as our great grandparents despite having wages that are much higher (this is a straight implication of the preferences that we have posed).
- It is not historically true, but almost.

OBTAINING AN EXPRESSION FOR HOURS

WITH TAXES WE HAD

$$\ell_i = \frac{1}{1+\theta} \frac{w_i(1-\tau_{\ell i}) - \theta T_i}{w_i(1-\tau_{\ell i})} = \frac{1}{1+\theta} \frac{w_i(1-\tau_{\ell i}) - \theta \xi_i (\tau_{\ell i} \ell_i w_i + \tau_{c i} c_i)}{w_i(1-\tau_{\ell i})}$$

- Because of $T_i = \xi_i (\tau_{\ell i} \ell_i w_i + \tau_{c i} c_i)$

- With $\tau_{c i} = 0$,

$$\ell_i = \frac{1}{1+\theta} \frac{(1-\tau_{\ell i}) - \tau_{\ell i} \theta \xi_i \ell_i}{(1-\tau_{\ell i})}$$

-

$$\ell_i = \frac{1-\tau_{\ell i}}{(1+\theta)(1-\tau_{\ell i}) + \tau_{\ell i} \theta \xi_i} < \frac{1}{1+\theta}, \quad \text{if } \theta \xi_i \text{ is not far from 2}$$

- Hours Worked Depends on Taxes
- With Consumption Taxes the Expressions get a bit more Complicated, but the same logic follows.
- We will use such an expression to actually compare across countries.

- Key for labor supply: tax wedge $\frac{(1-\tau_{\ell i})}{(1+\tau_{c i})}$ in the intratemporal optimality condition

$$\frac{\theta_{c i}}{1 - \ell_i} = \frac{(1 - \tau_{\ell i})}{(1 + \tau_{c i})} w_i$$

- Wages w_i ? Recall neoclassical production function operated by typical firm in the economy.

$$Y_i = A_i K_i^\alpha L_i^{1-\alpha}$$

- Profit maximization

$$\max_{(K_i, L_i)} A_i K_i^\alpha L_i^{1-\alpha} - w_i L_i - \rho_i K_i.$$

- Taking FOC with respect to L and setting it equal to 0 yields

$$\begin{aligned}(1 - \alpha)A_i K_i^\alpha L_i^{-\alpha} &= w_i \\ \frac{(1 - \alpha)A_i K_i^\alpha L_i^{1-\alpha}}{L_i} &= w_i \\ (1 - \alpha)\frac{Y_i}{L_i} &= w_i\end{aligned}$$

- Labor share equals $1 - \alpha$, capital share equals α .
- Use $w_i = (1 - \alpha)\frac{Y_i}{L_i}$ and equilibrium in labor market $L_i = \ell_i$ to obtain

$$\frac{\theta c_i}{1 - \ell_i} = \frac{(1 - \tau_{\ell i})}{(1 + \tau_{c i})} (1 - \alpha) \frac{Y_i}{\ell_i}$$

- Solving relates hours to taxes and Consumption to Output ratios.

$$\ell_i = \frac{1 - \alpha}{1 - \alpha + \frac{\theta(1 + \tau_{c i})}{(1 - \tau_{\ell i})} \frac{c_i}{Y_i}} \in (0, 1)$$

Country	GDP p.p.	Hours	GDP p.h.
Germany	74	75	99
France	74	68	110
Italy	57	64	90
Canada	79	88	89
United Kingdom	67	88	76
Japan	78	104	74
United States	100	100	100

Country	GDP p.p.	Hours	GDP p.h.
Germany	75	105	72
France	77	105	74
Italy	53	82	65
Canada	86	94	91
United Kingdom	68	110	62
Japan	62	127	49
United States	100	100	100

- GDP per capita, relative to the U.S. in Germany, France and Italy lags the U.S. by 25 – 40%, both in early 70's and mid 90's

- Early 70's due to lower productivity.

- In mid 90's: not due to lower productivity, but rather due to lower hours worked.

- Why do Europeans now work so much less than Americans? Proposed answer by Prescott (2004): taxes.

- Use

$$l_{it} = \frac{1 - \alpha}{1 - \alpha + \frac{\theta(1 + \tau_{cit})}{(1 - \tau_{\ell it})} \frac{c_{it}}{Y_{it}}}$$

to assess whether answer makes quantitative sense.

- $\frac{C_{it}}{Y_{it}}$ from NIPA accounts.
 - Assume that all but military government spending is yielding private consumption.
 - Indirect consumption taxes part of NIPA consumption, but not part of c in model.
 - This is what we meant by ξ .
- τ_{cit} is set to ratio between total indirect consumption taxes and total consumption expenditures in data.

- Labor income taxes

$$\tau_{\ell} = \tau_{ss} + \tau_{inc}$$

For τ_{ss} take payroll tax rates (currently 15.3%, shared by employers and employees). To compute marginal income tax rate τ_{inc} , compute average income taxes. by dividing total direct taxes by national income. Multiply by 1.6, to capture progressivity of tax code.

- Specify parameter values, θ and α .
 - Since α equals the capital share, set $\alpha = 0.3224$, the average across countries and time.
 - Parameter θ determines fraction of time worked. Choose θ such that in model number of hours spent working equals the average hours (across countries) in the data, which requires 1.54.

- Combined labor income and consumption tax rate relevant for the labor supply decision.

$$\frac{(1 - \tau_\ell)}{(1 + \tau_c)} = 1 - \tau$$

where $\tau = \frac{\tau_\ell + \tau_c}{1 + \tau_c}$.

- A person wanting to spend one dollar on consumption needs to earn x dollars as labor income, where x solves

$$\begin{aligned}x(1 - \tau) &= 1 \text{ or} \\x &= \frac{1}{1 - \tau}\end{aligned}$$

MODEL: 1990's

Country	Tax Rate τ	$\frac{c}{Y}$	Hours per Person per Week	
			Actual	Predicted
Germany	0.59	0.74	19.3	19.5
France	0.59	0.74	17.5	19.5
Italy	0.64	0.69	16.5	18.8
Canada	0.52	0.77	22.9	21.3
United Kingdom	0.44	0.83	22.8	22.8
Japan	0.37	0.68	27.0	29.0
United States	0.40	0.81	25.9	24.6

MODEL: 1970's

Country	Tax Rate τ	$\frac{c}{Y}$	Hours per Person per Week	
			Actual	Predicted
Germany	0.52	0.66	24.6	24.6
France	0.49	0.66	24.4	25.4
Italy	0.41	0.66	19.2	28.3
Canada	0.44	0.72	22.2	25.6
United Kingdom	0.45	0.77	25.9	24.0
Japan	0.25	0.60	29.8	35.8
United States	0.40	0.74	23.5	26.4

- Measured effective tax rates differ substantially by countries.
- Model does very well in explaining the cross-country differences in hours worked for the 90's.,
- Large part of the difference in hours worked between the U.S. and Europe (but not all of it) is explained by tax differences.

- Model is not quite as successful matching all countries for early 70's.
- Does predict that in the early 70's Germans and French did not work so much less than Americans, precisely because tax rates on labor were lower then than in the 90's in these countries.
- Two big failures of the model: Japan and Italy. What other than taxes depressed labor supply in these countries in this time period.

- History of U.S. system
- Taxes and Benefits under the current law
- Theoretical Analysis:
 - a) Effect of private savings,
 - b) Welfare consequences

HISTORY OF THE U.S. SOCIAL SECURITY SYSTEM

- Historical ancestor: system in Germany, introduced in 1889 by Bismarck. Benefits started at age 70.
- Response to rapid industrialization that transformed a largely agrarian society into a modern industrialized economy.
- Also a response to the growing popularity of the socialist movement and their demands for basic, publicly provided social insurance.
- 23 states in U.S. had introduced some public pension systems for needy elders in the early 1930's.
- A national old-age social insurance system started in 1935
- Various major forces responsible for the introduction of social security at that time.

- U.S. economy had undergone a dramatic transition from an agrarian to an industrialized economy.
- Share of employment in agriculture dropped from more than 50% in 1880 to less than 20% in 1935.
- Why was farm life less likely to leave the elders impoverished? Elders could perform less physically demanding tasks on family farms. Also, elders tended to own the farms. Second, employment opportunities in agriculture were less volatile than in the rest of the economy.

II: THE GREAT DEPRESSION

- Great depression in 1929-1932, the most severe recession in U.S. economic history, reduced unemployment opportunities of the elderly.
- Destroyed most of retirement wealth: September 1, 1929, value of stocks listed at NYSE was \$89.7 billion; in middle of 1932 it was \$15.6 billion, a decline of over 80%. In 1930 and 1931 over 3,000 banks suspended operations, deposits being lost were more than \$2 billion. Prices of wheat and cotton dropped by 66% and 75%, respectively, with it incomes and asset values in agricultural sector.
- Consequently, the great depression left an entire generation impoverished.

- Franklin D. Roosevelt's "New Deal" "It was an idea that all the political and practical forces of the community should and could be directed to making life better for ordinary people." (Francis Perkins)

- Several public programs arose out of this idea, one of which was social security. Designed to deal with the specific problems of the impoverished elders.

- The Elderly Population had started to grow as a result of increased life expectancy

- It is hard to coexist with large numbers of very poor elderly.

- Society cannot commit to let old citizens starve
- Governments want to make households save for their own retirement
- Social Security is a way to make savings for retirement compulsory
- A fully funded Social Security as in Chile is a pile of assets that various groups want to steal
- Hence unfunded (pay as you go) social security systems become the norm

- Social Security Act passed in 1935
- Original plan: use the 2% payroll tax for the accumulation of financial assets for retirement.
- Why special tax (rather than general tax revenues) to finance benefits?
 - These taxes were never a problem of economics. They are politics all the way through. We put those payroll contributions there so as to give the contributors a legal, moral, and political right to collect their pensions. With these taxes in there, no damn politician can ever scrap my social security program. [Franklin D. Roosevelt]*
- By 1939 it became clear that the widespread poverty of the old could be met by more than a funded system: It was changed to pay-as-you-go.

- Basically a pay-as-you-go system. Taxes paid by current workers are immediately used for paying benefits of current retirees.
- Fully funded system would save taxes of current workers, invest them in some assets and uses the returns to pay benefits when these current workers are old.
- U.S. social security system has accumulated the so-called trust fund, but with the expressed purpose of handling the retirement of the massive baby boom generation without having to increase payroll taxes.

- Payroll tax rate τ ,
- A maximum amount of earnings \bar{y} for which this payroll tax applies
- A benefit formula that calculates social security benefits as a function of the labor earnings over your lifetime.

- Currently, both employers and employees currently pay a proportional tax on labor income of $\tau = 6.2\%$, for a total of 12.4% of wages and salaries.
- Applies to all income below a threshold of \$168,600. (2024)
- Maximum amount an employee has to pay in 2024 is

$$0.062 * \$168,600 = \$10453$$

SOCIAL SECURITY TAXES OVER TIME

Year	Max. Taxable Ear.	Tax Rate
1937	\$3,000	2.00%
1950	\$3,000	3.00%
1960	\$4,800	6.00%
1970	\$7,800	8.40%
1980	\$29,700	10.16%
1990	\$51,300	12.40%
1998	\$68,400	12.40%
2007	\$97,500	12.40%
2012	\$110,100	12.40%
2017	\$127,200	12.40%
2019	\$132,900	12.40%
2021	\$142,800	12.40%
2024	\$168,600	12.40%

CALCULATION OF BENEFITS

- Consider a person that just turned 67 and retires in 2024
- Two steps.
 - Compute average indexed monthly earnings (AIME). Basically average monthly salary, where salaries early in life are adjusted by inflation and average wage growth.
 - Apply benefit formula

$$b = f(AIME)$$

- <https://www.ssa.gov/news/press/factsheets/colafacts2021.pdf>
- <https://www.ssa.gov/oact/cola/Benefits.html>

- Suppose household worked for 45 years, from age 23 to age 67, starting in 1977
- Let income in year t be denoted by y_t , for $t = 1980, 1981, \dots, 2024$.
- Denote maximal taxable earnings in year t by \bar{y}_t

- 1 For each year t define qualified earnings as

$$\hat{y}_t = \min\{y_t, \bar{y}_t\}$$

- 2 Adjust for inflation. Let P_{1980} denote CPI in 1980 and P_{2024} CPI in 2024. Then $\frac{P_{2024}}{P_{1980}}$ is the relative price of a typical basket of consumption goods in 2024, relative to 1980. Thus we take

$$\begin{aligned}\tilde{y}_{1980} &= \hat{y}_{1980} * \frac{P_{2021}}{P_{1980}} \\ \tilde{y}_t &= \hat{y}_t * \frac{P_{2024}}{P_t}\end{aligned}$$

- ① 3 Adjust by average wage growth. Define as the gross growth rate of average wages between 1980 and 2024

$$G_{1980,2024} = \frac{\bar{W}_{2024}}{\bar{W}_{1980}}$$
$$G_{t,2024} = \frac{\bar{W}_{2024}}{\bar{W}_t}$$

In addition to inflation earnings in early years of a persons's life are therefore adjusted in the following fashion

$$Y_t = \tilde{y}_t * G_{t,2024}$$

- ② 4 We arrive at 45 numbers, $\{Y_{1980}, Y_{1981}, \dots, Y_{2024}\}$. AIME equals the average of the 35 highest entries from the list (so if worked less than 35 years the AIME will be calculated with zero earnings for those years).

FROM AIME TO BENEFITS: PRIMARY INSURANCE AMOUNT (PIA).

- Benefit formula (monthly) for 2024

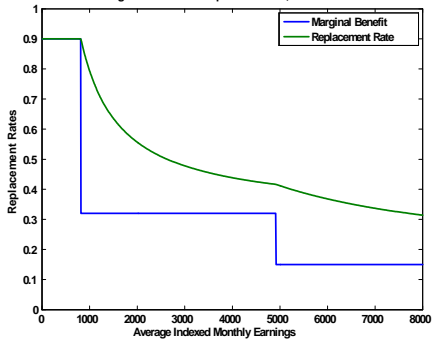
$$b = \begin{cases} 0.9AIME & \text{if } AIME \leq \$1,174 \\ 1,056 + 0.32(AIME - 1,174) & \text{if } \$1,174 < AIME \leq \$7,078 \\ 29,46 + 0.15(AIME - 7,078) & \text{if } \$7,078 < AIME \end{cases}$$

- With a maximum of \$3,822 (maximum AIME around \$10,600).
- Coefficients $\{0.9, 0.32, 0.15\}$ are determined by law. *Bend points* are adjusted every year.
- This gives household's benefits in 2024. From that point on benefits are indexed by inflation. Benefits are paid until death.

- Social security benefits are perfectly determined by average indexed monthly earnings, that is, by the best 35 working years.
- Rational forward-looking household understand that working more today will increase social security benefits, although the link becomes weaker the higher is income.
- Define the replacement rate as

$$rr(AIME) = \frac{b(AIME)}{AIME}$$

Marginal Benefits and Replacement Rate, 2014



- Tax rate now stands at 12.4%
- In the 1990's the situation and especially the future outlook of social security system deteriorated, due to demographic changes.
 - Life expectancy increased
 - Fertility rates decreased
 - Higher (predicted) dependency ratio (the ratio of people above 65 to the population aged 16-65)

WHAT REFORMS?

- Increase social security tax rates further
- Reduce benefits (e.g. increase the retirement age). Retirement age will gradually increase to age 67.
- Limit the scope of the program by reducing benefits and giving incentives to complement public pensions by private retirement accounts.

- Does a Pay-As-You-Go social security system reduce private savings rates?
- Under what conditions is the introduction of a Pay-As-You-Go social security system a good idea.
- Social Security as Insurance against Longevity Risk

- Household maximizes

$$\begin{aligned} \max_{c_1, c_2, s} \quad & \log(c_1) + \beta \log(c_2) \quad \text{s.t.} \\ c_1 + s \quad &= (1 - \tau)y \\ c_2 \quad &= (1 + r)s + b \end{aligned}$$

- Population grows at rate n , technical progress at rate g
- Social security system balances its budget

$$b = (1 + n)(1 + g)\tau y$$

- Rewrite the budget constraints of the household as

$$\begin{aligned} c_1 + s \quad &= (1 - \tau)y \\ c_2 \quad &= (1 + r)s + (1 + n)(1 + g)\tau y \end{aligned}$$

- Consolidate

$$c_1 + \frac{c_2}{1 + r} = (1 - \tau)y + \frac{(1 + n)(1 + g)\tau y}{1 + r} = I(\tau)$$

$$c_1 = \frac{I}{1 + \beta}$$

$$c_2 = \frac{\beta}{1 + \beta}(1 + r)I$$

$$s = (1 - \tau)y - \frac{I}{1 + \beta}$$

$$\begin{aligned}s &= (1 - \tau)y - \frac{l}{1 + \beta} \\ &= \frac{\beta y}{1 + \beta} - \frac{(1 + n)(1 + g) + \beta(1 + r)}{(1 + r)(1 + \beta)} * \tau y\end{aligned}$$

- which is obviously decreasing in τ . The larger the public pay-as-you-go system, the smaller are private savings.
- Because of its pay-as-you go nature of the system the social security system itself does not save, so total savings in the economy unambiguously decline with an increase in the size of the system as measured by τ .

- Is the introduction of PAYGO social security system good for households being born into the system?
- Social security tax rate only appears in $I(\tau)$, which is given as

$$I(\tau) = (1 - \tau)y + \frac{(1 + g)(1 + n)\tau y}{1 + r}.$$

- Under which $I(\tau)$ is strictly increasing in τ ?

$$\begin{aligned} I(\tau) &= y - \tau y + \frac{(1 + g)(1 + n)\tau y}{1 + r} \\ &= y + \left[\frac{(1 + g)(1 + n)}{1 + r} - 1 \right] \tau y \end{aligned}$$

- Pay-as-you go social security system is welfare improving if and only if $(1 + n)(1 + g) > 1 + r$.
- As good approximation

$$n + g > r$$

- If people save by themselves for their retirement, the return on their savings equals $1 + r$. If they save via a social security system (are forced to do so), their return to this forced saving consists of $(1 + n)(1 + g)$.
- May help to understand why in some countries the reform away from a PAYGO system is underway, in others not.
- But transition problem: there is one missing generation (since initial generation received benefits without paying taxes). If we abolish the system, either the currently young pay double, or we just default on the promises for the old.

- Modern social security systems provide some form of insurance to individuals, namely insurance against the risk of living longer than expected.
- Why: social security benefits paid as long as the person lives.
- People that live (unexpectedly) longer receive more over their lifetime than those that die prematurely.
- But: could also be done by private annuities.

- Household lives up to two periods, but die after the first period with probability $1 - p$. Normalize the utility of being dead to 0
- Household problem

$$\begin{aligned} \max_{c_1, c_2, s} \quad & \log(c_1) + p \log(c_2) \\ \text{s.t.} \quad & \\ c_1 + s = & y \\ c_2 = & (1 + r)s \end{aligned}$$

- Intertemporal budget constraint: $c_1 + \frac{c_2}{1+r} = y$
- Solution

$$\begin{aligned} c_1 &= \frac{1}{1+p}y \\ c_2 &= \frac{p(1+r)}{1+p}y \end{aligned}$$

- With social security: budget constraints

$$c_1 + s = (1 - \tau)y$$

$$c_2 = (1 + r)s + b$$

- Budget constraint of the social security administration

$$pb = (1 + n)(1 + g)\tau y$$

- Consolidating household budget constraints and substituting for b yields

$$c_1 + \frac{c_2}{1 + r} = y + \tau y \left(\frac{(1 + n)(1 + g)}{\rho(1 + r)} - 1 \right)$$

- Two reasons for social security

- If $(1 + n)(1 + g) > 1 + r$, the implicit return on social security is higher than the return on private assets, even absent the insurance aspect.

- As long as $p < 1$, even if $(1 + n)(1 + g) \leq 1 + r$ social security may be good, since the surviving individuals are implicitly insured by their dead brethren: the implicit return on social security is $\frac{(1+n)(1+g)}{p} > (1 + n)(1 + g)$.

- Focus on the insurance aspect and assume

$$(1 + n)(1 + g) = 1 + r$$

- Implicit return on social security is $\frac{(1+n)(1+g)}{p} = \frac{1+r}{p}$.
- Private insurance via annuities. An annuity is a contract where the household pays \$1 today, for the promise of the insurance company to pay you $\$(1 + r_a)$ as long as you live, from tomorrow on.

- If perfect competition among insurance companies, then zero profits.
- Insurance company takes \$1 today, which it can invest at the market interest rate $1 + r$
- Tomorrow it has to pay out with probability p . It has to pay out $1 + r_a$ per \$ of insurance contract. Thus zero profits imply

$$1 + r = p(1 + r_a)$$

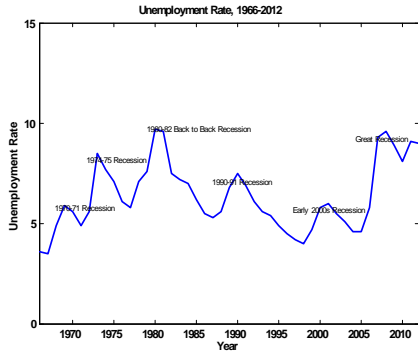
$$1 + r_a = \frac{1 + r}{p}$$

- Return on the annuity equals return via social security, as long as $(1 + n)(1 + g) = 1 + r$. Insurance against longevity can equally be provided by a social security system or by private annuity markets.

PUBLIC OR PRIVATE INSURANCE?

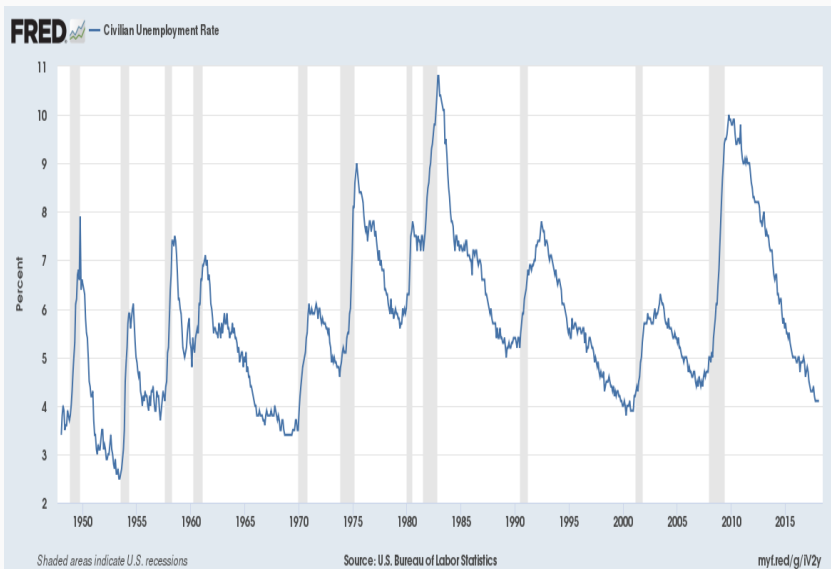
- Most countries provide this insurance publicly, with social security system?
Why?
 - If there is already a public system in place (for whatever reason), there are no strong incentives to purchase additional private insurance.
 - Adverse selection: individuals have better information about their life expectancy than insurance companies

- A variety of public insurance programs
- Goal: insuring citizens against the major risks of life
- Examples: unemployment insurance, welfare, food stamps, social security, public health insurance



The U.S. Unemployment Rate

U.S. UNEMPLOYMENT RATE 1950-2018



LENGTH OF UNEMPLOYMENT SPELLS OVER THE CYCLE

Unemployment Spell	2006	2010
< 5 weeks	37%	19%
5 - 14 weeks	30%	22%
15 - 26 weeks	15%	16%
> 26 weeks	18%	43%

INTERNATIONAL COMPARISON OF UNEMPLOYMENT RATES

	Unemployment (%)				≥ 1 Year		
	2000	2008	2011	2017	1999	2006	2011
France	9.0	7.8	9.7	9.0	38.7	41.9	41.4
Germany	8.0	7.5	5.9	3.7	51.7	56.4	48.0
Spain	11.7	11.3	21.6	15.6	46.3	21.7	41.6
Italy	10.1	6.7	8.4	11.0	61.4	49.6	51.9
Greece	11.2	7.7	17.7	20.7	55.3	54.3	49.6
Portugal	4.0	7.7	12.9	9.0	41.2	50.2	48.2
Sweden	5.6	6.2	7.5	6.3	30.1	13.0	17.2
UK	5.4	5.7	8.0	4.4	29.6	22.3	33.4
US	4.0	5.8	9.7	4.1	6.8	10.0	31.7
Tot. OECD	6.1	6.0	5.6	8.0	32.2	31.4	33.6

UNEMPLOYMENT BENEFITS: REPLACEMENT RATE

	Single			With Dependent Spouse		
	1. Y.	2.-3. Y.	4.-5. Y.	1. Y.	2.-3. Y.	4.-5. Y.
Belgium	79	55	55	70	64	64
France	79	63	61	80	62	60
Germany	66	63	63	74	72	72
Netherlands	79	78	73	90	88	85
Spain	69	54	32	70	55	39
Sweden	81	76	75	81	100	101
UK	64	64	64	75	74	74
US	34	9	9	38	14	14

EUROPEAN UNEMPLOYMENT DILEMMA OF 1980'S AND 1990'S: A POTENTIAL EXPLANATION

- Ljungqvist & Sargent: unemployment benefits & increased turbulence
- Increased turbulence in the 80's: laid-off workers faced a higher risk of losing their skills when becoming unemployed.
- Newly laid off worker in Europe has access to high and long-lasting unemployment compensation; on other hand, he may have lost his skill and thus is not offered new jobs that are attractive enough. Decides to stay unemployed, rather than accept a bad job. European unemployment dilemma.

- Why U.S. unemployment dilemma in/after Great Recession: Mitman and Rabinovich (2014) point to extension of unemployment benefits.
- Reduces incentives for workers to search for new job.
- Reduces incentives of firms to post new vacancies
- Europeization of U.S. labor market.

- Household lives for two periods. First period: job for sure, wage y_1 .
- Second period: with probability p he has a job and earns y_2^e ; with probability $1 - p$ he is unemployed and earns y_2^u .
- Interest rate $r = 0$.

- Utility function

$$\log(c_1) + p \log(c_2^e) + (1 - p) \log(c_2^u)$$

- Budget constraints are

$$c_1 + s = y_1$$

$$c_2^e = y_2^e + s$$

$$c_2^u = y_2^u + s$$

- Suppose that $y_1 = y$ and $y_2^e = y_2^u = y_1 = y$.
- Maximization problem

$$\begin{aligned} \max_{c_1, c_2^e, c_2^u, s} \quad & \log(c_1) + p \log(c_2^e) + (1 - p) \log(c_2^u) \\ \text{s.t.} \quad & \\ c_1 + s = & y \\ c_2^e = & y + s \\ c_2^u = & y + s \end{aligned}$$

- Solution

$$\begin{aligned} c_1 &= c_2^e = c_2^u = y \\ s &= 0 \end{aligned}$$

- Income is perfectly smooth and $\beta(1 + r) = 1$, so consumption simply equals income in every period.

- Let $y_1 = y$ and $p = 0.5$ and $y_2 = 2y_1 = 2y$. Mean-preserving spread, since

$$0.5 * 2y + 0.5 * 0 = y$$

- Maximization problem

$$\max \log(c_1) + 0.5 \log(c_2^e) + 0.5 \log(c_2^u)$$

$$c_1 + s = y$$

$$c_2^e = 2y + s$$

$$c_2^u = s$$

- Lagrangian

$$L = \log(c_1) + 0.5 \log(c_2^e) + 0.5 \log(c_2^u) + \lambda_1 (y - c_1 - s) \\ + \lambda_2 (2y + s - c_2^e) + \lambda_3 (s - c_2^u)$$

- First order conditions with respect to (c_1, c_2^e, c_2^u, s) yields

$$\begin{aligned} \frac{1}{c_1} - \lambda_1 &= 0 \\ \frac{0.5}{c_2^e} - \lambda_2 &= 0 \\ \frac{0.5}{c_2^u} - \lambda_3 &= 0 \\ -\lambda_1 + \lambda_2 + \lambda_3 &= 0 \end{aligned}$$

- Rewriting

$$\frac{1}{c_1} = \lambda_1$$

$$\frac{0.5}{c_2^e} = \lambda_2$$

$$\frac{0.5}{c_2^u} = \lambda_3$$

$$\lambda_2 + \lambda_3 = \lambda_1$$

- Substituting the first three equations into the last yields

$$\frac{0.5}{c_2^e} + \frac{0.5}{c_2^u} = \frac{1}{c_1}$$

- Use the budget constraints to obtain

$$\frac{0.5}{2y + s} + \frac{0.5}{s} = \frac{1}{(y - s)}$$

- Bringing the equation to one common denominator, $s * (2y + s) * (y - s)$, yields

$$\frac{0.5s(y - s)}{s(2y + s)(y - s)} + \frac{0.5(2y + s)(y - s)}{s(2y + s)(y - s)} = \frac{s(2y + s)}{s(2y + s)(y - s)}$$

or

$$\frac{0.5s(y - s) + 0.5(2y + s)(y - s) - s(2y + s)}{s(2y + s)(y - s)} = 0$$

- But this can only be 0 if the numerator is 0, or

$$0.5s(y - s) + 0.5(2y + s)(y - s) - s(2y + s) = 0$$

- Multiplying things out and simplifying a bit yields

$$s^2 + ys - \frac{1}{2}y^2 = 0$$

- Quadratic equation, has two solutions:

$$s_1 = -\frac{y}{2} - \left[\left(\frac{3}{4} \right) y^2 \right]^{0.5} = -\frac{1}{2}y (1 + 3^{0.5}) < 0$$

$$s_2 = -\frac{y}{2} + \left[\left(\frac{3}{4} \right) y^2 \right]^{0.5} = \frac{1}{2}y (3^{0.5} - 1) > 0$$

- Discard first solution on economic grounds, since

$$c_2^u = s = -\frac{1}{2}y (1 + 3^{0.5}) < 0$$

- Optimal consumption and savings choices with uncertainty satisfy

$$\hat{s} = \frac{1}{2}y (3^{0.5} - 1) > 0$$

$$\hat{c}_1 = y - \frac{1}{2}y (3^{0.5} - 1) = \frac{1}{2}y (3 - 3^{0.5}) < y$$

$$\hat{c}_2^e = 2y + \hat{s} = \frac{1}{2}y (3 + 3^{0.5})$$

$$\hat{c}_2^u = \frac{1}{2}y (3^{0.5} - 1)$$

- Note:

$$\hat{c}_1 = \frac{1}{2}y(3 - 3^{0.5}) < y = c_1$$
$$\hat{s} = \frac{1}{2}y(3^{0.5} - 1) > 0 = s$$

- Even though income in the first period and expected income in the second period have not changed, households increase their savings, compared to situation without uncertainty.
- This effect is called precautionary savings. Households save more with increased uncertainty.
- Precautionary saving behavior arises whenever $u'''(c) > 0$.
- Strict concavity of u (that is, risk-aversion, $u'' < 0$) is not enough for this result. If utility quadratic, $u(c) = -\frac{1}{2}(c - 100,000)^2$ then consumption and savings choice in first period identical to no uncertainty case.

- Suppose

$$u(c) = -\frac{1}{2}(c - 100,000)^2$$

- In this case the first order conditions become

$$\begin{aligned}-(c_1 - 100,000) &= \lambda_1 \\ -0.5(c_2^e - 100,000) &= \lambda_2 \\ -0.5(c_2'' - 100,000) &= \lambda_3 \\ \lambda_2 + \lambda_3 &= \lambda_1\end{aligned}$$

- Inserting the first three equations into the fourth yields

$$-(c_1 - 100,000) = -0.5(c_2^e - 100,000) - 0.5(c_2^u - 100,000)$$

or

$$c_1 = 0.5(c_2^e + c_2^u)$$

- Now using the budget constraints one obtains

$$y - s = 0.5(2y + s + s)$$

$$y - s = y + s$$

$$2s = 0$$

and thus the optimal savings choice with quadratic utility is $s = 0$, as in the case with no uncertainty.

- Optimal consumption choices exhibit “certainty equivalence”, that is, even with risk households make exactly the same choices as without uncertainty.
- But: realized consumption in period differs with and without uncertainty. With uncertainty one consumes $2y$ with probability 0.5 and 0 with probability 0.5, whereas under certainty one consumes y for sure.
- Even with quadratic utility households dislike risk, but they optimally don't change their saving behavior to hedge against it.
- It is easy to verify that with quadratic utility $u''' = 0$.

- Now the government levies unemployment insurance taxes on employed people in the second period at rate τ and pays benefits b to unemployed people
- Budget of the unemployment insurance system is balanced. There are many identical people, so fraction of employed in the second period is $p = 0.5$ and the fraction of unemployed is $1 - p = 0.5$
- Budget constraint of the unemployment administration reads as

$$0.5\tau y_2 = 0.5b$$

$$\tau y_2 = b$$

- For concreteness $\tau = 0.5$ and $y_2 = 2y_1 = 2y$ as before.
- Budget constraints in the second period of life become

$$c_2^e = y + s$$

$$c_2^u = y + s$$

- Unemployment system perfectly insures the unemployed: unemployment benefits are exactly as large as after tax income when being employed.
- Optimal consumption and savings choices: It immediately follows that

$$c_2^e = c_2^u = c_2$$

- Then maximization problem of household becomes

$$\begin{aligned} & \max \log(c_1) + 0.5 \log(c_2) + 0.5 \log(c_2) \\ & = \max \log(c_1) + \log(c_2) \\ & \text{s.t.} \end{aligned}$$

$$c_1 + s = y$$

$$c_2 = y + s$$

with obvious solution

$$c_1 = c_2 = y$$

$$s = 0.$$

- Exactly as without income uncertainty. When the government completely insures unemployment risk, private households make exactly the same choices as if there was no income uncertainty.

- With perfect unemployment insurance lifetime utility equals

$$V^{ins} = \log(y) + \log(y)$$

which exactly equals the lifetime utility without income uncertainty.

- Without unemployment insurance lifetime utility is $V^{no} =$

$$\log\left(\frac{1}{2}y(3 - 3^{0.5})\right) + 0.5 \log\left(\frac{1}{2}y(3 + 3^{0.5})\right) + 0.5 \log\left(\frac{1}{2}y(3^{0.5} - 1)\right)$$

- Easy to calculate that $V^{ins} > V^{no}$.

- Even with partial insurance, that is $0 < \tau < 0.5$, household would be better off with that partial insurance than without any insurance.

- Risk-averse individuals always benefit from public (or private) provision of actuarially fair insurance.

- But they prefer more insurance to less, absent any adverse selection or moral hazard problem.

WHY NOT FULL INSURANCE IN REAL WORLD

- No country provides full insurance against being unemployed. Why not?
- In the real world with perfect insurance a strong moral hazard problem arises. Why work if one get's the same money by not working.
- Trade-off between insurance and economic incentives. If the government could perfectly monitor individuals things would be easy: simply condition payment of benefits on good behavior. But with private information the complicated trade-off between efficiency and insurance arises.