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1 Introduction

A model is an artificial economy used to understand economic phenomena. The description of a model's environment includes specifying agents' preferences and endowments, technology available, information structure as well as property rights. One such example is the Neoclassical Growth Model. It is one of the workhorse frameworks of modern macroeconomics because it delivers some fundamental properties that are characteristics of industrialized economies. Kaldor (1957) summarizes these six stylized facts (the seventh was added later on):

1. Output per capita (Y/L) has grown at a roughly constant rate (of 2%).
2. The capital-output ratio (K/Y, where capital is measured using the perpetual inventory method) has remained roughly constant at around 2 (despite output per capita growth).
3. The capital-labor ratio (K/L) has grown at a roughly constant rate equal to the growth rate of output.
4. Labor income as a share of output (WL/Y) has remained roughly constant (0.66).
5. The wage rate has grown at a roughly constant rate equal to the growth rate of output.
6. The real interest rate has been stationary and, during long periods, roughly constant.
7. Hours worked per capita have been roughly constant.

A model is not complete without the notion of an equilibrium concept. Equilibrium can be defined as a prediction of what will happen in the economy, i.e. a mapping from environments to outcomes (allocations, prices, etc.). One equilibrium concept that we will deal with during the course is the Competitive Equilibrium (CE). Characterizing the equilibrium usually involves defining a commodity space and finding solutions to a system of an infinite number of equations.[1] There are generally three ways of getting around this challenge. The first is to invoke the first welfare theorem to solve for the allocation and then find the equilibrium prices associated with it. However, this may sometimes not

[1] As in Arrow-Debreu or Valuation Equilibrium.

work due to, say, the presence of externalities. The second way is to simply construct an equilibrium to reflect observed realities and work backwards in the model from there. But this defeats the purpose of writing down a model, which we do in order to learn something about the world. The third way is to resort to dynamic programming and study a Recursive Competitive Equilibrium (RCE), in which equilibrium objects are functions instead of sequences. We briefly review the first in the next section and then move on to recursive competitive equilibria. In any form of competitive (general) equilibria agents do their best and their actions are mutually compatible (i.e. market clearing holds). However, the last assumption can be quite restrictive: later we will explore equilibria with non-Walrasian features like search frictions.

2 Review: Neoclassical Growth Model

We review briefly the basic neoclassical growth model. The neoclassical growth has three main defining characteristics: exogenous technological change, Cobb-Douglas production technology, and balanced growth preferences. Balanced growth preferences means that preferences are compatible with a balanced growth path in the steady state, and requires that the substitution and income effects of wage increases cancel out. Two important examples of such preferences are Cobb-Douglas preferences, and log plus constant times Frisch preferences:

$$U(c,l) = \frac{(c^{\theta} l^{1-\theta})^{1-\sigma}}{1-\sigma} \tag{Cobb-Douglas}$$

$$U(c,n) = \log c + \chi \frac{n^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}} \tag{Log plus constant times Frisch}$$

where l is leisure and $n = 1 - l$ is labor.

2.1 The Neoclassical Growth Model (Without Uncertainty)

The commodity space is

$$\mathcal{L} = \{(l_1, l_2, l_3) : l_i = (l_{it})_{t=0}^{\infty}\ l_{it} \in \mathbb{R},\ \sup_t |l_{it}| < \infty,\ i = 1, 2, 3\}.$$

The consumption possibility set is

$$X(\bar{k}_0) = \{x \in \mathcal{L} :\ \exists (c_t, k_{t+1})_{t=0}^{\infty} \text{ s.th. } \forall t = 0, 1, \ldots$$
$$c_t, k_{t+1} \geq 0,\ x_{1t} + (1-\delta)k_t = c_t + k_{t+1},\ -k_t \leq x_{2t} \leq 0,\ -1 \leq x_{3t} \leq 0,\ k_0 = \bar{k}_0\}.$$

The production possibility set is $Y = \prod_t Y_t$, where

$$Y_t = \{(y_{1t}, y_{2t}, y_{3t}) \in \mathbb{R}^3 :\ 0 \leq y_{1t} \leq F(-y_{2t}, -y_{3t})\}.$$

Definition 1 *An Arrow-Debreu equilibrium is $(x^*, y^*) \in X \times Y$, and a continuous linear functional ν^* such that*

1. $x^* \in \arg\max_{x \in X, \nu^*(x) \leq 0} \sum_{t=0}^{\infty} \beta^t u(c_t(x), -x_{3t})$,
2. $y^* \in \arg\max_{y \in Y} \nu^*(y)$,
3. *and* $x^* = y^*$.

Note that in this definition we have added leisure. Now, let's look at the Social Planner's Problem of the one-sector growth model:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, -x_{3t}) \qquad (SPP)$$

$$\text{s.t.} \qquad c_t + k_{t+1} - (1-\delta)k_t = x_{1t}$$

$$-k_t \leq x_{2t} \leq 0$$

$$-1 \leq x_{3t} \leq 0$$

$$0 \leq y_{1t} \leq F(-y_{2t}, -y_{3t})$$

$$x = y$$

$$k_0 \text{ given.}$$

Suppose we know that a solution in sequence space exists for (SPP) and it is unique.

Exercise 1 *Clearly stating sufficient assumptions on utility and production function, show that (SPP) has a unique solution.*

Two important theorems show the relationship between CE allocations and Pareto optimal allocations:

Theorem 1 (FWT) *Suppose that for all $x \in X$ there exists a sequence $(x_k)_{k=0}^{\infty}$ such that for all $k \geq 0$, $x_k \in X$ and $U(x_k) > U(x)$. If (x^*, y^*, ν^*) is an Arrow-Debreu equilibrium, then (x^*, y^*) is a Pareto efficient allocation.*

Theorem 2 (SWT) *If X is convex, preferences are convex, U is continuous, Y is convex and has an interior point, then for any Pareto efficient allocation (x^*, y^*) there exists a continuous linear functional ν such that (x^*, y^*, ν) is a quasiequilibrium, that is:*
(i) for all $x \in X$ such that $U(x) \geq U(x^*)$ it implies $\nu(x) \geq \nu(x^*)$;
(ii) for all $y \in Y$, $\nu(y) \leq \nu(y^*)$.

Note that at the very basis of the CE definition and welfare theorems there is an implicit assumption of perfect commitment and perfect enforcement. Note also that the FWT implicitly assumes there is no externality or public goods (it achieves this implicit assumption by defining a consumer's utility function only on his own consumption set but no other points in the commodity space). The Greenwald-Stiglitz (1986) theorem establishes the Pareto inefficiency of market economies with imperfect information and incomplete markets.

From the FWT, we know that if a Competitive Equilibrium exists, it is Pareto Optimal. Moreover, if the assumptions of the SWT are satisfied and if the (SPP) has a unique solution, then the competitive equilibrium allocation is unique and is the same as the Pareto Optimal allocation. Prices can then be constructed using this allocation and first-order conditions.

Exercise 2 *Show that*

$$\frac{v_{2t}}{v_{1t}} = F_k(k_t, l_t) \text{ and } \frac{v_{3t}}{v_{1t}} = F_l(k_t, l_t).$$

One shortcoming of the Arrow-Debreu Equilibrium (ADE) is that all trade occurs at the beginning of time. This assumption is unrealistic. Modern economics is based on sequential markets. Therefore, we define another equilibrium concept, the Sequential Markets Equilibrium (SME). We can easily show that SME is equivalent to ADE by introducing Arrow-Debreu securities. All of our results still hold and SME is therefore the right problem to solve.

Exercise 3 *Define a Sequential Markets Equilibrium (SME) for the economy above. Prove that the objects we get from the AD equilibrium satisfy SME conditions and that the converse is also true. We should first show that a CE exists and therefore coincides with the unique solution of (SPP).*

Note that the (SPP) is hard to solve since we are dealing with an infinite number of choice variables. Instead, we can establish that this SPP problem is equivalent to the following dynamic problem (removing leisure from now on), which is easier to solve:

$$\begin{aligned} v(k) = \max_{c,k'} \quad & u(c) + \beta v(k') \qquad (RSPP) \\ \text{s.t.} \quad & c + k' = f(k). \end{aligned}$$

Exercise 4 *Compute the labor share in the neoclassical growth model under Cobb-Douglas and CES production function. Compare them.*

Exercise 5 *Check the degree of homogeneity of Cobb-Douglas preferences in wages.*

2.2 A Comment on the Welfare Theorems

Situations in which the welfare theorems would not hold include externalities, public goods, situations in which agents are not price-takers (e.g. monopolies), some legal systems, when markets are missing, which could rule out certain contracts that appear to be complete, or search frictions. What happens in such situations? The solutions to the Social Planner problem and the CE do not coincide, and so we cannot use the welfare theorems we have developed for dynamic programming. As we will see in this course, we can work with Recursive Competitive Equilibria. In general, we can prove that the solution to the RCE coincides with the one derived from the SME, but not the other way around (for example when we have multiple equilibria). However, in all the models we will see in this course, this equivalence will hold.

3 Recursive Competitive Equilibrium

3.1 A Simple Example

We have so far established the equivalence between the allocation of the SP problem, which gives the unique Pareto optimal allocation, and the allocations of the AD equilibrium and the SME. We can now solve for the very complicated equilibrium allocation by solving the relatively easier Dynamic Programming problem of the social planner. One handicap of this approach is that in many environments the equilibrium is not Pareto Optimal and hence not a solution of a social planner's problem (e.g. when taxes are distortionary or when externalities are present). Therefore, the recursive formulation of the problem (RSPP) would not be the right problem to solve. In some of these situations we can still write the problem in sequence form. However, we would lose the powerful computational techniques of dynamic programming. In order to resolve this issue we will define the Recursive Competitive Equilibrium equivalent to SME that we can always solve for.

We start with the household's problem. In order to write it recursively, we need to use equilibrium conditions that tells the household what prices are, in particular as functions of economy-wide aggregate state variables. Let aggregate capital be K and aggregate labor $N = 1$. Then from solving the firm's problem, factor prices are given by $w(K) = F_n(K, 1)$ and $R(K) = F_k(K, 1)$. Therefore, since households take prices as given, they need to know aggregate capital in order to make their decisions. A household who is choosing how much to consume and how much to work has to know the whole sequence of future prices in order to make her decision. That means that she needs to know the path of aggregate capital. Therefore, if she believes that aggregate capital changes according to the mapping G, such that $K' = G(K)$, then knowing aggregate capital today enables her to project the path of aggregate capital into the future and thus the path for prices. So, we can write the household's recursive problem given function $G(\cdot)$ as follows:

$$\begin{aligned} V(K,a;G) = \max_{c,a'} \quad u(c) \quad &+ \quad \beta V(K',a';G) \qquad\qquad (RCE) \\ \text{s.t. } c + a' \quad &= \quad w(K) + R(K)a \\ K' \quad &= \quad G(K), \\ c \quad &\geq \quad 0 \end{aligned}$$

The dynamic programming problem above is for a household that sees K in the economy, has a belief G about its evolution, and carries a units of assets from the past. The price functions $w(K), R(K)$ are obtained from the firm's FOCs. The solution of this problem yields policy functions $c(K, a; G)$ for consumption and $g(K, a; G)$ for next period asset holdings, as well as a value function $V(K, a; G)$, which must satisfy

$$\begin{aligned} u_c\left(c(K,a;G)\right) &= \beta V_{a'}\left(G(K), g(K,a;G);G\right) \\ V_a\left(K,a;G\right) &= R(K)\, u_c\left(c(K,a;G)\right) \end{aligned}$$

Now we can define the Recursive Competitive Equilibrium.

Definition 2 *A Recursive Competitive Equilibrium with arbitrary expectations G is a set of functions V, $g : \mathcal{K} \times \mathcal{A} \to \mathbb{R}$, and R, w, $G : \mathcal{K} \to \mathbb{R}_+$ such that:*[2]

1. *Given G, w, R, V and g solve the household's problem in (RCE),*
2. *$K' = G(K) = g(K, K; G)$ (representative agent condition),*
3. *$w\left(K\right) = F_n\left(K, 1\right)$, and*
4. *$R\left(K\right) = F_k\left(K, 1\right)$.*

Note that G represents some arbitrary expectations and do not have to necessarily be rational. Next, we define another notion of equilibrium in which the expectations of the household are consistent with what happens in the economy:

Definition 3 *A Rational Expectations Recursive Competitive Equilibrium (REE) is a set of functions V, g, R, w, G^*, such that:*

1. *Given w, R, $V(K, a; G^*)$ and $g(K, a; G^*)$ solve the household's problem in (RCE),*
2. *$K' = G^*(K) = g(K, K; G^*)$,*
3. *$w\left(K\right) = F_n\left(K, 1\right)$, and*
4. *$R\left(K\right) = F_k\left(K, 1\right)$.*

[2] Note that we could add the policy function for consumption $c(K, a; G)$.

What this means is that in a REE households optimize given what they believe is going to happen in the future and what happens in the aggregate is consistent with the households' decision. The proof that every REE can be used to construct a SME is left as an exercise. The reverse turns out not to be true. Notice that in a REE, function G^* projects next period's aggregate capital. If there is a multiplicity of SME, this would imply that we cannot construct such function G^*, since one value of capital today could imply more than one value of capital tomorrow, i.e. G^* is not a correspondence. From now on, we will focus on REE unless otherwise stated because it helps us select an equilibrium in case more than one exists.

Remark 1 *Note that unless otherwise stated, we will assume that the capital depreciation rate* δ *is 1, with the firm's profits given by* $F(K,1) - (r(K) + \delta)K - w(K)$*.* $R(K)$ *is the gross rate of return on capital, which is given by* $R(K) = F_k(K,1) + 1 - \delta$*. The net rate of return on capital is* $r(K) = F_k(K,1) - \delta$*.*

3.2 The Envelope Theorem and the Functional Euler equation

To solve for the RCE and, in particular, to derive the household's optimality conditions we use envelope theorem. This method is valid because of time consistency of consumption choice.

Take the household's problem given by

$$\begin{aligned} V(K,a) = \max_{c,a'} \quad & u(c) \quad + \quad \beta V(K',a') \\ \text{s.t.} \quad c + a' &= w(K) + R(K)a \\ K' &= G(K) \\ c &\geq 0 \end{aligned}$$

with decision rules for consumption and next period asset holdings given by $c = c(K,a)$ and $a' = g(K,a)$.

By taking the first-order conditions (assuming an interior solution since u is well behaved), we get:

$$-u_c(c) + \beta V_{a'}(K', a') = 0,$$

which evaluated at the optimum is

$$-u_c\left(w(K) + R(K)a - g(K, a)\right) + \beta V_{a'}\left(G(K), g(K, a)\right) = 0 \tag{1}$$

The problem with solving the functional Euler equation is that $V_{a'}$ is not known. However, we can write the value function as a function of current states and differentiate both sides with respect to a.[3] Since the Euler equation holds for all (a, K), we have

$$V(K, a) = u(w(K) + R(K)a - g(K, a)) + \beta V(G(K), g(K, a)) \tag{2}$$

and using the implicit function theorem we can get its derivative with respect to a:

$$\begin{aligned} V_a(K, a) &= u_c(w(K) + R(K)a - g(K, a))R(K) + \\ & \quad \frac{\partial g(K, a)}{\partial a}\left[-u_c(w(K) + R(K)a - g(K, a)) + \beta V_{a'}(G(K), g(K, a))\right] \end{aligned} \tag{3}$$

The term in square brackets in the right hand side is the first-order condition (1) and hence it is zero. So equation (3) simplifies to $V_a(K, a) = u_c(w(K)+R(K)a-g(K, a))R(K)$. Note, however, that we need $V_{a'}(G(K), g(K, a))$ to find the optimal asset holdings allocation. We would need to follow the same procedure for $V(G(K), g(K, a))$, but since equation 1 holds for all (a, K) next period's Euler equation is $u_{c'}(w(G(K))+R(G(K))g(K, a)-g(G(k), g(K, a))) = \beta V_{a''}\Big[G(G(K)), g(G(K), g(K, a))\Big]$. This in turn implies that $V_{a'}(G(K), g(K, a)) = u_{c'}(w(G(K))+R(G(K))g(K, a)-g(G(k), g(K, a)))\ R(G(K))$.

[3] Under some assumptions, V is differentiable. See p. 121 of Prof. Krueger's notes for details.

Finally, we can replace that in equation (1) and get the functional Euler equation

$$u_c\left(w(K)+R(K)a-g(K,a)\right)-\beta u_{c'}(w(G(K))+R(G(K))g(K,a)-g(G(k),g(K,a)))\ R(G(K))=0 \tag{4}$$

What kind of problems would time inconsistency create here? To illustrate, consider an individual who wants to lose weight and is deciding whether to start a diet or not. However, he would rather postpone the diet for tomorrow, and prefers to eat well today. Let 1 denote obeying the diet restrictions and 0 otherwise. Let his preference ordering be given by:

1. $(0,1,1,1...)$
2. $(1,1,1,1...)$
3. $(0,0,0,0,...)$

Even though he promises himself that he will start the diet tomorrow and chooses to eat well today, tomorrow he will face the same problem. So he will choose the same option again tomorrow. He will thus never start the diet and will end up with his least preferred option: $(0,0,0,0,...)$.

However, in our model that is not what happens. Agents' preferences are time consistent, so what an individual promises today has to be optimal for her tomorrow as well. And that is why we can use the envelope theorem.

3.3 Economies with Fiscal Policy

From here on, for notational simplicity we will drop the functional arguments to the value function, like the belief G the household has about the evolution of aggregate capital.

3.3.1 Lump-Sum Tax

The government levies each period T units of goods in a lump sum fashion and spends it in a public good, say, medals. Assume however that consumers do not care about medals. The household's problem is

$$\begin{aligned} V(K, a) = \max_{c,a'} \quad & u(c) \quad + \quad \beta V(K', a') \\ \text{s.t.} \quad c + a' \quad & = \quad w(K) + R(K)a - T \\ K' \quad & = \quad G(K) \\ c \quad & \geq \quad 0 \end{aligned}$$

Let the solution of this problem be given by policy function $g_a(K, a; M, T)$ and value function $V(K, a; M, T)$. The equilibrium can be characterized by $G^*(K; M, T) = g_a(K, K; G^*, M, T)$ and $M^* = T$ (the government budget constraint is balanced period by period). We will write a complete definition of equilibrium for a version with government debt (below).

Exercise 6 *Define the aggregate resource constraint as $C + K' + M = f(K, 1)$ for the planner. Show that the equilibrium is optimal when consumers do not care about medals.*

Note that if households cared about medals, then the equilibrium would not necessarily be optimal. The social planner would equate the marginal utility of consumption and of medals, while the agent would not.

3.3.2 Labor Income Tax

We have an economy in which the government levies a tax on labor income in order to purchase medals. Medals are goods that provide utility to the consumers.

$$\begin{aligned} V(K,a) = \max_{c,a'} \quad u(c,M) \quad &+ \quad \beta V(K',a') \\ \text{s.t.} \quad c + a' \quad &= \quad (1 - \tau(K))\, w(K) + R(K)a \\ K' \quad &= \quad G(K) \\ c \quad &\geq \quad 0 \end{aligned}$$

with $M = \tau(K) w\,(K)$ being the government budget constraint. Since leisure is not valued, the labor decision is trivial and there is not distortion from the introduction of taxes.

Exercise 7 *Suppose medals do not provide utility to agents but leisure does. Is the CE optimal? What if medals also provide utility?*

3.3.3 Capital Income Tax

Now let us look at an economy in which the government levies tax on capital in order to purchase medals. Medals provide utility to the consumers.

$$\begin{aligned} V(K,a) = \max_{c,a'} \quad u(c,M) \quad &+ \quad \beta V(K',a') \\ \text{s.t.} \quad c + a' \quad &= \quad w(K) + a\,[1 + r(K)\,(1 - \tau(K))] \\ K' \quad &= \quad G(K) \\ c \quad &\geq \quad 0 \end{aligned}$$

with $M = \tau(K) r\,(K)\, K$ and $R\,(K) = 1 + r\,(K)$. Now, the First Welfare Theorem is no longer applicable and the CE will therefore not be Pareto optimal anymore (if $\tau(K) > 0$ there will be a wedge, and the efficiency conditions will not be satisfied).

Exercise 8 *Derive the first order conditions in the above problem to see the wedge introduced by taxes.*

3.3.4 Taxes and Debt

Assume that the government can now issue debt and use taxes to finance its expenditures. Also assume that agents derive utility from these government expenditures.

A government policy consists of capital taxes, spending (medals) as well as bond issuance. When the aggregate states are K and B, as you will see why, then a government policy (in a recursive world) is

$$\tau(K,B)\,,\ M(K,B)\ \text{and}\ B'(K,B)\,.$$

For now, we shall assume these values are chosen so that the equilibrium exists. In this environment, debt issued is relevant for the household because it permits him to correctly infer the amount of taxes. Therefore the household needs to form expectations about the future level of debt from the government.

The government budget constraint now satisfies (with taxes on capital income):

$$B + M(K,B) = \tau(K,B)R(K)K + q(K,B)B'(K,B)$$

Notice that the household does not care about the composition of his portfolio as long as assets have the same rate of return, which is true because of the no arbitrage condition.

The problem of a household with assets a is given by:

$$\begin{aligned} V(K,B,a) \quad = \quad \max_{c,a'} \qquad & u(c, M(K,B)) \quad + \quad \beta V(K',B',a') \\ \text{s.t.} \quad c + a' &= w(K) + aR(K)\,(1-\tau(K,B)) \\ K' &= G(K,B) \\ B' &= H(K,B) \\ c &\geq 0 \end{aligned}$$

Let $g\,(K,B,a)$ be the policy function associated with this problem. Then, we can define a RCE as follows.

Definition 4 *A* Rational Expectations Recursive Competitive Equilibrium, *given policies $M(K,B)$ and $\tau(K,B)$, is a set of functions V, g, G, H, w, q, and R, such that*

1. *Given w and R, V and g solve the household's problem,*

2. *Factor prices are paid their marginal productivities*

   $w(K) = F_2(K,1)$ *and* $R(K) = F_1(K,1)$,

3. *Household wealth = Aggregate wealth*

   $$g(K,B,K+q(K^-,B^-)B) = G(K,B) + q(K,B)H(K,B),$$

4. *No arbitrage condition*

   $$\frac{1}{q(K,B)} = [1-\tau(G(K,B),H(K,B))]\,R(G(K)),$$

5. *Government's budget constraint holds*

   $$B + M(K,B) = \tau(K,B)R(K)K + q(K,B)H(K,B),$$

6. *Government debt is bounded; i.e.* $\exists$ *some* $\bar{B}$*, such that for all* $K \in \left[0, \widetilde{k}\right)$ *and* $B \leq \bar{B}$*,* $H(K, B) \leq \bar{B}$*.*

4 Some Other Examples

4.1 A Few Popular Utility Functions

Consider the following three utility forms:

1. $u(c, c^-)$: this function is called *habit formation* utility function. The utility is increasing in consumption today, but decreasing in the deviations from past consumption (e.g. $u(c, c^-) = v(c) - (c - c^-)^2$). Under habit persistence, an increase in current consumption lowers the marginal utility of consumption in the current period ($u''_{1,1} < 0$) and increases it in the next period ($u''_{1,2} > 0$). Intuitively, the more the agent eats today, the hungrier she will be tomorrow. The aggregate states in this setup are K and C^-, while the individual states are a and c^-.

   Definition 5 *A Recursive Competitive Equilibrium is a set of functions* V*,* g*,* G*,* w*, and* R*, such that*

   (a) Given w *and* R*,* V *and* g *solve the household's problem,*

   (b) Factor prices are paid their marginal productivities

   $$w(K, C^-) = F_2(K, 1) \text{ and } R(K, C^-) = F_1(K, 1),$$

   (c) Household wealth = Aggregate wealth

   $$g(K, C^-, K, F(G^{-1}(K), 1) - K) = G(K, C^-).$$

   Exercise 9 *Is the equilibrium optimal in this case?*

2. $u(c, C^{-})$: this form is called *catching up with Jones*. There is an externality from aggregate consumption to the agent's payoff. Intuitively, agents care about what their neighbors consume. The aggregate states in this case are K and C^{-}, while c^{-} is no longer an individual state.

   Exercise 10 *How does the agent know C?*

   Exercise 11 *Is the equilibrium optimal in this case?*

3. $u(c, C)$: this form is called *keeping up with Jones*. The aggregate state is K and C is no longer a predetermined variable.

   Exercise 12 *How does the agent know C?*

   Exercise 13 *Is the equilibrium optimal in this case?*

4.2 An Economy with Capital and Land

Consider an economy with capital and land but without labor. A firm in this economy buys and installs capital, and owns one unit of land that is used in production, according to the technology $F(K, L)$. In other words, a firm is a "chunk of land of area one" (e.g. farmland), in which it installs its own capital (e.g. tractors). The firm's shares are traded in a stock market, which are bought by households.

A household's problem in this economy is given by:

$$\begin{aligned} V(K, a) = \max_{c, a'} \quad & u(c) \quad + \quad \beta V(K', a') \\ \text{s.t.} \quad c + P(K)a' \quad & = \quad a\left[D(K) + P(K)\right] \\ K' \quad & = \quad G(K) \end{aligned}$$

where a are shares held by the household, $P(K)$ is their price, and $D(K)$ are dividends per share.

The firm's problem is given by

$$
\begin{aligned}
\Omega(K,k) = \max_{d,k'} \quad d \quad &+ \quad q(K')\Omega(K',k') \\
\text{s.t.} \quad F(k,1) \quad &= \quad d + k' \\
K' \quad &= \quad G(K)
\end{aligned}
$$

Ω here is the *value of the firm,* measured in units of output today. The value of the firm tomorrow must be discounted into units of output today, which is done by the discount factor $q(K')$. Note that the firm needs to know K', using the aggregate law of motion G to do so.

Definition 6 *A Recursive Competitive Equilibrium consists of functions, V, Ω, h, g, d, q, D, P, and G so that:*

1. *Given prices, V and h solve the household's problem,*
2. *Ω, g, and d solve the firm's problem,*
3. *Representative household holds all shares of the firm*

   $$h(K,1) = 1,$$

4. *The capital of the firm when it is representative must equal the aggregate stock of capital*

   $$g(K,K) = G(K),$$

5. *Value of a representative firm must equal its price and dividends*

   $$\Omega(K,K) = D(K) + P(K),$$

6. *The dividends of the representative firm must equal aggregate dividends*

   $$d(K,K) = D(K)$$

Exercise 14 *One condition is missing in the definition of the RCE above. Find it!* [Hint: it relates the discount factor of the firm $q(G(K))$ with the price and dividends households receive $(P(K), P(G(K)),$ and $D(G(K)))$.]

Exercise 15 *Define the RCE if a were savings paying $R(K)$ as opposed to $shares$ of the firm.*

5 Adding Heterogeneity

In the previous section we looked at situations in which recursive competitive equilibria (RCE) were useful. In particular these were situations in which the welfare theorems failed and so we could not use the standard dynamic programming techniques learned earlier. In this section we look at another way in which RCE are helpful, in particular in models with heterogeneous agents.

5.1 Heterogeneity in Wealth

First, let us consider a model in which we have two types of households that differ only in the amount of wealth they own. Say there are two types of agents, labeled type R (for rich) and P (for poor), of measure μ and $1-\mu$ respectively. Agents are identical other than their initial wealth position and there is no uncertainty in the model. The problem of an agent with wealth a is given by

$$\begin{aligned} V(K^R, K^P, a) = \max_{c,a'} \quad & u(c) \quad + \quad \beta V(K^{R'}, K^{P'}, a') \\ \text{s.t.} \quad c + a' \quad & = \quad w(\mu K^R + (1-\mu)K^P) + aR(\mu K^R + (1-\mu)K^P) \\ K^{i'} \quad & = \quad G^i(K^R, K^P) \quad \text{for } i = R, P. \end{aligned}$$

Remark 2 $K = \mu K^R + (1-\mu)K^P$ *is a sufficient statistic for factor prices but not for* $K^{i'}$. *Note that (in general) the decision rules of the two types of agents are not linear (even though they might be* almost *linear); therefore, we cannot add the two states,* K^R *and* K^P, *to write the problem with one aggregate state, in the recursive form.*

Definition 7 *A Recursive Competitive Equilibrium is a set of functions V, g, w, R, G^1, and G^2 such that that:*

1. *Given prices, V solves the household's functional equation, with g as the associated policy function,*

2. *w and R are the marginal products of labor and capital, respectively (watch out for arguments!),*

3. *Consistency: representative agent conditions are satisfied, i.e.*

$$g(K^R, K^P, K^R) = G^R(K^R, K^P),$$

and

$$g(K^R, K^P, K^P) = G^P(K^R, K^P).$$

Remark 3 *Note that $G^R\left(K^R, K^P\right) = G^P\left(K^P, K^R\right)$ (look at the arguments carefully). Why? (How are rich and poor different?) What about value functions?*

Remark 4 *This is a variation of the simple neoclassical growth model. What does the neoclassical growth model say about inequality? In the steady state, the Euler equations for the two different types simplify to*

$$u'\left(c^{R^*}\right) = \beta R u'\left(c^{R^*}\right), \text{ and } u'\left(c^{P^*}\right) = \beta R u'\left(c^{P^*}\right).$$

and we thus have $\beta R = 1$, where

$$R = F_K\left(\mu K^{R^*} + (1-\mu)K^{P^*}, 1\right).$$

Finally, by the household's budget constraint, we must have:

$$c^i + a^i = w + a^i R \quad \text{for } i = R, P$$

where $a^i = K^i$ by the representative agent's condition. Therefore, we have three equations (budget constraints and Euler equation) and four unknowns (a^{i^*} and c^{i^*} for $i = R, P$). This implies that this theory is silent about the distribution of wealth in the steady state!

This is an important implication of the aggregation property. In fact, in the neoclassical growth model with n agents that only differ in their initial endowments, with homothetic preferences, there is a continuum with dimension $n - 1$ of steady state wealth distributions.

As we will see throughout the course, heterogeneity will matter in various situations. In the setup we have discussed above, however, wealth heterogeneity did not matter. This aggregation property applied to our macroeconomic context (see Gorman's aggregation theorem for further details) states that if agents' individual savings decision is linear in their individual state (i.e. $g(K, a) = \mu^i(K) + \lambda(K)a$, with $\lambda(K)$ being the marginal propensity to save common to all agents) provided that they all have the same preferences, then aggregate capital can be expressed as the choice of a representative agent (with savings decision given by $g(K, K) = \bar{\mu}(K) + \lambda(K)K$).

Remark 5 *Does this property hold when discount factors or coefficients of relative risk aversion are heterogeneous?*

5.2 Heterogeneity in Skills

Now, consider a slightly different economy in which type i has labor skill ϵ_i. Measures of agents' types, μ^1 and μ^2, satisfy $\mu^1\epsilon_1 + \mu^2\epsilon_2 = 1$ (below we will consider the case in which $\mu^1 = \mu^2 = 1/2$).

The question we have to ask ourselves is: would the value functions of the two types remain the same, as in the previous subsection? The answer turns out to be no!

The problem of the household $i \in \{1,2\}$ can be written as follows:

$$
\begin{aligned}
V^i(K^1,K^2,a) = \max_{c,a'} \quad & u(c) \quad + \quad \beta V^i(K^{1'},K^{2'},a') \\
\text{s.t.} \quad c + a' \quad & = \quad w\left(\frac{K^1+K^2}{2}\right)\epsilon_i + aR\left(\frac{K^1+K^2}{2}\right) \\
K^{i'} \quad & = \quad G^i(K^1,K^2) \quad \text{for } i = 1,2.
\end{aligned}
$$

Notice that we have indexed the value function by the agent's type and thus the policy function is also indexed by i. The reason is that the marginal product of the labor supplied by each of these types is different (think of $w^i\left(\frac{K^1+K^2}{2}\right) = w\left(\frac{K^1+K^2}{2}\right)\epsilon_i$).

Exercise 16 *Define the RCE.*

Remark 6 *We can also rewrite this problem as*

$$
\begin{aligned}
V^i(K,\lambda,a) = \max_{c,a'} \quad & \left\{u(c) + \beta V^i(K',\lambda',a')\right\} \\
s.t. \quad & c + a' = R(K)a + W(K)\epsilon_i \\
& K' = G(K,\lambda) \\
& \lambda' = H(K,\lambda),
\end{aligned}
$$

where K is the total capital in this economy, and λ is the share of one type in total wealth (e.g. type 1).

Then, if g^i is the policy function of type i, then the consistency conditions of the RCE must be:

$$
G(K,\lambda) = \frac{1}{2}\left[g^1(K,\lambda,2\lambda K) + g^2(K,\lambda,2(1-\lambda)K)\right],
$$

and

$$
H(K,\lambda) = \frac{g^1(K,\lambda,2\lambda K)}{2G(K,\lambda)}.
$$

5.3 An International Economy Model

In an international economy model the definition of country is an important one. We can introduce the idea of different locations or geography, countries can be victims of different policies, trade across countries maybe more difficult due to different restrictions.

Here we will focus on a model with two countries, 1 and 2, where labor is not mobile between the countries, but capital markets perfect and thus investment can flow freely across countries. However, in order to use it in production, it must have been installed in advanced. Traded goods flow within the period.

The aggregate resource constraint is:

$$C^1 + C^2 + K^{1'} + K^{2'} = F(K^1, 1) + F(K^2, 1)$$

Suppose that there is a mutual fund that owns the firms in each country and chooses labor in each country and capital to be installed. Its shares are traded in the market and thus, as in the economy with capital and land, individuals own shares of this mutual fund.

The first question to ask, as usual, is *what are the appropriate states in this world?* As it is apparent from the resource constraint and production functions, we need the capital in each country. Moreover, we need to know total wealth in each country. Therefore, we need an additional variable as the aggregate state: the shares owned by country 1 is a sufficient statistic.

We can then write the country i's household problem as:

$$\begin{aligned} V^i\left(K^1, K^2, A, a\right) = \max_{c,a'} \quad & u(c) + \beta V^i\left(K^{1'}, K^{2'}, A', a'\right) \\ s.t. \quad & c + Q(K^1, K^2, A)a' = w^i\left(K^i\right) + a\Phi(K^1, K^2, A) \\ & K^{i'} = G^i\left(K^1, K^2, , A\right), \quad \text{for } i = 1, 2 \\ & A' = H\left(K^1, K^2, A\right) \end{aligned}$$

where A is the total amount of shares in the mutual fund that individuals in country 1 own and a is the amount of shares that an individual owns in country i.

Since labor is immobile and capital is installed in advanced, the wage is country-specific and is simply given by the marginal product of labor: $w^i(K^i) = F^i_N(K^i, 1)$.

Now let's look at the problem of the mutual fund:

$$\begin{aligned}
\Phi(K^1, K^2, A, k^1, k^2) &= \max_{k^{1'}, k^{2'}, n^1, n^2} \sum_i \left[F^i(k^i, n^i) - n^i w^i(K_i) - k^{i'} \right] + \\
&\qquad \frac{1}{R(K^{1'}, K^{2'}, A)} \Phi(K^{1'}, K^{2'}, A', k^{1'}, k^{2'}) \\
s.t. \qquad & K^{i'} = G^i\left(K^1, K^2, , A\right), \quad \text{for } i = 1, 2 \\
& A' = H\left(K^1, K^2, A\right)
\end{aligned}$$

Definition 8 *A Recursive Competitive Equilibrium for the (world's) economy is a list of functions,* $\{V^i, h^i, g^i, n^i, w^i, G^i\}_{i=1,2}$, Φ, H, Q, *and* R, *such that the following conditions hold:*

1. *Given prices and aggregate laws of motion,* V^i *and* h^i *solve the household's problem in country* i *(for* $i \in \{1, 2\}$*),*
2. *Given prices and aggregate laws of motion,* Φ, $\{g^i, n^i\}_{i=1,2}$ *solve the mutual fund problem,*
3. *Labor markets clear*

$$n^i(K^1, K^2, A, K^1, K^2) = 1 \qquad \text{for } i = 1, 2,$$

4. *Consistency (MF)*

$$g^i(K^1, K^2, A, K^1, K^2) = G^i(K^1, K^2, A) \qquad \text{for } i = 1, 2,$$

5. *Consistency (Households)*

$$h^1(K^1, K^2, A, A) = H(K^1, K^2, A)$$

and

$$h^1(K^1, K^2, A, A) + h^2(K^1, K^2, A, 1 - A) = 1$$

6. *No arbitrage*

$$Q(K^1, K^2, A) = \frac{1}{R(K^{1'}, K^{2'}, A')}\Phi(K^{1'}, K^{2'}, A', K^{1'}, K^{2'})$$

Exercise 17 *Solve for the mutual fund's decision rules. Is next period capital in each country chosen by the mutual fund priced differently? What about labor?*

6 Stochastic Economies

6.1 A Review

6.1.1 Markov Processes

From now on, we will focus on stochastic economies, in which productivity shocks affects the economy. The stochastic process for productivity that we assume is a first-order Markov Process that takes on a finite number of values in the set $Z = \{z^1 < \cdots < z^{n_z}\}$. A first order Markov process implies

$$\mathsf{Pr}(z_{t+1} = z^j | h_t) = \Gamma_{ij}, \qquad z_t(h_t) = z^i$$

where h_t is the history of previous shocks. Γ is a Markov chain with the property that the elements of each rows sum to 1.

Let μ be a probability distribution over initial states, i.e.

$$\sum_i \mu_i = 1$$

and $\mu_i \geq 0 \; \forall i = 1, ..., n_z$.

For next periods the probability distribution can be found by $\mu' = \Gamma^T \mu$.

If Γ is "nice" then $\exists$ a unique μ^* s.t. $\mu^* = \Gamma^T \mu^*$ and $\mu^* = \lim_{m \to \infty} (\Gamma^T)^m \mu_0$, $\forall \mu_0 \in \Delta^i$.

Γ induces the following probability distribution conditional on the initial draw z_0 on $h_t = \{z^0, z^1, ..., z^t\}$:

$\Pi(\{z^0, z_1\}) = \Gamma_{i,\cdot} \qquad$ for $z^0 = z_i$.

$\Pi(\{z^0, z_1, z_2\}) = \Gamma^T \Gamma_{i,\cdot} \qquad$ for $z^0 = z_i$.

Then, $\Pi(h_t)$ is the probability of history h_t conditional on z^0. The expected value of z' is $\sum_{z'} \Gamma_{zz'} z'$ and $\sum_{z'} \Gamma_{zz'} = 1$.

6.1.2 Problem of the Social Planner

Let productivity affect the production function in a multiplicative fashion; i.e. technology is $zF(K, N)$, where z is a shock that follows a Markov chain on a finite state-space. Recall that the problem of the social planner problem (SPP) in sequence form is

$$\max_{\{c_t(z^t), k_{t+1}(z^t)\} \in X(z^t)} \quad \sum_{t=0}^{\infty} \sum_{z^t} \beta^t \pi(z^t) u(c_t(z^t))$$
$$s.t. \quad c_t(z^t) + k_{t+1}(z^t) = z_t F(k_t(z^{t-1}), 1),$$

where z_t is the realization of the shock in period t, and z^t is the history of shocks up to (and including) time t. $X(z^t)$ is similar to the consumption possibility set defined earlier but this is after history z^t has occurred and is for consumption and next period capital.

We can then formulate the stochastic SPP in a recursive fashion as

$$V(z_i, K) = \max_{c,K'} \quad \left\{ u(c) + \beta \sum_j \Gamma_{ij} V(z_j, K') \right\}$$
$$s.t. \quad c + K' = z_i F(K, 1),$$

where Γ is the Markov transition matrix. The solution to this problem gives us a policy function of the form $K' = G(z, K)$.

In a decentralized economy, the Arrow-Debreu equilibrium can be defined by:

$$\max_{\{c_t(z^t), k_{t+1}(z^t), x_{1t}(z^t), x_{2t}(z^t), x_{3t}(z^t)\} \in X(z^t)} \quad \sum_{t=0}^{\infty} \sum_{z^t} \beta^t \pi(z^t) u(c_t(z^t))$$
$$s.t. \quad \sum_{t=0}^{\infty} \sum_{z^t} p_t(z^t).x_t(z^t) \leq 0,$$

where $X(z^t)$ is again a variant of the consumption possibility set after history z^t has occurred. Ignore the overloading of notation. Note that we are assuming the markets are dynamically complete; i.e. there is a complete set of securities for every possible history.

By the same procedure as before, the SME can be written in the following way:

$$\max_{\{c_t(z^t), b_{t+1}(z^t, z_{t+1}), k_{t+1}(z^t)\}} \quad \sum_{t=0}^{\infty} \sum_{z^t} \beta^t \pi(z^t) u(c_t(z^t))$$
$$s.t. \quad c_t(z^t) + k_{t+1}(z^t) + \sum_{z_{t+1}} q_t(z^t, z_{t+1}) b_{t+1}(z^t, z_{t+1})$$
$$= k_t(z^{t-1}) R_t(z^t) + w_t(z^t) + b_t(z^{t-1}, z_t)$$
$$b_{t+1}\left(z^t, z_{t+1}\right) \geq -B.$$

To replicate the AD equilibrium, we have introduced Arrow securities to allow agents to trade with each other against possible future shocks.

Note that when there is no heterogeneity, there will be no trade in equilibrium, i.e $b_{t+1}(z^t, z_{t+1}) = 0$ for any z^t, z_{t+1}. Moreover, we have two ways of delivering the goods specified in an Arrow security contract: *after production* and *before production*. In an after production setting, the goods will be delivered after production takes place and can only be consumed or saved for the next period. This is the above setting. It is also possible to allow the consumer to rent the Arrow security income as capital to firms, which will correspond to the before production setting.

An important condition that must hold true in the *before production setting* is the no-arbitrage condition:

$$\sum_{z_{t+1}} q_t(z^t, z_{t+1}) = 1$$

Exercise 18 *Every equilibrium achieved in AD settings can also be achieved in a SM setting, by the relation where*

$$q_t(z^t, z_{t+1}) = \frac{p_{1t+1}(z^t, z_{t+1})}{p_{1t}(z^t)},$$

$$R_t(z^t) = \frac{p_{2t}(z^t)}{p_{1t}(z^t)},$$

and

$$w_t(z^t) = \frac{p_{3t}(z^t)}{p_{1t}(z^t)}.$$

Check from the FOC's that the we get the same allocations in the two settings.

Exercise 19 *The problem above assumes state contingent goods are delivered in terms of consumption goods. Instead of this assume they are delivered in terms of capital goods. Show that the same allocation would be achieved in both settings.*

6.1.3 Recursive Competitive Equilibrium

Assume that households can trade state contingent assets, as in the sequential markets case above. Then, we can write a household's problem in recursive form as:

$$\begin{aligned} V(K,z,a) = \max_{c,k',b'(z')} \quad & u(c) + \beta \sum_{z'} \Gamma_{zz'} V(K', z', a'(z')) \\ s.t. \quad & c + k' + \sum_{z'} q(K,z;z') b'(z') = w(K,z) + aR(K,z) \\ & K' = G(K,z) \\ & a'(z') = k' + b'(z'). \end{aligned}$$

Exercise 20 *Write the FOC's for this problem, given prices and the law of motion for aggregate capital.*

Definition 9 *A Recursive Competitive Equilibrium is a collection of functions V, g^k, $g^b_{z'}$, w, R, $q_{z'}$, and G so that*

1. *Given G, w, and R, V solves the household's functional equation, with g^k and g^b as the associated policy function,*
2. *$g^b(K,z,K;z') = 0$, for all z',*
3. *$g^k(K,z,K) = G(K,z)$,*
4. *$w(K,z) = zF_n(K,1)$ and $R(K,z) = zF_k(K,1)$,*
5. *and $\sum_{z'} q(K,z;z') = 1$.*

The last condition is known as the no-arbitrage condition (recall that we had this equation in the case of sequential markets as well). To see why this is a necessary equation in the equilibrium, note that an agent can either save in the form of capital or through Arrow securities. However, these two choices must cost the same, which implies Condition 5 above.

Remark 7 *Note that in the SME version of the household problem, in order for households not to achieve infinite consumption, we need a no-Ponzi condition. Such condition is*

$$\lim_{t\to\infty} \frac{a_t}{\prod_{s=0}^{t} R_s} < \infty.$$

This is the weakest condition that imposes no restrictions on the first order conditions of the household's problem. It is harder to come up with its analogue for the recursive case. One possibility is to assume that a' lies in a compact set $\mathcal{A}$, or a set that is bounded from below.[4]

6.2 A Stochastic International Economy Model

We revisit the international economy model studied before and we now add country-specific shocks. Let z_1 and z_2 represent productivity shocks in country 1 and 2, respectively. The aggreagte state variables are now the productivity shocks, the aggregate stocks of capital in each country, and the amount of shares owned by country 1 in the mutual fund.

The problem of an household in country i is:

$$\begin{aligned} V^i\left(\vec{z}, \vec{K}, A, a\right) = \max_{c, a'(\vec{z}')} \quad & u(c) + \beta \sum_{\vec{z}'} \Gamma_{\vec{z}\vec{z}'} V^i\left(\vec{z}', \vec{K}', A'(\vec{z}'), a'(\vec{z}')\right) \\ s.t. \quad & c + \sum_{\vec{z}'} q(\vec{z}, \vec{K}, A; \vec{z}') a'(\vec{z}') = w^i(z_i, K_i) + a\Phi(\vec{z}, \vec{K}, A) \\ & K_i' = G_i\left(\vec{z}, \vec{K}, A\right), \quad \text{for } i = 1, 2 \\ & A'(\vec{z}') = H\left(\vec{z}, \vec{K}, A; \vec{z}'\right) \quad \forall \vec{z}' \end{aligned}$$

Let decision rule for next period asset holdings be $a'(\vec{z}') = h(\vec{z}, \vec{K}, A, a; \vec{z}') \quad \forall \vec{z}'$. Note the financial market structure assumed here. The agent is fully insured against all possible states of the world. As before, labor is immobile and thus wages are country-specific and given by $w^i(z_i, K_i) = z_i F_N(K_i, 1)$.

4 We must specify $\mathcal{A}$ such that the borrowing constraint implicit in $\mathcal{A}$ is never binding.

Exercise 21 *Write this economy with state-contingent claims in own country only.*

Exercise 22 *Write this economy where individuals can move freely in advance, but with incomplete markets.*

Now let's look at the net present value of the mutual fund in equilibrium:

$$\begin{aligned}\Phi(\vec{z}, \vec{K}, A) &= \sum_{z_i} \left[z_i F(K_i, 1) - w^i(z_i, K_i)\right] - \sum_i G_i(\vec{z}, \vec{K}, A) + \\ &\sum_{\vec{z}'} \Gamma_{\vec{z}\vec{z}'} Q(\vec{z}', G(\vec{z}, \vec{K}, A), H(\vec{z}, \vec{K}, A; \vec{z}')) \Phi(\vec{z}', G(\vec{z}, \vec{K}, A), H(\vec{z}, \vec{K}, A; \vec{z}')) \qquad (5)\end{aligned}$$

where Q (or $\frac{1}{R}$) represents intertemporal prices, which in equilibrium should satisfy $\forall \vec{z}'$:

$$q(\vec{z}, \vec{K}, A; \vec{z}') = \Gamma_{\vec{z}\vec{z}'} Q(\vec{z}', G(\vec{z}, \vec{K}, A), H(\vec{z}, \vec{K}, A; \vec{z}')) \Phi(\vec{z}', G(\vec{z}, \vec{K}, A), H(\vec{z}, \vec{K}, A; \vec{z}'))$$

Exercise 23 *There is one more condition for G_i that equates expected return in each country. What is it?*

Definition 10 *A Recursive Competitive Equilibrium for the (world's) economy is a set of functions V^i, $h^i_{\vec{z}}$, Φ, w^i, Q, $q_{\vec{z}}$, G_i, and H, for $i \in \{1, 2\}$, such that the following conditions hold:*

1. *Given prices and laws of motion, V^i and h^i solve the household's problem in country i for $i \in \{1, 2\}$,*
2. *The mutual fund's value Φ satisfies equation 5*
3. *$w^i(z_i, K_i)$ is equated to the marginal products of labor in each country i for $i \in \{1, 2\}$*
4. *The expected rates of return on capital is equalized across countries*
5. *The market for shares in the mutual fund clears*
   $h^1\left(\vec{z}, \vec{K}, A, A; \vec{z}'\right) + h^2\left(\vec{z}, \vec{K}, A, 1 - A; \vec{z}'\right) = 1 \quad \forall \vec{z}'$,

6. *The representative agent condition must hold*
$h^1\left(\vec{z}, \vec{K}, A, A; \vec{z}'\right) = H\left(\vec{z}, \vec{K}, A, A; \vec{z}'\right) \quad \forall \vec{z}'$,

7. *No arbitrage*

$$q(\vec{z}, \vec{K}, A; \vec{z}') = \Gamma_{\vec{z}\vec{z}'} Q(\vec{z}', G(\vec{z}, \vec{K}, A), H(\vec{z}, \vec{K}, A; \vec{z}'))\Phi(\vec{z}', G(\vec{z}, \vec{K}, A), H(\vec{z}, \vec{K}, A; \vec{z}')) \quad \forall \vec{z}'$$

8. *The aggregate resource constraint must hold:*

$$\sum_i \Big[z_i F(K_i, 1) - G_i(\vec{z}, \vec{K}, A) - \Big(w^i(z_i, K_i) + A_i \Phi(\vec{z}, \vec{K}, A) - \sum_{\vec{z}'} q(\vec{z}, \vec{K}, A; \vec{z}') h^i\left(\vec{z}, \vec{K}, A, A_i; \vec{z}'\right) \Big) \Big] = 0$$

where $A_1 = A$ *and* $A_2 = 1 - A$.

6.3 Heterogeneity in Wealth and Skills with Complete Markets

Now, let us consider a model in which we have two types of households, with equal measure $\mu_i = 1/2$, that care about leisure, but differ in the amount of wealth they own as well as their labor skill. There is also uncertainty and Arrow securities like we have seen before.

Let A^1 and A^2 be the aggregate asset holdings of the two types of agents. These will now be state variables for the same reason K^1 and K^2 were state variables earlier. The problem of an agent $i \in \{1, 2\}$

with wealth a is given by

$$
\begin{aligned}
V^i\left(z,A^1,A^2,a\right) = \max_{c,n,a'(z')} \quad & u\left(c,n\right) + \beta \sum_{z'} \Gamma_{zz'} V^i\left(z',A^{1'}(z'),A^{2'}(z'),a'(z')\right) \\
s.t. \quad & c + \sum_{z'} q\left(z,A^1,A^2,z'\right) a'\left(z'\right) = R\left(z,K,N\right) a + W\left(z,K,N\right) \epsilon_i n \\
& A^{i'}(z') = G^i\left(z,A^1,A^2,z'\right), \quad \text{for } i = 1,2, \forall z' \\
& N = H\left(z,A^1,A^2\right) \\
& K = \frac{A^1 + A^2}{2}.
\end{aligned}
$$

Let $g^i(z,A^1,A^2,a^i)$ and $h^i(z,A^1,A^2,a^i)$ be the asset and labor policy functions be the solution of each type i to this problem. Then, we can define the RCE as below.

Definition 11 *A* Recursive Competitive Equilibrium with Complete Markets *is a set of functions V^i, g^i, h^i, G^i for $i \in \{1,2\}$, R, w, H, and q, such that:*

1. *Given prices and laws of motion, V^i, g^i and h^i solve the problem of household i for $i \in \{1,2\}$,*
2. *Labor markets clear:*
   $H\left(z,A^1,A^2\right) = \epsilon_1 h^1\left(z,A^1,A^2,A^1\right) + \epsilon_2 h^2\left(z,A^1,A^2,A^2\right)$,
3. *The representative agent condition:*
   $G^i\left(z,A^1,A^2,z'\right) = g^i\left(z,A^1,A^2,A^i,z'\right)$ *for* $i = 1,2,\forall z'$
4. *The average price of the Arrow security must satisfy:*
   $\sum_{z'} q\left(z,A^1,A^2,z'\right) = 1$,
5. $G^1\left(z,A^1,A^2,z'\right) + G^2\left(z,A^1,A^2,z'\right)$ *is independent of z' (due to market clearing).*
6. *R and W are the marginal products of capital and labor.*

Exercise 24 *Write down the household problem and the definition of RCE with non-contingent claims instead of complete markets.*

7 Asset Pricing: Lucas Tree Model

We now turn to the simplest of all models in term of allocations as they are completely exogenous, called the *Lucas tree model*. We want to characterize the properties of prices that are capable of inducing households to consume the stochastic endowment.

7.1 The Lucas Tree with Random Endowments

Consider an economy in which the only asset is a tree that gives fruit. The agent's problem is to choose consumption c and the amount of shares of the tree to hold s' according to

$$\begin{aligned} V(z,s) = \max_{c,s'} \quad & u(c) + \beta \sum_{z'} \Gamma_{zz'} V(z',s') \\ s.t. \quad & c + p(z)\, s' = s\,[p(z) + d(z)], \end{aligned}$$

where $p(z)$ is the price of the shares (to the tree), in state z, and $d(z)$ is the dividend associated with state z.

Definition 12 *A Rational Expectations Recursive Competitive Equilibrium is a set of functions, V, g, d, and p, such that*

1. *V and g solve the household's problem given prices,*
2. *$d(z) = z$, and,*
3. *$g(z,1) = 1$, for all z.*

To explore the problem further, note that the FOC for the household's problem imply the equilibrium condition

$$u_c(c(z,1)) = \beta \sum_{z'} \Gamma_{zz'} \left[\frac{p(z') + d(z')}{p(z)} \right] u_c(c(z',1)).$$

Define $u_c(z) := u_c(c(z,1))$. Then this simplifies to

$$p(z)\,u_c(z) = \beta \sum_{z'} \Gamma_{zz'} u_c(z')\,[p(z') + z'] \qquad \forall z.$$

Exercise 25 *Derive the Euler equation for household's problem to show the result above.*

Note that this is just a system of n_z equations with n_z unknowns $\{p(z_i)\}_{i=1}^{n}$. We can use the power of matrix algebra to solve the system. To do so, let:

$$\mathbf{p} := \begin{bmatrix} p(z_1) \\ \vdots \\ p(z_n) \end{bmatrix}_{(n_z \times 1)},$$

and

$$\mathbf{u}_c := \begin{bmatrix} u_c(z_1) & & 0 \\ & \ddots & \\ 0 & & u_c(z_n) \end{bmatrix}_{(n_z \times n_z)}.$$

Then

$$\mathbf{u}_c.\mathbf{p} = \begin{bmatrix} p(z_1)\,u_c(z_1) \\ \vdots \\ p(z_n)\,u_c(z_n) \end{bmatrix}_{(n_z \times 1)},$$

and

$$\mathbf{u}_c.\mathbf{z} = \begin{bmatrix} z_1 u_c(z_1) \\ \vdots \\ z_n u_c(z_n) \end{bmatrix}_{(n_z \times 1)},$$

Now, rewrite the system above as

$$\mathbf{u}_c\mathbf{p} = \beta\Gamma\mathbf{u}_c\mathbf{z} + \beta\Gamma\mathbf{u}_c\mathbf{p},$$

where Γ is the transition matrix for z, as before. Hence, the price for the shares is given by

$$\left(\mathbf{I}_{\mathbf{n_z}} - \beta\Gamma\right)\mathbf{u}_c\mathbf{p} = \beta\Gamma\mathbf{u}_c\mathbf{z},$$

or

$$\mathbf{p} = \left(\left[\mathbf{I}_{\mathbf{n_z}} - \beta\Gamma\right]\mathbf{u}_c\right)^{-1}\beta\Gamma\mathbf{u}_c\mathbf{z},$$

where $\mathbf{p}$ is the vector of prices that clears the market.

7.2 Asset Pricing

Consider our simple model of Lucas tree with fluctuating output. What is the definition of an asset in this economy? It is "a claim to a chunk of fruit, sometime in the future."

If an asset, a, promises an amount of fruit equal to $a_t\left(z^t\right)$ after history $z^t = \left(z_0, z_1, \ldots, z_t\right)$ of shocks, after a set of (possible) histories in H_t, the price of such an entitlement in date $t = 0$ is given by:

$$p\left(a\right) = \sum_t \sum_{z^t \in H_t} q_t^0\left(z^t\right) a_t\left(z^t\right),$$

where $q_t^0\left(z^t\right)$ is the price of one unit of fruit after history z^t in units of today's fruit; this follows from a no-arbitrage argument. If we have the date $t = 0$ prices, $\{q_t\}$, as functions of histories, we can *replicate any possible asset by a set of state-contingent claims* and use this formula to price that asset.

To see how we can find prices at date $t=0$, consider a world in which the agent wants to solve

$$\max_{c_t(z^t)} \quad \sum_{t=0}^{\infty} \beta^t \sum_{z^t} \pi_t\left(z^t\right) u\left(c_t\left(z^t\right)\right)$$
$$s.t. \quad \sum_{t=0}^{\infty} \sum_{z^t} q_t^0\left(z^t\right) c_t\left(z^t\right) \leq \sum_{t=0}^{\infty} \sum_{h^t} q_t^0\left(z^t\right) z_t.$$

This is the familiar Arrow-Debreu market structure, where the household owns a tree, and the tree yields $z \in Z$ amount of fruit in each period. The FOC for this problem imply:

$$q_t^0\left(z^t\right)=\beta^t \pi_t\left(z^t\right) \frac{u_c\left(z_t\right)}{u_c\left(z_0\right)}.$$

This enables us to price the good in each history of the world and price any asset accordingly.

To see how we can price an asset given today's shock is z, consider *the option to sell it tomorrow* at price P as an example. The price of such an asset today is

$$\hat{q}(z, P)=\sum_{z'} q\left(z, z'\right) \max \left\{P-p\left(z'\right), 0\right\},$$

where the agent has the option not to sell it. The American option to sell at price P either tomorrow or the day after tomorrow is priced as:

$$\widetilde{q}(z, P)=\sum_{z'} q\left(z, z'\right) \max \left\{P-p\left(z'\right), \hat{q}\left(z', P\right)\right\}.$$

Similarly, an European option to buy the asset at price P the day after tomorrow is priced as:

$$\bar{q}(z, P)=\sum_{z'} \sum_{z''} \max \left\{p\left(z''\right)-P, 0\right\} q\left(z', z''\right) q\left(z, z'\right).$$

Note that $R(z)=\left[\sum_{z'} q\left(z, z'\right)\right]^{-1}$ is the gross risk free rate, given today's shock is z. The unconditional gross risk free rate is then given by $R^f=\sum_z \mu_z^* R(z)$ where μ^* is the steady-state distribution of the shocks.

The average gross rate of return on the stock market is $\sum_z \mu_z^* \sum_{z'} \Gamma_{zz'} \left[\frac{p(z')+z'}{p(z)}\right]$ and the risk premium is the difference between this rate and the unconditional gross risk free rate (i.e. given by $\sum_z \mu_z^* \left(\sum_{z'} \Gamma_{zz'} \left[\frac{p(z')+z'}{p(z)}\right] - R(z)\right)$).

Exercise 26 *Use the expressions for p and q and the properties of the utility function to show that risk premium is positive.*

7.3 Taste Shocks

Consider an economy in which the only asset is a tree that gives fruits. The fruit is constant over time (normalized to 1) but the agent is subject to preference shocks for the fruit each period given by $\theta \in \Theta$, where θ evolves according to Markov transition probability $\Gamma_{\theta\theta'}$. The agent's problem in this economy is

$$
\begin{aligned}
V(\theta, s) = \max_{c, s'} \quad & \theta u(c) + \beta \sum_{\theta'} \Gamma_{\theta\theta'} V(\theta', s') \\
s.t. \quad & c + p(\theta) s' = s \left[p(\theta) + d(\theta)\right].
\end{aligned}
$$

The equilibrium is defined as before. The only difference is that, now, we must have $d(\theta) = 1$ since z is normalized to 1. What does it mean that the output of the economy is constant (fixed at one), but the tastes for this output change? In this setting, the function of the price is to convince agents to keep their consumption constant even in the presence of taste shocks. All the analysis follows through as before once we write the FOC's characterizing the prices of shares, $p(\theta)$, and state-contingent prices $q(\theta, \theta')$.

This is a simple model, in the sense that the household does not have a real choice regarding consumption and savings. Due to market clearing, household consumes what nature provides her. In each period, according to the state of productivity z and taste θ, prices adjusts such that household would like to consume z, which is the amount of fruit that the nature provides. In this setup, output is equal to z. If we look at the business cycle in this economy, the only source of output fluctuations is caused by nature. Everything is determined by the supply side of the economy and the demand side has no impact on output.

In next section, we are going to introduce search frictions to incorporate a role for the demand side into our model.

Exercise 27 *Find the state contingent prices in the Lucas Tree model with preference shocks.*

8 Endogenous Productivity in a Product Search Model

We will model the situation in which households need to find the fruit before consuming it.[5] Assume that households have to find the tree in order to consume the fruit. Finding trees is characterized by a constant returns to scale (increasing in both arguments) matching function $M(T, D)$,[6] where T is the *number of trees* in the economy and D is the aggregate *shopping effort* exerted by households when searching. The probability that a tree finds a shopper is given by $\frac{M(T,D)}{T}$, i.e. the total number of matches divided by the number of trees. The probability that a unit of shopping effort finds a tree is given by $\frac{M(T,D)}{D}$, i.e. the total number of matches divided by the economy's effort level.

Let's assume that $M(T, D)$ takes the form $D^{\varphi}T^{1-\varphi}$ and denote $\frac{1}{Q} := \frac{D}{T}$, i.e. the ratio of shoppers per trees, as capturing *the market tightness* (and thus $Q = \frac{T}{D}$). The probability of a household finding a tree is given by $\Psi^h(Q) := \frac{M(T,D)}{D} = Q^{1-\varphi}$ and thus the higher the number of people searching, the smaller the probability of a household finding a tree. The probability of a tree finding a household is then given by $\Psi^f(Q) := \frac{M(T,D)}{T} = Q^{-\varphi}$, and thus the higher the number of people searching, the higher the probability of a tree finding a shopper. Note that in this economy the number of trees is

[5] Think of fields in *The Land of Apples*, full of apples that are owned by firms; agents have to buy the apples. In addition, they have to search for them as well!

[6] What does the fact that M is constant returns to scale imply?

constant and equal to one.[7]

Let us assume households face a demand side shock θ and a supply side shock z. They follow independent Markov processes with transition probabilities $\Gamma_{\theta\theta'}$ and $\Gamma_{zz'}$, respectively. Households choose the consumption level c, the search effort exerted to get the fruit d, and the shares of the tree to hold next period s'. The household's problem can be written as

$$V(\theta, z, s) = \max_{c,d,s'} \quad u(c, d, \theta) + \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} V(\theta', z', s') \tag{6}$$

$$s.t. \quad c + P(\theta, z) s' = P(\theta, z) \left[s \left(1 + \hat{R}(\theta, z) \right) \right] \tag{7}$$

$$c = d \, \Psi^h (Q(\theta, z)) z. \tag{8}$$

where P is the price of the tree relative to that of consumption and $\hat{R}$ is the dividend income (in units of the tree). Note that the equation 7 is our standard budget constraint, while equation 8 corresponds to the shopping constraint.

Note some notation conventions here. $P(\theta, z)$ is in terms of consumption goods, while $\hat{R}(\theta, z)$ is in terms of shares of the tree (that's why we are using the hat). We could also write the household budget constraint in terms of the price of consumption relative to that of the tree. To do so, let's define $\hat{P}(\theta, z) = \frac{1}{P(\theta, z)}$ as the price of consumption goods in terms of the tree. Then the budget constraint can be defined as:

$$c\hat{P}(\theta, z) + s' = s \left(1 + \hat{R}(\theta, z) \right)$$

[7] It is easy to find the statements for Ψ^h and Ψ^f, given the Cobb-Douglas matching function:

$$\Psi^h(Q) = \frac{D^{\varphi} T^{1-\varphi}}{D} = \left(\frac{T}{D} \right)^{1-\varphi} = Q^{1-\varphi},$$

$$\Psi^f(Q) = \frac{D^{\varphi} T^{1-\varphi}}{T} = \left(\frac{T}{D} \right)^{-\varphi} = Q^{-\varphi}.$$

The question is: is Cobb-Douglas an appropriate choice for the matching function, or its choice is a matter of simplicity?

Let's maintain our notation with $P(\theta, z)$ and $\hat{R}(\theta, z)$ from now on. We can substitute the constraints into the objective and solve for d in order to get the Euler equation for the household. Using the market clearing condition in equilibrium, the problem reduces to one equation and two unknowns, $P(\theta, z)$ and $Q(\theta, z)$ (other objects, C,D and $\hat{R}$ are known functions of P and Q, and the amount shares of the tree in equilibrium is 1 as before). We thus still need another functional equation to solve for the equilibrium of this economy, i.e. we need to specify the search protocol. We now turn to one way of doing so.

Exercise 28 *Derive the Euler equation of the household from the problem defined above.*

8.1 Competitive Search

Competitive search is a particular search protocol of what is called non-random (or directed) search.[8] To understand this protocol, consider a world consisting of a large number of islands. Each island has a sign that displays two numbers, $P(\theta, z)$ and $Q(\theta, z)$. $P(\theta, z)$ is the price on the island and $Q(\theta, z)$ is a measure of market tightness in that island (or if the price is a wage rate W, then Q is the number of workers on the island divided by the number of job opportunities in that island). Both individuals and firms have to decide to go to one island. For instance, in an island with a higher wage, the worker might have a higher income conditional on finding a job. However, the probability of finding a job might be low on that island given the tightness of the labor market on that island. The same story holds for the job owners, who are searching to hire workers.

In our economy, both firms and workers search for specific markets indexed by price P and a market tightness Q.[9] An island, or a pair of (P, Q), is operational if there exists some consumer and firm choosing that market. Therefore, an agent should choose P and Q such that it gives sufficient profit to the firm, so that it wants to be in that island as opposed to doing something else, which will be determined in the equilibrium. Competitive search is magic in the sense that it does not presuppose a particular pricing protocol that other search protocols need (e.g. bargaining). In addition, it has the

[8] This is in contrast to random (or indirect) search. The difference stands in the timing and type of the protocol: in this case the main assumption is that the terms of the contract are decided after the match is sealed and barganiing theory is used to discipline how the surplus is shared.

[9] From now on, we will drop the arguments of P and Q.

desirable property of producing a unique equilibrium outcome.

Maintaining the demand shock θ and supply side shock z we introduced before, we can then define the household problem with competitive search as follows:

$$V(\theta, z, s) = \max_{c,d,s',P,Q} \quad u(c, d, \theta) + \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} V(\theta', z', s') \tag{9}$$

$$s.t. \quad c + Ps' = P\left[s\left(1 + \hat{R}(\theta, z)\right)\right], \tag{10}$$

$$c = d\ \Psi^{h}(Q)\, z \tag{11}$$

$$\frac{z\Psi^{f}(Q)}{P} \geq \hat{R}(\theta, z) \tag{12}$$

Let $u(c, d, \theta) = u(\theta c, d)$ from here on. The first two constraints were defined above, while the last is the firm's participation constraint, which is the condition that states that firms would prefer this market to other markets in which they would get $\hat{R}(\theta, z)$.

To solve the problem, let's take the first order conditions. One way to do this is to first plug the first two constraints into the objective function (expressing c and s' as functions of d) and then take the derivative with respect to d (recall that $\Psi^{h} = Q^{1-\varphi}$) to get:

$$\theta Q^{1-\varphi} z u_c(\theta d Q^{1-\varphi} z, d) + u_d(\theta d Q^{1-\varphi} z, d) = \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} V_3\left(\theta', z', s(1 + \hat{R}(\theta, z)) - \frac{dQ^{1-\varphi} z}{P}\right) \frac{Q^{1-\varphi} z}{P} \tag{13}$$

To find V_3 consider the original problem where constraints are not plugged into the objective function. Using the envelope theorem we get:

$$V_3(\theta, z, s) = \left[\theta u_c(\theta d Q^{1-\varphi} z, d) + \frac{u_d(\theta d Q^{1-\varphi} z, d)}{Q^{1-\varphi} z}\right] P(1 + \hat{R}(\theta, z))$$

Combining these two gives the Euler equation:

$$\theta u_c(\theta d Q^{1-\varphi} z, d) + \frac{u_d(\theta d Q^{1-\varphi} z, d)}{Q^{1-\varphi} z} = \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} \frac{P'(1+\hat{R}(\theta', z'))}{P} \left[\theta' u_c(\theta' d' Q'^{1-\varphi} z', d') + \frac{u_d(\theta' d' Q'^{1-\varphi} z', d')}{Q'^{1-\varphi} z'} \right] \quad (14)$$

Observe that this equation is the same as the Euler equation from the random search model. This gives us the optimal search and saving behavior for a given island (i.e. a market tightness $1/Q$ and price level P) by relating dynamically consuming today versus consuming tomorrow, adjusting for the effort costs of searching. To understand which market to search in, we need to look at the FOC with respect to Q and P. Let λ denote the Lagrange multiplier on the firm's participation constraint, then the FOC with respect to Q and P are respectively:

$$\theta d(1-\varphi) Q^{-\varphi} z u_c(\theta d Q^{1-\varphi} z, d) = \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} V_3 \left(\theta', z', s(1+\hat{R}(\theta, z)) - \frac{d Q^{1-\varphi} z}{P} \right) \frac{d(1-\varphi) Q^{-\varphi} z}{P} - \lambda \frac{\varphi Q^{-\varphi-1} z}{P} \quad (15)$$

and

$$\beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} V_3 \left(\theta', z', s(1+\hat{R}(\theta, z)) - \frac{d Q^{1-\varphi} z}{P} \right) dQ = -\lambda \quad (16)$$

Combining these two equation gives us:

$$\theta u_c(\theta d Q^{1-\varphi} z, d) = \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} V_3 \left(\theta', z', s(1+\hat{R}(\theta, z)) - \frac{d Q^{1-\varphi} z}{P} \right) \left[\frac{1}{(1-\varphi) P} \right] \quad (17)$$

Recall that we had defined $V_3(\cdot, \cdot, \cdot)$ above and thus this Euler equation simplifies to

$$(1-\varphi)\theta u_c(\theta d Q^{1-\varphi} z, d) = \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} \frac{P'(1+\hat{R}(\theta', z'))}{P} \left[\theta' u_c(\theta' d' Q'^{1-\varphi} z', d') + \frac{u_d(\theta' d' Q'^{1-\varphi} z', d')}{Q'^{1-\varphi} z'} \right]$$

$$(18)$$

Or by equations (14) and (18), we get:

$$\theta u_c(\theta d Q^{1-\varphi} z, d) + \frac{u_d(\theta d Q^{1-\varphi} z, d)}{Q^{1-\varphi} z} = (1-\varphi)\theta u_c(\theta d Q^{1-\varphi} z, d) \tag{19}$$

Rearranging,

$$\varphi \theta z Q^{1-\varphi} u_1(\theta c, d) + u_2(\theta c, d) = 0$$

This equation is static, and relates the marginal utility of consumption to the marginal disutility of search. How does this relate to the social planner's problem?

The social planner faces a static maximization problem, because there is no capital or other resource that can be saved: you can think of the fruit as going rotten after the period is over.

What is the aggregate resource constraint in this economy? Well, given some number of trees T and search effort D, the "factors of production" in this economy, the matching function $M(T, D)$ tells us how many resources are "produced." Multiplying the number of matches by the productivity z gives us the total amount of fruit available for consumption in the economy.

Thus, the social planner's problem is as follows:

$$\max_{C,D} u(\theta C, D)$$
$$s.t. \quad C = zM(T, D) = zD^{\varphi}$$

where we've substituted in the fact that $T = 1$ in this economy. Substituting in the budget constraint, this becomes a simple one-variable static problem:

$$\max_{D} u(\theta z D^{\varphi}, D)$$

Taking first-order conditions, we obtain the condition for optimality in this economy:

$$\varphi \theta z D^{\varphi-1} u_1(\theta C, D) + u_2(\theta C, D) = 0 \tag{2}$$

This is the same condition as we obtained for the competitive search equilibrium, so this shows that the equilibrium is optimal.[10]

Now we can define the equilibrium:

Definition 13 *An equilibrium with competitive search consists of functions V, c, d, s', P, Q, and R that satisfy:*

1. *Household's budget constraint, (condition 10)*
2. *Household's shopping constraint, (condition 11)*
3. *Household's Euler equation, (condition 14)*
4. *Market condition, (condition 18)*
5. *Firm's participation constraint, (condition 12), which gives us that the dividend payment is the profit of the firm,* $\hat{R}(\theta, z) = \frac{zQ^{-\varphi}}{P}$*,*
6. *Market clearing, i.e.* $s' = 1$ *and* $Q = 1/d$*.*

Note that if you had solved the problem by replacing c and d as functions of s', then the Euler equations

[10] This is in stark difference with random search, where the equilibrium is not always optimal but only for a particular value of the bargaining power parameter. This is because firms have incentives to charge higher prices given that matches are already given (after exerting effort).

(14) and (18) would be given by:

$$\theta u_c + \frac{u_d}{Q^{1-\varphi}z} = \beta \sum_{\theta',z'} \Gamma_{\theta\theta'}\Gamma_{zz'}\frac{P'(1+\hat{R}(\theta',z'))}{P}\left[\theta' u_c' + \frac{u_d'}{Q'^{1-\varphi}z'}\right] \tag{20}$$

and

$$\theta u_c + \frac{u_d}{Q^{1-\varphi}z} = -\frac{(1-\varphi)}{\varphi}\frac{u_d}{Q^{-1-\varphi}z} \tag{21}$$

where now $u_c = u_c\left(\theta P\left[s\left(1+\hat{R}\right)-s'\right], \frac{P\left[s\left(1+\hat{R}\right)-s'\right]}{Q^{1-\varphi}z}\right)$ and $u_d = u_d\left(\theta P\left[s\left(1+\hat{R}\right)-s'\right], \frac{P\left[s\left(1+\hat{R}\right)-s'\right]}{Q^{1-\varphi}z}\right)$.

Also, if the agent's budget constraint would be defined as $c + P(\theta,z)s' = s\left(P(\theta,z) + R\left(\theta,z\right)\right)$, then the firm's participation constraint is given by $z\Psi^f\left(Q(\theta,z)\right) \geq R(\theta,z)$ and the equilibrium conditions are

$$\theta u_c + \frac{u_d}{Q^{1-\varphi}z} = \beta \sum_{\theta',z'} \Gamma_{\theta\theta'}\Gamma_{zz'}\frac{P'+R'}{P}\left[\theta' u_c' + \frac{u_d'}{Q'^{1-\varphi}z'}\right] \tag{22}$$

and

$$\left(\theta u_c + \frac{u_d}{Q^{1-\varphi}z}\right)\left[s\left(1-\varphi\frac{R}{Q}\right)-s'\right] = (1-\varphi)\,Q^{-\varphi}\left(\frac{s\left[P+R\right]-Ps'}{z}\right)u_d \tag{23}$$

where now $u_c = u_c\left(\theta\left[s\left(P+R\right)-Ps'\right], \frac{s[P+R]-Ps'}{Q^{1-\varphi}z}\right)$ and $u_d = u_d\left(\theta\left[s\left(P+R\right)-Ps'\right], \frac{s[P+R]-Ps'}{Q^{1-\varphi}z}\right)$.

Exercise 29 *Define the recursive equilibrium with competitive search for this last setup.*

8.1.1 Firms' Problem

Note that in any given period a firm maximizes its returns to the tree by choosing the appropriate market, Q. Note that, by choosing a market Q, the firm is effectively choosing a price. Let the numeraire be the price of trees, then $\hat{P}\left(Q\right)$ is price of consumption.

Since there is nothing dynamic in the choice of a market (note that, we are assuming firms can choose a different market in each period), we can write the problem of a firm as:

$$\pi = \max_{Q} \hat{P}(Q)\Psi^{f}(Q)z. \tag{24}$$

The first order condition for the optimal choice of Q is

$$\hat{P}'(Q)\Psi^{f}(Q) + \hat{P}(Q)\Psi^{f\prime}(Q) = 0, \tag{25}$$

which then determines $\hat{P}(Q)$ as

$$\frac{\hat{P}'(Q)}{\hat{P}(Q)} = -\frac{\Psi^{f\prime}(Q)}{\Psi^{f}(Q)}. \tag{26}$$

The first order condition characterizes the set of prices (given Q) such that firms get the same profits. In other words, it tells us by how much prices have to adjust such that firms are indifferent between markets with different Qs.

Exercise 30 *Show that the equilibrium is Pareto efficient.*

9 Measure Theory

This section will be a quick review of measure theory, so that we are able to use it in the subsequent sections. In macroeconomics we encounter the problem of aggregation often and it's crucial that we do it in a reasonable way. Measure theory is a tool that tells us when and how we could do so. Let us start with some definitions on sets.

Definition 14 *For a set S, $\mathcal{S}$ is a family of subsets of S, if $B \in \mathcal{S}$ implies $B \subseteq S$ (but not the other way around).*

Remark 8 *Note that in this section we will assume the following convention*

1. *small letters (e.g. s) are for elements,*

2. *capital letters (e.g. S) are for sets, and*

3. *fancy letters (e.g. $\mathcal{S}$) are for a set of subsets (or families of subsets).*

Definition 15 *A family of subsets of S, $\mathcal{S}$, is called a σ-algebra in S if*

1. *$S, \emptyset \in \mathcal{S}$;*

2. *if $A \in \mathcal{S} \Rightarrow A^c \in \mathcal{S}$ (i.e. $\mathcal{S}$ is closed with respect to complements and $A^c = S \backslash A$); and,*

3. *for $\{B_i\}_{i \in \mathbb{N}}$, if $B_i \in \mathcal{S}$ for all $i \Rightarrow \bigcap_{i \in \mathbb{N}} B_i \in \mathcal{S}$ (i.e. $\mathcal{S}$ is closed with respect to countable intersections and by De Morgan's laws, $\mathcal{S}$ is closed under countable unions).*

Example 1

1. *The power set of S (i.e. all the possible subsets of a set S), is a σ-algebra in S.*

2. *$\{\emptyset, S\}$ is a σ-algebra in S.*

3. *$\left\{\emptyset, S, S_{1/2}, S_{2/2}\right\}$, where $S_{1/2}$ means the lower half of S (imagine S as an closed interval in $\mathbb{R}$), is a σ-algebra in S.*

4. *If $S = [0, 1]$, then*

$$\mathcal{S} = \left\{\emptyset, \left[0, \frac{1}{2}\right), \left\{\frac{1}{2}\right\}, \left[\frac{1}{2}, 1\right], S\right\}$$

is not *a σ-algebra in S. But*

$$\mathcal{S} = \left\{\emptyset, \left\{\frac{1}{2}\right\}, \left\{\left[0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right]\right\}, S\right\}$$

is a σ-algebra in S.

Exercise 31 *Show the previous statement.*

Why do we need the σ-algebra? Because it defines which sets may be considered as "events": things that have positive probability of happening. Elements not in it may have no properly defined measure. Basically, a σ-algebra is the "patch" that lets us avoid some pathological behaviors of mathematics, namely non-measurable sets. We are now ready to define a measure.

Definition 16 *Suppose $\mathcal{S}$ is a σ-algebra in S. A measure is a real-valued function $x : \mathcal{S} \to \mathbb{R}_+$, that satisfies*

1. $x(\emptyset) = 0$;
2. *if* $B_1, B_2 \in \mathcal{S}$ *and* $B_1 \cap B_2 = \emptyset \Rightarrow x(B_1 \cup B_2) = x(B_1) + x(B_2)$ *(additivity); and,*
3. *if* $\{B_i\}_{i\in\mathbb{N}} \in \mathcal{S}$ *and* $B_i \cap B_j = \emptyset$ *for all* $i \neq j \Rightarrow x(\cup_i B_i) = \sum_i x(B_i)$ *(countable additivity).*[11]

Put simply, a measure is just a way to assign each possible "event" a non-negative real number. A set S, a σ-algebra on it ($\mathcal{S}$), and a measure on $\mathcal{S}$ (x) define a measurable space, $(S, \mathcal{S}, x)$.

Definition 17 *A Borel σ-algebra is a σ-algebra generated by the family of all open sets $\mathfrak{B}$ (generated by a topology). A Borel set is any set in $\mathfrak{B}$.*

Since a Borel σ-algebra on $\mathbb{R}$ contains all the subsets generated by the intervals, you can recognize any "reasonable" subset of $\mathbb{R}$ using a Borel σ-algebra. In other words, a Borel σ-algebra corresponds to complete information.

Definition 18 *A probability measure is a measure with the property that $x(S) = 1$ and thus $(S, \mathcal{S}, x)$ is now a probability space. The probability of an event is then given by $x(A)$, where $A \in \mathcal{S}$.*

Definition 19 *Given a measurable space $(S, \mathcal{S}, x)$, a real-valued function $f : S \to \mathbb{R}$ is measurable*

[11] Countable additivity means that the measure of the union of countable disjoint sets is the sum of the measure of these sets.

(with respect to the measurable space) if, for all $a \in \mathbb{R}$, we have

$$\{b \in S \mid f(b) \leq a\} \in \mathcal{S}.$$

Given two measurable spaces $(S, \mathcal{S}, x)$ and $(T, \mathcal{T}, z)$, a function $f : S \to T$ is measurable if for all $A \in \mathcal{T}$, we have

$$\{b \in S \mid f(b) \in A\} \in \mathcal{S}.$$

One way to interpret a σ-algebra is that it describes the information available based on observations, i.e. a structure to organize information. Suppose that S is comprised of possible outcomes of a dice throw. If you have no information regarding the outcome of the dice, the only possible sets in your σ-algebra can be $\emptyset$ and S. If you know that the number is even, then the smallest σ-algebra given that information is $\mathcal{S} = \{\emptyset, \{2, 4, 6\}, \{1, 3, 5\}, S\}$. Measurability has a similar interpretation. A function is measurable with respect to a σ-algebra $\mathcal{S}$, if it can be evaluated under the current measurable space $(S, \mathcal{S}, x)$.

Example 2 *Suppose $S = \{1, 2, 3, 4, 5, 6\}$. Consider a function f that maps the element 6 to the number 1 (i.e. $f(6) = 1$) and any other elements to -100. Then f is NOT measurable with respect to $\mathcal{S} = \{\emptyset, \{1, 2, 3\}, \{4, 5, 6\}, S\}$. Why? Consider $a = 0$, then $\{b \in S \mid f(b) \leq a\} = \{1, 2, 3, 4, 5\}$. But this set is not in $\mathcal{S}$.*

We can also generalize Markov transition matrices to any measurable space, which is what we do next.

Definition 20 *Given a measurable space $(S, \mathcal{S}, x)$, a function $Q : S \times \mathcal{S} \to [0, 1]$ is a transition probability if*

1. *$Q(s, \cdot)$ is a probability measure for all $s \in S$; and,*
2. *$Q(\cdot, B)$ is a measurable function for all $B \in \mathcal{S}$.*

Intuitively, for $B \in \mathcal{S}$ and $s \in S$, $Q(s, B)$ gives the probability of being in set B tomorrow, given that the state is s today. Consider the following example: a *Markov chain* with transition matrix given by

$$\Gamma = \begin{bmatrix} 0.2 & 0.2 & 0.6 \\ 0.1 & 0.1 & 0.8 \\ 0.3 & 0.5 & 0.2 \end{bmatrix},$$

on the set $S = \{1, 2, 3\}$, with the σ-algebra $\mathcal{S} = P(S)$ (where $P(S)$ is the power set of S). If Γ_{ij} denotes the probability of state j happening, given the current state i, then

$$Q(3, \{1, 2\}) = \Gamma_{31} + \Gamma_{32} = 0.3 + 0.5 .$$

As another example, suppose we are given a measure x on $\mathcal{S}$ with x_i being the fraction of type i, for any $i \in S$. Given the previous transition function, we can calculate the fraction of types that will be in i tomorrow using the following formulas:

$$\begin{aligned} x_1' &= x_1\Gamma_{11} + x_2\Gamma_{21} + x_3\Gamma_{31}, \\ x_2' &= x_1\Gamma_{12} + x_2\Gamma_{22} + x_3\Gamma_{32}, \\ x_3' &= x_1\Gamma_{13} + x_2\Gamma_{23} + x_3\Gamma_{33}. \end{aligned}$$

In other words

$$\mathbf{x}' = \Gamma^T \mathbf{x},$$

where $\mathbf{x}^T = (x_1, x_2, x_3)$.

To extend this idea to a general case with a general transition function, we define an *updating operator* as $T(x,Q)$, which is a measure on S with respect to the σ-algebra $\mathcal{S}$, such that

$$\begin{aligned} x'(B) =& T(x,Q)(B) \\ =& \int_S Q(s,B)\,x(ds), \quad \forall B \in \mathcal{S}, \end{aligned}$$

where we integrated over all the possible current states s to get the probability of landing in set B tomorrow.

A stationary distribution is a fixed point of T, that is x^* such that

$$x^*(B) = T(x^*,Q)(B), \quad \forall B \in \mathcal{S}.$$

We know that, if Q has nice properties (monotone, Feller property, and enough mixing),[12] then a unique stationary distribution exists (for instance, we discard alternating from one state to another) and we have that

$$x^* = \lim_{n\to\infty} T^n(x_0,Q),$$

for any x_0 in the space of probability measures on $(S,\mathcal{S})$.

Exercise 32 *Consider unemployment in a very simple economy (in which the transition matrix is exogenous). There are two states of the world: being employed and being unemployed. The transition matrix is given by*

$$\Gamma = \begin{pmatrix} 0.95 & 0.05 \\ 0.50 & 0.50 \end{pmatrix}.$$

Compute the stationary distribution corresponding to this Markov transition matrix.

Exercise 33 *Using the Markov transition above, determine how many periods it would take for an economy that starts with 100% employment to reach an unemployment rate of 5%.*

[12] See Chapters 11/12 in Stockey, Lucas, and Prescott (1989) for more details.

10 Industry Equilibrium

10.1 Preliminaries

Now we are going to study an equilibrium model of an industry developed by Hopenhayn (1992). We abandon the general equilibrium framework from the previous sections to study the dynamics of the distribution of firms in a partial equilibrium environment. We formally drop the household sector and assume an exogenously given demand function for the good produced in the industry.

To motivate things, let's start with the problem of a single firm that produces a good using labor n as an input according to a technology described by the production function $f(n)$. Let us assume that this function is increasing, strictly concave, and with $f(0) = 0$. A firm that hires n units of labor, at price w, is able to produce output $sf(n)$, where s is the firm's productivity, and sell it on the market at price p. Markets are assumed to be competitive and the good is homogeneous, so a firm takes prices (p and w) as given. A firm then chooses n in order to maximize its profits according to

$$\pi(s,p) = \max_{n \geq 0} \{psf(n) - wn\}. \tag{27}$$

The first order condition implies that in the optimum n^* solves

$$psf_n(n^*) = w. \tag{28}$$

Let us denote the solution to this problem as the function $n^*(s;p)$.[13] Given the above assumptions, n^* is an increasing function of both s (i.e. more productive firms have more workers) and p (i.e. the higher the output price, the more workers the firm will hire).

Exercise 34 *Prove that $n^*(s,p)$ is an increasing function.*

[13] As we declared in advance, this is a partial equilibrium analysis. Hence, we ignore the dependence of the solution on w to focus on the determination of p.

Suppose now there is a mass of firms in that industry, each characterized by a productivity parameter $s \in S \subset \mathbb{R}_{+}$, where $S := [\underline{s}, \bar{s}]$. Let $\mathcal{S}$ denote a σ-algebra on S (a Borel σ-algebra, for instance). Let x be a probability measure defined over the space $(S, \mathcal{S})$, which describes the cross-sectional distribution of productivity among firms. Then, for any $B \subset S$ with $B \in \mathcal{S}$, $x(B)$ is the mass of firms having productivities in S. As we will see later, the measure x will be useful to compute statistics at the industry level.

The aggregate supply of the industry corresponds to the sum of each firm's output. Since firm-level supply after choosing labor is given by $sf(n^*(s;p))$, we can write the aggregate supply Y^S as[14]

$$Y^S(p) = \int_S sf(n^*(s;p))\, x(ds). \tag{29}$$

Observe that Y^S is only a function of the price p. For any price p, $Y^S(p)$ gives us the supply curve in this economy.

Suppose now that the aggregate demand for that industry's good is described by some function $Y^D(p)$. Then the industry's equilibrium price p^* is determined by the market clearing condition

$$Y^D(p^*) = Y^S(p^*). \tag{30}$$

So far, everything is too simple to be interesting. The ultimate goal here is to understand how the object x is determined by the fundamentals of the industry. Hence, we will be adding tweaks to this basic environment in order to obtain a theory of firms' distribution in a competitive environment. Let's start by allowing firms to die.

[14] S in Y^S stands for supply.

10.2 A Simple Dynamic Environment

Consider now a dynamic environment, in which firms face the above problem in every period. Firms discount profits at rate r_t, which is exogenously given. In addition, assume that a firm faces a probability $(1-\delta)$ of exiting the market in each period. Notice first that the firm's decision problem is still a static problem. We can easily write the value of an incumbent firm with productivity s as

$$\begin{aligned} V(s;p) &= \sum_{t=0}^{\infty}\left(\frac{\delta}{1+r}\right)^t \pi(s;p) \\ &= \left(\frac{1+r}{1+r-\delta}\right)\pi(s;p) \end{aligned}$$

Note that we are considering that p is fixed (that is why we use a semicolon and therefore we can omit it from the expressions above).

In what follows, we will focus on *stationary equilibria*, i.e. an equilibrium in which the price of the final output p, the rate of return, r, and the productivity of firm, s, stay constant through time. Note however that every period there is a positive mass of firms that die $(1-\delta)$. Then, how can this economy be in a stationary equilibrium? To achieve that, we have to assume that there is a constant flow of firms entering the economy in each period as well, so that the mass of firms that disappears is exactly replaced by new entering firms.

As before, let x be the measure of firms within the industry. The mass of firms that die in any given period is given by $(1-\delta)x(s)$. We will allow these firms to be replaced by new entrants. These entrants draw a productivity parameter s from a probability distribution γ over $(S, \mathcal{S})$.

One might ask what keeps these firms out of the market in the first place? If $\pi(s;p) = psf(n^*(s;p)) - wn^*(s;p) > 0$, which is the case for the firms operating in the market (since $n^* > 0$), then all the (potential entering) firms with productivities in S would want to enter the market. We can fix this by assuming that there is a fixed entry cost that each firm must pay in order to operate in the market, denoted by c^E. Moreover, we will assume that the entrant has to pay this cost before learning s. Hence the value of a new entrant is given by the following

$$V^E(p) = \int_S V(s;p)\,\gamma(ds) - c^E. \tag{31}$$

Entrants will continue to enter if $V^E \geq 0$ and decide not to enter if $V^E < 0$ (since the option value from staying out of the market is 0). As a result, stationarity occurs when V^E is exactly equal to zero (which is the *free-entry* condition).[15] You can think of this condition as a "*no money left on the table*" condition, which ensures additional entrants find it unprofitable to operate in the industry.

Let's analyze how this environment shapes the distribution of firms in the market. Let x_t be the cross-sectional distribution of firms in any given period t. For any $B \subset S$, a fraction $1-\delta$ of firms with productivity $s \in B$ will die and newcomers will enter the market (the mass of which is m). Hence, next period's measure of firms on set B will be given by

$$x_{t+1}(B) = \delta x_t(B) + m\gamma(B),. \tag{32}$$

That is, the mass m of firms would enter the market in $t+1$, but only fraction $\gamma(B)$ of them will have productivities in the set B. As you might suspect, this relationship must hold for every $B \in \mathcal{S}$. Moreover, since we are interested in a stationary equilibrium, the previous expression tells us that the cross-sectional distribution of firms will be completely determined by γ.

Define the updating operator T be defined as

$$Tx(B) = \delta x(B) + m\gamma(B), \quad \forall B \in \mathcal{S}. \tag{33}$$

Then, a stationary distribution is the fixed point of the mapping T, i.e. x^* such that $Tx^* = x^*$, which implies the following stationary measure of firms on set B

$$x^*(B;m) = \frac{m}{1-\delta}\gamma(B), \quad \forall B \in \mathcal{S}. \tag{34}$$

[15] We are assuming that there is an infinite number (mass) of prospective firms willing to enter the industry.

Note that the demand and supply condition in equation (30) takes the form

$$Y^D\left(p^*\left(m\right)\right) = \int_S sf\left(n^*\left(s;p^*\right)\right)dx^*\left(s;m\right), \tag{35}$$

whose solution $p^*\left(m\right)$ is a continuous function under regularity conditions stated in Stockey, Lucas, and Prescott (1989).

Exercise 35 *Prove that p^* is an increasing function.*

We now have two equations, (31) and (35), and two unknowns, p and m. Thus, we can defined the equilibrium as follows

Definition 21 *A stationary equilibrium consists of functions V, π^*, n^*, p^*, x^*, and m^* such that*

1. *Given prices, V, π^*, and n^* solve the incumbent firm's problem;*
2. *Supply equals demand: $Y^D\left(p^*\left(m\right)\right) = \int_S sf\left(n^*\left(s;p^*\right)\right)dx^*\left(s;m\right)$;*
3. *Free-entry condition: $\int_s V\left(s;p^*\right)\gamma\left(ds\right) = c^E$; and,*
4. *Stationary distribution: $x^*\left(B\right) = \delta x^*\left(B\right) + m^*\gamma\left(B\right), \quad \forall B \in \mathcal{S}$.*

10.3 Introducing Exit Decisions

Next, we want to introduce more economic content by making the exit of firms endogenous (i.e. a decision of the firm). As a first step, we let the productivity of firms follow a Markov process governed by the transition function Γ. This changes the mapping T in Equation (33) to

$$Tx\left(B\right) = \delta\int_S \Gamma\left(s,B\right)x\left(ds\right) + m\gamma\left(B\right), \quad \forall B \in \mathcal{S}. \tag{36}$$

This alone doesn't add much economic content to our environment; firms still don't make any (interesting) decision.

Another ingredient that we introduce in the model is to let firms face operating costs. Suppose firms now have to pay c^v each period in order to stay in the market. In this case, when s is low, the firm's profit might not cover its operating cost. The firm might therefore decide to leave the market. Note, however, that the firm has already paid (the sunk cost of) c^E from entering the market and since s follows a first-order Markov process, the prospects of future profits might deter the firm from exiting the market. Therefore, having negative profits in one period does not necessarily imply that the firm's optimal choice is to leave the market.

By adding such a minor change, the solution will feature a reservation productivity property (under some conditions to be discussed in the comment below). There will be a minimum productivity, $s^* \in S$, above which it is profitable for the firm to stay in the market (and below which the firm decides to exit).

To see this, note that the value of a firm currently operating in the market with productivity $s \in S$ is given by

$$V(s;p) = \max \left\{ 0, \pi(s;p) + \frac{1}{(1+r)} \int_S V(s';p) \Gamma(s, ds') - c^v \right\}. \tag{37}$$

Exercise 36 *Show that the firm's decision takes the form of a reservation productivity strategy, in which, for some $s^* \in S$, $s < s^*$ implies that the firm would leave the market.*

Then the law of motion of the distribution of firms on S is given by

$$x'(B) = \int_{s^*}^{\bar{s}} \Gamma(s, B \cap [s^*, \bar{s}]) x(ds) + m\gamma(B \cap [s^*, \bar{s}]), \quad \forall B \in \mathcal{S}, \tag{38}$$

and a stationary distribution, x^*, is the fixed point of this equation.

Example 3 *How productive does a firm have to be, to be in the top 10% largest firms in this economy (in the stationary equilibrium)? The answer to this question is the solution to the following equation*

$$\frac{\int_{\hat{s}}^{\bar{s}} x^*(ds)}{\int_{s^*}^{\bar{s}} x^*(ds)} = 0.1,$$

where $\hat{s}$ is the productivity level above which a firm is in the top 10% largest firm. Then, the fraction of the labor force in the top 10% largest firms in this economy, is

$$\frac{\int_{\hat{s}}^{\bar{s}} n^*(s,p)\, x^*(ds)}{\int_{s^*}^{\bar{s}} n^*(s,p)\, x^*(ds)}.$$

Exercise 37 *Compute the average growth rate of the smallest one third of the firms. What would be the fraction of firms in the top 10% largest firms in the economy that remain in the top 10% in next period? What is the fraction of firms younger than five years?*

10.4 Stationary Equilibrium

Now that we have all the ingredients in the table, let's define the equilibrium formally for the economy with the exit decision.

Definition 22 *A stationary equilibrium consists of a list of functions Φ, V^E, π^*, n^*, a productivity threshold s^*, a price p^*, a measure x^*, and mass m^* such that*

1. *Given p^*, the functions Φ, π^*, n^* solve the problem of the incumbent firm*
2. *The reservation productivity s^* is such that the firm stays in the market if $s \geq s^*$*
3. *Free-entry condition:*

$$V^E(p^*) = 0$$

4. *For any $B \in \mathcal{S}$,*

$$x^*(B) = m^* \gamma (B \cap [s^*, \bar{s}]) + \int_{s^*}^{\bar{s}} \Gamma (s, B \cap [s^*, \bar{s}])\, x^*(ds)$$

5. *Market clearing:*

$$Y^d(p^\star) = \int_{s^\star}^{\bar{s}} s f(n^\star(s;p^\star)) x^\star(ds)$$

We can now use this model to compute interesting statistics about this industry. For example, the average output of the firm is given by

$$\frac{Y}{N} = \frac{\int_{s^\star}^{\bar{s}} s f(n^*(s)) x^*(ds)}{\int_{s^\star}^{\bar{s}} x^*(ds)}$$

Suppose that we want to compute the share of output produced by the top 1% of firms. To do so, we first need to find $\widetilde{s}$ such that

$$\frac{\int_{\widetilde{s}}^{\bar{s}} x^*(ds)}{N} = .01$$

where N is the total measure of firms as defined above. Then the share of output produced by these firms is given by

$$\frac{\int_{\widetilde{s}}^{\bar{s}} s f(n^*(s)) x^*(ds)}{\int_{s^\star}^{\bar{s}} s f(n^*(s)) x^*(ds)}$$

Suppose now that we want to compute the fraction of firms in the top 1% two periods in a row. If s is a continuous variable, this is given by

$$\int_{s \geq \widetilde{s}} \int_{s' \geq \widetilde{s}} \Gamma_{ss'} x^*(ds)$$

or if s is discrete, then

$$\sum_{s \geq \widetilde{s}} \sum_{s' \geq \widetilde{s}} \Gamma_{ss'} x^*(s)$$

10.5 Adjustment Costs

To end with this chapter it is useful to think about environments in which firms' productive decisions are no longer static. A simple way of introducing dynamics is by adding adjustment costs. Since firms only use labor, we will consider labor adjustment costs.[16]

Consider a firm that enters period t with n_{t-1} units of labor hired in the previous period and has to choose how many units of labor n_t to hire today. Let the adjustment costs be denoted by $c(n_t, n_{t-1})$ as a result of hiring n_t units of labor in t. We will consider three different specifications for these costs:

- *Convex Adjustment Costs*: if the firm wants to vary the units of labor, it has to pay $\alpha(n_t - n_{t-1})^2$ units of the numeraire good. The cost here depends on the size of the adjustment.
- *Training Costs or Hiring Costs*: if the firm wants to increase labor, it has to pay $\alpha[n_t - (1-\delta)n_{t-1}]^2$ units of the numeraire good only if $n_t > n_{t-1}$. We can write this as

  $$\mathbf{1}_{\{n_t > n_{t-1}\}} \alpha [n_t - (1-\delta) n_{t-1}]^2 ,$$

  where $\mathbf{1}$ is the indicator function and δ measures the exogenous attrition of workers in each period.
- *Firing Costs*: the firm has to pay to reduce the number of workers.

The recursive formulation of the firm's problem can now be expressed as

$$V(s, n_-; p) = \max \left\{ 0, \max_{n \geq 0} \; p\, s f(n) - wn - c^v - c(n, n_-) + \frac{1}{(1+r)} \int_{s' \in S} V(s', n; p)\, \Gamma(s, ds') \right\}, \tag{39}$$

where last period labor is a state and the function $c(\cdot,\cdot)$ gives the specified cost of adjusting n_- to n. Note that we are assuming limited liability for the firm since its exit value is 0 and not $-c(0, n_-)$.

[16] These costs work pretty much like capital adjustment costs, as one might suspect.

Now a firm is characterized by both its current productivity s and labor in the previous period n_-. Note that since the production function f has decreasing returns to scale, there exists an amount of labor $\bar{N}$ such that none of the firms hire labor greater than $\bar{N}$. So, $n_- \in N := [0, \bar{N}]$, with $\mathcal{N}$ as a σ-algebra on N. If the labor policy function is $n = g(s, n_-; p)$, then the law of motion for the measure of firms becomes

$$x'\left(B^S, B^N\right) = m\gamma\left(B^S \cap [s^*, \bar{s}]\right) \mathbf{1}_{\{0 \in B^N\}} + \int_{s^*}^{\bar{s}} \int_0^{\bar{N}} \mathbf{1}_{\{g(s,n_-;p) \in B^N\}} \Gamma\left(s, B^S \cap [s^*, \bar{s}]\right) x\left(ds, dn_-\right), \quad \forall\ B^S \in \mathcal{S}, \forall\ B^N \in \mathcal{N}. \tag{40}$$

Exercise 38 *Write the first order conditions for the problem in* (39)*. Define the recursive competitive equilibrium for this economy.*

Exercise 39 *Another example of labor adjustment costs is when the firm has to post vacancies to attract labor. As an example of such case, suppose the firm faces a firing cost according to the function c. The firm also pays a cost κ to post vacancies and after posting vacancies, it takes one period for the workers to be hired. How can we write the problem of firms in this environment?*

Exercise 40 *Write the problem of a firm with capital adjustment costs.*

Exercise 41 *Write the problem of a firm with R&D expenditures that uses labor to improve its productivity.*

11 Non-stationary Equilibria

Until now we have focused on *stationary equilibria*, where prices, aggregates, and the distribution of firms are invariant. Next we will delve into non-stationary equilibria and examine how the distribution of firms or individuals shifts over time as a result of some aggregate change. We will do so both in the sequence space as well as in the functional space.

11.1 Sequence vs. Recursive Industry Equilibrium

Up to now the distribution of firms was invariant so that the share of firms that enter and exit the industry is equal. A more interesting case is to look at what happens in a *non-stationary equilibrium* with demand shifters $z_{t+1} = \phi(z_t)$ so that aggregate demand is $D(p_t, z_t)$.

A few remarks regarding the shock. In general, z_t can be *deterministic* or *stochastic*. Deterministic shocks are fully anticipated by agents in the economy, while stochastic shocks are random and agents only know the random process that governs them. Solving the model with deterministic shocks is almost as easy as solving the transitional path of the model without shocks. But models with stochastic shocks are much harder to solve given the possible realizations of the shock going forward. We will consider for now that the shock z_t is deterministic and thus focus on the perfect foresight equilibrium.

Let's first start with the sequential case introducing a sequence of aggregate shocks $\{z_t\}_{t=0}^{\infty}$. We maintain our baseline model (with entry and exit, but no adjustment costs) and consider the economy starting with some (arbitrary) initial distribution of incumbent firms x_0. Without any shocks, the firm distribution would converge to the stationary equilibrium distribution x^* defined in section 10.4. On the transitional path towards the stationary equilibrium, firms would face a sequence of prices $\{p_t\}_{t=0}^{\infty}$, which are going to be pinned down by equating the endogenous aggregate industry supply and the exogenous aggregate demand in each period.

The problem of a firm in sequential form is now given by

$$
\begin{aligned}
V_t(s) &= \max\left\{0, \pi_t(s) + \frac{1}{1+r}\int_S V_{t+1}(s')\Gamma(s, ds')\right\} \\
\text{s.t.} \qquad & \pi_t(s) = \max_{n_t \geq 0}\ p_t s f(n_t) - w n_t - c^v \qquad (41) \\
& x_0 \text{ given} \qquad (42)
\end{aligned}
$$

We can maintain the cutoff property of the decision rule to exit the market given the regularity conditions assumed above regarding the matrix of transition probabilities. Let's denote the exit cutoff productivity as s_t^*. In order to solve this problem we need to know how the measure of firms in the industry evolves over time. The law of motion of the measure of firms is such that for each $B \in \mathcal{S}$ we have

$$x_{t+1}(B) = m_{t+1}\gamma(B \cap [s_{t+1}^*, \bar{s}]) + \int_{s_t^*}^{\bar{s}} \Gamma(s, B \cap [s_{t+1}^*, \bar{s}])x_t(ds) \tag{43}$$

where m_{t+1} is the mass of firms that enter at the beginning of period $t+1$, which is pinned down by the free-entry condition

$$\int_S V_t(s)\gamma(ds) \leq c^e, \tag{44}$$

with strict equality if $m_t > 0$. The distribution of productivity among entrants γ and the entry cost c^e are exogenously given (and for simplicity constant through time). Finally, the market clearing condition will close the model by pinning down price p_t from

$$D(p_t, z_t) = \int_{s_t^*}^{\bar{s}} p_t s f(n_t(s))x_t(ds) \tag{45}$$

We can now define the perfect foresight equilibrium in sequence space as follows

Definition 23 *Given a path of shocks $\{z_t\}_{t=0}^{\infty}$ and a initial measure of firms x_0, a perfect foresight equilibrium (PFE) in sequence space consists of a sequence of numbers $\{p_t, x_t, s_t^*\}_{t=0}^{\infty}$, of measures $\{x_t\}_{t=0}^{\infty}$, and of functions $\{V_t(s), n_t(s)\}_{t=0}^{\infty}$ that satisfy:*

1. *Optimality: Given $\{p_t\}_{t=0}^{\infty}$, $\{V_t, n_t, s_t^*\}$ solve the firm's problem (41) for each period t.*
2. *Free-entry: $\int_S V_t(s)\gamma(ds) \leq c^e$, with strict equality if $m_t > 0$.*
3. *Law of motion: $x_{t+1}(B) = m_{t+1}\gamma(B \cap [s_{t+1}^*, \bar{s}]) + \int_{s_t^*}^{\bar{s}} \Gamma(s, B \cap [s_{t+1}^*, \bar{s}])x_t(ds)$, $\forall B \in \mathcal{S}$.*
4. *Market clearing: $D(p_t, z_t) = \int_{s_t^*}^{\bar{s}} p_t s f(n_t(s))x_t(ds)$.*

Next we can define the same problem in the functional space using the tools of dynamic programming we have developed so far. The time subscripts disappear and we now have to add two aggregate states

$\{z, x\}$ and a function G that tells us how the distribution of firms changes, mapping a measure of firms and state today into a measure of firms tomorrow.

Let's rewrite the problem of the firm recursively for the deterministic economy where $z' = \phi(z)$

$$V(s, z, x) = \max \left\{0, \pi(s, z, x) + \frac{1}{1+r} \int_S V(s', \phi(z), G(z, x))\Gamma(s, ds')\right\} \tag{46}$$

$$\text{s.t.} \quad \pi(s, z, x) = \max_{n \geq 0} \; p(z, x)sf(n) - wn - c^v$$

Note that firms take output prices as given and those must clear the market as before. We can now define the PFE in functional space for this economy.

Definition 24 *Given ϕ, a perfect foresight equilibrium defined recursively is a list of functions $\{V(s, z, x), n(s, z, x), s^*(s, z, x)\}$ for the firm, pricing function $p(z, x)$, and functions $G(z, x)$ for the measure's law of motion and $m(z, x)$ for the mass of entrants, such that*

1. *Given prices and law of motion, $\{V(s, z, x), n(s, z, x), s^*(s, z, x)\}$ solve the firm's problem* (46).
2. *Free-entry condition holds*

$$\int_S V(s, z, x)\gamma(ds) \leq c^e, \text{ with strict equality if } m(z, x) > 0.$$

3. *The measure evolves according to $\forall B \in \mathcal{S}$*

$$G(z, x)(B) = m(z, x)\gamma(B \cap [s^*(s, z, x), \bar{s}]) + \int_{s^*(s,z,x)}^{\bar{s}} \Gamma(s, B \cap [s^*(s, z, x), \bar{s}])x(ds).$$

4. *Market clearing:*

$$D(p(z, x), z) = \int_{s^*(s,z,x)}^{\bar{s}} p(z, x)sf(n(s, z, x))\, x(ds).$$

We can see that finding this high-dimensional object G is computationally troublesome. That is why we will go back to the problem in sequence space to solve for the aggregate transition in non-stationary environments. The natural next step is to solve the stochastic equilibrium, in which ϕ is random. For simplicity, we will assume that the shock follows an AR1 process and thus $z_{t+1} = \rho z_t + \varepsilon_{t+1}$. Although the definition of equilibrium resembles the one above, it is a much harder problem to solve. We will resort to some notion of linearization to achieve it in the context of the Neoclassical Growth model in the next subsection.

Exercise 42 *What happens if demand doubles? Sketch an algorithm to find the equilibrium prices.*

Exercise 43 *Write the stochastic version of the non-stationary economy above in both the sequential and recursive forms.*

11.2 Linear Approximation in the Neoclassical Growth model

Solving the previous problem with stochastic shocks can be challenging. To do so, we will resort to a solution method proposed by Boppart et al. (2018), who study the equilibrium response to an MIT shock by exploring the idea that linearization can provide a good approximation for equilibria of economies with aggregate shocks and heterogeneous agents.

To better understand linearization, we will first look at the standard deterministic growth model and approximate the solution linearly. Consider the social planner's problem (with full depreciation)

$$\begin{aligned} V(k_t) &= \max_{c_t, k_{t+1}} u(c_t) + \beta V(k_{t+1}) \qquad (47) \\ & \text{s.t. } c_t + k_{t+1} \leq f(k_t), \ \forall\, t \geq 0 \\ & c_t, k_{t+1} \geq 0, \ \forall\, t \geq 0 \\ & k_0 > 0 \text{ given.} \end{aligned}$$

We can show that $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ is a solution to the above social planner's problem if and only if

$$u'(c_t) = \beta u'(c_{t+1}) f'(k_{t+1}), \forall\, t \geq 0 \tag{48}$$

$$c_t + k_{t+1} = f(k_t), \forall\, t \geq 0 \tag{49}$$

$$\lim_{t\to\infty} \beta^t u'(c_t) k_{t+1} = 0 \tag{50}$$

Exercise 44 *Derive the above equilibrium conditions.*

We will focus on cases where a steady state k^* exists. The goal is to do a linear approximation of the Euler equation. Note that the above necessary and sufficient conditions give us a second order difference equation (which we denote as $\psi(k_t, k_{t+1}, k_{t+2}) = 0$), with exactly two boundary conditions (given by k_0 and the steady state k^*). This second order difference equation is the residual function from the Euler equation, i.e.

$$\psi(k_t, k_{t+1}, k_{t+2}) = u_c(f(k_t) - k_{t+1}) - \beta u_c(f(k_{t+1}) - k_{t+2}) f_k(k_{t+1}) = 0. \tag{51}$$

So how do we solve this model? One obvious option is to find the global solution. For instance, you can guess a k_1, use k_0 and $\psi(k_t, k_{t+1}, k_{t+2}) = 0$ to get $k_2, k_3, \ldots$ forward up until some k_T, and adjust k_1 to make sure k_T is close enough to the steady state k^* (this is called forward shooting). Or you can guess a k_{T-1} and do it backward (which is called reverse shooting). Or you can guess and adjust the whole path (which is called the extended path method). Finally, you can also solve for the path by casting the whole sequence as an almost-diagonal system of equations. All these methods will give you a numerical solution for the path of k starting from the arbitrary k_0 (that's why we call it a *global* solution).

As you may have done it in another course, the above process can be computationally time consuming. Linearization is a short cut that can yield a good approximation of the solution *locally*, that is, around the neighborhood of some point (usually, around the steady state). The idea is simple. We know the true solution is in the form of $k_{t+1} = g(k_t)$. Let's conjecture a linear function to approximate the true solution $g(\cdot)$ and denote it by $\hat{g}(\cdot)$. Since we are interested in a linear approximation, we assume that $k_{t+1} = \hat{g}(k_t) = a + bk_t$. Then we only need to figure out two numbers: a and b. We thus need two conditions to pin them down.

Since we know the steady state is k^* and that the solution must hold in the steady state, we have that $k^* = a + bk^*$ and therefore $a = (1-b)k^*$. So we got one condition for free. How do we find the second condition regarding b? We can find it in ψ and our criteria is that we are going to choose b such that the slope of $\hat{g}$ exactly matches the slope of the true decision rule g *at the steady state* k^*. So we take a first order Taylor expansion of the residual function $\psi[k_t, g(k_t), g(g(k_t))]$ around k^* and obtain

$$\psi[k, g(k_t), g(g(k_t))] \approx \psi(k^*, k^*, k^*) + \psi_k(k^*, k^*, k^*)\underbrace{(k_t - k^*)}_{\hat{k}_t} \tag{52}$$

where $\hat{k}_t$ is deviation of capital from its steady state value. We know the residual function $\psi[k, g(k), g(g(k))] = 0$ by definition since the Euler equation must hold for all k. For k_t in the neighborhood of k^*, we have that the derivative of the second order difference equation satisfies

$$\psi_k(k^*, k^*, k^*) = \psi_1^* + \psi_2^* g'(k^*) + \psi_3^* g'(g(k^*))g'(k^*) = 0. \tag{53}$$

Solving this equation gives us $g'(k^*)$, which is exactly what we need (note ψ_1, ψ_2, and ψ_3 may also involve $\hat{g}'(k^*)$). We then let $b = g'(k^*) = \hat{g}'(k)$. We now have all the ingredients to compute $\hat{g}(k_t) = a + bk_t$ to approximate the solution near the steady state. From any k_0 we can now compute the full of $\left\{\hat{k}_{t+1}\right\}_{t=0}^{\infty}$ for the deterministic economy.

Exercise 45 *Suppose* $f(k_t) = k_t^{\alpha}$*,* $u(c_t) = \ln c_t$*. Verify that the solution to the social planner's problem is* $k_{t+1} = \alpha\beta k_t^{\alpha}$*. Get the linearized solution around the steady state and compare it with the closed form solution. How precise is the linear approximation?*

With stochastic productivity shocks, the second order difference equation would include the current as well as next period's realization of the shock and thus $\psi(k_t, k_{t+1}, k_{t+2}, z_t, z_{t+1}) = 0$. The idea above follows through with some additional notation. What Boppart et al. (2018) suggest is to use the path of $\left\{\hat{k}_{t+1}\right\}_{t=0}^{\infty}$ and a sequence of shocks $\{\varepsilon_{t+1}\}_{t=0}^{\infty}$ to approximate the impulse response of the fully stochastic economy $\left\{\widetilde{k}_{t+1}\right\}_{t=0}^{\infty}$ (where $z_{t+1} = \rho z_t + \varepsilon_{t+1}$).

To see this, consider the model above. The resource constraint is now $c_t + k_{t+1} = z_t f(k_t)$, with $z_0 = 1$. We want to solve first for the deterministic transition with $\varepsilon_t = 0\ \forall t$ and thus $z_{t+1} = \rho^{t+1}$. In that case, the second order difference equation is

$$\psi(k_t, k_{t+1}, k_{t+2}, z_t) = u_c(z_t f(k_t) - k_{t+1}) - \beta u_c(\rho z_t f(k_{t+1}) - k_{t+2})\rho z_t f_k(k_{t+1}) = 0, \tag{54}$$

where $z_t = \rho^t$ and $z_{t+1} = \rho^{t+1} = \rho z_t$. As before, we conjecture an approximation of the solution $k_{t+1} = g(k_t, z_t)$ using the linear function $\hat{g}(k_t, z_t)$ given by

$$k_{t+1} = a + bk_t + fz_t. \tag{55}$$

Now we need to pin down three numbers: $a, b,$ and f. We can use the linear rule evaluated at the steady state to find $a = (1 - b)k^* - f$. Next, we proceed as before by taking a first-order Taylor expansion of $\psi(k_t, g(k_t), g(g(k_t)), z_t)$, with b and f satisfying the following expression

$$\psi_1^* \hat{k}_t + (\psi_2^* + b\psi_3^*)\,\hat{k}_{t+1} + (\psi_3^* k^* f\rho + \psi_4^*)\,\hat{z}_t = 0, \tag{56}$$

which we can solve for $\hat{k}_{t+1}$. Recall that $\hat{k}_{t+1} = k_{t+1} - k^*$ and using the approximating function $\hat{g}$ we have $\hat{k}_{t+1} = b\hat{k}_t + f\hat{z}_t$. We now have two equations and two unknowns to pin down b and f.

Exercise 46 *Describe a log-linear approximation of the Euler equation of the growth model.*

Now that we have found the linear approximation for the path of aggregate capital in the perfect foresight model, we can use that path to solve for the sequence of capital in the stochastic economy given any sequence of randomly drawn innovations ε_t. Denote the deviation of capital from steady state in the stochastic economy by $\widetilde{k}_t(\left\{\hat{k}_t\right\}; \{\varepsilon_t\}, k_0, z_0)$. We can now compute the impulse response to a sequence of shocks as

$$\begin{aligned}
\widetilde{k}_1 &= \varepsilon_0 \hat{k}_1 \\
\widetilde{k}_2 &= \varepsilon_0 \hat{k}_2 + \varepsilon_1 \hat{k}_1 \\
&\vdots \\
\widetilde{k}_{t+1} &= \sum_{\tau=0}^{t+1} \varepsilon_\tau \hat{k}_{t+1-\tau}
\end{aligned}$$

or in the MIT shock scenario with $\varepsilon_0 = 1$ standard deviation and $\varepsilon_t = 0 \; \forall t \neq 0$.

Thus, the model with aggregate shocks can be obtained by a simple simulation based on the deterministic path of the aggregate variable. The solution corresponds to the superposition of non-linear impulse response functions derived from the PFE. This can then be applied to our non-stationary industry equilibria or any other economy with heterogeneous agents. The computational cost is linear (not exponential) in the number of shocks. Yet, we do not know how to use this procedure for asymmetric shocks (e.g. when there is downward wage rigidity).

Exercise 47 *Describe an algorithm to approximate the solution to an industry subject to demand shocks that follow an AR1 process.*

Exercise 48 *Describe how to compute the evolution of the Gini index or the Herfindahl index of an industry over the first fifteen periods.*

12 Incomplete Market Models

We now turn to models with incomplete asset markets and thus agents will not be able to fully insure in all possible states of the world.

12.1 A Farmer's Problem

We start with a simple Robinson Crusoe economy with coconuts that can be stored. Consider the problem of a farmer given by

$$\begin{aligned} V(s,a) = \max_{c,a'} \quad & u(c) + \beta \sum_{s'} \Gamma_{ss'} V(s', a') \\ s.t. \quad & c + qa' = a + s \\ & c \geq 0 \\ & a' \geq 0, \end{aligned} \tag{57}$$

where a is his holding of coconuts, which can only take positive values, c is his consumption, and s is amount of coconuts that nature provides. The latter follows a Markov chain, taking values in a finite set S, and q is the fraction of coconuts that can be stored to be consumed tomorrow. Note that the constraint on the holdings of coconuts tomorrow (a') is a constraint imposed by nature. Nature allows the farmer to store coconuts at rate $1/q$, but it does not allow him to *transfer coconuts* from tomorrow to today (i.e. borrow).

We are going to consider this problem in the context of a partial equilibrium setup in which q is given. What can be said about q?

Remark 9 *Assume there are no shocks in the economy, so that s is a fixed number. Then, we could write the problem of the farmer as*

$$V(a) = \max_{c,a' \geq 0} \{u(a + s - qa') + \beta V(a')\}. \tag{58}$$

We can derive the first order condition as

$$q\, u_c \geq \beta u_c' \tag{59}$$

If u is assumed to be logarithmic, the FOC for this problem simplifies to

$$\frac{c'}{c} \geq \frac{\beta}{q}, \tag{60}$$

and with equality if $a' > 0$. Therefore, if $q > \beta$ (i.e. nature is more stingy, or the farmer is less patient than nature), then $c' < c$ and the farmer dis-saves (at least, as long as $a' > 0$). But, when $q < \beta$, consumption grows without bound. Finally, for $q = \beta$ then $c' < c$ (with noise and $u_{ccc} > 0$ grows without bound). For that reason, we impose the assumption that $\beta/q < 1$ in what follows.

A crucial assumption to bound the asset space is that $\beta/q < 1$, which states that agents are sufficiently impatient so that they want to consume more today and thus decumulate their assets when they are richer and far away from the non-negativity constraint, $a' \geq 0$. However, this does not mean that when faced with the possibility of very low consumption, agents would not save (even though the rate of return, $1/q$, is smaller than the rate of impatience $1/\beta$).

The first order condition for farmer's problem (57) with s stochastic is given by

$$u_c\left(c\left(s,a\right)\right) \geq \frac{\beta}{q}\sum_{s'} \Gamma_{ss'} u_c\left(c\left(s', a'\left(s,a\right)\right)\right), \tag{61}$$

with equality when $a'\left(s,a\right) > 0$, where $c\left(\cdot\right)$ and $a'\left(\cdot\right)$ are policy functions from the farmer's problem. Notice that $a'\left(s,a\right) = 0$ is possible for an appropriate stochastic process. Specifically, it depends on the value of $s_{\min} := \min_{s_i \in S}\ s_i$. Note that $a \gg g(s,a), \quad \forall s$ for sufficiently large a, so $\exists \bar{a}$, s.t. $a' \in A = [0, \bar{a}]$

The solution to the problem of the farmer, for a given value of q, implies a distribution of coconut holdings in each period. This distribution, together with the Markov chain describing the evolution of s, can be summed together as a single probability measure for the distribution of shocks and assets (coconut holdings) over the product space $E = S \times \mathbb{R}_+$, and its σ-algebra, $\mathcal{B}$. We denote that measure by X. The evolution of this probability measure is then given by

$$X'(B) = \sum_{s} \int_{0}^{\bar{a}} \sum_{s' \in B_s} \Gamma_{ss'} \mathbf{1}_{\{a'(s,a) \in B_a\}} X(s, da), \quad \forall B \in \mathcal{B}, \tag{62}$$

where B_s and B_a are the S-section and $\mathbb{R}_+$-section of B (projections of B on S and $\mathbb{R}_+$), respectively, and $\mathbf{1}$ is an indicator function. Let $\widetilde{T}(\Gamma, a', \cdot)$ be the mapping associated with (62) (the adjoint operator), so that

$$X'(B) = \widetilde{T}(\Gamma, a', X)(B), \quad \forall B \in \mathcal{B}. \tag{63}$$

Define $\widetilde{T}^n(\Gamma, a', \cdot)$ as

$$\widetilde{T}^n(\Gamma, a', X) = \widetilde{T}\left(\Gamma, a', \widetilde{T}^{n-1}(\Gamma, a', X)\right). \tag{64}$$

Exercise 49 *Prove that the one above is indeed a transition. That is, it satisfies the properties to guarantee that there exists a stationary distribution.*

Exercise 50 *Show that the policy function $g(s, a)$ is monotonic. Show that under the assumptions above there exists an upper bound $\bar{a}$ such that $a' \in A = [0, \bar{a}]$.*

Then, we can define the following theorem.

Theorem 3 *Under some conditions on $\widetilde{T}(\Gamma, a', \cdot)$,*[17] *there is a unique probability measure X^*, so that:*

$$X^*(B) = \lim_{n \to \infty} \widetilde{T}^n(\Gamma, a', X_0)(B), \quad \forall B \in \mathcal{B}, \tag{65}$$

for all initial probability measures X_0 on $(E, \mathcal{B})$.

A condition that makes things considerably easier for this theorem to hold is that E is a compact set.

[17] As in the previous section, we need Γ to be monotone, enough mixing in the distribution, and that $\widetilde{T}$ maps the space of bounded continuous functions to itself.

Then, we can use Theorem (12.12) in Stokey, Lucas, and Prescott (1989) to show this result holds. Given that S is finite, this is equivalent to a compact support for the distribution of asset holdings. We discuss this in further detail in Appendix A.

12.2 Huggett Economy

Now we modify the farmer's problem in (57) a little bit, in line with Huggett (1993). Look carefully at the borrowing constraint in what follows

$$\begin{aligned} V(s,a) = \max_{c,a'} \quad & u(c) + \beta \sum_{s'} \Gamma_{ss'} V(s',a') \\ s.t. \quad & c + qa' = a + s \\ & c \geq 0 \\ & a' \geq \underline{\mathsf{a}}, \end{aligned} \tag{66}$$

where $\underline{\mathsf{a}} < 0$. Now farmers can borrow and lend among each other, but up to a borrowing limit. We continue to make the same assumption on q; i.e. that $\beta/q < 1$. As before, solving this problem gives the policy function $a'(s,a)$. It is easy to extend the analysis in the last section to show that there is an upper bound on the asset space, which we denote by $\bar{a}$, so that for any $a \in A := [\underline{\mathsf{a}}, \bar{a}]$, $a'(s,a) \in A$, for any $s \in S$.

Remark 10 *One possibility for $\underline{\mathsf{a}}$ is what we call the natural borrowing limit. This limit ensures the agent can pay back his debt with certainty, no matter what the nature unveils (i.e. whatever sequence of idiosyncratic shocks is realized). This is given by*

$$a^n := -\frac{s_{\min}}{(q-1)}. \tag{67}$$

If we impose this constraint on (66)*, the farmer can fully pay back his debt in the event of receiving an infinite sequence of bad shocks by setting his consumption equal to zero forever.*

But, what makes this problem more interesting is to tighten this borrowing constraint more. The natural borrowing limit is very unlikely to be binding. One way to restrict borrowing further is to assume no borrowing at all, as in the previous section. Another case is to choose $0 > \underline{a} > a^n$*, which we will consider in this section.*

Now suppose there is a (unit) mass of farmers with distribution function $X(\cdot)$, where $X(D,B)$ denotes the fraction of people with shock $s \in D$ and $a \in B$, where D is an element of the power set of S, $P(S)$ (which, when S is finite, is the natural σ-algebra over S), and B is a Borel subset of $\mathcal{A}$ ($B \in \mathcal{A}$). Then the distribution of farmers tomorrow is given by

$$X'(D', B') = \sum_{s \in S} \int_A \mathbf{1}_{\{a'(s,a) \in B'\}} \sum_{s' \in D'} \Gamma_{ss'} X(s, da), \tag{68}$$

for any $D' \in P(S)$ and $B' \in \mathcal{A}$.

Implicitly this defines an operator T such that $X' = T(X)$. If T is *sufficiently nice* (as defined above and in the previous footnote), then there exits a unique X^* such that $X^* = T(X^*)$ and $X^* = \lim_{n\to\infty} T^n(X_0)$ for any initial distribution over the product space $S \times A$, X_0. Note that the decision rule is obtained for a given price q. Hence, the resulting stationary distribution X^* also depends on q. So, let us denote it by $X^*(q)$.

To determine the equilibrium value of q in a general equilibrium setting consider the following variable (as a function of q):

$$\int_{A \times S} a \, dX^*(q). \tag{69}$$

This expression give us the average asset holdings, given the price q (assuming s is a continuous variable). What we want to do is to determine the endogenous q that clears the asset market. Recall that we assumed that there is no storage technology so that the supply of assets is 0 in equilibrium. Hence, the price q should be such that the asset demand equals asset supply, i.e.

$$\int_{A \times S} a \, dX^*(q) = 0. \tag{70}$$

In this sense, the equilibrium price q is the price that generates the stationary distribution of asset holdings that clears the asset market.

We can now show that a solution exists by invoking the intermediate value theorem. We need to ensure that the following three conditions are satisfied (note that $q \in [\beta, \infty]$)

1. $\int_{A\times S} a\, dX^*(q)$ is a continuous function of q;

2. $\lim\limits_{q\to\beta} \int_{A\times S} a\, dX^*(q) \to \infty$; (As $q \to \beta$, the interest rate $R = 1/q$ increases up to $1/\beta$, which is the steady state interest rate in the representative agent economy. Hence, agents would like to save more. Adding to this the precautionary savings motive, agents would want to accumulate an unbounded amount of assets in the stationary equilibrium); and,

3. $\lim\limits_{q\to\infty} \int_{A\times S} a\, dX^*(q) < 0$. (This is also intuitive: as $q \to \infty$, the interest rate $R = 1/q$ converges to 0. Hence, everyone would rather borrow).

12.3 Aiyagari Economy

The Aiyagari (1994) economy is one of the workhorse models of modern macroeconomics. It features incomplete markets and an endogenous wealth distribution, which allows us to examine interactions between heterogeneous agents and distributional effects of public polices. The setup will similar to the one above, but now physical capital is introduced. Then, the average asset holdings in the economy that we computed above must be equal to the average amount of (physical) capital K. Keeping the notation from the previous section (i.e. the stationary distribution of assets is X^*), then we have that

$$\int_{A\times S} a\, dX^*(q) = K, \tag{71}$$

where A is the support of the distribution of wealth. (It is not difficult to see that this set is compact.)

We will now assume that the shocks affect labor income. We can think of these shocks as fluctuations in the employment status of individuals. Now the restriction for the existence of a stationary equilibrium is $\beta(1+r) < 1$. Thus, the problem of an individual in this economy can be written as

$$\begin{aligned} V(s,a) = \max_{c,a'} \quad & u(c) + \beta \int_{s'} V(s',a')\,\Gamma(s, d\,s') \\ s.t. \quad & c + a' = (1+r)\,a + ws \\ & c \geq 0 \\ & a' \geq \underline{\mathsf{a}}, \end{aligned} \tag{72}$$

where r is the return on savings and w is the wage rate. Then,

$$\int_{A\times S} s\,dX^*(q) \tag{73}$$

gives the average labor in this economy. If agents are endowed with one unit of time, we can think of the expression as determining the *effective labor supply*.

We also assume the standard constant returns to scale production technology for the firm as

$$F(K,L) = AK^{1-\alpha}L^{\alpha}, \tag{74}$$

where A is TFP and L is the average amount of labor in the economy. Let δ be the rate of depreciation of capital. Hence, solving for the firm's FOC we have that factor prices satisfy

$$\begin{aligned} r &= F_k(K,L) - \delta \\ &= (1-\alpha)\,A\left(\frac{K}{L}\right)^{-\alpha} - \delta \\ &=: r\left(\frac{K}{L}\right), \end{aligned}$$

and

$$\begin{aligned} w &= F_l(K,L) \\ &= \alpha A\left(\frac{K}{L}\right)^{1-\alpha} \\ &=: w\left(\frac{K}{L}\right). \end{aligned}$$

The prices faced by agents are functions of the capital-labor ratio. As a result, we may write the stationary distribution of assets as a function of the capital-labor ratio as well and thus $X^*\left(\frac{K}{L}\right)$. The equilibrium condition now becomes

$$\frac{K}{L} = \frac{\int_{A\times S} a\, dX^*\left(\frac{K}{L}\right)}{\int_{A\times S} s\, dX^*\left(\frac{K}{L}\right)}. \tag{75}$$

Using this condition, one can solve for the equilibrium capital-labor ratio and study the distribution of wealth in this economy.

Remark 11 *Note that relative to Huggett (1993), the price of assets q is now given by*

$$q = \frac{1}{(1+r)} = \frac{1}{[1+F_k(K,L)-\delta]}. \tag{76}$$

Exercise 51 *Show that aggregate capital is higher in the stationary equilibrium of the Aiyagari economy than it is the standard representative agent economy.*

Remark 12 *When leisure choice is introduced into the Aiyagari economy, the precautionary savings results do not necessarily hold. The reason is that agents may use their labor effort to act as insurance against poor shocks, creating a tradeoff between precautionary savings and "precautionary labor." This means that under certain parametrizations, capital could be lower than in the complete markets economy. In addition, labor productivity will tend to be lower: agents work more when their productivity shock is lower.*

12.3.1 Policy Changes and Welfare

Let the model parameters in an In Aiyagari or Huggett economy be summarized by $\theta = \{u, \beta, s, \Gamma, F\}$. The value function $V(s, a; \theta)$ as well as $X^*(\theta)$ can be obtained in the stationary equilibrium as functions of the model parameters, where $X^*(\theta)$ is a mapping from the model parameters to the stationary distribution of agent's asset holding and shocks. Suppose now there is a policy change that shifts θ to $\hat{\theta} = \{u, \beta, s, \hat{\Gamma}, F\}$. Associated with this new environment there is a new value function $V\left(s, a; \hat{\theta}\right)$ and a new distribution $X^*\left(\hat{\theta}\right)$. Now define $\eta(s, a)$ to be the solution of

$$V\left(s, a + \eta(s, a); \hat{\theta}\right) = V(s, a; \theta), \tag{77}$$

which corresponds to the transfer necessary to make the agent indifferent between living in the old environment and living in the new one (say from an initial steady state to a final steady state). Hence, the total transfer needed to compensate the agent for this policy change is given by

$$\int_{A\times S} \eta(s, a)\, dX^*(\theta). \tag{78}$$

Remark 13 *Notice that the changes do not take place when the government is trying to compensate the households and that is why we use the original stationary distribution associated with θ to aggregate the households ($X^*(\theta)$).*

If $\int_{A\times S} V(s, a)\, dX^*\left(\hat{\theta}\right) > \int_{A\times S} V(s, a)\, dX^*(\theta)$, does this necessarily mean that households are willing to accept this policy change? Not necessarily! Recall that comparing welfare requires us to compute the transition from one world to the other. Then, during the transition to the new steady state, the welfare losses may be very large despite agents being better off in the final steady state.

Remark 14 *Another way to think about this comparison of steady states is in the context of the neoclassical growth model. We know that the capital income tax $\tau > 0$ is suboptimal, but a positive capital income tax can produce a higher level of steady state consumption than a zero capital income tax. This shows that comparing steady states does not equate to a welfare comparison.*

12.4 Business Cycles in an Aiyagari Economy

12.4.1 Aggregate Shocks

In this section, we consider an economy that is subject to both aggregate and idiosyncratic shocks. Consider the Aiyagari economy again, but with a production function that is subject to an aggregate shock z so that we have $zF\left(K, \bar{N}\right)$.

Then the current aggregate capital stock is given by

$$K = \int a \, dX\left(s, a\right). \tag{79}$$

and next period aggregate capital is

$$K' = G\left(z, K\right) \tag{80}$$

The question is what are the sufficient statistics to predict the aggregate capital stock and, consequently, prices tomorrow? Are z and K sufficient to determine capital tomorrow? The answer to these questions is no, in general. It is only true if, and only if, the decision rules are linear. Therefore, X, the distribution of agents in the economy becomes a state variable (even in the stationary equilibrium).[18]

[18] Note that with X we can compute aggregate capital.

Then, the problem of an individual becomes

$$
\begin{aligned}
V(z, X, s, a) = \max_{c,a'} \quad & u(c) + \beta \sum_{z',s'} \Pi_{zz'} \Gamma_{ss'}^{z'} V(z', X', s', a') \\
s.t. \quad & c + a' = azf_k(K, \bar{N}) + szf_n(K, \bar{N}) \\
& K = \int a dX(s, a) \\
& X' = G(z, X) \\
& c, a' \geq 0,
\end{aligned} \tag{81}
$$

where we replaced factor prices with marginal productivities. Computationally, this problem is a beast! So, how can we solve it? To fix ideas, we will first consider an economy with *dumb* agents!

Consider an economy in which people are stupid. By stupid, we mean that people believe tomorrow's capital depends only on K and not on X. This, obviously, is not an economy with rational expectations. The agent's problem in such a setting is

$$
\begin{aligned}
\widetilde{V}(z, X, s, a) = \max_{c,a'} \quad & u(c) + \beta \sum_{z',s'} \Pi_{zz'} \Gamma_{ss'}^{z'} \widetilde{V}(z', X', s', a') \\
s.t. \quad & c + a' = azf_k(K, \bar{N}) + szf_n(K, \bar{N}) \\
& K = \int a dX(s, a) \\
& X' = \widetilde{G}(z, K) \\
& c, a' \geq 0.
\end{aligned} \tag{82}
$$

The next step is to allow people to become slightly smarter, by letting them use extra information, such as the mean and variance of X, to predict X'. Does this economy work better than our *dumb benchmark*? Computationally no! This answer, as stupid as it may sound, has an important message: agents' decision rules are approximately linear. It turns out that the approximations are quite reliable in the Aiyagari economy!

12.4.2 Linear Approximation Revisited

Let's now revisit our discussion of linear approximation in the context of the Aiyagari economy. As we can see in section 12.4.1, solving the heterogeneous agent model with aggregate shocks is computationally hard. We need to guess a reduced form rule to approximate the distribution for agents to forecast future prices, and when the model has frictions on several dimensions, there is little we can say on how to choose such a rule.

We can, however, use a linear approximation to obtain the model's solution around the steady state. The idea is as follows: starting from the steady state, we obtain the the impulse responses of the perfect foresight economy given a sequence of small deterministic shocks. Then, we use these responses to approximate the behavior of the main aggregates in the economy with heterogeneous agents by adding small stochastic shocks around the steady state. This method was recently proposed by Boppart, Krusell, and Mitman (2018).[19].

To fix ideas, let's consider the above Aiyagari economy with a TFP shock z. Let $\log(z_t)$ follow an AR(1) process with ρ as the autocorrelation parameter as $\log(z_t) = \rho \log(z_{t-1}) + \epsilon_t$. First, compute the path of the (log of the) shock by letting ϵ will go up by, say, one unit in period 0. Rewriting the process in its MA form, we have the full sequence of values $(1, \rho, \rho^2, \rho^3, \dots)$ to pin down the TFP path. Then, we can compute the transition path in the deterministic economy, with the agent taking as given the sequence of prices. Thus, solving the deterministic path is straightforward: we guess a path for price (or else we could also guess the path for an aggregate variable), solve the household's problem backwards from the final steady state back to initial steady state, and then derive the aggregate implications of the households' behavior and update our guess for the price path. This iterative procedure is also standard and fully nonlinear.

After solving the PFE, we have a sequence of aggregates we care about. We choose one of those,

[19] The description of the method below is from that paper with minor modifications.

call it x, and we thus have a sequence $\{x_0, x_1, x_2, \dots\}$. Now consider the same economy subject to recurring aggregate shocks to z. Now we want to approximate the object of interest in that economy, call it $\hat{x}$. The key assumption behind this procedure is that we regard the $\hat{x}$ as well approximated by a linear system of the sequence of x computed as a response to the one-time shock. A linear system means that the effects of shocks are linearly scalable and additive so that the *level* of $\hat{x}$ at some future time T, after a sequence of random shocks to z is given by

$$\hat{x}_T \approx x_0\epsilon_T + x_1\epsilon_{T-1} + x_2\epsilon_{T-2} + \dots$$

or in deviation from steady state

$$(\hat{x}_T - x_{ss}) \approx (x_0 - x_{ss})\epsilon_T + (x_1 - x_{ss})\epsilon_{T-1} + (x_2 - x_{ss})\epsilon_{T-2} + \dots$$

where ϵ_t is the innovation to $\log(z_t)$ at period t. Thus, the model with aggregate shocks can be obtained by mere simulation based on the one deterministic path. It corresponds to the superposition of non-linear impulse response functions derived from the PFE.

12.5 Aiyagari Economy with Job Search

In the Aiyagari model we have seen, the labor market is assumed to be competitive and everybody is employed at the wage rate w. Now, we want to add the possibility of agents being in one of two labor market status: $\varepsilon = \{0, 1\}$, where 0 stands for unemployment and 1 for the case in which the agent is employed.

Agents can now exert effort h while searching for a job. Although exerting search effort provides some disutility, it also increases the probability of finding a job. Call that probability $\phi(h)$, with $\phi' > 0$. An employed worker, on the other hand, does not need to search for a new job and so $h = 0$, but his job can be destroyed with some exogenous probability δ. Let s be an employed worker's stochastic labor productivity, which is a first order Markov process with transition probabilities given by Γ.

The unemployed worker's problem is given by

$$V(s,0,a) = \max_{c,a',h} u(c,h) + \beta \sum_{s'} \Gamma_{ss'} \left[\phi(h)V(s',1,a') + (1-\phi(h))V(s',0,a')\right] \tag{83}$$

$$s.t. \quad c + a' = (1+r)a$$

$$a' \geq 0.$$

Similarly, the employed worker's problem is as follows

$$V(s,1,a) = \max_{c,a'} u(c,0) + \beta \sum_{s'} \Gamma_{ss'} \left[\delta V(s',0,a') + (1-\delta)V(s',1,a')\right] \tag{84}$$

$$s.t. \quad c + a' = sw + (1+r)a$$

$$a' \geq 0.$$

Exercise 52 *Solve for the FOC and define the stationary equilibrium for this economy.*

12.6 Two-Sided Undirected Search in Aiyagari Economy

Let us consider an alternative scenario, where there is only one job market and workers are identical. In order to hire workers, firms need to create vacant positions. Creating a vacancy requires a machine of size κ, that could be interpreted as an entry cost, and a periodic cost of posting a vacancy c^{κ}. Unemployed workers meet a vacancy according to a matching function $M(H,T)$, where H is the aggregate household search effort and T is the number of vacancies created. Let us assume that $M(H,T)$ is a constant return to scale function, and denote by $Q = \frac{T}{H}$ the market tightness of the market. Under this assumption, the probability that a vacancy meets an unemployed worker is $\Psi^{\kappa} = \frac{M(H,T)}{T}$. Notice the similarities of this problem to the competitive search framework presented in section (8). In that framework, there was a continuum of markets indexed by a pair of values (P,Q), and the household could choose to go in the market that maximizes her utility. Such a framework is defined competitive (or directed) search. In this economy, on the other hand, there is only one job market. This means that the households take the market tightness Q as given, and chooses the optimal search effort. After choosing the search effort, unemployed workers meet a random vacancy with probability $\Psi^H = \frac{M(H,T)}{H}$, and then bargain with the firm to determine the wage. Such a framework is often called random (or undirected) search.

Let us define by Ω^1 and Ω^0, respectively, the value of filled and an unfilled vacancy. Furthermore, assume that when a filled vacancy produces an output z each period. Then:

$$\Omega^0 = -c^\kappa + \frac{1}{1+r}(\Psi^f(Q)\Omega^1 + (1 - \Psi^f(Q))\Omega^0) \tag{85}$$

$$\Omega^1 = z - w + \frac{1}{1+r}((1-\delta)\Omega^1 + \delta\Omega^0) \tag{86}$$

where w is the wage and δ is the exogenous probability that a match get destroyed from one period to the next.

The problem of the household is identical to the previous section, with the exception that now the probability of finding a job is equivalent to $\phi(h) = h\Psi^h(Q)$, and there aren't idiosyncratic productivity shocks.

In this model, firms are all identical and are considered as machines. To solve the model, we need a free-entry condition that pins down the measure of new vacancies posted in each period. The free-entry condition is $\kappa = \Omega^0$.

Finally, to characterize the equilibrium of the model, we need to specify a bargaining protocol that allows to determine the wage offered to the worker when a vacancy is filled. A standard protocol that is used in models of random search is the so called Nash Bargaining solution. That is, when a worker and a vacancy meet, they bargain over the total surplus of the match to decide how to split it. Assume that the bargaining power of the workers is given by the exogenous parameter η. Then, according to the NB solution, the wage setting is given by the following maximization problem:

$$w^* = \arg\max_w \{[V(w;1,a) - V(0,a)]^\eta [\Omega^1(w) - \Omega^0]^{1-\eta}\}$$

where the value functions $V(1,a), V(0,a)$ come from the household maximization problem without idiosyncratic shocks.

Exercise 53 *Solve for the FOC of the household problem. Find the optimal wage setting implied by the Nash Bargaining protocol.*

Exercise 54 *Define a RCE in this economy.*

Exercise 55 *Rewrite the model when workers are subject to idiosyncratic productivity shocks s, and the output produces by a filled vacancy with a worker with productivity s is equal to zs. Define a REC for this economy.*

12.7 Aiyagari Economy with Entrepreneurs

Next, we will introduce entrepreneurs into the Aiyagari world. Suppose every period agents choose an occupation: to be either an entrepreneur or a worker. Entrepreneurs run their own business, by managing a project that combines her entrepreneurial ability (ϵ), capital (k), and labor(n); while workers supply labor in the market.

Let's denote $V^w(s,\epsilon,a)$ the value of a worker labor productivity s, and entrepreneurial ability ϵ, and wealth a. Similarly, denote $V^e(s,\epsilon,a)$ the value of an entrepreneur. The worker's problem is to choose tomorrow's occupation and wealth level as well as today's consumption for a given wage rate w and interest rate r.

$$V^w(s,\epsilon,a) = \max_{c,a',d\in\{0,1\}} u(c) + \beta \sum_{s',\varepsilon'} \Gamma_{ss'}\Gamma_{\varepsilon\varepsilon'} \left[dV^w(s',\epsilon',a') + (1-d)V^e(s',\epsilon',a')\right] \tag{87}$$

$$s.t. \quad c + a' = ws + (1+r)a$$

$$a' \geq 0$$

Similarly, the entrepreneur's problem can be formulated as follows

$$V^e(s,\epsilon,a) = \max_{c,a',d\in\{0,1\}} u(c) + \beta \sum_{s',\varepsilon'} \Gamma_{ss'}\Gamma_{\varepsilon\varepsilon'} \left[dV^w(s',\epsilon',a') + (1-d)V^e(s',\epsilon',a')\right] \tag{88}$$

$$s.t. \quad c + a' = \pi(s,\epsilon,a)$$

$$a' \geq 0$$

Note the entrepreneur's income is from profits $\pi(a,s,\epsilon)$ rather than wage. We assume entrepreneurs have access to a DRS technology f that produces output using as inputs (k,n). After paying factors and loans needed to operate the business, the entrepreneurs' profits are given by

$$\pi(s,\epsilon,a) = \max_{k,n} \epsilon f(k,n) + (1-\delta)k - (1+r)(k-a) - w\max\{n-s,0\} \tag{89}$$

$$s.t. \quad k - a \leq \phi a$$

The constraint here reflects the fact that entrepreneurs can only make loans up to a fraction ϕ of his total wealth. A limit of this model is that entrepreneurs never make an operating loss within a period, as they can always choose $k = n = 0$ and earn the risk free rate on saving. In this model, agents with high entrepreneurial ability have access to an investment technology f that provides higher returns than workers with high labor productivity and therefore the entrepreenurs accumulate wealth faster.

So, who is going to be an entrepreneur in this economy? In a world without financial constraints, wealth will play no role. There would be a threshold ϵ^* above which an agent would decide to become an entrepreneur. With financial constraints, this changes and wealth now plays an important role. Wealthy individuals with high entrepreneurial ability will certainly be entrepreneurs, while the poor with low entrepreneurial ability will become workers. For the other cases, it depends. If the entrepreneurial ability is persistent, poor individuals with high entrepreneurial ability will save to one day become entrepreneurs, while rich agents with low entrepreneurial ability will lend their assets and become workers.

Exercise 56 *Solve for the FOC and define the RCE for this economy.*

12.8 Unsecured Credit and Default Decisions

There are many more interesting applications. One of these is the economy with unsecured credit and default decisions. The price of lending will now incorporate the possibility of default. For simplicity, assume that if the agent decides to default, she live in autarky forever after. In that case, she is excluded from the financial market and has to consume as much as her labor earnings allow. Let the individual's budget constraint in the case of no default be given by

$$c + q(a')a' = a + ws,$$

where s is labor productivity with transition probabilities given by $\Gamma_{ss'}$. The problem of an agent is thus given by

$$V(s,a) = \max \Bigg\{ \underbrace{u(ws) + \beta \sum_{s'} \Gamma_{ss'} \bar{V}(s')}_{\text{default}}, \underbrace{\max_{a'} u(ws + a - q(a')a') + \beta \sum_{s'} \Gamma_{ss'} V(s', a')}_{\text{repayment}} \Bigg\} \quad (90)$$

where $\bar{V}(s') = \frac{1}{1-\beta} u(ws')$ is the value of autarky.

What determines the price $q(a')$? Well, we have a zero-profit condition on lenders, and thus the probability of default determines the price of borrowing.

In this economy, the agent's consumption-savings decision looks like the following (from the FOC):

$$qu_c + a'q_{a'}u_c = \beta \sum_{s'} \Gamma_{ss'} V_a$$

What's unique here is that the agent takes into account both her marginal utility of consumption and the effect that her amount of savings has on the price q.

An issue with this problem is that the agent's preferences are *time inconsistent*: she may decide to borrow today based on a commitment to not default tomorrow, but a shock tomorrow could cause defaulting to be optimal tomorrow. As discussed earlier, time inconsistency can create issues with applying the Envelope Theorem and solving for the RCE. This issue is mostly ignored in this literature.

13 New Keynesian Framework

13.1 Benchmark Monopolistic Competition

The two most important macroeconomic variables are perhaps output and inflation (movement in the aggregate price). In this section we take a first step in building a theory of aggregate price. To achieve this, we need a framework in which firms can choose their own prices and yet the aggregate price is well defined and easy to handle. The setup of Dixit and Stigltitz (1977) with monopolistic competition is such a framework.

In an economy with monopolistic competition, firms are sufficiently "different" so that they face a downward sloping demand curve and thus price discriminate, but also sufficiently small so that they ignore any strategic interactions with their competitors. We thus assume there are infinitely many measure 0 firms, each producing one variety of goods. Varieties span on the $[0, n]$ interval and are imperfect substitutes. Consumers have a "taste for variety" in that they prefer to consume a diversified bundle of goods (this gives firms some market power as we want). The consumer's utility function will have the constant elasticity of substitution (CES) form

$$u\left(\{c(i)\}_{i\in[0,n]}\right) = \left(\int_0^n c(i)^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{\sigma}{\sigma-1}}$$

where σ is the elasticity of substitution, which is a constant (as the name CES suggests), and $c(i)$ is the quantity consumed of variety i. For simplicity, we will rename $c(i) = c_i$. For now, we will assume the agent receives the exogenous *nominal* income I and is endowed with one unit of time.

We can now solve the household problem

$$\max_{\{c_i\}_{i\in[0,n]}} \left(\int_0^n c_i^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{\sigma}{\sigma-1}}$$
$$s.t. \quad \int_0^n p_i c_i di \leq I$$

and derive the FOC, which relates the demand for any varieties i and j as

$$c_i = c_j \left(\frac{p_i}{p_j}\right)^{-\sigma}$$

Multiplying both sides by p_i and integrating over i, we get the downward sloping demand curve faced by an individual firm producing variety i as

$$c_i^* = \frac{I}{\int_0^n p_j^{1-\sigma} dj} p_i^{-\sigma}$$

We can see that the demand for variety i depends both on the price of variety i, some measure of "aggregate price" and total expenditures I. It is actually convenient to define the aggregate price index P as follows

$$P = \left(\int_0^n p_j^{1-\sigma} dj\right)^{\frac{1}{1-\sigma}}$$

and thus the demand faced by the firm producing variety i can be reformulated as

$$c_i^* = \frac{I}{P}\left(\frac{p_i}{P}\right)^{-\sigma}$$

where the first term is real income and the second is a measure of relative price of variety i.

Exercise 57 *Show the following within the monopolistic competition framework above:*

1. *σ is the elasticity of substitution between varieties.*
2. *Price index P is the expenditure to purchase a unit-level utility for consumers.*
3. *Consumer utility is increasing in the number of varieties n.*
4. *Is there a missing n?*

We are now ready to characterize the firm's problem. Let's assume that the production technology is linear in its inputs and so one unit of output is produced with one unit of labor linearly, i.e., $f(\ell_j) = \ell_j$. Let the nominal wage rate be given by W. Also, recall that the quantity of variety j demanded by the representative agent is such that $f(\ell_j) = c_j^*$. Then, the firm producing variety j solves the following problem

$$\max_{p_j} \pi(p_j) = p_j c_j^*(p_j) - W c_j^*(p_j)$$
$$s.t. \quad c_j^* = \frac{I}{P}\left(\frac{p_i}{P}\right)^{-\sigma}$$

Recall that we assume firms are sufficiently small so they would ignore the effect of their own pricing strategies on aggregate price index P, which greatly simplify the algebra. By solving for the FOC, we get the straightforward pricing rule

$$p_j^* = \frac{\sigma}{\sigma - 1} W \qquad \forall j$$

where $\frac{\sigma}{\sigma-1}$ is a constant mark-up over the marginal cost, which reflects the elasticity of substitution of consumers. When varieties are very close substitutes ($\sigma \to \infty$), price just converge to the factor price W. Not that all firms follow the same pricing strategy, which is independent of the variety j.

We can now define an equilibrium for this simple economy.

Definition 25 *Set the wage as the numeraire. An equilibrium consists of prices $\{p_i^*\}_{i\in[0,n]}$, the aggregate price index P, household's consumption $\{c_i^*\}_{i\in[0,n]}$, income I, firm's labor demand $\{\ell_i^*\}_{i\in[0,n]}$ and profits $\{\pi_i^*\}_{i\in[0,n]}$, such that*

1. *Given prices, $\{c_i^*\}_{i\in[0,n]}$ solves the household's problem*
2. *Given P and I, p_i^* and π_i^* solve the firm's problem $\forall i \in [0,n]$*
3. *The aggregate price index satisfies*

$$P = \left(\int_0^n p_j^{1-\sigma} dj\right)^{\frac{1}{1-\sigma}}$$

4. *Markets clear*

$$\begin{aligned} \int \ell_i^* di &= 1 \\ 1 + \int \pi_i^* di &= I \end{aligned}$$

Note that in a symmetric equilibrium we have $c_i^* = \bar{c}$, $p_i^* = \bar{p}$, $n_i^* = \bar{n}$, $\pi_i^* = \bar{\pi}$ for all i.

13.2 Price Rigidity

We now have a simple theory of aggregate price P, which is ultimately shaped by the consumer's elasticity of substitution across varieties. However, we are still silent on inflation. To study inflation, and to have meaningful interactions between output and inflation, we need i) a dynamic model and ii) some source of nominal frictions.

Nominal frictions mean that nominal variables (things measured in dollars, say, quantity of money) can affect real variables. The most popular friction used is called *price rigidity*. With price rigidity, firms cannot adjust their prices freely. Two commonly used specifications to achieve this in the model are *Rotemberg pricing* (menu costs) and *Calvo pricing* (fairy blessing).

In Rotemberg pricing, firms face adjustment cost $\phi(p_j, p_j^-)$ when changing their prices p_j each period from p_j^-. Let S summarize the aggregate state and let $I(S), W(S)$, and $P(S)$ be aggregate nominal income, nominal wages and price index, respectively. Then the firm's problem under Rotemberg pricing in a dynamic setup is as follows:

$$\Omega(S, p_j^-) = \max_{p_j} p_j c_j^*(p_j) - W(S)c_j^*(p_j) - \phi(p_j, p_j^-) + \mathbb{E}\left[\frac{1}{R(G(S))}\Omega(G(S), p_j)\right], \tag{91}$$

where demand is taken as given and thus $c_j^*(p_j) = \left(\frac{p_j}{P(S)}\right)^{-\sigma}\frac{I(S)}{P(S)}$. Each period, firms choose the price that maximizes the expected present discounted value of the flow profit. Without capital, the aggregate state S includes P^- and the aggregate shocks. Rotemberg is easy in terms of algebra when we assume a quadratic price adjustment cost.

Another popular version of price rigidity is Calvo pricing. Instead of facing adjustment costs, there is some positive probability $(1-\theta)$ so that the firm cannot adjust its price. When setting the price, the firm now needs to incorporate the possibility of not being allowed to adjust its price. The value function of a firm that can change its price is given by

$$\begin{aligned} \Omega^1(S, p_j^-) &= \max_{p_j} p_j c_j^*(p_j) - W(S)c_j^*(p_j) + (1-\theta)\mathbb{E}\left[\frac{1}{R(G(S))}\Omega^0(G(S), p_j)\right] \\ &\quad + \theta\mathbb{E}\left[\frac{1}{R(G(S))}\Omega^1(G(S), p_j)\right], \end{aligned} \tag{92}$$

where demand is taken as given and thus $c_j^*(p_j) = \left(\frac{p_j}{P(S)}\right)^{-\sigma}\frac{I(S)}{P(S)}$, and $\Omega^0(G(S), p_j)$ is the value of firm that cannot change its price:

$$\begin{aligned}\Omega^0(S,p_j^-) &= p_j^- c_j^*(p_j^-) - W(S)c_j^*(p_j^-) + (1-\theta)\mathbb{E}\left[\frac{1}{R(G(S))}\Omega^0(G(S),p_j^-)\right] \\ &\quad +\theta\mathbb{E}\left[\frac{1}{R(G(S))}\Omega^1(G(S),p_j^-)\right]. \end{aligned} \tag{93}$$

Note that the state variable p_j^- is not really necessary for the firm that can change its price; it has full flexibility so its decision only depends upon today's states. We retain it for notational consistency.

Notice that the Calvo price setting imposes a nasty restriction on firms that cannot update their prices, which is that at some point they could sell at a loss and cannot do anything to counteract that. That is even worse if the Calvo fairy imposes wage rigidity, forcing workers to work at a loss.

Exercise 58 *Derive the following for the dynamic model with Calvo pricing*

1. *Solve the firm's problem in sequence space and write the firm's equilibrium pricing $p_{j,t}$ as a function of present and future aggregate prices, wages, and endowments: $\{P_t, W_t, I_t\}_{t=0}^{\infty}$.*

2. *Show that under flexible pricing ($\theta = 1$), the firm's pricing strategy is identical to the static model.*

3. *Show that with price rigidity ($\theta < 1$), the firm's pricing strategy is identical to the static model in the steady state with zero inflation.*

13.3 Aggregate Price Dynamics

From now we will consider price rigidities modelled using Calvo frictions. Define the set of firms not reoptimizing their posted price in period t as $A(t) \in [0,1]$. Using the definition of the aggregate price level and the fact that all firms resetting orices will choose the same P^{*},

$$P = \left[\int_{A(t)} \left(P^{-}\right)^{1-\sigma} di + (1-\theta)\left(p^{*}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}$$
$$= \left[\theta\left(P^{-}\right)^{1-\sigma} + (1-\theta)\left(P^{*}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}$$

where the second equality follows from the fact that the distribution of prices among firms not adjusting in period t corresponds to the distribution of effective prices in $t-1$ with reduced mass θ. Finally, dividing both sides by P^{-} we get

$$\Pi^{1-\sigma} = \theta + (1-\theta)\left(\frac{P^{*}}{P^{-}}\right) \tag{94}$$

which describes aggregate price dynamics, where $\Pi = P/P^{-}$ is the gross rate of inflation between $t-1$ and t. Moreover, log-linearizing teh aggregate price index around the steady state yields

$$\pi = (1-\theta)\left(p^{*} - p^{-}\right) \tag{95}$$

where lower case varibales denote log transformed variables. Equivalently

$$p = \theta p^{-} + (1-\theta)p^{*} \tag{96}$$

i.e. the current price level is a weighted average of last period's price level and the newly chosen one (in logs) where the weights are dictated by the fraction of adjusting firms.

Exercise 59 *Show equations (95) and (96) by taking a log-linearization of the price index around the steady state level.*

13.4 Optimal Price Setting

A firm reoptimizing in period t will choose the price P^* that maximizes the current market value of the profits generated while that price remains effective. The optimality condition associated with the problems is (after using representative agent condition)

$$P^* = \frac{\sigma}{\sigma-1}\frac{E\left\{\sum_\tau (\theta\beta)^\tau u_c P_\tau^{\sigma-1}\varphi_\tau y_\tau\right\}}{E\left\{\sum_\tau (\theta\beta)^\tau u_c P_\tau^{\sigma-1} y_\tau\right\}}$$

where φ_τ is nominal marginal cost. Log-linearizing around zero-inflation steady state we get (excluding staeady state terms)

$$p^* = (1-\theta\beta)E\left\{\sum_\tau (\theta\beta)^\tau \left[mc_\tau + p_\tau\right]\right\} \tag{97}$$

where $\mu = \log\frac{\sigma}{\sigma-1}$ and where mc is log real marginal cost.

Exercise 60 *Show the the optimal price setting satisfies eq. (97), after taking a log-linearization around the steady state.*

14 Extreme Value Shocks

Extreme value shocks are a tool commonly used in empirical micro. They are popular because of their mathematical convenience when working with discrete choice models, where agents choose between a finite number of unordered or ordered options. Obviously, in these situations we could simply go to the data and calculate what fraction of agents pick which option and be done, but we want to say a little more about the dependence of these choice probabilities on the characteristics of the options. Extreme value shocks in a random utility model give us a convenient way of doing so.

In macro models, extreme value shocks can play a role in turning decision rules into decision densities. Typically, decision rules are deterministic functions of states, even in models where we have uncertainty:

agents form expectations about future states, and any agent in the same state will make the same choice. But sometimes we want to allow the agents to make mistakes. For instance, in environments with private information it may be hard to distinguish between decision makers, and there may be issues with pooling or separating equilibria. In this regard extreme value shocks make all equilibria pooling because all agents have a positive probability of making any choice, making it infeasible to separate agents.

Extreme value shocks also give a natural way to deal with off-equilibrium behavior since any type of behavior can now be on the equilibrium path with some positive probability.

14.1 Discrete Choice Problems

Consider the following discrete choice setting. An agent is choosing between I options, which each come with a cost z^i and a utility of $u^i + v(c)$, where $v(c)$ is the utility of all other consumption. Here we will allow the agent to have an exogenous income y (measured in units of the consumption good) and thus $c = y - z^i$.

Now, let's say that we have some data on these choices, and we observe various fractions of agents making each of the choices $i \in \{1, \dots, I\}$. So far, our model can't account for that at all; every agent looks the same and is thus going to make the same choice! Thus, we need some sort of probabilistic shock ϵ^i that affects the utility of each choice such that agents will make each choice with some positive probability p^i. This gives us the following model:

$$\max_i \; u^i + \epsilon^i + v(y - z^i) = \max_i \; u^i + \beta z^i + \epsilon^i,$$

where in the second formulation we've simply modeled $v(\cdot)$ in a reduced-form manner. This model will allow us to make sense of two things:

1. the percentage of choices being of each option i; and

2. how agents respond to changes in prices z^i (elasticities).

In order to use the model for these two purposes, we need to find an expression for p^i, or the probability of an agent making a choice i. If we assume the shocks are iid with arbitrary cdf $F(\cdot)$ and pdf $f(\cdot)$, we can find these probabilities as follows:

$$\begin{aligned} p^i &= Pr(u^i + \beta z^i + \epsilon^i \geq u^j + \beta z^j + \epsilon^j, \ \ \forall j \neq i) \\ &= Pr(\epsilon^j \leq u^i - u^j + \beta(z^i - z^j) + \epsilon^i, \ \ \forall j \neq i) \\ &= \int_{-\infty}^{\infty} \prod_{j \neq i} F(u^i - u^j + \beta(z^i - z^j) + \epsilon^i) f(\epsilon^i) d\epsilon^i \end{aligned}$$

This is a complicated expression which does not have an analytical form for many common distributions $F(\cdot)$, including the normal distribution. However, if we assume that the shocks are distributed according to a Gumbel, or Type 1 extreme value, distribution, some mathematical magic happens:

$$\begin{aligned} p^i &= \int_{-\infty}^{\infty} \prod_{j \neq i} F(u^i - u^j + \beta(z^i - z^j) + \epsilon^i) f(\epsilon^i) d\epsilon^i \\ &= \frac{\exp(u^i + \beta z^i)}{\sum_{j=1}^{I} \exp(u^j + \beta z^j)} \end{aligned}$$

We'll go over exactly what this Gumbel distribution looks like in a later section, but suffice it to say for now that it's similar to a normal distribution but with fatter tails. What is important here is that we have a very simple analytic expression to characterize these choice probabilities, which we can easily take to the data and estimate using maximum likelihood.

There are some subtler problems with this model that we won't dwell on too much here. For example, the assumption that the errors are independent may not be so benign, and also the implicit assumption of Independence of Irrelevant Alternatives (IIA) can cause issues when two or more options are essentially the same (see the red bus/blue bus problem). It will suffice to say for our purposes that these issues can be addressed in specific circumstances by making modifications to the original model, such as the nested logit.

14.2 Discrete and Continuous Choices

In macro settings, we typical model agents making both discrete and continuous choices; for example, an agent may choose how much to save (continuous) and whether to retire or not to retire (discrete). If we consider a static setting with a continuous choice and two discrete options, we can write the problem as $\max\{u(y,0), u(y-q,1)\}$, where 0 is the first option and 1 is the second option. If the preferences $u(\cdot)$ are separable and strictly concave, we will find a simple threshold rule: choose 0 if y is lower than some threshold $\overline{y}$ and 1 otherwise. Generally this results in an overall drop in consumption.

However, when we write this problem in a dynamic setting, the discontinuities propagate across periods. This makes this type of problem very challenging to solve. Thus, we introduce extreme value shocks ϵ^i to produce decision probabilities rather than decision rules and write the problem as follows:

$$\begin{aligned} V(s,a) &= \max\{V^0(a), V^1(a)\} \\ &= \max\{\max_{a'} u(Ra+s-a',0) + \epsilon^0 + \beta EV(s',a'), \\ &\qquad\qquad \max_{a'} u(Ra+s-a'-q,1) + \epsilon^1 + \beta EV(s',a')\} \end{aligned}$$

Since both the choices occur with positive probability for all a, this model gets rid of the discontinuities that arise from the deterministic model.

14.3 The Gumbel Distribution

In the previous sections, we assume that ϵ^i are iid according to the Gumbel, or Type 1 extreme value, distribution. We'll denote this distribution as $\epsilon^i \sim G(\mu, \alpha)$, where μ is the location parameter and α is the scale parameter. Here are a few important statistics of this distribution:

$$\begin{aligned}
\mathbb{E}(\epsilon^i) &= \mu + \alpha\gamma \\
\mathbb{V}(\epsilon^i) &= \frac{\pi^2\alpha^2}{6} \\
\text{Median}(\epsilon^i) &= \mu - \alpha \ln(\ln(2)) \\
\text{Mode}(\epsilon^i) &= \mu \\
\text{cdf}(\epsilon^i) &= e^{-e^{-\frac{\epsilon^i - \mu}{\alpha}}}
\end{aligned}$$

Here γ is the Euler-Mascheroni constant, $\gamma \approx 0.5772$.

One of the most useful properties of Gumbel distributed random variables is that the expected maximum of these random variables has an analytic form. This matters for the kinds of discrete choice problems that we analyze, where agents maximize over choices with Gumbel distributed shocks. In order to see this, let's define the following terms:

$$\begin{aligned}
X^N &= \max\{\epsilon^1, \ldots, \epsilon^N\} \\
V^N &= \mathbb{E}[X^N]
\end{aligned}$$

Amazingly, X^N, the maximum of N Gumbel distributed random variables, is also Gumbel distributed:

$$X^N \sim G(\mu + \alpha \ln N, \alpha)$$

And V^N has the following formula, derived from the expectation of a Gumbel distributed random variable:

$$V^N = \mu + \alpha \ln N + \alpha\gamma$$

Notice that V^N depends upon the number of choices, and increases as this number goes up. If we want V^N to be a specific number regardless of the number of choices, we can adjust α, the scale parameter, to keep this expectation constant in N:

$$V^N = \overline{V} \Rightarrow \alpha(N) = \frac{\overline{V} - \mu}{\gamma + \ln N}$$

Another way is to adjust the location parameter μ:

$$V^N = \overline{V} \Rightarrow \mu(N) = \overline{V} - \alpha \ln N - \alpha\gamma$$

What if the location parameters of the shocks are heterogeneous? We can handle this as well. Suppose $\epsilon^i \sim G(\delta^i + \mu, \alpha)$. Then, we have the following:

$$X^N \sim G\left(\mu + \alpha \ln \sum_i e^{\frac{\delta^i}{\alpha}}, \alpha\right)$$
$$V^N = \mu + \alpha \ln \sum_i e^{\frac{\delta^i}{\alpha}} + \alpha\gamma$$

If we want to make V^N independent of the number of choices again, it's a bit more difficult. For the location parameter, we still have an analytical solution:

$$\mu(N) = \overline{V} - \alpha \left(\gamma + \mu + \alpha \ln \sum_i e^{\frac{\delta^i}{\alpha}}\right)$$

For the scale parameter, we have to solve the following equation:

$$\alpha(N) = \frac{\overline{V} - \mu}{\gamma + \ln \sum_i e^{\frac{\mu + \delta^i}{\alpha(N)}}}$$

This equation has no closed-form solution, and thus needs to be solved numerically.

14.4 A Continuum of Choices

Suppose we wish to consider a continuum of choices c in an interval $C = [0, \overline{c}]$, each with a utility shock $\epsilon(c) \sim G(0, \alpha(c))$. We want to have an expectation of the maximum utility from this continuum of choices, like the following:

$$V^C = \mathbb{E}[\max_{c \in C} \epsilon(c)], \quad \text{for some } V^C > 0$$

The problem is that the expected maximum of Gumbel distributed random variables increases with the number of variables, and does not converge to some finite number as that number of variables increases to infinity. Thus, instead of having an infinite number of shocks, we'll create a finite grid over C with N equally-spaced points, each of which has a shock $\epsilon^i \sim G(0, \alpha(i))$. Then we can define $X^N = \max\{\epsilon^1, \ldots, \epsilon^N\}$ and $V^N = \mathbb{E}[X^N]$ in the typical fashion. We will choose $\alpha(V^C, N)$ such that $V^N = V^C$, where we treat V^C as a target parameter. This choice of scale parameter behaves according to the following function:

$$\alpha(V^C, N) = \frac{V^C}{\ln N + \gamma}$$

Now, some agents may not be able to afford all portions of the choice set C, meaning that they may find themselves facing a restricted choice set $[0, \widetilde{c}]$ where $\widetilde{c} < \overline{c}$. As we've seen before, V^N is increasing in the number of choices, so if the number of grid points below $\widetilde{c}$ is less than N, the expected utility for any given choice c will be lower, even if it's the same choice as an agent with a larger choice set would have chosen.

The question is, does this represent something fundamental economically? It may or may not, but assuming it does, it introduces a new type of precautionary savings motivation: agents save in order to have a larger choice set in the future. You can think of this as a utility bonus for "freedom" or "capriciousness" on the part of agents with larger choice sets: they enjoy being less restricted in their number of options, regardless of whether they choose those options. We can then think of V^C as a fundamental parameter that determines the size of the utility bonus for the richest agent, or the one with the largest choice set.

How do we calculate expected utility for an agent with a restricted choice set $[0, \widetilde{c}]$, when $\widetilde{c}$ isn't exactly on top of one of the grid points? We need to first keep track of the highest grid point to the left of $\widetilde{c}$, which we will denote as $N^{\widetilde{c}}$. Then the lowest grid point to the right of $\widetilde{c}$ is $N^{\widetilde{c}} + 1$. We will then say that the probability that our agent draws $N^{\widetilde{c}}$ ϵ's is as follows:

$$\underline{p}(\widetilde{c}) = \frac{N^{\widetilde{c}}+1}{N^{\overline{c}}} - \frac{\widetilde{c}}{\overline{c}}$$

And the probability that the agent draws $N^{\widetilde{c}}+1$ ϵ's:

$$\overline{p}(\widetilde{c}) = \frac{\widetilde{c}}{\overline{c}} - \frac{N^{\widetilde{c}}}{N^{\overline{c}}}$$

According to these formulae, some agents may draw zero ϵ's with positive probability; we will define the expected utility from this outcome to be zero. For all other situations, we define expected utility as:

$$V^{\widetilde{c}} = \underline{p}(\widetilde{c})V^{N^{\widetilde{c}}} + \overline{p}(\widetilde{c})V^{N^{\widetilde{c}}+1}$$

where

$$V^{n} = \alpha(V^{\overline{c}}, N^{\overline{c}})(\ln n + \gamma), \ \ \forall n.$$

When we attempt to solve this problem algorithmically with respect to some discretized state space s, we have to account for both the density of grid points implied by the state space and the density of grid points in the consumption space. To do this, we calculate the maximum consumption attainable from each state, and evaluate the expected value at that maximum consumption according to the above formula, and then calculate the appropriate scale parameter. Then we can iterate on the value function, including the utility bonus for the larger choice set.

15 Endogenous Growth and R&D

So far, we have seen the *neoclassical growth model* as our benchmark model, and built on it for the analysis of more interesting economic questions. One peculiar characteristic of our benchmark model, unlike its name suggests, is the lack of growth (after reaching the steady state), whereas, many interesting questions in economics are related to the cross-country differences of growth rates. To see why this is the case, consider the standard neoclassical technology:

$$F(K,N) = AK^{\theta_1}L^{\theta_2},$$

for some $\theta_1, \theta_2 \geq 0$. We already know that the only possible case that is consistent with the notion of competitive equilibrium is that $\theta_1 + \theta_2 = 1$. However, this implies a decreasing marginal rate of product for capital. Given a fixed quantity for labor supply, in the presence of depreciation, this implies a maximum sustainable capital stock, and puts a limit on the sustainable growth; economy converges to some steady-state, without exhibiting any balanced growth.

So if our economy is to experience sustainable growth for a long period of time, we either give up the curvature assumption on our technology, or we have to be able to shift our production function upwards. Given a fixed amount of labor, this shift is possible either by an increasing (total factor) productivity parameter or increasing labor productivity. We will cover a model that will allow for growth, so that we will be able to attempt to answer such questions.

Exercise 61 *Assume that the production function takes the form $F = AK$. This is often called the AK model. Compute the growth rate in a balance growth path.*

Consider the following economy due to the highly cited model of endogenous growth. There are three sectors in the economy: a final good sector, an intermediate goods sector, and an R&D sector. Final goods are produced using labor (as we will see there is only one wage, since there is only one type of labor) and intermediate goods according to the production function

$$N_{1,t}^{\alpha}\int_0^{A_t} x_t(i)^{1-\alpha}\,di$$

where $x(i)$ denotes the utilization of intermediate good of variety $i \in [0, A_t]$.[20] Note that marginal contribution of each variety is decreasing (since $\alpha < 1$), however, an increase in the number of varieties

[20] The function that aggregates consumption of intermediate goods is often referred as Dixit-Stiglitz aggregator.

would increase the output. We will assume that the final good producers operate in a competitive market.

Exercise 62 *If the price of all varieties are the same, what is the optimal choice of input vector for a producer?*

What if they do not have the same amount? Would a firm decide not to use a variety in the production?

Intermediate producers are monopolists that have access to a differentiated technology of the form:

$$x(i) = \frac{k(i)}{\eta}.$$

Therefore, they can end up charging a mark up above the marginal cost for their product. This is the main force behind research and development in this economy; developer of a new variety is the sole proprietor of the blue print that allows him earn profit. It is easy to observe that the aggregate demand of capital from the intermediate sector is $\int_0^{A_t} \eta x(i)\, di$.

The R&D sector in the economy is characterized by a flow of intermediate goods in each period; a new good is a new variety of the intermediate good. The flow of the new intermediate goods is created by using labor, according to the following production technology:

$$\frac{A_{t+1}}{A_t} = 1 + \xi N_{2,t}.$$

Notice that, after some manipulation, one can express growth in the stock of intermediate goods as follows:

$$A_{t+1} - A_t = A_t \xi N_{2,t}. \tag{98}$$

Hence, the flow of new intermediate goods is determined by the current stock of them in the economy. This type of externality in the model is the key propeller in the model. This assumption provides us with a constant returns to scale technology in the R&D sector. In what follows, we will assume that the inventors act as price takers in the economy.

Remark 15 *The reason we see A_t on the right hand side of* (98) *as an externality is that it is indeed used as an input in the process of R&D, while, it is not paid for. Thus, inventors, in a sense, do not pay the investors of the previous varieties, while using their inventions. They only pay for the labor they hire. Perhaps, the basic idea of this production function might be traced back to* Isaac Newton*'s quote:* "If I have seen further, it is only by standing on the shoulders of giants".

The preferences of the consumers are represented by the following utility function:

$$\sum_{t=0}^{\infty} \beta^t u(c_t),$$

and their budget constraint in period t is give by:

$$c_t + k_{t+1} \leq r_t k_t + w_t + (1-\delta) k_t.$$

Remark 16 *In this economy, GDP, in terms of gross product, is given by:*

$$Y_t = W_t + r_t K_t + \pi_t,$$

where π_t is the net profits. On the other hand, in terms of expenditures, GDP is:

$$Y_t = C_t + K_{t+1} - (1-\delta) K_t + \pi_t,$$

where $K_{t+1} - (1-\delta) K_t$ is the investment in physical capital. At last, in terms of value added, it is given by:

$$Y_t = N_t^{\alpha} \int_0^{A_t} x_t(i)^{1-\alpha} \, di + p_t (A_{t+1} - A_t).$$

Certainly, this is not a model that one can map to the data. Instead it has been carefully crafted to deliver what is desired and it provides an interesting insight in thinking about endogenous growth.

Solving the Model Let's first consider the problem of a final good producer; in every period, he chooses $N_{1,t}$ and $x_t(i)$, for every $i \in [0, A_t]$, in order to solve:

$$\max\ N_{1,t}^{\alpha} \int_0^{A_t} x_t(i)^{1-\alpha}\, di - w_t N_{1,t} - \int_0^{A_t} q_t(i)\, x_t(i)\, di,$$

where $q_t(i)$ is the price of variety i in period t. First order conditions for this problem are:

1. $N_{1,t}$: $\alpha N_{1,t}^{\alpha-1} \int_0^{A_t} x_t(i)^{1-\alpha}\, di = w_t$; and,

2. $x_t(i)$: $(1-\alpha) N_{1,t}^{\alpha} x_t(i)^{-\alpha} = q_t(i)$, for all $i \in [0, A_t]$.

From the second condition, one obtains:

$$x_t(i) = \left(\frac{(1-\alpha)}{q_t(i)}\right)^{\frac{1}{\alpha}} N_{1,t}, \tag{99}$$

which, given N_{1t}, is the *demand function* for variety i, by the final good producer.

Next, let's consider the problem of an intermediate firm; these firms acts as price setters. The reason is the ownership of a differentiated patent, whose sole owner is the intermediate good producer of variety i. In addition, as long as $\alpha < 1$, this variety does not have a perfect substitute, and always demanded in the equilibrium. Therefore, their problem is to choose $q_t(i)$, in order to solve:

$$\begin{aligned} \pi_t(i) = \max \quad & q_t(i)\, x_t(q_t(i)) - r_t \eta x_t(q_t(i)) \\ s.t. \quad & x_t(q_t(i)) = \left(\frac{(1-\alpha)}{q_t(i)}\right)^{\frac{1}{\alpha}} N_{1,t}, \end{aligned}$$

where $x_t(q_t(i))$ is the demand function, substituted from (99). Notice that we have substituted for the technology of the monopolist, $x(i) = k(i)/\eta$. First order condition for this problem, with respect to $q_t(i)$, is:

$$x_t(q_t(i)) + (q_t(i) - r_t\eta) \frac{\partial x_t(q_t(i))}{\partial q_t(i)} = 0,$$

which can be written as

$$\frac{(1-\alpha)^{\frac{1}{\alpha}}}{q_t(i)^{\frac{1}{\alpha}}}N_{1,t}=\frac{(q_t(i)-r_t\eta)}{\alpha}\frac{(1-\alpha)^{\frac{1}{\alpha}}}{q_t(i)^{\frac{1+\alpha}{\alpha}}}N_{1,t}.$$

Rearranging yields:

$$q_t(i)=\frac{1}{(1-\alpha)}r_t\eta. \tag{100}$$

This is the familiar pricing function of a monopolist; price is marked-up above the marginal cost.

By substituting (100) into (99), we get:

$$x_t(i)=\left[\frac{(1-\alpha)^2}{r_t\eta}\right]^{\frac{1}{\alpha}}N_{1,t}, \tag{101}$$

and the demand for capital services is simply $\eta x_t(i)$. In a symmetric equilibrium, where all the intermediate good producers choose the same pricing rule, we have:

$$\int_0^{A_t}x_t(i)\,di=A_tx_t=\frac{k_t}{\eta},$$

where x_t is the common supply of intermediate goods. Therefore:

$$x_t=\frac{k_t}{\eta A_t}.$$

Moreover, if we let Y_t be the production of the final good; by plugging (101) we get:

$$Y_t=N_{1,t}A_t\left[\frac{(1-\alpha)^2}{r_t\eta}\right]^{\frac{1-\alpha}{\alpha}}. \tag{102}$$

Hence the model displays constant returns to scale in $N_{1,t}$ and A_t.

Let us study the problem of the R&D firms, next; a representative firm in this sector (recall that this is a competitive sector) chooses $N_{2,t}$, in order to solve the following problem:

$$\max_{N_{2,t}} \quad p_t A_t \xi N_{2,t} - w_t N_{2,t}.$$

The first order condition for this problem implies:

$$p_t = \frac{w_t}{A_t \xi}.$$

In summary, there are two equations to be solved form; one relating the choice of consumption versus saving (or capital accumulation), and one dividing labor demand for R&D, and that for final good production. Consumption-investment decisions result from solving household's problem in equilibrium, and the corresponding Euler equation:

$$u'(c_t) = \beta u'(c_{t+1})\left[r_{t+1} + (1-\delta)\right].$$

For determining the labor choices $N_{1,t}$ and $N_{2,t}$, first note that the demand for patterns produced by R&D sector, is from the prospect monopolists. As long as there is positive profit from buying demand, the new monopolists would keep entering markets in a given period. This fact derives the profits of prospect monopolists to zero. So, the lifetime profit of the monopolist, must be equal to the price he pays for the blueprints;

$$p_t = \sum_{s=t}^{\infty}\left(\prod_{\tau=t}^{s}\frac{1}{1+r_\tau-\delta}\right)\pi_s.$$

This completes the solution to the model. Notice that output can grow at the same rate as A_t, from Equation (102). In addition, K_t grows at the same rate. As a result, the rate of growth of A_t would be the important aspect of equilibrium. For instance, if A_t grows at rate γ in the long-run, we have a balanced growth path in equilibrium. This growth comes from the externality in the R&D sector. Without that, we cannot get sustained growth in this model. The nice thing about this model is how neat it is in delivering the balanced growth, with just enough structure imposed on the economy.

Exercise 63 *Define a decentralized equilibrium for this economy.*

Exercise 64 *Characterize as far as you can the balanced growth path of this economy.*

15.1 Growth Model With Many Firms

In this section we will introduce a different model, a growth model with many firms that is suitable for the pandemic. One of the most salient economic consequences of the pandemic was the destruction of small firms, and we'd like to have a model that incorporates the financial and technological reasons for this phenomenon. The model will incorporate heterogeneous firm sizes/types, but the emphasis will be on small business creation and not inequality so we will use a representative household. Unlike the previous model, this model doesn't explicitly incorporate any epidemiological phenomena so its results can be applied to a more general business cycle.

As in Quadrini (2000), we will have two sectors in the economy: a corporate and a non-corporate sector. The corporate sector will produce the consumption good using capital and labor via the aggregate production function $F(K, N)$. The non-corporate sector is where the heterogeneity will be incorporated; we will have firms with sizes or types $i \in \{1, \ldots, I\}$, who produce the consumption good according to production function $f^i(n)$. This production function will be increasing in labor, $f_n^i > 0$, and production will only occur if the firm has the required number of managers λ^i (we'll return to this later when discussing the household's problem).

Firms will need to be created (by households), and have a cost of creation ξ^i. In addition, firms can invest in maintenance m, which affects the probability of survival $q^i(m)$. This probability function has the following properties: $q^i(0) = 0, q^i(\infty) < 1, q_m^i > 0$. Finally, we wish to allow for demand-side factors when it comes to firm profits so we will say that a portion of the goods produced $\psi(S) < 1$ will be sold, where S is the aggregate state of the economy.

The firm's problem is therefore as follows:

$$\Omega^i(S) = \max_{n\geq 0, m\leq f^i(n)-wn} \psi(S)f^i(n) - wn - m + \frac{q^i(m)}{R(S')}\Omega^i(S')$$

Notice how financial constraints are incorporated into the model: the firm has no access to financing and thus must choose maintenance investment such that it makes only positive profits in each period.

Now for the household. Each household will own a measure of type i firms x^i. Households can choose to create b^i new firms of each type at cost ξ^i each. In addition, households can save using corporate capital a. Households also must allocate their members to be managers, workers, or enjoyers of leisure such that the labor constraint is satisfied:

$$n + \sum_i \lambda^i x^i + l = 1$$

Managers must choose between allocating firm resources towards maintenance or profits. Note that λ^i is not a choice made by the household; it is a managerial constraint where each firm of type i requires λ^i managers to be operational in a given period.

Finally, households have preferences over consumption and leisure given by utility function $u(c, l)$, and discount the future at rate β. The problem of the household thus reads as follows:

$$V(S, a, x^1, \ldots, x^I) = \max_{c,n,b^1,\ldots,b^I,a} \quad u(c, 1-n-\sum_i \lambda^i x^i) + \beta V(S', a', x^{1'}, \ldots, x^{I'})$$

$$s.t. \;\; c + \sum_i b^i \xi^i + a' = nw(S) + aR(S) + \sum_i \pi^i(S)x^i$$

$$x^{i'} = q^i(M^i)x^i + b^i, \qquad \forall i \in \{1, \ldots, I\}$$

The last constraint is the law of motion of the firms of type i.

In the aggregate, the feasibility constraint for the economy looks like the following:

$$C + K' - (1-\delta)K + \sum_i X^i M^i + \sum_i B^i \xi^i = \sum_i X^i f^i(N^i) + F(K, N)$$

Exercise 65 *Derive the FOC's for the firm, first assuming that m is unrestricted, and then assuming that $m \leq f^i(n) - wn$.*

Exercise 66 *Derive the FOC's for the household, first assuming that $\lambda^i = 0$ and $\pi > 0$, and then assuming that $\lambda^i > 0$.*

16 An Integrated Analysis Model of Climate Change

We turn now to the analysis of a simple model of climate change. Energy is an important input for economic production. However, the use of energy in the private sector generates a negative externality for the society as a whole. For instance, using fossil fuel for energy creates carbon dioxide in the environment. This, in turn, negatively affects the quality of life for the entire society: higher levels of carbon dioxide in the atmosphere contributes to global warming, which in turn causes damages like production shortfalls, poor health or deaths, capital destruction and much more. This is a classic example of negative externality, since firms don't internalize the negative effect of carbon dioxide on the environment. The goal of this section is to build a model where energy is a required input for production, and characterized the optimal taxation policy that makes the decentralized equilibrium socially efficient.

We assume that there is a representative firm that produces with technology $Y_t = F_t(K_t, N_t, E_t, S_t)$, where E_t is a vector of energy sources used in the production. There are many types of energy inputs $E_{j,t} = 1, ..., J$. The first $J_g - 1$ sectors are *dirty*, and the last one is *clean* energy. Let us assume that one unit of dirty $E_{j,t}$ produces one unit of carbon. Then, total emissions are $\sum_{j+1}^{J_g - 1} E_{j,t}$. Furthermore, we assume that the efficacy of the energy sources in production is given by a vector of parameters α_j. Then, the actual amount of energy used in production is given by $E_t = \sum_{j=1}^{J} E_{j,t}\alpha_j$.

Additionally, assume that some energy resources have a finite stock, and are subject to the constraint $R_{j,t+1} = R_{j,t} - E^{j}_{j,t} \geq 0$, where $R_{j,t}$ is the stock of source available in period t. Lastly, assume that a unit of dirty energy in the production has constant cost ξ_j, while a clean energy has convex cost $\xi_J(E_{J,T})$.

The climate variable S_t is the amount of carbon in the atmosphere, and depends on past emissions. Define a function $\widetilde{S}_t$ that maps the history of man made pollution into the current level of carbon dioxide:

$$S_t = \widetilde{S}_t \left(\sum_{j=1}^{J_g - 1} E_{j,-T}, \sum_{j=1}^{J_g - 1} E_{j,-T+1}, ..., \sum_{j=1}^{J_g - 1} E_{j,t} \right)$$

where $-T$ is defined as the start of industrialization.

To analyze the model, we make the following functional assumptions:

1. $U(C) = \log(C)$
2. We postulate a damage function. That is, carbon reduces output proportionally to a function $D_t(S_t)$. Therefore, what is left is given by:

$$F_t(K_t, N_t, E_t, S_t) = [1 - D_t(S_t)]\, \widetilde{F}_t(K_t, N_t, E_t)$$

3. The damage function takes the specific form:

$$[1 - D_t(S_t)] = \exp\{-\gamma_t(S_t - \bar{S})\}$$

4. The function $\widetilde{S}_t$ is linear and has the depreciation structure:

$$S_t - \bar{S} = \sum_{s=0}^{t+T} \sum_{j=1}^{J_g - 1} E_{j,t-s}$$

We now turn to the solution of the social planner problem, in order to characterize the socially optimal allocation. The social planner problem is the following:

$$\max_{\substack{\{C_t,N_t,K_{t+1},R_{j,t+1},\\ E_{j,t},S_t\}_{t=0}^{\infty}\geq 0}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(C_t) \qquad \text{s.t.}$$

$$C_t + K_{t+1} = F_t(K_t, N_t, E_t, S_t) + (1-\delta)K_t \qquad \text{FB}$$

$$E_t = \sum_j E_{j,t}\ \alpha^j \qquad \text{AGE}$$

$$R_{j,t+1} = R_{j,t} - E_{j,t} \geq 0 \qquad \text{for all } j \qquad ExE$$

$$S_t = \widetilde{S}_t\left(\sum_{j=1}^{J_g-1} E_{j,-T}, \sum_{j=1}^{J_g-1} E_{j,-T+1}, ..., \sum_{j=1}^{J_g-1} E_{j,t}\right) \qquad CC$$

We can define the marginal externality damage from using a *dirty* source of energy as follows:

$$\Lambda_t^s = \mathbb{E}\sum_{i=0}^{\infty} \beta^i \frac{U'(C_{t+i})}{U'(C_t)} \frac{\partial F_{t+i}}{\partial S_{t+i}} \frac{\partial S_{t+i}}{\partial E_{j,t}} \tag{103}$$

Intuitively, the marginal externality is a result of the fact that using a dirty energy increases the amount of carbon in the atmosphere in all future periods ($\frac{\partial S_{t+i}}{\partial E_{j,t}}$), and that in turn the amount of carbon has a negative impact on future production ($\frac{\partial F_{t+i}}{\partial S_{t+i}}$). The marginal negative externality is the sum of all the negative effects on future production, discounted by the sequence of stochastic discount factors ($\beta^i \frac{U'(C_{t+i}}{U'(C_t)}$). Notice that, under our functional assumptions, equation (103) simplifies to:

$$\Lambda_t^s = \mathbb{E}\sum_{i=0}^{\infty} \beta^i C_t \frac{Y_{t+i}}{C_{t+i}} \gamma_{t+i}(1-d_i)$$

Exercise 67 *Solve for the FOCs of the social planner problem. Find the optimality conditions as a function of* Λ_t^s*.*

The decentralized equilibrium consists of a consumer and a firms' problem. The problem of the consumers is the following:

$$\max_{\{C_t, N_t, K_{t+1}\}_{t=0}^{\infty}} \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(C_t)$$

$$\text{s.t.} \quad \mathbb{E}_0 \sum_{t=0}^{\infty} q_t(C_t + K_{t+1}) = \mathbb{E}_0 \sum_{t=0}^{\infty} q_t((1 + r_t - \delta)K_t + w_t N_t + T_t) + \Pi_t.$$

The firms' problem is the following:

$$\Pi_0 = \max_{\{K_t, N_t, E_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} q_t \left[F_t(K_t, N_t, E_t, S_t) - r_t K_t - w_t N_t - \sum_{j=1}^{J} p_{j,t} E_{j,t} \right]$$

Suppose that we solved for the optimality conditions for both the planner and the decentralized problem. Then, we could characterized the optimal tax policy that equalizes the conditions of the planner to the one of the decentralized problem. Under our functional forms, it can be shown that the optimal tax is $\tau_{j,t} = \Lambda_t^s$ on the use of *dirty* energy sources, and $\tau_{j,0}$ on *clean* energy. Intuitively, by taxing dirty energy sources, the firms decrease their use of energy, and the amount of carbon in the atmosphere drops. Setting the optimal tax rate is equivalent to make the firms internalize the negative externality caused by dirty energy.

Exercise 68 *Show that the optimal tax rate is given by* $\tau_{j,t} = \Lambda_t^s$. *That is, solve for the FOCs of the decentralized problem when usage of energy is taxed, and compare the conditions with the solution of the planner problem.*

A A Farmer's Problem: Revisited

Consider the following problem of a farmer that we studied in class:

$$V(s,a) = \max_{c,a'} \left\{ u(c) + \beta \sum_{s'} \Gamma_{ss'} V(s',a') \right\} \tag{104}$$

$$\begin{aligned} s.t. \quad & c + qa' = a + s \\ & c \geq 0 \\ & a' \geq 0. \end{aligned}$$

As we discussed, we are in particular interested in the case where $\beta/q < 1$. In what follows, we are going to show that, under monotonicity assumption on the Markov chain governing s, the optimal policy associated with (104) implies a finite support for the distribution of asset holding of the farmer, a.[21]

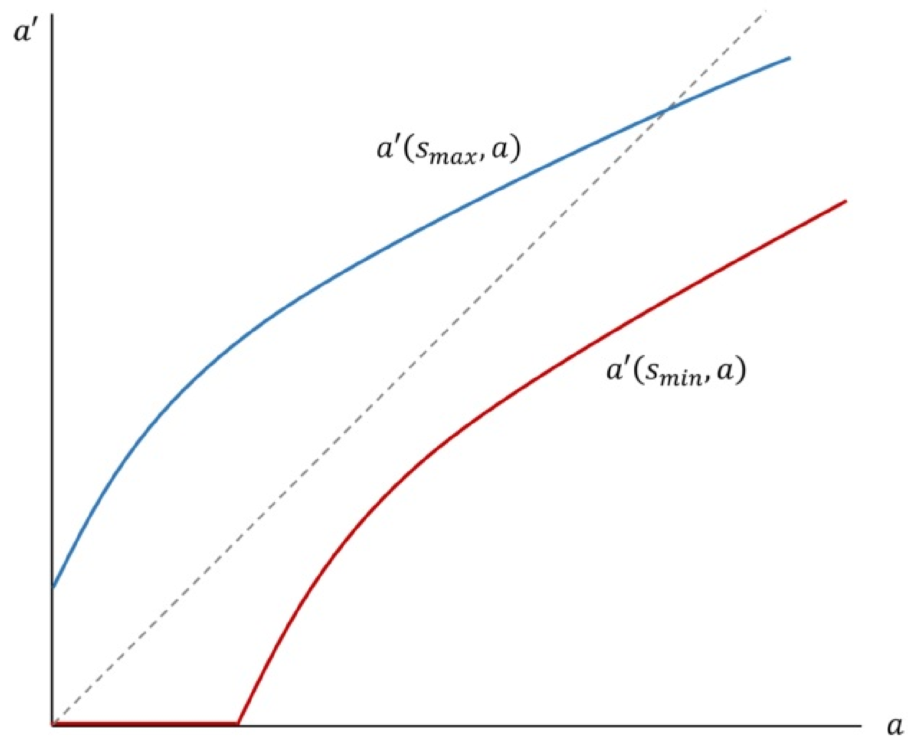
Before we start the formal proof, suppose $s_{\min} = 0$, and $\Gamma_{ss_{\min}} > 0$, for all $s \in S$. Then, the agent will optimally always choose $a' > 0$. Otherwise, there is a strictly positive probability that the agent enters tomorrow into state $s_{\min}$, where he has no *cash in hand* ($a' + s_{\min} = 0$) and is forced to consume 0, which is extremely painful to him (e.g. when Inada conditions hold for the instantaneous utility). Hence he will raise his asset holding a' to insure himself against such risk.

If $s_{\min} > 0$, then the above argument no longer holds, and it is indeed possible for the farmer to choose zero assets for tomorrow.

Notice that the borrowing constraint $a' \geq 0$ is affecting agent's asset accumulation decisions, even if he is away from the zero bound, because he has an incentive to ensure against the risk of getting a series of bad shocks to s and is forced to 0 asset holdings. This is what we call *precautionary savings motive*.

[21] This section was prepared by *Keyvan Eslami*, at the *University of Minnesota*. This section is essentially a slight variation on the proofs found in Huggett. However, he accepts the responsibility for the errors.

Figure 1: Policy function associated with farmer's problem.



Next, we are going to prove that the policy function associated with (104), which we denote by $a'(\cdot)$, is similar to that in Figure 1. We are going to do so, under the following assumption.

Assumption 1 *The Markov chain governing the state s is monotone; i.e. for any $s_1, s_2 \in S$, $s_2 > s_1$ implies $E(s|s_2) \geq E(s|s_1)$.*

It is straightforward to show that, the value function for Problem (104) is concave in a, and bounded. Now, we can state our intended result as the following theorem.

Theorem 4 *Under Assumption 1, when $\beta/q < 1$, there exists some $\hat{a} \geq 0$ so that, for any $a \in [0, \hat{a}]$, $a'(s, a) \in [0, \hat{a}]$, for any realization of s.*

To prove this theorem, we proceed in the following steps. In all the following lemmas, we will assume that the hypotheses of Theorem 4 hold.

Lemma 1 *The policy function for consumption is increasing in a and s;*

$$c_a(a,s) \geq 0 \text{ and } c_s(a,s) \geq 0.$$

Proof 1 *By the first order condition, we have:*

$$u'(c(s,a)) \geq \frac{\beta}{q} \sum_{s'} \Gamma_{ss'} V_a\left(s', \frac{a+s-c(s,a)}{q}\right),$$

with equality, when $a+s-c(s,a) > 0$.

For the first part of the lemma, suppose a increases, while $c(s,a)$ decreases. Then, by concavity of u, the left hand side of the above equation increases. By concavity of the value function, V, the right hand side of this equation decreases, which is a contradiction.

For the the second part, we claim that $V_a(s,a)$ is a decreasing function of s. To show this is the case, firs consider the mapping T as follows:

$$\begin{aligned} Tv(s,a) = \max_{c,a'} \quad & \left\{ u(c) + \beta \sum_{s'} \Gamma_{ss'} v(s', a') \right\} \\ s.t. \quad & c + qa' = a + s \\ & c \geq 0 \\ & a' \geq 0. \end{aligned}$$

Suppose $v_a^n(s,a)$ is decreasing in its first argument; i.e. $v_a^n(s_2,a) < v_a^n(s_1,a)$, for all $s_2 > s_1$ and $s_1, s_2 \in S$. We claim that, $v^{n+1} = Tv^n$ inherits the same property. To see why, note that for $a^{n+1}(s,a) = a'$ (where a^{n+1} is the policy function associated with n'th iteration) we must have:

$$u'(a+s-qa') \geq \frac{\beta}{q} \sum_{s'} \Gamma_{ss'} v_a^n(s', a'),$$

with strict equality when $a' > 0$. For a fixed value of a', an increase in s leads to a decrease in both sides of this equality, due to the monotonicity assumption of Γ, and the assumption on v_a^n. As a result, *we must have*

$$u'\left(a+s_2-qa^{n+1}(s_2,a)\right)\leq u'\left(a+s_1-qa^{n+1}(s_1,a)\right),$$

for all $s_2>s_1$. *By Envelope theorem, then:*

$$v_a^{n+1}(s_2,a)\leq v_a^{n+1}(s_1,a).$$

It is straightforward to show that v^n *converges to the value function* V *point-wise. Therefore,*

$$V_a(s_2,a)\leq V_a(s_1,a),$$

for all $s_2>s_1$.

Now, note that, by envelope theorem:

$$V_a(s,a)=u'(c(s,a)).$$

As s *increases,* $V_a(s,a)$ *decreases. This implies* $c(s,a)$ *must increase.*

Lemma 2 *There exists some* $\hat{a}\in\mathbb{R}_+$*, such that* $\forall a\in[0,\hat{a}]$*,* $a'(a,s_{\min})=0$.

Proof 2 *It is easy to see that, for* $a=0$*,* $a'(a,s_{\min})=0$*. First of all, note the first order condition:*

$$u_c(c(s,a))\geq\frac{\beta}{q}\sum_{s'}\Gamma_{ss'}u_c(c(s',a'(s,a))),$$

with equality when $a'(s,a) > 0$. Under the assumption that $\beta/q < 1$, we have:

$$\begin{aligned} u_c\left(c\left(s_{\min},0\right)\right) =& \frac{\beta}{q}\sum_{s'}\Gamma_{s_{\min}s'}u_c\left(c\left(s',a'\left(s_{\min},0\right)\right)\right) \\ <& \sum_{s'}\Gamma_{ss'}u_c\left(c\left(s',a'\left(s_{\min},0\right)\right)\right). \end{aligned}$$

By Lemma 1, if $a' = a'(0,s_{\min}) > a = 0$, then $c(s',a') > c(s_{\min},0)$ for all $s' \in S$, which leads to a contradiction.

Lemma 3 *$a'(s_{\min},a) < a$, for all $a > 0$.*

Proof 3 *Suppose not; then $a'(s_{\min},a) \geq a > 0$ and as we showed in Lemma 2:*

$$u_c\left(c\left(s_{\min},a\right)\right) < \sum_{s'}\Gamma_{ss'}u_c\left(c\left(s',a'\left(s_{\min},a\right)\right)\right).$$

Contradiction, since $a'(s_{\min},a) \geq a$, and $s' \geq s_{\min}$, and the policy function in monotone.

Lemma 4 *There exits an upper bound for the agent's asset holding.*

Proof 4 *Suppose not; we have already shown that $a'(s_{\min},a)$ lies below the 45 degree line. Suppose this is not true for $a'(s_{\max},a)$; i.e. for all $a \geq 0$, $a'(s_{\max},a) > a$. Consider two cases.*

In the first case, suppose the policy functions for $a'(s_{\max},a)$ and $a'(s_{\min},a)$ diverge as $a \to \infty$, so that, for all $A \in \mathbb{R}_+$, there exist some $a \in \mathbb{R}_+$, such that:

$$a'\left(s_{\max},a\right) - a'\left(s_{\min},a\right) \geq A.$$

Since S is finite, this implies, for all $C \in \mathbb{R}_+$, there exist some $a \in \mathbb{R}_+$, so that

$$c\left(s_{\min},a\right) - c\left(s_{\max},a\right) \geq C,$$

which is a contradiction, since c is monotone in s.

Next, assume $a'(s_{\max}, a)$ and $a'(s_{\min}, a)$ do not diverge as $a \to \infty$. We claim that, as $a \to \infty$, c must grow without bound. This is quite easy to see; note that, by envelope condition:

$$V_a(s,a) = u'(c(s,a)).$$

The fact that V is bounded, then, implies that V_a must converge to zero as $a \to \infty$, implying that $c(s,a)$ must diverge to infinity for all values of s, as $a \to \infty$. But, this implies, if $a'(s_{\max}, a) > a$,

$$u_c(c(s_{\max}, a'(s_{\max}, a))) \to \sum_{s'} \Gamma_{s_{\max}s'} u_c(c(s', a'(s_{\max}, a))).$$

As a result, for large enough values of a, we may write:

$$\begin{aligned} u_c(c(s_{\max}, a)) =& \frac{\beta}{q} \sum_{s'} \Gamma_{s_{\max}s'} u_c(c(s', a'(s_{\max}, a))) \\ <& \sum_{s'} \Gamma_{s_{\max}s'} u_c(c(s', a'(s_{\max}, a))) \\ \approx& u_c(c(s_{\max}, a'(s_{\max}, a))). \end{aligned}$$

But, this implies:

$$c(s_{\max}, a) > c(s_{\max}, a'(s_{\max}, a)),$$

which, by monotonicity of policy function, means $a > a'(s_{\max}, a)$, and this is a contradiction.

B Linearization and Log-linearization

The equilibrium conditions of a model at time t can generally be described by the following nonlinear system of $M \in \mathbb{N}^*$ equations[22]:

$$E_t \begin{bmatrix} g_1(x_t, x_{t+1}) \\ \vdots \\ g_M(x_t, x_{t+1}) \end{bmatrix} = 0,$$

where $x_t \in \mathbb{R}^n$ is a vector of variables including choice variables (e.g., C_t, K_t, L_t, but also possibly their lags) as well as exogenous variables like A_t, and the $g_m, 1 \leqslant m \leqslant M$, are some functions $g_m : (x_t, x_{t+1}) \in \mathbb{R}^{2n} \to \mathbb{R}$.[23]

B.1 Linearization

We start with simple linearization. The idea is simply to do a first-order Taylor expansion of function G around some natural steady state $\bar{x} \in \mathbb{R}^n$. Consider equilibrium condition m, we get the following expression, up to the first order:

$$E_t \left[g_m(\bar{x}, \bar{x}) + \nabla g_m(\bar{x}, \bar{x}) \cdot \begin{pmatrix} x_t - \bar{x} \\ x_{t+1} - \bar{x} \end{pmatrix} \right] = 0$$

In general, the steady state $\bar{x}$ will correspond to some non-stochastic steady state. In that case, we impose the additional requirement that

$$g_m(\bar{x}, \bar{x}) = 0 \text{ for all } 1 \leqslant m \leqslant M.$$

[22] This is not the most general form equilibrium conditions will take, since we are only allowing for one-period ahead variables to matter. It is straightforward to extend this to more general cases

[23] This section was prepared by *Luigi Falasconi* (UPenn).

Denoting $\Delta x_t = x_t - \bar{x}$, we end up with the following linear system of equations:

$$E_t \begin{bmatrix} \nabla g_1(\bar{x}, \bar{x}) \cdot (\Delta x_t, \Delta x_{t+1})' \\ \vdots \\ \nabla g_M(\bar{x}, \bar{x}) \cdot (\Delta x_t, \Delta x_{t+1})' \end{bmatrix} = 0,$$

which is a (possibly forward looking) linear system of equations. We will learn later how to solve those.

Example 1 *The linearization of the Cobb-Douglas production function $Y_t = K_t^{\alpha} N_t^{1-\alpha}$ around some point $\bar{Y} = \bar{K}^{\alpha} \bar{N}^{1-\alpha}$ is*

$$\Delta Y_t = \alpha \bar{K}^{\alpha-1} \bar{N}^{1-\alpha} \Delta K_t + (1-\alpha) \bar{K}^{\alpha} \bar{N}^{1-\alpha} \Delta N_t = \alpha \frac{\bar{Y}}{\bar{K}} \Delta K_t + (1-\alpha) \frac{\bar{Y}}{\bar{N}} \Delta N_t$$

Example 2 *The linearization of the law of motion for capital $K_{t+1} = (1-\delta)K_t + I_t$ around the steady state $\bar{K} = \bar{I}/\delta$ is*

$$\Delta K_{t+1} = (1-\delta)\Delta K_t + \Delta I_t$$

B.2 Log-linearization

Linearization is nice when we are dealing with additive equilibrium conditions like resource constraints. However, many of the models we use are close to log-linear and contain various multiplicative terms. It becomes then useful to express equilibrium conditions not as a function of level deviations from steady state, Δx_t, but as percent deviations from it, i.e., $\frac{\Delta x_t}{\bar{x}}$. By doing so, we may obtain lighter expressions and we may be able to relate equilibrium conditions to elasticities, which have a nice economic content. This is what we call log-linearization. Let the log-deviation of variable x_t from steady state $\bar{x} \neq 0$ be

[24]

$$\hat{x}_t \equiv \frac{\Delta x_t}{\bar{x}} = \frac{x_t - \bar{x}}{\bar{x}} \simeq \Delta \log x_t$$

In practice, there are different ways to proceed:

Method 1. Simply linearize around the steady state $\bar{x}$ and then multiply/divide each term to make the log-deviation terms appear:

$$g_m(\bar{x}, \bar{x}) + \nabla g_m(\bar{x}, \bar{x}) \cdot (x_t - \bar{x}, x_{t+1} - \bar{x})' = g_m(\bar{x}, \bar{x}) + \sum_{i=1}^{n} \frac{\partial g_m}{\partial x_{i,t}}(\bar{x}, \bar{x})\bar{x}_i \cdot \left(\frac{x_{it} - \bar{x}_i}{\bar{x}_i}\right) + \sum_{i=1}^{n} \frac{\partial g_m}{\partial x_{i,t+1}}(\bar{x}, \bar{x})\bar{x}_i \cdot \left(\frac{x_{i,t+1} - \bar{x}_i}{\bar{x}_i}\right)$$

assuming that $x_t = (x_{1,t}, \ldots, x_{n,t})'$. Under the additional assumption that $g_m(\bar{x}, \bar{x}) = 0$, we end up with the log-linearized equilibrium condition

$$E_t \left[\sum_{i=1}^{n} \frac{\partial g_m}{\partial x_{i,t}}(\bar{x}, \bar{x})\bar{x}_i \cdot \left(\frac{x_{it} - \bar{x}_i}{\bar{x}_i}\right) + \sum_{i=1}^{n} \frac{\partial g_m}{\partial x_{i,t+1}}(\bar{x}, \bar{x})\bar{x}_i \cdot \left(\frac{x_{i,t+1} - \bar{x}_i}{\bar{x}_i}\right)\right] = 0.$$

Example 3 *Recall that the linearization of the Cobb-Douglas production function $Y_t = K_t^{\alpha} N_t^{1-\alpha}$ around some point $\bar{Y} = \bar{K}^{\alpha} \bar{N}^{1-\alpha}$ was*

$$\Delta Y_t = \alpha \frac{\bar{Y}}{\bar{K}} \Delta K_t + (1 - \alpha) \frac{\bar{Y}}{\bar{N}} \Delta N_t$$

The corresponding log-linearized expression is

$$\hat{Y}_t = \frac{\Delta Y_t}{\bar{Y}} = \alpha \frac{\Delta K_t}{\bar{K}} + (1 - \alpha) \frac{\Delta N_t}{\bar{N}} = \alpha \hat{K}_t + (1 - \alpha) \hat{N}_t.$$

[24] Note that log-deviations are only well defined if the steady state $\bar{x}$ is not 0 . So what do we do when $\bar{x} = 0$? You can either reformulate your problem and log-linearize it around another non-zero variable, for instance $1 + x_t$, or you can do a simple linearization. It is perfectly fine to use a mix of linearization/log-linearization.

Method 2. Similarly to the way we compute derivatives, you can apply the chain rule and the following set of elementary operations:

1. $\hat{a} = 0$ where a is a constant;
2. $\widehat{ax_t} = \hat{x}_t$;
3. $\widehat{x_t y_t} = \hat{x}_t + \hat{y}_t$;
4. $\widehat{x_t / y_t} = \hat{x}_t - \hat{y}_t$;
5. $\widehat{x_t + y_t} = \frac{\bar{x}}{\bar{x}+\bar{y}}\hat{x}_t + \frac{\bar{y}}{\bar{x}+\bar{y}}\hat{y}_t$;
6. $\widehat{x_t - y_t} = \frac{\bar{x}}{\bar{x}-\bar{y}}\hat{x}_t - \frac{\bar{y}}{\bar{x}-\bar{y}}\hat{y}_t$;
7. $\widehat{x_t^a} = a\hat{x}_t$;
8. $\widehat{f(x_t)} = \varepsilon_f \hat{x}_t$ where f is a function and ε_f is its elasticity, $\varepsilon_f = \frac{\partial \log f}{\partial \log x}(\bar{x}) = \frac{f'(\bar{x})\bar{x}}{f(\bar{x})}$.

Example 4 *Go back to the Cobb-Douglas case. This case is quite straightforward:*

$$\hat{Y}_t = \widehat{K_t^\alpha N_t^{1-\alpha}} = \widehat{K_t^\alpha} + \widehat{N_t^{1-\alpha}} = \alpha\hat{K}_t + (1-\alpha)\hat{N}_t.$$

Method 3. Another approach is to express our equilibrium conditions directly as a function of the log-deviations $\hat{x}_t$ by noticing that $x_t = e^{\log x_t} = e^{\log \bar{x}} e^{\log x_t - \log \bar{x}} = \bar{x}e^{\hat{x}t}$. In that case, we re-express our equilibrium conditions as functions of $\hat{x}_t$ and then do a Taylor expansion around $\hat{x}_t = 0$. For instance,

$$\begin{aligned} g_m(x_t, x_{t+1}) &= g_m\left(\bar{x}e^{\hat{x}_t}, \bar{x}e^{\hat{x}_{t+1}}\right) \\ &\simeq g_m(\bar{x},\bar{x}) + \sum_{i=1}^{n} \frac{\partial g_m}{\partial x_{i,t}}(\bar{x},\bar{x}) \cdot \frac{\partial}{\partial \hat{x}_{it}}\left(\bar{x}_i e^{\hat{x}_{it}}\right)(0) \cdot (\hat{x}_{it} - 0) \\ &+ \sum_{i=1}^{n} \frac{\partial g_m}{\partial x_{i,t+1}}(\bar{x},\bar{x}) \cdot \frac{\partial}{\partial \hat{x}_{it+1}}\left(\bar{x}_i e^{\hat{x}_{it+1}}\right)(0) \cdot (\hat{x}_{it+1} - 0) \\ &= g_m(\bar{x},\bar{x}) + \sum_{i=1}^{n} \frac{\partial g_m}{\partial x_{i,t}}(\bar{x},\bar{x}) \cdot \bar{x}_i\hat{x}_{it} + \sum_{i=1}^{n} \frac{\partial g_m}{\partial x_{i,t+1}}(\bar{x},\bar{x})\bar{x}_i\hat{x}_{it+1} \end{aligned}$$

where we recognize the same expression we had derived with method 1.

Example 5 *Go back to the Cobb-Douglas example and let's use method 3 :*

$$\begin{aligned} Y_t = K_t^\alpha N_t^{1-\alpha} &= \bar{K}^\alpha e^{\alpha \hat{K}_t} \bar{N}^{1-\alpha} e^{(1-\alpha)\hat{N}_t} \\ &= \bar{K}^\alpha \bar{N}^{1-\alpha} + \alpha \bar{K}^\alpha \bar{N}^{1-\alpha} \hat{K}_t + (1-\alpha) \bar{K}^\alpha \bar{N}^{1-\alpha} \hat{N}_t \end{aligned}$$

Throw $\bar{Y} = \bar{K}^\alpha \bar{N}^{1-\alpha}$ to the other side and divide everything by $\bar{Y}$, we recover

$$\hat{Y}_t = \frac{Y_t - \bar{Y}}{\bar{Y}} = \alpha \hat{K}_t + (1-\alpha)\hat{N}_t.$$

C Solutions to Recitation Exercises

Exercise 1 *If output per capita grows at 2 percent per year, how often does it double?*

Proof 1 (Solution) *Let's call the growth rate g (here, $g = 0.02$). If output per capita is x in year 0, it will be $x(1+g)$ in year 1, $x(1+g)(1+g)$ in year 2, and so on. Thus, in year t, output per capita is $x(1+g)^t$. So, if output per capita doubles every t years, t must satisfy the following equality:*

$$x(1+g)^t = 2x$$

Dividing both sides by x,

$$(1+g)^t = 2$$

We want to solve for t. So, taking logs:

$$\begin{aligned} t \log(1+g) &= \log 2 \\ t &= \frac{\log 2}{\log(1+g)} \end{aligned}$$

Substituting in $g = 0.02$, we get $t \approx 35$, which means that output per capita doubles every 35 years.

There's another, easier way to do this: the "rule of 70." The rule states: If a quantity grows at rate g each year, the amount of time it will take to double is approximately $70/(100 \times g)$. We use $100 \times g$ because it is equivalent to g measured in percentages. This formula also immediately gives us that output per capita doubles every 35 years if it grows at 2 percent per year.

This formula is simple to derive. We know that $t = \frac{\log 2}{\log(1+g)}$. Note that $\log 2 \approx 0.7$, and $\log(1+g) \approx g$. Thus, $t = \frac{\log 2}{\log(1+g)} \approx \frac{0.7}{g}$. We can then multiply numerator and denominator by 100 to obtain the formula.

Exercise 2 *Build a growth model with exogenous technical change, and show that the balanced growth path of this model satisfies the Kaldor facts. Use either Cobb-Douglas or log plus constant times Frisch preferences.*

Proof 2 (Solution) *We'll provide a solution for Cobb-Douglas preferences.*

The balanced growth path means that in the steady state, all variables are growing at a constant rate g - with some notable exceptions. The interest rate r is constant, and the aggregate labor N is also constant. The reason aggregate labor is constant is because we are restricting the time endowment for agents to be 1 and not allowing population growth. So in order to still have this model satisfy the Kaldor facts, we will make all the exogenous technical change be changes in labor productivity, which we'll call X.

The (recursive) problem of the household is:

$$V(K,a) = \max_{c,l,a'} \frac{(c^{\theta} l^{1-\theta})^{1-\sigma}}{1-\sigma} + \beta V(K',a')$$
$$s.t. \;\; c + a' = w(1-l) + a(1+r)$$

The problem of the firm is (letting $\delta = 1$):

$$\max_{K,L} AK^{\alpha}(XL)^{1-\alpha} - (r+1)K - wL$$

We assume exogenous technical change occurs in X, the labor productivity term, which grows at a rate of g. First order conditions for the firm give us the factor price equations:

$$\alpha A \left(\frac{K}{XL}\right)^{\alpha-1} = r+1$$
$$X(1-\alpha)A\left(\frac{K}{XL}\right)^{\alpha} = w$$

First-order conditions and the Envelope Theorem give us the following two equations for the household:

$$\theta c^{\theta-1} l^{1-\theta}(c^{\theta}l^{1-\theta})^{-\sigma} = \beta\theta(1+r')c'^{\theta-1}l'^{1-\theta}(c'^{\theta}l'^{1-\theta})^{-\sigma} \tag{1}$$
$$(1-\theta)c^{\theta}l^{-\theta}(c^{\theta}l^{1-\theta})^{-\sigma} = \theta w c^{\theta-1}l^{1-\theta}(c^{\theta}l^{1-\theta})^{-\sigma} \tag{2}$$

The second, intratemporal, equation simplifies to the following result:

$$w = \frac{1-\theta}{\theta}\frac{c}{l}$$

The first, intertemporal, equation doesn't immediately simplify. However, under the balanced growth path, we know that $c' = g \cdot c$, $l' = l$, and $r' = r$. Then, we can simplify (1) to:

$$r^* = \frac{1}{\beta}g^{1-\theta(1+\sigma)} - 1$$

This constant interest rate is Kaldor fact six.

Equating this with the factor price equations from the firm, we find:

$$\frac{K}{L} = \left(\frac{\alpha A \beta}{g^{1-\theta(1+\sigma)}}\right)^{\frac{1}{1-\alpha}} X$$

The right-hand side of this equation is simply a constant times X. Since X grows at rate g, it follows *that the capital-labor ratio grows at rate g. This shows that the model satisfies Kaldor fact three.*

What about Kaldor fact one (the output-labor ratio)? Let's examine this ratio in the context of this model:

$$\frac{Y}{L} = \frac{AK^{\alpha}L^{1-\alpha}X^{1-\alpha}}{L} = A\left(\frac{K}{L}\right)^{\alpha}X^{1-\alpha}$$

Then, iterating forward a period and plugging in the growth rate for the relevant variables,

$$\begin{aligned}\frac{Y'}{L'} &= A\left(g\frac{K}{L}\right)^{\alpha}g^{1-\alpha}X^{1-\alpha}\\ &= gA\left(\frac{K}{L}\right)^{\alpha}X^{1-\alpha}\\ &= g\frac{Y}{L}\end{aligned}$$

So Kaldor fact one is satisfied: output per capita grows at a constant rate.

Now let's look at the capital-output ratio:

$$\begin{aligned}\frac{K}{Y} &= \frac{K}{AK^{\alpha}L^{1-\alpha}X^{1-\alpha}}\\ &= A^{-1}\left(\frac{K}{L}\right)^{1-\alpha}X^{\alpha-1}\end{aligned}$$

Then, iterating forward a period and plugging in the growth rate for the relevant variables,

$$\begin{aligned}\frac{K'}{Y'} &= A^{-1}g^{1-\alpha}\left(\frac{K}{L}\right)^{1-\alpha}g^{\alpha-1}X^{\alpha-1}\\ &= A^{-1}\left(\frac{K}{L}\right)^{1-\alpha}X^{\alpha-1}\\ &= \frac{K}{Y}\end{aligned}$$

So the capital-output ratio is constant, and Kaldor fact two is satisfied.

What about the wage? Well, we can rewrite the firm's wage condition as follows:

$$w = (1-\alpha)LY$$

L is constant, $(1-\alpha)$ is constant, and Y grows at rate g. Thus, wages grow at rate g, satisfying Kaldor fact five.

Now, since w grows at rate g, L is constant, and Y grows at rate g, it follows that wL/Y is constant over time. This satisfies Kaldor fact four.

Finally, (2) gives us the condition $l = \frac{1-\theta}{\theta}\frac{c}{w}$. c and w grow at rate g, so their ratio is constant, and l is constant. This satisfies Kaldor fact seven.

Exercise 3 *Prove that the RCE with capital income tax is not optimal.*

Proof 3 (Solution) *The household's problem is as follows (suppressing arguments for pricing functions and taxes):*

$$V(K,a) = \max_{c,a'} u(c,M) + \beta V(K',a')$$

$$s.t. \;\; c + a' = w + a(1 + r(1-\tau))$$

$$K' = G(K)$$

Substituting in the budget constraint, taking first order conditions, and using the Envelope Theorem we obtain the following:

$$\beta\frac{u_1(c',M')}{u_1(c,M)} = \frac{1}{1+r'(1-\tau)}$$

Now, the firm's problem gives us $F_1(K,1) = r+1$, so we find the following in equilibrium:

$$\beta\frac{u_1(c',M')}{u_1(c,M)} = \frac{1}{1+(F_1(K',1)-1)(1-\tau)}$$

What is the social planner's problem? In recursive form, it is as follows:

$$V(K) = \max_{C,K'} u(C) + \beta V(K')$$
$$s.t. \quad C + K' = F(K,1)$$

Substituting in the resource constraint and using FOC's and the Envelope Theorem we obtain:

$$\beta \frac{u_1(C',M')}{u_1(C,M)} = \frac{1}{F_1(K',1)}$$

Comparing the social planner's problem to the RCE with capital income tax, we see that the RCE is optimal whenever the following equation holds true:

$$1 + (F_1(K',1) - 1)(1-\tau) = F_1(K',1)$$

Clearly, this is only true if $\tau = 0$, or if there are no taxes. Otherwise, the RCE is not optimal.

Exercise 4 *Define an RCE with habit formation utility. Is it optimal?*

Proof 4 (Solution) Definition 26 *A Recursive Competitive Equilibrium is a set of functions V, g, G, w, and R, such that*

1. *Given w and R, V and g solve the household's problem,*
2. *Factor prices are paid their marginal productivities*

   $$w(K,C^-) = F_2(K,1) \text{ and } R(K,C^-) = F_1(K,1),$$

3. *Household wealth = Aggregate wealth*

   $$g(K,C^-,K,F(G^{-1}(K),1) - K) = G(K,C^-).$$

Now to establish optimality, let's restate the household's problem:

$$V(K, C^-, a, c^-) = \max_{c,a'} u(c, c^-) + \beta V(K', C, a', c)$$
$$s.t. \;\; c + a' = R(K, C^-)a + w(K, C^-)$$
$$K' = G(K)$$

From now on, I'll suppress the arguments for the price functions. Instead of substituting in the budget constraint here, I prefer to write out the Lagrangian for this problem:

$$V(K, C^-, a, c^-) = \max_{c,a',\lambda} u(c, c^-) + \beta V(K', C, a', c) + \lambda(Ra + w - c - a')$$

The first order conditions are as follows:

$$u_1(c, c^-) + \beta V_4(K', C, a', c) = \lambda \tag{1}$$
$$\beta V_3(K', C, a', c) = \lambda \tag{2}$$
$$Ra + w - c - a' = 0 \tag{3}$$

From the Envelope Theorem, we obtain the following two equations:

$$V_4(K', C, a', c) = u_2(c', c) \tag{4}$$
$$V_3(K', C, a', c) = \lambda' R' \tag{5}$$

In order to get rid of the multiplier, we can set the LHS's of (1) and (2) equal to each other:

$$u_1(c, c^-) + \beta V_4(K', C, a', c) = \beta V_3(K', C, a', c)$$

And substitute in (4) and (5) for the value function derivatives:

$$u_1(c, c^-) + \beta u_2(c', c) = \beta \lambda' R'$$

But unfortunately, we're still left with next period's multiplier λ' in this equation. To get rid of this, we'll substitute in the LHS of (1), but iterated forward a period, and the value function derivative from (4), also iterated forward a period:

$$u_1(c, c^-) + \beta u_2(c', c) = \beta R'(u_1(c', c) + \beta u_2(c'', c'))$$

Rearranging, we arrive at the following expression:

$$\beta \frac{u_1(c', c) + \beta u_2(c'', c')}{u_1(c, c^-) + \beta u_2(c', c)} = \frac{1}{R'} \tag{6}$$

This may look a little weird relative to what we normally see for the household's intertemporal Euler equation. But if you think about what the lifetime utility function is for the household, it should become clearer:

$$U(c^t) = \cdots + \beta^t u(c_t, c_{t-1}) + \beta^{t+1} u(c_{t+1}, c_t) + \beta^{t+2} u(c_{t+2}, c_{t+1}) + \cdots$$

When we take the derivative of this function with respect to consumption in period t, there are two terms instead of the usual one: it is $\beta^t u_1(c_t, c_{t-1}) + \beta^{t+1} u_2(c_{t+1}, c_t)$. So hopefully now you can recognize why ratio on the LHS of (6) has two terms in the numerator and denominator: it is simply the MRS for the household, but accounting for the double effect that consumption in each period has on utility.

Finally, substituting in the marginal product of capital for the rate of return R' in equilibrium, we find the typical $MRS = MRT$ condition still holds. Thus, this equilibrium is optimal.

Exercise 5 *Write down the Euler equations of the household and firm in the capital and land economy.*

Proof 5 (Solution) *The household's problem is as follows:*

$$\begin{aligned} V(K, a) &= \max_{c, a'} u(c) + \beta V(K', a') \\ s.t. \;\; c + P(K)a' &= a(D(K) + P(K)) \\ K' &= G(K) \end{aligned}$$

For the remainder of the problem, we will suppress the argument of the pricing functions. The easiest way to solve this problem is to substitute in the budget constraint:

$$V(K,a) = \max_{a'} u(a(D+P) - Pa') + \beta V(K',a')$$

The first order condition, with the envelope theorem, gives us the following:

$$\begin{aligned} Pu_1(c) &= \beta(D'+P')u_1(c') \\ \beta\frac{u_1(c')}{u_1(c)} &= \frac{P}{D'+P'} \end{aligned} \tag{1}$$

The firm's problem is as follows:

$$\begin{aligned} \Omega(K,k) &= \max_{d,k'} d + q'\Omega(K',k') \\ s.t.\ \ F(k,1) &= d + k' \\ K' &= G(K) \end{aligned}$$

Substituting in the budget constraint and taking first order conditions:

$$1 = q'F_1(k',1)$$

Rearranging and using the representative firm condition,

$$F_1(K',1) = \frac{1}{q'} \tag{2}$$

Now, one condition is missing from the equilibrium in the notes: the no arbitrage condition. It's a bit difficult to see given the way we've written down this problem, but there are actually two types of assets that are traded in this economy: shares of the firm, and savings in terms of the consumption good. No arbitrage means that the gross rates of return on these two types of assets must be equal.

We know what the gross rate of return is on shares of the firm: it's just $\frac{D'+P'}{P}$*, or the dividends plus the* *selling price of the share tomorrow, divided by the price you pay for the share today. But what about the price of a consumption good tomorrow in terms of units of the good today (remember, consumption good is the numeraire in this economy)? Well, the firm has to discount according to this conversion, because firms don't have an internal discount factor. So the rate of return on the consumption good is* $\frac{1}{q'}$.

Setting these equal, we can see that the no arbitrage condition in this economy looks like:

$$\frac{1}{q'} = \frac{D' + P'}{P}$$

This allows us to set the LHS's of (1) and (2) equal (after taking reciprocals) in equilibrium to find

$$\beta \frac{u_1(c')}{u_1(c)} = \frac{1}{F_1(K', 1)},$$

which is the typical $MRS = MRT$ *condition that lets us know that this equilibrium is optimal.*

Exercise 6 *Solve for the decision rules of the mutual fund in the international economy model.*

Proof 6 (Solution) *The problem of the mutual fund is as follows:*

$$\Phi(K^1, K^2, A, k^1, k^2) = \max_{k^{1'}, k^{2'}, n^1, n^2} \sum_i (F^i(k^i, n^i) - n^i w^i(K^i) - k^{i'}) + \frac{1}{R(K^{1'}, K^{2'}, A)} \Phi(K^{1'}, K^{2'}, A', k^{1'}, k^{2'})$$

From now on, we'll suppress the arguments of the pricing functions. The first order conditions give us:

$$R' = \Phi_{k^{i'}}(K^{1'}, K^{2'}, A', k^{1'}, k^{2'})$$
$$w^i = F_2^i(k^i, n^i)$$

where $\Phi_{k^{i'}}$ *is the derivative of the value function with respect to* $k^{i'}$.

The Envelope theorem gives us the following results for the value function derivatives:

$$\Phi_{k^{i'}}(K^{1'}, K^{2'}, A', k^{1'}, k^{2'}) = F_1^i(k^{i'}, n^{i'})$$

Substituting this into the first (pair of) first order condition, we obtain the following pricing functions:

$$R' = F_1^1(k^{1'}, n^{1'}) = F_1^2(k^{2'}, n^{2'})$$
$$w^1 = F_2^1(k^1, n^1)$$
$$w^2 = F_2^2(k^2, n^2)$$

If we make these conditions representative and use labor market clearing, we obtain the following:

$$R' = F_1^1(K^{1'}, 1) = F_1^2(K^{2'}, 1)$$
$$w^1 = F_2^1(K^1, 1)$$
$$w^2 = F_2^2(K^2, 1)$$

This tells us something about how the mutual fund chooses to employ capital and labor in each country. For labor, we see that it prices labor differently in each country because labor is immobile, and these prices depend only upon the capital in the country in which that labor is employed. For capital, we see that it prices capital the same in both countries, because capital is mobile. Thus, the marginal products of capital in countries 1 and 2 must be equal. Now, this doesn't mean that the mutual fund allocates equal amounts of capital to both countries; in fact, this would only be guaranteed if the production technologies in both countries were identical.

Exercise 7 *Write down the Euler equation for the Lucas tree economy with competitive search. How does it compare to the (static) social planner's problem?*

Proof 7 (Solution) *Let's rewrite the problem the household faces:*

$$
\begin{aligned}
V(\theta, z, s) = \max_{c,d,s',P,Q} u(\theta c, d) + \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} V(\theta', z', s') \\
s.t. \quad c + Ps' = P(s(1 + \hat{R}(\theta, z))) \\
c = d\Psi^h(Q)z \\
\frac{z\Psi^f(Q)}{P} \geq \hat{R}(\theta, z)
\end{aligned}
$$

Recall that $\Psi^h(Q) = Q^{1-\varphi}$ and $\Psi^f(Q) = Q^{-\varphi}$, since we assume a Cobb-Douglas matching function $M(T, D) = D^{\varphi} T^{1-\varphi}$.

The easiest way to solve this problem is to substitute in the first two budget constraints for c and s', and to add the firm's participation constraint into the maximization problem with a Lagrange multiplier λ. Doing this, we obtain the following:

$$
\begin{aligned}
V(\theta, z, s) = \max_{d,P,Q} u(\theta d Q^{1-\varphi} z, d) + \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} V\left(\theta', z' \frac{Ps(1 + \hat{R}(\theta, z)) - dQ^{1-\varphi} z}{P}\right) \\
+ \lambda \left(\hat{R}(\theta, z) - \frac{zQ^{-\varphi}}{P}\right)
\end{aligned}
$$

First-order conditions are as follows (after substituting c and s' back in to keep notation simple):

$$
\begin{aligned}
&[d] : \theta Q^{1-\varphi} z u_1(\theta c, d) + u_2(\theta c, d) = \frac{Q^{1-\varphi} z}{P} \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} V_3(\theta', z', s') \\
&[P] : \frac{dQ^{1-\varphi} z}{P^2} \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} V_3(\theta', z' s') + \frac{Q^{-\varphi} z}{P^2} \lambda = 0 \\
&[Q] : (1 - \varphi)\theta d z Q^{-\varphi} u_1(\theta c, d) - \frac{dz(1 - \varphi)Q^{-\varphi}}{P} \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} V_3(\theta', z', s') + \frac{\varphi z Q^{-\varphi-1}}{P} \lambda = 0
\end{aligned}
$$

We can use the Envelope Theorem to obtain V_3 (though it's not entirely necessary). We substitute the first and second budget constraints into the objective in a slightly different way, solving for c and d in *terms of s'. Then, differentiating, we obtain the following:*

$$V_3(\theta, z, s) = \theta P(1 + \hat{R}(\theta, z))u_1(\theta c, d) + \frac{P(1 + \hat{R}(\theta, Z))}{Q^{1-\varphi} z} u_2(\theta c, d)$$

If we substitute this equation into the first-order condition for d, the search effort, we find the household's intertemporal Euler equation from the endogenous productivity model without competitive search.

However, competitive search adds the choices of P and Q for the household into the problem, which allows us to pin down the equilibrium. Using the first-order condition for P, we obtain the following expression for λ:

$$\lambda = -dQ\beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} V_3(\theta', z', s')$$

We can then substitute this in for λ in the first-order condition for Q. Doing this and rearranging, we find:

$$(1 - \varphi)\theta z Q^{1-\varphi} u_1(\theta c, d) = \frac{Q^{1-\varphi} z}{P} \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} V_3(\theta', z', s')$$

Note that the RHS of this equation is exactly the same as the RHS of the first-order condition for d! So we can set the LHS's equal and obtain the following:

$$\theta Q^{1-\varphi} z u_1(\theta c, d) + u_2(\theta c, d) = (1 - \varphi)\theta z Q^{1-\varphi} u_1(\theta c, d)$$

Taking it a step further, we can move the RHS to the other side of the equation and simplify:

$$\varphi \theta z Q^{1-\varphi} u_1(\theta c, d) + u_2(\theta c, d) = 0$$

This one simple equation relates the marginal utility of consumption to the marginal disutility of searching, and thus governs the household's optimal search behavior for a particular productivity level z, hunger θ, and market tightness Q.

In equilibrium, we know that there is only one unit of trees, and thus $Q = T/D = 1/D$. Letting C be the aggregate consumption and D be aggregate search effort, we have the following, using the assumption that the agent is representative:

$$\varphi\theta z D^{\varphi-1} u_1(\theta C, D) + u_2(\theta C, D) = 0 \tag{1}$$

Now, let's take a look at the social planner's problem and see how the results compare. The social planner faces a static maximization problem, because there is no capital or other resource that can be saved: you can think of the fruit as going rotten after the period is over.

What is the aggregate resource constraint in this economy? Well, given some number of trees T and search effort D, the "factors of production" in this economy, the matching function $M(T, D)$ tells us how many resources are "produced." Multiplying the number of matches by the productivity z gives us the total amount of fruit available for consumption in the economy.

Thus, the social planner's problem is as follows:

$$\max_{C,D} u(\theta C, D)$$
$$s.t. \quad C = zM(T, D) = zD^{\varphi}$$

where we've substituted in the fact that $T = 1$ in this economy. Substituting in the budget constraint, this becomes a simple one-variable static problem:

$$\max_{D} u(\theta z D^{\varphi}, D)$$

Taking first-order conditions, we obtain the condition for optimality in this economy:

$$\varphi\theta z D^{\varphi-1} u_1(\theta C, D) + u_2(\theta C, D) = 0 \tag{2}$$

Recall the condition for optimality from competitive search from (1): it's exactly the same as equation (2)! So the competitive search protocol gives us a unique equilibrium prediction which is exactly the *same as if there were a single agent, knowing θ, z, and T, who simply chooses how much to search and consume each period in order to balance out marginal utility of consuming fruit and marginal disutility of searching for it.*

Exercise 8 *Analyze the McCall search model.*

Proof 8 (Solution) *In this labor search model, the agent maximizes expected discounted utility $\mathbb{E}\sum_{t=0}^{\infty}\beta^t x_t$. Here, x_t is a measure of period utility which depends upon employment status in the following manner:*

$$x_t = \begin{cases} w & \text{if employed} \\ b & \text{if unemployed} \end{cases}$$

In each period t, the unemployed agent receives one offer from the wage distribution $F(w)$. The wage draws are iid, and the wage distribution is constant. The agent has two choices in each period: accept the offer and be employed next period, or reject the offer and be unemployed the next period. There is no recall of offers, and the jobs last forever, so once an agent is employed, they remain employed indefinitely.

The value function for the employed agent is quite simple; the agent receives wage w this period and remains employed the next period.

$$W(w) = w + \beta W(w)$$

Therefore, we can easily solve for W analytically:

$$W(w) = \frac{w}{1-\beta}$$

This corresponds to the discounted sum of wages in all future periods.

The unemployed agent receives b in the current period, and then chooses between remaining unemployed next period or being employed at wage w drawn from distribution $F(w)$. The form is as follows:

$$U = b + \beta \int_0^\infty \max\{U, W(w)\} dF(w)$$

Substituting in the functional form for W, we find:

$$U = b + \beta \int_0^\infty \max\left\{U, \frac{w}{1-\beta}\right\} dF(w)$$

Note that there are no state variables for the unemployed agent, so his value function U is actually just a number! Thus, since $\frac{w}{1-\beta}$ is strictly increasing in w, the agent will accept the job if the wage is higher than some reservation wage w_R. In other words, he follows a threshold decision rule. Also, it follows that w_R is the point at which the value of being employed equals the value of being unemployed, so we have that the value function for the unemployed agent is:

$$U = \frac{w_R}{1-\beta}$$

We still would like to characterize w_R a bit further. Substituting our result for U into the Bellman equation, we find:

$$\begin{aligned}
\frac{w_R}{1-\beta} &= b + \beta \int_0^\infty \max\left\{\frac{w_R}{1-\beta}, \frac{w}{1-\beta}\right\} dF(w) \\
&= b + \frac{\beta}{1-\beta} \int_0^{w_R} w_R dF(w) + \frac{\beta}{1-\beta} \int_{w_R}^\infty w dF(w) \\
&= b + \frac{\beta}{1-\beta} \int_0^{w_R} w_R dF(w) + \frac{\beta}{1-\beta} \int_{w_R}^\infty w_R dF(w) + \frac{\beta}{1-\beta} \int_{w_R}^\infty w dF(w) - \frac{\beta}{1-\beta} \int_{w_R}^\infty w_R dF(w) \\
&= b + \frac{\beta}{1-\beta} w_R + \frac{\beta}{1-\beta} \int_{w_R}^\infty (w - w_R) dF(w)
\end{aligned}$$

Moving the first two terms of the RHS to the LHS and simplifying,

$$w_R - b = \frac{\beta}{1-\beta} \int_{w_R}^\infty (w - w_R) dF(w)$$

This equation has a nice economic interpretation. The left hand side is the cost of passing up on this period's offer of w_R and searching one more time. The right hand side is the expected profit from searching another period, discounted to put it in units of today's good. Thus, this tells us that w_R solves a typical marginal cost = marginal benefit equation.

Exercise 9 *In the industry equilibrium model, show that the n^* function is increasing in both arguments.*

Proof 9 (Solution) *The profit function of the firm is as follows:*

$$\pi(s,p) = \max_{n \geq 0} \{psf(n) - wn\}$$

Taking first order conditions, we find the following equation:

$$f'(n^*) = \frac{w}{ps}$$

Now, $f(n)$ is increasing and strictly concave, so $f'(n)$ is strictly decreasing and thus has an inverse $f^{'-1}$ which is also strictly decreasing. Thus, we can write:

$$n^*(s,p) = f^{'-1}\left(\frac{w}{ps}\right)$$

Now, as s increases, the fraction $\frac{w}{ps}$ decreases, and since $f^{'-1}$ is strictly decreasing, $n^*(s,p)$ increases. The exact same argument holds for p, and thus $n^*(s,p)$ is increasing in both arguments.

Exercise 10 *In the industry equilibrium model, what is the output produced by the top 10% largest firms?*

Proof 10 (Solution) *Recall that the distribution of firms in the productivity space S is captured by probability measure x. We noted that aggregate output of the entire industry can be calculated as follows:*

$$Y^S(p) = \int_{\underline{s}}^{\overline{s}} sf(n^*(s;p))x(ds)$$

In other words, we sum output for each and every firm, weighted by how many firms with that level of productivity there are in the economy. In order to calculate the output of the 10% largest firms, we simply need to sum over the restricted range of productivities corresponding to those firms. Thus, we need to find a $\hat{s}$ level of productivity such that the mass of firms above this level of productivity is 0.1*. $\hat{s}$ solves the following equation:*

$$\int_{\hat{s}}^{\overline{s}} x(ds) = 0.1$$

Then, the output of the top 10% largest firms is:

$$\int_{\hat{s}}^{\overline{s}} sf(n^*(s;p))x(ds)$$

Exercise 11 *In the industry equilibrium model, calculate the following statistics.*

1. *Compute the average growth rate of the smallest third of the firms.*

   Proof 11 (Solution) *We'll compute the growth rate of the profits, but you can easily extend this to the growth rate of some other measurement. The growth rate of a given firm's profits can be expressed as follows:*

$$\frac{\pi(s', p) - \pi(s, p)}{\pi(s, p)}$$

In other words, the change in profits relative to the original profits. Now, we need to characterize the smallest third of firms. The smallest third of firms are characterized by a maximum productivity level $\hat{s}$ where $\hat{s}$ solves the following:

$$\frac{\int_{s^*}^{\hat{s}} x^*(ds)}{\int_{s^*}^{\overline{s}} x^*(ds)} = \frac{1}{3}$$

Now we simply need to average the growth rate formula over both the distribution of productivities today for the smallest third of firms, and the distribution of productivities for the same firms tomorrow. We do this using the following double integral:

$$\int_{s^*}^{\hat{s}} \int_{s^*}^{\overline{s}} \frac{\pi(s', p) - \pi(s, p)}{\pi(s, p)} \Gamma(s, ds') x^*(ds)$$

x^* gives us the distribution of productivities today, and $\Gamma(s, ds')$ gives us the distribution of productivities tomorrow. Thus, this gives us the average growth rate for the smallest third of firms.

2. *Compute the fraction of firms in the top 10% that remain in the top 10% next period.*

Proof 12 (Solution) *To characterize the top 10% of firms, we need another threshold productivity $\hat{s}$, which in this case solves the following equation:*

$$\frac{\int_{\hat{s}}^{\overline{s}} x^*(ds)}{\int_{s^*}^{\overline{s}} x^*(ds)} = 0.1$$

Now that we have $\hat{s}$, we can easily distinguish firms in the top 10%. x^* gives us the distribution of firms both today and tomorrow by stationarity, but in order to keep track of the same firms in both periods we need to use the Markov transition probability Γ. This gives us the following formula:

$$\frac{\int_{\hat{s}}^{\overline{s}} \int_{\hat{s}}^{\overline{s}} \Gamma(s, ds') x^*(ds)}{\int_{\hat{s}}^{\overline{s}} x^*(ds)}$$

We divide it by the measure of firms in the top 10% this period in order to get the fraction of *these firms that remain in the top 10% next period.*

3. *Compute the fraction of firms younger than five years old.*

Proof 13 (Solution) *The easiest way to solve this problem is to define a transition of the measure γ of firms entering which keeps track of the measure of firms who entered in a period t that are still in the market some number of periods later. The transition is defined as follows:*

$$\gamma^{(t+1)}(B) = \int_{\underline{s}}^{\overline{s}} \Gamma(s, B \cap [s^*, \overline{s}])\gamma^{(t)}(ds)$$

Now, each period m firms enter the market. These firms are zero years old, satisfying the condition. Last period, m firms entered the market, and $m\gamma^{(0)}([s^*, \overline{s}])$ firms remain from those, letting $\gamma^{(0)} = \gamma$. For the firms who entered two periods ago, $m\gamma^{(1)}([s^*, \overline{s}])$ remain. This continues similarly, and in order to calculate the total measure of firms in the market that entered in the last four years, we simply sum them up.

Therefore, the fraction of firms younger than five years old in the market is as follows:

$$\frac{m + m\sum_{i=1}^{4} \gamma^{(i)}([s^*, \overline{s}])}{\int_{s^*}^{\overline{s}} x^*(ds)}$$

4. *Compute the Gini coefficient.*

Proof 14 (Solution) *The Gini coefficient is defined as one-half of the relative mean absolute difference. In our case, this translates to the following formula:*

$$G = \frac{\int_{s^*}^{\overline{s}} \int_{s^*}^{\overline{s}} |s - t| x^*(ds) x^*(dt)}{2\int_{s^*}^{\overline{s}} s x^*(ds)}$$

Exercise 12 *Write the first order conditions for the firm with labor adjustment costs.*

Proof 15 (Solution) *Remember, the problem of the firm with labor adjustment costs is:*

$$V(s,n_-,p) = \max\left\{0, \max_{n\geq 0}\ psf(n) - wn - c^v - c(n,n_-) + \frac{1}{1+r}\int_{s'} V(s',n,p)\Gamma(s,ds')\right\}$$

Assuming the firm doesn't decide to exit this period, it has one first order condition:

$$[n]: psf'(n) - w - c_1(n,n_-) + \frac{1}{1+r}\int_{s'} V_2(s',n,p)\Gamma(s,ds') = 0$$

We can also characterize V_2 using the Envelope Theorem:

$$V_2(s,n_-,p) = -c_2(n,n_-)$$

However, this isn't entirely accurate, because we've assumed the firm doesn't exit next period. That occurs with positive probability that isn't necessarily one, and depends both on the firm's labor decision this period, n, and the firm's productivity next period, s'. It is characterized by a threshold productivity above which the firm will stay in the market, but that threshold productivity necessarily depends upon the labor state n. Thus, we write it as $s^*(n)$.

Substituting these into the first order condition and rearranging, we obtain:

$$w = psf'(n) - c_1(n,n_-) - \frac{1}{1+r}c_2(n',n)\Gamma(s,[s^*(n),\bar{s}])$$

This equation relates the wage to the marginal productivity of labor, net of the marginal adjustment cost of hiring that labor and the expected discounted marginal adjustment cost of hiring next period's labor, accounting for the possibility of exit.

Exercise 13 *Another example of labor adjustment costs is when the firm has to post vacancies to attract labor. As an example of such case, suppose the firm faces a firing cost according to the function c. The firm also pays a cost κ to post vacancies and after posting vacancies, it takes one* *period for the workers to be hired. How can we write the problem of firms in this environment?*

Proof 16 (Solution) *The firing costs are according to the function $c(n, n_-)$, but they only apply when we're reducing the labor below last period's labor. Thus, the firing cost function is $\mathbf{1}_{\{n<n_-\}}c(n, n_-)$.*

Each period, the firm enters with a certain number of workers already hired, n_-, and v vacancies posted last period that are now ready to be filled this period. The firm chooses how much labor to hire this period, n, but they can only hire up to $n_- + v$ units of labor since vacancies take a period to fill. They can post v' vacancies to fill next period at a cost of $\kappa v'$.

This gives us the following problem for the firms:

$$V(s, n_-, v, p) = \max \Bigg\{ 0, \max_{n \leq n_- + v, v' \geq 0} psf(n) - wn - c^v - \kappa v' - \mathbf{1}_{\{n<n_-\}}c(n, n_-) + \frac{1}{1+r} \int_{s'} V(s', n, v', p)\Gamma(s, ds') \Bigg\}$$

Exercise 14 *Write the problem of a firm with capital adjustment costs.*

Proof 17 (Solution) *Now the firm's production function is $f(k, n)$. The problem is mostly the same as that with labor adjustment costs, but now the firm chooses both capital and labor each period:*

$$V(s, k_-, p) = \max \left\{ 0, \max_{n \geq 0, k \geq 0} psf(k, n) - wn - (r+\delta)k - c(k, k_-) + \frac{1}{1+r} \int_{s'} V(s', k, p)\Gamma(s, ds') \right\}$$

Exercise 15 *Write the problem of a firm with R&D expenditures that uses labor to improve its productivity.*

Proof 18 (Solution) *There are perhaps several ways to model this type of behavior, but we will model* *it by separating the productivity multiplier into two parts: the exogenous part s and the endogenous part ϵ. We will allow ϵ to accumulate over time as the firm invests more in R&D. The two productivity multipliers will interact in a multiplicative fashion.*

Firm investment in R&D will require the firm to use a portion $(1-\alpha)$ of its hired labor to work on R&D, meaning that only αn units of labor will be used in production of the good. The resulting law of motion for ϵ will be defined as $\epsilon' = \epsilon + g((1-\alpha)n)$, where g is the production function of the R&D.

Then, the firm's problem is as follows:

$$V(s,\epsilon,p) = \max\left\{0, \max_{n\geq 0, \alpha\in[0,1]} p\epsilon s f(\alpha n) - wn - c^v + \frac{1}{1+r}\int_{s'} V(s', \epsilon + g((1-\alpha)n), p)\Gamma(s, ds')\right\}$$

In words, the firm enters each period with exogenous productivity s and accumulated R&D productivity ϵ, and chooses how much labor to hire and what portion to use in production in order to maximize expected discounted profits.

Exercise 16 *What happens in the industry equilibrium model if demand doubles from the stationary equilibrium? Sketch an algorithm to find the equilibrium prices.*

Proof 19 (Solution) *Suppose demand suddenly doubles from a stationary equilibrium in a one-time, unanticipated MIT shock. In order for market clearing to occur, the industry supply must double, which means the price will rise. In response to this price increase, the threshold for staying in the market s^* decreases, and therefore much fewer firms exit the market. In addition, more firms enter the market due to the increase in expected profits.*

In the following periods, the price will begin to decrease as entry into the market lowers expected profits, until the free entry condition is met. Thus the number of firms entering the market will decrease. The threshold for staying in the market will also increase as the price decreases and more firms enter the market. Eventually, the economy will reach a new stationary equilibrium at the higher level of demand.

To solve for the prices, we perform the following steps:

0) Guess a path of prices $\{p_t\}_0^T$*, where* p_0 *is the price in the old stationary equilibrium, and* p_T *is the price in the new stationary equilibrium.*

1) Solve for all of the equilibrium objects in period 1 in terms of the guessed price p_1*.*

2) Set the resulting industry supply equal to demand, and solve for the p_1^* *which solves this expression.*

3) Continue through each time period, solving for a new price p_t^* *each time. Now you have an updated path of prices.*

4) Go back to step 1, using the new vector of prices as your guess.

5) Repeat until the path converges.

Exercise 17 *Derive FOC's of the Farmer's Problem, and describe why the asset space is bounded.*

Proof 20 (Solution) *The Farmer's problem is as follows:*

$$V(s,a) = \max_{c,a'} u(c) + \beta \sum_{s'} \Gamma_{ss'} V(s',a')$$

$$s.t. \quad c + qa' = a + s$$

$$a' \geq 0$$

We can substitute in the first constraint, and write the second with a multiplier μ*:*

$$V(s,a) = \max_{a'} u(a + s - qa') + \beta \sum_{s'} \Gamma_{ss'} V(s',a') + \mu a'$$

The first order condition is:

$$-qu'(c) + \beta \sum_{s'} \Gamma_{ss'} V_2(s', a') + \mu = 0$$

where $\mu \geq 0$. We can also write it as an inequality:

$$-qu'(c) + \beta \sum_{s'} \Gamma_{ss'} V_2(s', a') \leq 0$$

The Envelope Theorem gives us the value function derivative:

$$V_2(s', a') = u'(c')$$

Which we can substitute in and simplify to obtain:

$$qu'(c) \geq \beta \sum_{s'} \Gamma_{ss'} u'(c')$$

We can also rewrite this inequality in the following way:

$$\frac{u'(c)}{\sum_{s'} \Gamma_{ss'} u'(c')} \geq \frac{\beta}{q}$$

We generally assume that β/q is less than one in this setting. If it weren't, you can see from the above that as long as utility is concave in consumption, you would generally see consumption growing every period, without bound. Thus, we have to make the assumption that households are sufficiently impatient in order to bound the asset space.

To show that the asset space is bounded, we need to take a few intermediate steps that I won't fully expound upon here (you can find them in Appendix A of the lecture notes). Essentially, we first show that the consumption policy function is increasing in both arguments, which is a simple consequence of concavity and the envelope theorem. This gives us the next result, which is that at the lowest state s_{min} the optimal choice for savings is always zero given a is sufficiently low. Then we can show that the policy function for assets is below the 45-degree line everywhere for the lowest state. Finally, we argue *that the asset space is bounded using these facts and the assumption that $\beta/q < 1$, which bounds consumption.*

Exercise 18 *Solve for the demand for a good i in the monopolistic competition model.*

Proof 21 (Solution) *The household's problem is as follows:*

$$\max_{\{c_i\}} \left(\int_0^n c_i^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{\sigma}{\sigma-1}}$$
$$s.t. \int_0^n p_i c_i di \leq I$$

We can write it in the Lagrangian form:

$$\max_{\{c_i\}} \left(\int_0^n c_i^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{\sigma}{\sigma-1}} + \lambda \left(I - \int_0^n p_i c_i di\right)$$

Now we take the derivative with respect to one of the goods c_i. While this may seem a bit tricky given that we're integrating over c_i, remember to treat the integral like it's a summation. When we do this, it's easy to see that we can ignore ever other good c_j in the integral, and simply take the derivative of $c_i^{\frac{\sigma-1}{\sigma}}$. The same applies to the budget constraint integral/sum, where the relevant function is $p_i c_i$.

Doing this results in the following FOC:

$$\left(\int_0^n c_i^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{1}{\sigma-1}} c_i^{-\frac{1}{\sigma}} = \lambda p_i$$

Now we have to get rid of the λ multiplier somehow. The easiest way is to first recognize that the FOC for any other good c_j looks remarkably similar:

$$\left(\int_0^n c_i^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{1}{\sigma-1}} c_j^{-\frac{1}{\sigma}} = \lambda p_j$$

Then we can divide the first FOC by the second, and solve for c_i:

$$c_i = \left(\frac{p_i}{p_j}\right)^{-\sigma} c_j$$

This expresses the relative demand for good i in terms of the demand for good j and the relative prices. It has the properties we would expect: increase price p_i relative to p_j, and the demand for good i will decrease relative to demand for good j.

Now, we can also use the budget constraint to solve for the absolute demand of good c_i, only in terms of the income and prices. Rearranging the relative demand, we can find:

$$\begin{aligned} p_i^\sigma c_i &= p_j^\sigma c_j \\ &= p_j^{\sigma-1} p_j c_j \end{aligned}$$

Moving the $p_j^{\sigma-1}$ to the other side,

$$p_i^\sigma c_i p_j^{1-\sigma} = p_j c_j$$

Integrating over j and then substituting in the budget constraint,

$$\begin{aligned} \int_0^n p_i^\sigma c_i p_j^{1-\sigma} dj &= \int_0^n p_j c_j dj \\ p_i^\sigma c_i \int_0^n p_j^{1-\sigma} dj &= I \end{aligned}$$

Solving for c_i,

$$c_i = \frac{I}{\int_0^n p_j^{1-\sigma} dj} p_i^{-\sigma}$$

Defining $P = \left(\int_0^n p_j^{1-\sigma} dj\right)^{\frac{1}{1-\sigma}}$, we find:

$$c_i = \frac{I}{P} \left(\frac{p_i}{P}\right)^{-\sigma}$$

Exercise 19 *Show that σ is the elasticity of substitution between varieties.*

Proof 22 (Solution) *First, let's rearrange the relative demand equation to make it easier for what we're about to do:*

$$\frac{c_i}{c_j} = \left(\frac{p_j}{p_i}\right)^{\sigma}$$

From the definition of elasticity of substitution, we find the following:

$$\varepsilon = \frac{d(c_i/c_j)}{d(p_j/p_i)} \frac{p_j/p_i}{c_i/c_j} = \sigma \left(\frac{p_j}{p_i}\right)^{\sigma-1} \frac{\frac{p_j}{p_i}}{\left(\frac{p_j}{p_i}\right)^{\sigma}} = \sigma,$$

which gives us the desired result.

Exercise 20 *Show that the price index P is the expenditure required to purchase one unit of utility.*

Proof 23 (Solution) *In order to show this, we'll want to use the expenditure function $e(\{p_i\}, u)$ in this demand system. However, what we've solved for so far gives us only the utility function, demand function, and indirect utility function. So we need to use the duality relationship we learn in micro theory:*

$$v(p, e(p, u)) \equiv u$$

In our case, since we want the expenditure for a unit level of utility, we'll be using the following relationship:

$$v(p, e(p, 1)) \equiv 1$$

Let's solve for the LHS of this equation first by substituting the demand function into the utility:

$$\begin{aligned}
v(\{p_i\}, e) &= \left(\int_0^n \left(\frac{e}{P} \left(\frac{p_i}{P} \right)^{-\sigma} \right)^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}} \\
&= \left(\left(\frac{e}{P^{1-\sigma}} \right)^{\frac{\sigma-1}{\sigma}} \int_0^n p_i^{1-\sigma} di \right)^{\frac{\sigma}{\sigma-1}} \\
&= \frac{e}{P^{1-\sigma}} \left(\left(\int_0^n p_i^{1-\sigma} di \right)^{\frac{1}{1-\sigma}} \right)^{-\sigma} \\
&= \frac{e}{P^{1-\sigma}} P^{-\sigma}
\end{aligned}$$

Setting this equal to 1 and solving, we find:

$$e = P,$$

which shows us the desired result: the consumer must spend P in order to attain one level of utility.

There's a bit more than this that we can show. If we solve for the expenditure function for an arbitrary level of utility u in the same way, we find the following relationship:

$$e(\{p_i\}, u) = Pu$$

This form for the expenditure function tells us that for each extra unit of utility, the consumer must spend P more. Thus, in this economy P is truly a cost-of-living index, which is why it is useful for keeping track of inflation.

Exercise 21 *Show that consumer utility is increasing in the number of varieties n.*

Proof 24 (Solution) *For this problem, we will assume a symmetric equilibrium: $p_i = p_j = \overline{p}$, $c_i = c_j = \overline{c}$. The calculations also go through in the asymmetric case; it's just less intuitive. Let's write* *down the indirect utility function, which we found from the previous problem:*

$$v(P, I) = \frac{I}{P}$$

We need to write this using the number of varieties n, so we expand this expression and solve the integral:

$$\begin{aligned} v(P, I) &= I \left(\int_0^n \overline{p}^{1-\sigma} di \right)^{\frac{1}{\sigma - 1}} \\ &= I(n\overline{p}^{1-\sigma})^{\frac{1}{\sigma-1}} \\ &= n^{\frac{1}{\sigma-1}} \overline{p}^{-1} I \end{aligned}$$

Since $\sigma > 1$, clearly indirect utility is increasing in the number of varieties. Additionally, we can see that as σ gets closer to 1, consumer will like increases in the number of varieties more (the degree of substitutability is lower, so having more varieties matters more). The opposite is true as $\sigma \to \infty$; we can see that the limit is $\lim_{\sigma \to \infty} v(P, I) = \overline{p}^{-1} I$. As σ goes to ∞, the consumers don't care about the number of varieties at all!

Exercise 22 *In the dynamic model with Calvo pricing, solve the firm's problem in sequence space and write the firm's equilibrium pricing p_{jt} as a function of present and future aggregate prices, wages, and endowments: $\{P_t, W_t, I_t\}_{t=0}^{\infty}$.*

Proof 25 (Solution) *The firm's recursive problem is:*

$$\begin{aligned} \Omega^1(S, p_j^-) &= \max_{p_j} p_j c_j^* - W(S) c_j^* + (1 - \theta)\mathbb{E}\left[\frac{1}{R(G(S))} \Omega^0(G(S), p_j) \right] \\ &\quad + \theta \mathbb{E}\left[\frac{1}{R(G(S))} \Omega^1(G(S), p_j) \right], \end{aligned} \tag{105}$$

where $\Omega^0(\cdot)$ is the value of the firm who cannot change its price.

Now, notice that the price decision (or lack thereof) from the previous period actually doesn't factor into the firm's decision this period. Each period that the firm gets an opportunity to make a pricing decision, the past decisions are irrelevant except insofar as they affect the aggregate state S. So all we need to keep track of each time the firm makes a decision is how it affects this period and all future periods in which the firm cannot *make a decision. So the problem in sequence space can be written as follows:*

$$\max_{p_{jt}} \mathbb{E}\left[\sum_{s=0}^{\infty}\left(\frac{1-\theta}{R(S_{t+s})}\right)^s (p_{jt}c^*_{j,t+s} - W(S_{t+s})c^*_{j,t+s})\right]$$

Substituting in the functional form for the demand,

$$\max_{p_{jt}} \mathbb{E}\left[\sum_{s=0}^{\infty}\left(\frac{1-\theta}{R(S_{t+s})}\right)^s \left(p_{jt}\left(\frac{p_{jt}}{P(S_{t+s})}\right)^{-\sigma}\frac{I(S_{t+s})}{P(S_{t+s})} - W(S_{t+s})\left(\frac{p_{jt}}{P(S_{t+s})}\right)^{-\sigma}\frac{I(S_{t+s})}{P(S_{t+s})}\right)\right]$$

Taking first order conditions,

$$(1-\sigma)p_{jt}^{-\sigma}\mathbb{E}\sum_{s=0}^{\infty}\left(\frac{1-\theta}{R(S_{t+s})}\right)^s\frac{I(S_{t+s})}{P(S_{t+s})^{1-\sigma}} = -\sigma p_{jt}^{-\sigma-1}\mathbb{E}\sum_{s=0}^{\infty}\left(\frac{1-\theta}{R(S_{t+s})}\right)^s\frac{I(S_{t+s})}{P(S_{t+s})^{1-\sigma}}W(S_{t+s})$$

And solving for p_{jt},

$$p^*_{jt} = \frac{\sigma}{\sigma-1}\frac{\mathbb{E}\left[\sum_{s=0}^{\infty}\left(\frac{1-\theta}{R(S_{t+s})}\right)^s\frac{I(S_{t+s})}{P(S_{t+s})^{1-\sigma}}W(S_{t+s})\right]}{\mathbb{E}\left[\sum_{s=0}^{\infty}\left(\frac{1-\theta}{R(S_{t+s})}\right)^s\frac{I(S_{t+s})}{P(S_{t+s})^{1-\sigma}}\right]}$$

This gives us the pricing rule for the firm.

Exercise 23 *Show that under flexible pricing ($\theta = 1$), the firm's pricing strategy is identical to the static model.*

Proof 26 (Solution) *Substituting $\theta = 1$ and letting $0^0 = 1$, it follows fairly easily that:*

$$p^*_{jt} = \frac{\sigma}{\sigma - 1} \frac{\frac{I}{P^{1-\sigma}} W}{\frac{I}{P^{1-\sigma}}} = \frac{\sigma}{\sigma - 1} W$$

which is the same as that of the static model. This is fairly easy to see intuitively; if $\theta = 1$ then the firm never has to worry about what happens in future periods because its pricing decision only affects the present period.

Exercise 24 *Show that with price rigidity ($\theta < 1$), the firm's pricing strategy is identical to the static model in a steady state with zero inflation.*

Proof 27 (Solution) *Let's assume a deterministic steady state in this model: $I(S_{t+s}) = I, P(S_{t+s}) = P, R(S_{t+s}) = R, W(S_{t+s}) = W$. Then we can write the firm's pricing decision as follows:*

$$\begin{aligned}
p^*_{jt} &= \frac{\sigma}{\sigma - 1} \frac{\mathbb{E}\left[\sum_{s=0}^{\infty} \left(\frac{1-\theta}{R}\right)^s \frac{IW}{P}\right]}{\mathbb{E}\left[\sum_{s=0}^{\infty} \left(\frac{1-\theta}{R}\right)^s \frac{I}{P}\right]} \\
&= \frac{\sigma}{\sigma - 1} \frac{\frac{1}{1-\frac{1-\theta}{R}} \frac{IW}{P}}{\frac{1}{1-\frac{1-\theta}{R}} \frac{I}{P}} \\
&= \frac{\sigma}{\sigma - 1} W
\end{aligned}$$

This is also fairly intuitive: if there are no changes in any of the aggregate variables, the firm's pricing decision today is also optimal for tomorrow, the day after, and all periods in the future. Thus, it is as if the firm is solving a static problem.

References