Macro 7210 Lectures

Preliminary

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Introduction

- A model is an artificial economy used to ask questions.
- The description of a model's environment includes specifying agents' preferences and endowments, technology available, information structure as well as property rights.
- The workhorse model in Macro is the Neoclassical Growth Model.
- It delivers some fundamental properties that are characteristics of industrialized economies. Kaldor (1957) summarizes six (plus one) stylized facts.

- A model requires an equilibrium concept.
- Equilibrium is a prediction of what will happen in the economy, i.e. a mapping from environments to outcomes (allocations, prices, etc.).
- One such equilibrium concept is Competitive Equilibrium (CE).
- Characterizing equilibrium usually involves finding solutions to a system of an infinite number of equations. Three ways around it

    1. To invoke the first welfare theorem to solve for the allocation and then find the equilibrium prices associated with it (not so general: market incompleteness, externalities, distortions, heterogeneity (Negishi)).
    2. Construct the equilibrium (not good to learn about the world)
    3. Recursive Competitive Equilibrium (RCE) directly.

1. Output per capita has grown at a roughly constant rate
2. The capital-output ratio has remained roughly constant (capital measured using the perpetual inventory method)
3. The capital-labor ratio has grown at a roughly constant rate (same rate as output)
4. Wage rate has grown at roughly the same rate as output
5. The real interest rate has been stationary
6. Labor income as a share of output has remained roughly constant
7. Hours worked per capita have been roughly constant.

1. Exogenous Technical Change (there is no systematic variation of growth rates that really calls for a theory of growth rates)

2. Cobb-Douglas Technology not other (ot at least aggregates to Cobb-Douglas)

3. Balanced growth Preferences

   - Cobb-Douglas:

$$u(c, \ell) = \frac{\left[c^{\theta} \ \ell^{1-\theta}\right]^{1-\sigma}}{1-\sigma}$$

   - Log plus Constant Frisch: :

$$u(c, 1-\ell) = u(c, n) \log c + \chi \frac{n^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}}$$

Recursive Equilibria without Distortions

- A natural extension of the dynamic programming problem.
- It requires the definition of state variables
  - Aggregate K
  - Individual a
- In addition to decision rules we need
  - Pricing Functions (of aggregate variables)
  - Laws of motion of aggregate states
  - Equilibrium Conditions/ Representative Agent Conditions

- Aggregate State K with law of motion $K' = G(K)$
- Individual State a
- Equilibrium Prices $w(K)$, $R(K)$

$$
\begin{aligned}
V(K, a; G) = \max_{c, a'} \quad u(c) \quad &+ \quad \beta V(K', a'; G) \\
\text{s.t. } c + a' \quad &= \quad w(K) + R(K)a \\
K' \quad &= \quad G(K), \\
c \quad &\geq \quad 0
\end{aligned}
$$

- $c = c(K, a; G)$, $a' = g(K, a; G)$, $V(K, a; G)$ satisfy (use envelope)

$$
u_c\left[c(K, a; G)\right] \quad = \quad \beta V_{a'}\left[G(K), g(K, a; G); G\right]
$$

$$
V_a\left(K, a; G\right) \quad = \quad R(K)\, u_c\left[c(K, a; G)\right]
$$

- The Rep Agent Equilibrium Condition requires

$$G(K) = g(K, K; G)$$

- The most convenient is to summarize all conditions by successive substitution
- Yields a functional equation in K (after using marginal productivities)

$$u_c\left[w(K) + R(K)K - G(K)\right] =$$

$$\beta\ u_{c'}\{w[G(K)] + R[G(K)]G(K) - G[G(K)]\}\ R[G(K)]$$

- In this case we can use the $G(K)$ that comes out of the social planner's dynamic programming problem as the candidate for RCE.

Economies with Distortions and Heterogeneity

- Wedges: Externalities, Governments, Heterogeneity
- Just define Equilibria directly.
- Lump sum Taxes $T(K)$ levied for Parks. Government has a period by period balance budget constraint.

$$\begin{aligned} V(K,a;T,P,G) = \max_{c\geq 0, a'} \quad & u[c,P(K)] \quad + \quad \beta V(K',a';T,P,G) \\ \text{s.t.} \quad c + a' \quad & = \quad w(K) + R(K)a - T(K) \\ K' \quad & = \quad G(K) \end{aligned}$$

with solution $a' = g(K,a;T,P,G)$.

- Equilibrium requires

$$\begin{aligned} G^*(K) &= g(K,K;T,P^*,G^*), \\ P^*(K) &= T(K). \end{aligned}$$

- If labor income tax, substitute $T(K)$ with $\tau(K)\ w(K)$.

$$
\begin{aligned}
V(K, a; \tau, G) = \max_{c \geq 0, a'} \quad & u(c, P) \quad + \quad \beta\, V(K', a'; \tau, G) \\
\text{s.t.} \quad c + a' \quad & = \quad w(K) + a\left[1 + r(K)\left(1 - \tau(K)\right)\right] \\
K' \quad & = \quad G(K) \\
P \quad & = \quad P(K).
\end{aligned}
$$

- Eq Cond: $P^*(K) = \tau(K) r^*(K)\, K$, and $R(K) = 1 + r(K)$ plus Rep Agent.
- The First Welfare Theorem fails and the RCE is not Pareto optimal. (if $\tau(K) > 0$ there will be a wedge, and the efficiency conditions will not be satisfied).

Exercise

Derive the first order conditions in the above problem to see the wedge introduced by taxes.

- Aggregate State is K and B
- Government policy (for now assume it can):

$$\tau(K,B),\ P(K,B) \text{ and } B'(K,B).$$

- The government budget constraint reduces the degrees of freedom

$$B + P(K,B) = \tau(K,B)R(K)K + q(K,B)B'(K,B)$$

- The household does not care about the composition of his portfolio as long as assets have the same rate of return, which is true because of the no arbitrage condition.
- So individual state is just a

- The household needs to know the evolution of capital and *debt*

$$
\begin{aligned}
V(K,B,a) &= \max_{c\geq 0,a'} \quad u(c,P(K,B)) + \beta V(K',B',a') \\
\text{s.t.} \quad c+a' &= w(K)+aR(K)\left(1-\tau(K,B)\right) \\
K' &= G(K,B) \\
B' &= H(K,B)
\end{aligned}
$$

with solution $g\left(K,B,a\right)$.

Definition

A *Rational Expectations Recursive Competitive Equilibrium given* $P(K,B)$ *and* $\tau(K,B)$, *are functions* V, g, G, H, w, q, *and* R, *s.t.*

1. *Given* w *and* R, V *and* g *solve the household's problem,*
2. *Factor prices are paid their marginal productivities:* $w(K) = F_2(K,1)$ *and* $R(K) = F_1(K,1)$.
3. *Rep agent condition*

$$g[K, B, K + q(K^-, B^-)B] = G(K,B) + q(K,B)\ H(K,B),$$

4. *No arbitrage*

$$\frac{1}{q(K,B)} = [1 - \tau(G(K,B), H(K,B))]\ R(G(K))$$

5. *Gov b constr:* $B + P(K,B) = \tau(K,B)R(K)K + q(K,B)H(K,B)$
6. *Government debt is bounded:* $\exists$ *some* $\bar{B}$, *such that for all* $K \in \left[0, \tilde{k}\right)$ *and* $B \leq \bar{B}$, $H(K,B) \leq \bar{B}$.

1. *Habit formation:* $u(c, c^-)$, *increasing in* c, *decreasing in* c^- *(e.g.* $u(c, c^-) = v(c) - (c - c^-)^2$*). Agg. state* $\{K, C^-\}$, *individual* $\{a, c^-\}$.

Exercise

Define it. Is the equilibrium optimum in this case?

2. *Catching up with the Jones* $u(c, C^-)$. *Externality from aggregate consumption. Aggregate state* $\{K, C^-\}$, *while* c^- *is not a state.*

Exercise

How does the agent know C*? Is the equilibrium optimum?*

3. *Keeping up with the Jones* $u(c, C)$:

Exercise

How does the agent know C*? Is the equilibrium optimum?*

- Rep firms buy and install capital; own one unit of land used to produce $F(K, L)$.
- Firm's shares are publicly traded and bought by households.
- Agg State is K, ind state is shares a. The hhold solves

$$\begin{aligned} V(K,a) = \max_{c,a'} \quad u(c) \quad &+ \quad \beta V[G(K), a'] \\ \text{s.t.} \quad c + P(K)a' \quad &= \quad a\,[D(K) + P(K)] \end{aligned}$$

- $P(K)$ is shares price; $D(K)$ dividends per share. Soltn, $a' = h(K, a)$
- Firm solves

$$\begin{aligned} \Omega(K,k) = \max_{d,k'} \quad d \quad &+ \quad q[G(K)]\, \Omega[G(K), k'] \\ \text{s.t.} \quad F(k,1) \quad &= \quad d + k' \end{aligned}$$

- d dividends (solution $d(K, k)$), $q\,[G(K)]$ is price of good tomorrow.

Definition

A Rec Comp Eq are functions, V, Ω, h, g, d, q, D, P, G so that:

1. Given prices, V and h solve the household's problem,
2. Ω, g, and d solve the firm's problem,
3. Representative household holds all shares: $h(K, 1) = 1$
4. Rep Firm

$$\begin{aligned} F(K,K) - d(K,K) &= G(K) \\ d(K,K) &= D(K) \end{aligned}$$

5. Value of a representative firm equals price plus dividends

$$\Omega(K,K) = D(K) + P(K),$$

Exercise

Find missing condition. [Hint: it relates $q(G(K))$ with the price and dividends ($P(K), P(G(K))$, and $D(G(K))$).]

Exercise

Define the RCE if a were savings paying $R(K)$ instead of shares.

- Two types of households differing only in wealth: R (rich) and P (poor) with measures μ and $1-\mu$. Otherwise identical.

$$
\begin{aligned}
V(K^R, K^P, a) = \max_{c,a'} \quad & u(c) \quad + \quad \beta V(K^{R'}, K^{P'}, a') \\
\text{s.t.} \qquad c + a' \quad & = \quad w\left[(\mu K^R + (1-\mu)K^P\right] + aR\left[\mu K^R + (1-\mu)K^P\right] \\
K^{i'} \quad & = \quad G^i(K^R, K^P) \qquad \text{for } i = R, P.
\end{aligned}
$$

Remark

Decision rules are not linear (even though they might be almost linear); therefore, we need two states, K^1 and K^2, not aggregate K.

Definition

A Rec Comp Equil are functions V, g, w, R, G^1, and G^2 such that:

1. Given prices, V and g solve the household's probl
2. w and R are the marginal products of labor and capital, respectively
3. Consistency: representative agent conditions are satisfied, i.e.

$$
\begin{aligned}
g(K^R, K^P, K^R) &= G^R(K^R, K^P) \\
g(K^R, K^P, K^P) &= G^P(K^R, K^P).
\end{aligned}
$$

Remark

Note that $G^R\left(K^R, K^P\right) = G^P\left(K^P, K^R\right)$ *(look at the arguments carefully). Why? (How are rich and poor different?)*

- In steady state, the Euler equations for the two types simplify to

$$u'\left(c^{R^*}\right) = \beta R \; u'\left(c^{R^*}\right), \text{ and } u'\left(c^{P^*}\right) = \beta R \; u'\left(c^{P^*}\right).$$

$$\text{so } \beta R = 1, \text{ where } R = F_K\left(\mu K^{R^*} + (1-\mu)K^{P^*}, 1\right).$$

- Using hhold's budget constraint and $a^i = K^i$ because of the rep agent's condition

$$c^i + a^i = w + a^i R \quad \text{for } i = R, P$$

- We have three equations (2 budget constraints and Euler equation) and four unknowns (a^{i^*} and c^{i^*} for $i = R, P$).
- The theory is silent about the steady state distribution of wealth!
- If savings are linear in a state (i.e. $g(K, a) = \mu^i(K) + \lambda(K)a$, and all have the same preferences, then aggregate capital can be expressed as the choice of a representative agent (with savings decision given by $g(K, K) = \bar{\mu}(K) + \lambda(K)K$).

- Type i has labor skill ϵ_i, $\mu^1 = \mu^2 = 1/2$. $\mu^1\epsilon_1 + \mu^2\epsilon_2 = 1$.

- The value functions are now indexed by type:

$$V^i(K^1, K^2, a) = \max_{c,a'} \quad u(c) \quad + \quad \beta V^i(K^{1'}, K^{2'}, a')$$

$$\text{s.t.} \quad c + a' \quad = \quad w\left(\frac{K^1 + K^2}{2}\right)\epsilon_i + aR\left(\frac{K^1 + K^2}{2}\right)$$

$$K^{i'} \quad = \quad G^i(K^1, K^2) \quad \text{for } i = 1, 2.$$

with solution $g^i(K^1, K^2, a)$.

Exercise

Define the RCE.

Remark

We can also rewrite this problem as

$$V^i(K,\lambda,a) = \max_{c,a'} \left\{ u(c) + \beta V^i(K',\lambda',a') \right\}$$

$$\text{s.t.} \quad c + a' = R(K)\,a + W(K)\,\epsilon_i$$

$$K = G(K,\lambda)$$

$$\lambda' = H(K,\lambda),$$

where K is aggregate capital, and λ is the share of type 1.

Then the consistency conditions of the RCE must be:

$$G(K,\lambda) = \frac{1}{2}\left[g^1(K,\lambda,2\lambda K) + g^2(K,\lambda,2(1-\lambda)K)\right],$$

$$H(K,\lambda) = \frac{g^1(K,\lambda,2\lambda K)}{2G(K,\lambda)}.$$

- Have to define what is a country.
    - A Place?
    - A Technology?
    - A Policy?
    - A set of Trade Restrictions?
- Today, two countries, 1 and 2, labor is immobile, but capital markets perfect. Traded goods flow within the period. Different technology.
- Aggregate resource constraint is:

$$C^1 + C^2 + K^{1'} + K^{2'} = F^1(K^1, 1) + F^2(K^2, 1)$$

- There are mutual funds that own all firms countries. They choose labor and installs capital. Shares are traded in the world market.
- *What are the appropriate aggregate states in this world?*
    - Capital in each country.
    - Need also a variable for wealth distribution, say, shares in country 1.

- Hhold Probl. A are shares held by country 1 hholds. a are own shares.

$$V^i\left(K^1, K^2, A, a\right) = \max_{c, a'(z)} \quad u(c) + \beta V^i\left(K^{1'}, K^{2'}, A', a'\right)$$

$$s.t. \quad c + Q(K^1, K^2, A)a' = w^i\left(K^i\right) + a\Phi(K^1, K^2, A)$$

$$K^{i'} = G^i\left(K^1, K^2, , A\right), \quad \text{for } i = 1, 2$$

$$A' = H\left(K^1, K^2, A\right)$$

- Mutual Funds' problem (note wages are country specific)

$$\Phi(K^1, K^2, A, k^1, k^2) = \max_{k^{1'}, k^{2'}, n^1, n^2} \sum_i \left[F^i(k^i, n^i) - n^i w^i(K_i) - k^{i'}\right] + \frac{1}{R(K^{1'}, K^{2'}, A)} \Phi(K^{1'}, K^{2'}, A', k^{1'}, k^{2'})$$

$$s.t. \quad K^{i'} = G^i\left(K^1, K^2, , A\right), \quad \text{for } i = 1, 2$$

$$A' = H\left(K^1, K^2, A\right)$$

Definition

Rec Comp Equil: $\{V^i, h^i, g^i, n^i, w^i, G^i\}_{i=1,2}$, Φ, H, Q, and R, S.t.:

1. Given prices and aggregate laws of motion, V^i and h^i solve hholds' probl
2. Samo: Φ, $\{g^i, n^i\}_{i=1,2}$ solve mutual funds' probl,
3. Labor markets clear $n^i(K^1, K^2, A, K^1, K^2) = 1 \qquad \text{for } i = 1, 2,$
4. Consistency (MF)
$$g^i(K^1, K^2, A, K^1, K^2) = G^i(K^1, K^2, A) \qquad \text{for } i = 1, 2,$$
5. Consistency (Households)
$$\begin{aligned} h^1(K^1, K^2, A, A) &= H(K^1, K^2, A) \\ h^1(K^1, K^2, A, A) + h^2(K^1, K^2, A, 1-A) &= 1 \end{aligned}$$
6. No arbitrage $\quad Q(K^1, K^2, A) = \frac{1}{R(K^{1'}, K^{2'}, A')} \Phi(K^{1'}, K^{2'}, A', K^{1'}, K^{2'})$

Exercise

Solve for the mutual fund's decision rules. Is next period capital in each country chosen by the mutual fund priced differently? What about labor?

The Lucas Tree

- The Purpose: To Price Assets so they do the right thing
- The Environment:
  - Goods: A measure one of trees that give fruit, z, that follows a Markov Process with transition matrix $\Gamma_{zz'}$.
  - Preferences: $E \sum_t \beta^t u(c_t)$.
  - Markets: Hholds buy shares s' of trees in stock markets at price $p(z)$, and consume fruit. They receive dividends $d(z)$ and have shares.
- State Variables
  - Aggregate z
  - Individual s

$$V(z,s) = \max_{c,s'} \quad u(c) + \beta \sum_{z'} \Gamma_{zz'} V(z', s')$$
$$s.t. \quad c + p(z)\, s' = s\,[p(z) + d(z)]\,,$$

Definition

A Rational Expectations Recursive Competitive Equilibrium is a set of functions, V, g, d, and p, such that

1. V and g solves the household's problem given prices,
2. $d(z) = z$, and,
3. $g(z, 1) = 1$, for all z.

- Recall

$$u_c\left(c\left(z,s\right)\right)=\beta\sum_{z'}\Gamma_{zz'}\left[\frac{p\left(z'\right)+d\left(z'\right)}{p\left(z\right)}\right]\ u_c\left(c\left(z',s'\right)\right).$$

- In equilibrium $s=1$ and $c(z,1)=z$ so we have $u_c\left(z\right):=u_c\left(c\left(z,1\right)\right)$. The

$$p\left(z\right)u_c\left(z\right)=\beta\sum_{z'}\Gamma_{zz'}u_c\left(z'\right)\left[p\left(z'\right)+z'\right]\qquad\forall z.$$

- A system of n_z equations. Denote $\mathrm{p}:=\left[p\left(z_1\right)\vdots p\left(z_n\right)\right]_{(n_z\times 1)}$ and

$$\mathrm{u}_c:=\begin{bmatrix} u_c\left(z_1\right) & & 0 \\ & \ddots & \\ 0 & & u_c\left(z_n\right)\end{bmatrix}_{(n_z\times n_z)}.$$

- Then

$$\mathrm{u}_c.\mathrm{p} = \begin{bmatrix} p(z_1)\, u_c(z_1) \\ \vdots \\ p(z_n)\, u_c(z_n) \end{bmatrix}_{(n_z \times 1)},$$

- Now, rewrite the system above as

$$\mathrm{u}_c\mathrm{p} = \beta\Gamma\mathrm{u}_c\mathrm{z} + \beta\Gamma\mathrm{u}_c\mathrm{p},$$

- where Γ is the transition matrix for z, as before. Hence, share prices are

$$\left(\mathrm{I}_{\mathbf{n_z}} - \beta\Gamma\right)\mathrm{u}_c\mathrm{p} = \beta\Gamma\mathrm{u}_c\mathrm{z},$$

- or

$$\mathrm{p} = \left(\left[\mathrm{I}_{\mathbf{n_z}} - \beta\Gamma\right]\mathrm{u}_c\right)^{-1}\beta\Gamma\mathrm{u}_c\mathrm{z},$$

- An asset is "a claim to a chunk of fruit, sometime in the future."

- An asset that promises $m_t\left(z^t\right)$ after history $z^t = (z_0, z_1, \ldots, z_t) \in H^t$. The price of such an asset is the price of what it entitles its owner to.

- This follows from a no-arbitrage argument.

$$p^m(z_0) = \sum_t \sum_{z^t \in H^t} q_t^0\left(z^t\right)\; a_t\left(z^t\right),$$

$q_t^0\left(z^t\right)$ is the price of one unit of fruit after z^t in time zero's goods.

- Given the $\left\{q_t^0(z^t)\right\}$, we can *replicate any possible asset by a set of state-contingent claims* and use this formula to price that asset.

- To find those q^0 consider a world where agents solve

$$\max_{c_t(z^t)} \quad \sum_{t=0}^{\infty} \beta^t \sum_{z^t} \pi_t\left(z^t\right) u\left(c_t\left(z^t\right)\right)$$

$$s.t. \quad \sum_{t=0}^{\infty} \sum_{z^t} q_t^0\left(z^t\right) c_t\left(z^t\right) \leq \sum_{t=0}^{\infty} \sum_{h^t} q_t^0\left(z^t\right) z_t.$$

- The $\pi(z^t)$ are the prob and can be constructed recursively with Γ.
- Note that this is the familiar Arrow-Debreu market structure, where the household owns a tree, and the tree yields $z \in Z$ amount of fruit in each period). The FOC for this problem imply:

$$q_t^0\left(z^t\right) = \beta^t \pi_t\left(z^t\right) \frac{u_c\left(z_t\right)}{u_c\left(z_0\right)}.$$

- This enables us to price the good in each history of the world and price any asset accordingly.

- Hhold Probl

$$V(z,s,b) = \max_{c,s',b'(z')} \quad u(c) + \beta \sum_{z'} \Gamma_{zz'} V\left(z', s', b'\left(z'\right)\right)$$

$$s.t. \quad c + p(z)\, s' + \sum_{z'} q\left(z,z'\right) b'\left(z'\right) = s\left[p(z) + z\right] + b.$$

- A characterization of q can be obtained by the FOC, evaluated at the equilibrium, and thus written as:

$$q\left(z,z'\right) u_c(z) = \beta \Gamma_{zz'} u_c\left(z'\right).$$

- We can thus price *all types* of securities using p and q in this economy.

- To sell the tree tomorrow at price P

$$\widehat{q}(z, P) = \sum_{z'} q\left(z, z'\right) \max\left\{P - p\left(z'\right), 0\right\},$$

- The (American) option to sell either tomorrow or the day after

$$\tilde{q}(z, P) = \sum_{z'} q\left(z, z'\right) \ \max\left\{P - p\left(z'\right), \widehat{q}\left(z', P\right)\right\}.$$

- The European option to buy the day after tomorrow is

$$\bar{q}(z, P) = \sum_{z'} \sum_{z''} \max\left\{p\left(z''\right) - P, 0\right\} q\left(z', z''\right) q\left(z, z'\right).$$

- If today's shock is z, the gross risk free rate

$$R(z) = \left[\sum_{z'} q(z, z')\right]^{-1}$$

- The unconditional gross risk free rate is

$$R^f = \sum_{z} \mu_z^* R(z)$$

where μ^* is the steady-state distribution of the shocks.

- The average gross rate of return on the stock market is

$$\sum_z \mu_z^* \sum_{z'} \Gamma_{zz'} \left[\frac{p\left(z'\right) + z'}{p\left(z\right)} \right]$$

- The Risk Premia is

$$\sum_z \mu_z^* \left(\sum_{z'} \Gamma_{zz'} \left[\frac{p\left(z'\right) + z'}{p\left(z\right)} \right] - R(z) \right).$$

- Use the expressions for p and q and the properties of the utility function to show that risk premium is positive.

- The fruit is constant over time (normalized to 1)
- The agent is subject to preference shocks for the fruit each period given by $\theta \in \Theta$ with transition Γ^{θ}.

$$V(\theta, s) = \max_{c,s'} \quad \theta u(c) + \beta \sum_{\theta'} \Gamma_{\theta\theta'} V(\theta', s')$$
$$s.t. \quad c + p(\theta) s' = s [p(\theta) + d(\theta)].$$

- The equilibrium is defined as before.
- In Eq $d(\theta) = 1$
- Discussion of Demand vs Supply Shocks and what RBC vs Lucas trees are.

An Introduction to Search with a Particular Application: Endogenous Productivity in a Product Search Model

- Most of Economics posts Supply and Demand
- This is NOT the only wat to think.
- There are Trades all the time (houses jobs). What does it mean to clear the market?
- Search theory models decentralised exchanges: Trades require pairwise meetings of buyers and sellers (workers, firms, prospective couples) which do not happen automatically:
- Difficulties in meeting partners.
- After meeting, trades may happen or not.

- So far
    - Hholds own the tree
    - Purchase Shares
    - To access the fruit they JUST have to Purchase it.
- Now They also have to FIND the fruit

- There is matching function $M(T, D)$: Trees and Search Effort.
  - Constant Returns to Scale, e.g. $D^{\varphi} T^{1-\varphi}$. Let $\frac{1}{Q} := \frac{D}{T}$, i.e. the ratio of shoppers per trees, *the market tightness*.
  - Other more natural matching functions $D \leq M(T, D) \leq D$.
- The probability that a unit of shopping effort finds a tree is

$$= \Psi^{h}(Q) := \frac{M(T, D)}{D} = Q^{1-\varphi}$$

- The probability that a tree finds a shopper is

$$\Psi^{f}(Q) := \frac{M(T, D)}{T} = Q^{-\varphi}$$

- Here $T = 1$. The number of trees is constant.

- A hunger (demand) shock θ with transition matrix $\Gamma_{\theta\theta'}$
- A Productvity (TFP, supply) shock z with transition matrix $\Gamma_{zz'}$
- We look for a Lucas tree *type* Equilibrium
- State Variables
    - Aggregate θ, z
    - Individual s

$$V(\theta, z, s) = \max_{c,d,s'} \quad u(c, d, \theta) + \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} \; V(\theta', z', s')$$

$$s.t. \quad c + P(\theta, z)\, s' = P(\theta, z) \left[s \left(1 + \widehat{R}(\theta, z) \right) \right]$$

$$c = d \; \Psi^h (Q(\theta, z))\, z$$

- $P(\theta, z)$ is the price of the tree relative to that of consumption
- $\widehat{R}(\theta, z)$ is the dividend income (in units of the tree).
- $Q(\theta, z)$ is market tightness.

- Substitute the constraints into the objective, solve for d and get the Euler equation for the household.

- Using THEN the market clearing condition in equilibrium, the problem is reduced to one equation and two unknowns, $P(\theta, z)$ and $Q(\theta, z)$

- Still need another functional equation.

- We need to specify the search protocol (how it happens).

Exercise

Derive the Euler equation of the household from the problem defined above.

- It is a particular search protocol of what is called non-random (or directed) search.
- Ex-ante Commitment to the terms of trade (in other search protocols it is not the case)
- Consider a world consisting of a large number of islands. Each island has a sign that displays two numbers, $P(\theta, z)$ and $Q(\theta, z)$. (price and market tightness)
- Searchers and (trees and household effort) choose which island to go to. They have different trade-offs of price versus tightness.
- Equilibrium determines which island (Optimal so unique).

$$V(\theta, z, s) = \max_{c,d,s',P,Q} \quad u(\theta\ c, d) + \beta \sum_{\theta',z'} \Gamma_{\theta\theta'} \Gamma_{zz'} V\left(\theta', z', s'\right) \tag{1}$$

$$s.t. \quad c + Ps' = P\left[s\left(1 + \widehat{R}(\theta, z)\right)\right], \tag{2}$$

$$c = d\ \Psi^{h}(Q)\, z \tag{3}$$

$$\frac{z\Psi^{f}(Q)}{P} \geq \widehat{R}(\theta, z) \tag{4}$$

- The last constraint states that for a market to exist firms have to be guaranteed $\widehat{R}(\theta, z)$.

Plug the first two constraints into the objective function (c and s' as functions of d) and (recall that $\Psi^h = Q^{1-\varphi}$) :

$$\theta Q^{1-\varphi} z u_c(\theta d Q^{1-\varphi} z, d) + u_d(\theta d Q^{1-\varphi} z, d) = \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} V_3\left(\theta', z', s(1+\widehat{R}(\theta, z)) - \frac{dQ^{1-\varphi} z}{P}\right) \frac{Q^{1-\varphi} z}{P} \quad (5)$$

Get rid of V_3 using original problem and use the envelope theorem

$$V_3(\theta, z, s) = \left[\theta u_c(\theta d Q^{1-\varphi} z, d) + \frac{u_d(\theta d Q^{1-\varphi} z, d)}{Q^{1-\varphi} z}\right] P(1+\widehat{R}(\theta, z))$$

Combining these two gives the Euler equation:

$$\theta u_c(\theta d Q^{1-\varphi} z, d) + \frac{u_d(\theta d Q^{1-\varphi} z, d)}{Q^{1-\varphi} z} = \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} \frac{P'(1+\widehat{R}(\theta', z'))}{P} \left[\theta' u_c(\theta' d' Q'^{1-\varphi} z', d') + \frac{u_d(\theta' d' Q'^{1-\varphi} z', d')}{Q'^{1-\varphi} z'}\right] \quad (6)$$

FOC with respect to Q and P.

λ: Lagrange multiplier on the firm's participation constraint, then

$$\theta d(1-\varphi)Q^{-\varphi} z u_c(\theta d Q^{1-\varphi} z, d) = \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} V_3 \left(\theta', z', s(1+\widehat{R}(\theta, z)) - \frac{dQ^{1-\varphi} z}{P}\right) \frac{d(1-\varphi)Q^{-\varphi} z}{P} - \lambda \frac{\varphi Q^{-\varphi-1} z}{P} \quad (7)$$

and

$$\beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} V_3 \left(\theta', z', s(1+\widehat{R}(\theta, z)) - \frac{dQ^{1-\varphi} z}{P}\right) dQ = -\lambda \quad (8)$$

Combining these two equation gives us:

$$\theta u_c(\theta d Q^{1-\varphi} z, d) = \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'}$$
$$V_3\left(\theta', z', s(1+\widehat{R}(\theta, z)) - \frac{dQ^{1-\varphi} z}{P}\right)\left[\frac{1}{(1-\varphi)P}\right] \quad (9)$$

Recall $V_3(\cdot,\cdot,\cdot)$ so

$$(1-\varphi)\theta u_c(\theta d Q^{1-\varphi} z, d) = \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'}$$
$$\frac{P'(1+\widehat{R}(\theta', z'))}{P}\left[\theta' u_c(\theta' d' Q'^{1-\varphi} z', d') + \frac{u_d(\theta' d' Q'^{1-\varphi} z', d')}{Q'^{1-\varphi} z'}\right] \quad (10)$$

Definition

An Eq with competitive search is functions $\{V, c, d, s', P, Q, \widehat{R}\}$ that:

1. Household's budget constraint, (condition 2)
2. Household's shopping constraint, (condition 3)
3. Household's Euler equation, (condition 6)
4. Market condition, (condition 10)
5. Firm's participation constraint, (condition 4), which gives us that the dividend payment is the profit of the firm, $\widehat{R}(\theta, z) = \frac{zQ^{-\varphi}}{P}$,
6. Market clearing, i.e. $s' = 1$ and $Q = 1/d$.

Firms maximize returns by choosing market, Q, P. It helps to use trees as numeraire, so $\widehat{P}(Q) = 1/P$ is the price of consumption. We want to characterize the set of available markets for firms, $\widehat{P}(Q)$ by looking at the implications for firms that face it:

$$\pi = \max_{Q} \ \widehat{P}(Q) \ \Psi^{f}(Q) \ z$$

with FOC

$$\widehat{P}'(Q)\,\Psi^{f}(Q) + \widehat{P}(Q)\,\Psi^{f'}(Q) = 0,$$

The set of pairs P a that satisfies FOC yields a relation of indifference between the firms the pairs $\{P, Q\}$ for the firms that implicitly determines $\widehat{P}(Q)$ as

$$\frac{\widehat{P}'(Q)}{\widehat{P}(Q)} = -\frac{\Psi^{f'}(Q)}{\Psi^{f}(Q)}.$$

Measure Theory

Measure theory is a tool that helps us aggregate.

Definition

For a set S, $\mathcal{S}$ is a family of subsets of S, if $B \in \mathcal{S}$ implies $B \subseteq S$ (but not the other way around).

Remark

Note that in this section we will assume the following convention

1. *small letters (e.g. s) are for elements,*
2. *capital letters (e.g. S) are for sets, and*
3. *fancy letters (e.g. $\mathcal{S}$) are for a set of subsets (or families of subsets).*

Definition

A family of subsets of S, $\mathcal{S}$, is called a σ-algebra in S if

1. $S, \emptyset \in \mathcal{S}$;
2. if $A \in \mathcal{S} \Rightarrow A^c \in \mathcal{S}$ (i.e. $\mathcal{S}$ is closed with respect to complements and $A^c = S \backslash A$); and,
3. for $\{B_i\}_{i \in \mathbb{N}}$, if $B_i \in \mathcal{S}$ for all $i \Rightarrow \bigcap_{i \in \mathbb{N}} B_i \in \mathcal{S}$ (i.e. $\mathcal{S}$ is closed with respect to countable intersections.

Example

1. The power set of S and $\{\emptyset, S\}$ are σ-algebras in S.
2. $\{\emptyset, S, S_{1/2}, S_{2/2}\}$, where $S_{1/2}$ means the lower half of S (imagine S as an closed interval in $\mathbb{R}$), is a σ-algebra in S.
3. If $S = [0, 1]$, then $\mathcal{S} = \left\{\emptyset, \left[0, \frac{1}{2}\right), \left\{\frac{1}{2}\right\}, \left[\frac{1}{2}, 1\right], S\right\}$ is *not* a σ-algebra in S. But $\mathcal{S} = \left\{\emptyset, \left\{\frac{1}{2}\right\}, \left\{\left[0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right]\right\}, S\right\}$ is.

It allows us to define sets where things happen and we can *weigh* those sets (avoiding math troubles)

Definition

Suppose $\mathcal{S}$ is a σ-algebra in S. A measure is a real-valued function $x : \mathcal{S} \to \mathbb{R}_+$, that satisfies

1. $x(\emptyset) = 0$;
2. if $B_1, B_2 \in \mathcal{S}$ and $B_1 \cap B_2 = \emptyset \Rightarrow x(B_1 \cup B_2) = x(B_1) + x(B_2)$ (additivity); and,
3. if $\{B_i\}_{i \in \mathbb{N}} \in \mathcal{S}$ and $B_i \cap B_j = \emptyset$ for all $i \neq j \Rightarrow x(\cup_i B_i) = \sum_i x(B_i)$ (countable additivity).

A set S, a σ-algebra in it ($\mathcal{S}$), and a measure on $\mathcal{S}$ x, define a measurable space, $(S, \mathcal{S}, x)$.

Definition

A Borel σ-algebra is a σ-algebra generated by the family of all open sets $\mathfrak{B}$ (generated by a topology). A Borel set is any set in $\mathfrak{B}$.

A Borel σ-algebra corresponds to complete information.

Definition

A probability measure is measure where $x(S) = 1$. $(S, \mathcal{S}, x)$ is a probab space. The probab of an event is then given by $x(A)$, where $A \in \mathcal{S}$.

Definition

Given a m'able space $(S, \mathcal{S}, x)$, a real-valued function $f : S \to \mathbb{R}$ is m'able (with respect to the m'able space) if, for all $a \in \mathbb{R}$, we have

$$\{b \in S \mid f(b) \leq a\} \in \mathcal{S}.$$

Interpret σ-algebras as describing available information.

Similarly, a function is m'able wrt a σ-algebra $\mathcal{S}$, if it can be evaluated

Example

Suppose $S = \{1, 2, 3, 4, 5, 6\}$. Consider a function f that maps the element 6 to the number 1 (i.e. $f(6) = 1$) and any other elements to -100. Then f is NOT measurable with respect to $\mathcal{S} = \{\emptyset, \{1, 2, 3\}, \{4, 5, 6\}, S\}$. Why? Consider $a = 0$, then $\{b \in S \mid f(b) \leq a\} = \{1, 2, 3, 4, 5\}$. But this set is not in $\mathcal{S}$.

Extend the notion of Markov stuff to any measurable space

Definition

Given a measurable space $(S, \mathcal{S}, x)$, a function $Q : S \times \mathcal{S} \to [0, 1]$ is a transition probability if

1. $Q(s, \cdot)$ is a probability measure for all $s \in S$; and,
2. $Q(\cdot, B)$ is a measurable function for all $B \in \mathcal{S}$.

Intuitively, for $B \in \mathcal{S}$ and $s \in S$, $Q(s, B)$ gives the probability of being in set B tomorrow, given that the state is s today.

1. A *Markov chain* with transition matrix given by

$$\Gamma = \begin{bmatrix} 0.2 & 0.2 & 0.6 \\ 0.1 & 0.1 & 0.8 \\ 0.3 & 0.5 & 0.2 \end{bmatrix},$$

on $S = \{1, 2, 3\}$, with the the power set being the σ-algebra of S).

$$Q(3, \{1, 2\}) = \Gamma_{31} + \Gamma_{32} = 0.3 + 0.5 \ .$$

2. Consider a measure x on $\mathcal{S}$. x_i is the fraction of type i. Then

$$\begin{aligned} x_1' &= x_1\Gamma_{11} + x_2\Gamma_{21} + x_3\Gamma_{31}, \\ x_2' &= x_1\Gamma_{12} + x_2\Gamma_{22} + x_3\Gamma_{32}, \\ x_3' &= x_1\Gamma_{13} + x_2\Gamma_{23} + x_3\Gamma_{33}. \end{aligned}$$

In other words: $\mathrm{x}' = \Gamma^T \mathrm{x}$, where $\mathrm{x}^T = (x_1, x_2, x_3)$.

From a measure x today to one tomorrow x'

$$\begin{aligned} x'(B) &= T(x,Q)(B) \\ &= \int_S Q(s,B)\,x(ds), \quad \forall B \in \mathcal{S}, \end{aligned}$$

we integrated over all $s \in S$ to get the prob of B tomorrow.

A stationary distribution is a fixed point of T, that is x^* such that

$$x^*(B) = T(x^*,Q)(B), \quad \forall B \in \mathcal{S}.$$

Theorem

If Q has nice properties (American Dream and Nightmare) then $\exists$ a unique stationary distribution x^* and

$$x^* = \lim_{n\to\infty} T^n(x_0,Q), \qquad \textit{for any } x_0.$$

Exercise

Consider unemployment in a very simple economy (in which the transition matrix is exogenous). There are two states of the world: being employed and being unemployed. The transition matrix is given by

$$\Gamma = \begin{pmatrix} 0.95 & 0.05 \\ 0.50 & 0.50 \end{pmatrix}.$$

Compute the stationary distribution corresponding to this Markov transition matrix.

Industry Equilibrium

- Study the dynamics of the distribution of firms in partial equilibrium
- A single firm produces a good using labor:
- Output is $sf(n)$ (f increasing, strictly concave, $f(0) = 0$, s is productivity.
- Markets are competitive, (p and $w = 1$) as given.
- A firm solves

$$\pi(s, p) = \max_{n \geq 0} \{psf(n) - wn\}. \tag{11}$$

- With FOC

$$psf_n(n^*) = 1. \tag{12}$$

  Solution is $n^*(s, p)$.

- n^* is an increasing function of both arguments. Prove it.

- A mass of firms in the industry, indexed by $s \in S \subset \mathbb{R}_+$, $S := [\underline{s}, \bar{s}]$.
- $\mathcal{S}$ is a σ-algebra on S (a Borel σ-algebra, for instance).
- x is a measure on $(S, \mathcal{S})$, which describes the cross-sectional distribution of productivity among firms.
- Use x to define statistics of the industry: Since individual supply is $sf(n^*(s,p))$, then the aggregate supply

$$Y^S(p) = \int_S sf(n^*(s,p))\, x(ds). \tag{13}$$

Y^S is a function of the price p only.

- Let Demand $Y^D(p)$. Then p^* clears the market:

$$Y^D(p^*) = Y^S(p^*). \tag{14}$$

Where does x come from?

- Price p and output Y are constant over time.
- Firms face the problem above every period and discount profits at exogenous r.
- Each firm faces a probability $1-\delta$ of disappearing in each period.
- The choice is static. The value of an s firm is

$$V(s;p) = \sum_{t=0}^{\infty} \left(\frac{\delta}{1+r} \right)^{t} \pi(s,p) = \left(\frac{1+r}{1+r-\delta} \right) \pi(s,p)$$

- Every period a mass of firms die. To achieve a stationary equilibrium we need firms entry: assume that there is a constant flow of firms entering the economy in each as well, so that entry equals exit.
- x is the measure of firms. Firms that die are $(1-\delta)x(S)$.
- Entrants draw s from probability measure γ over $(S, \mathcal{S})$.

- What keeps other firms out of the market in the first place?

- (if $\pi(s;p) = psf(n^*(s;p)) - wn^*(s;p) > 0$, then any firm with $s \in S$ would enter.

- Assume a fixed entry cost, c^E before learning s. Value of an entrant

$$V^E(p) = \int_S V(s;p)\gamma(ds) - c^E. \tag{15}$$

  If $V^E > 0$ there will be entry.

- Equilibrium requires $V^E = 0$

- x_t : cross-sectional distribution of firms. For any $B \subset S$, fraction $1-\delta$ of firms with $s \in B$ die and mass m of newcomers enter. Next period's measure of firms on set B is

$$x_{t+1}(B) = \delta x_t(B) + m\gamma(B). \tag{16}$$

- Mass m of firms would enter $t+1$, with fraction $\gamma(B)$ having $s \in B$, $\forall B \in \mathcal{S}$.

- Cross-sectional distribution of firms completely determined by γ.

- Consider an updating operator T

$$Tx(B) = \delta x(B) + m\gamma(B), \quad \forall B \in \mathcal{S}, \tag{17}$$

a stationary dbon is a fixed point, i.e. x^* such that $Tx^* = x^*$, so

$$x^*(B; m) = \frac{m}{1-\delta}\gamma(B), \quad \forall B \in \mathcal{S}. \tag{18}$$

- Demand and supply condition in equation (14) is

$$Y^D\left(p^*\left(m\right)\right) = \int_S s\ f\left[n^*\left(s;p\right)\right]\ dx^*\left(s;m\right), \tag{19}$$

whose solution $p^*\left(m\right)$ is a continuous function

- We have two equations, (15) and (19), and two unknowns, p and m.

Definition

A stationary distribution for this environment consists of functions V, π^*, n^*, p^*, x^*, and m^*, that satisfy:

1. Given prices, V, π^*, and n^* solve the incumbent firm's problem;
2. $Y^D\left(p^*\left(m\right)\right) = \int_S s\ f\left[n^*\left(s;p\right)\right]\ dx^*\left(s;m\right)$;
3. $\int_s V\left(s;p\right)\gamma\left(ds\right) - c^E = 0$; and,
4. $x^*\left(B\right) = \delta x^*\left(B\right) + m^*\gamma\left(B\right), \quad \forall B \in \mathcal{S}$.

- Assume s follows a Markov process with transition Γ. This would change the mapping T in Equation (17) to

$$Tx\left(B\right)=\delta\int_{S}\Gamma\left(s,B\right)x\left(ds\right)+m\gamma\left(B\right),\quad\forall B\in\mathcal{S}. \tag{20}$$

But no firm exits (c^E is a sunk cost). Still not much Econ.

- Suppose now an operating cost c^v each period.
  - when s is low, firm's profits maybe negative and firm exits
  - But it is not enough. Assume Γ satisfies stochastic dominance: $s^1 > s^2$ implies $\sum_{s'=1}^{\widehat{s}}\Gamma_{s^{1},s'}<\sum_{s'=1}^{\widehat{s}}\Gamma_{s^{2},s'}$.
  - Then $\exists$ a threshold, $s^*\in S$, below which firms exit and above stay.

$$V\left(s;p\right)=\max\left\{0,\pi\left(s;p\right)+\frac{1}{(1+r)}\int_{S}V\left(s';p\right)\Gamma\left(s,ds'\right)-c^{v}\right\}. \tag{21}$$

- Updating operator becomes

$$x'(B) = \int_{s^*}^{\bar{s}} \Gamma(s, B \cap [s^*, \bar{s}])\, x(ds) + m\gamma(B \cap [s^*, \bar{s}]), \quad \forall B \in \mathcal{S}. \tag{22}$$

A stationary distribution of the firms in this economy, x^*, is the fixed point of this equation.

- With x^* we get all class of statistics:
  - Threshold for being in top 10% by size? Solve for $\widehat{s}$

$$\frac{\int_{\hat{s}}^{\bar{s}} x^*(ds)}{\int_{s^*}^{\bar{s}} x^*(ds)} = 0.1,$$

  - Fraction of workers in largest top 10% of firms

$$\frac{\int_{\hat{s}}^{\bar{s}} n^*(s,p)\, x^*(ds)}{\int_{s^*}^{\bar{s}} n^*(s,p)\, x^*(ds)}.$$

Exercise

Compute the average growth rate of the smallest one third of the firms.

Exercise

What would be the fraction of firms in the top 10% largest firms in the economy that remain in the top 10% in next period?

Exercise

What is the fraction of firms younger than five uears?

Definition

π^*, n^*, d^*, s^*, V, a price p^*, a measure x^*, and mass m^* such that

1. Given p^*, the functions V, π^*, n^*, d^* solve the firm's
2. The reservation productivity s^* satisfies $d^*(s; p^*) = \begin{cases} 1 & \text{if } s \geq s^* \\ 0 & \text{otherwise} \end{cases}$.
3. Free-entry condition: $\quad V^E(p^*) = 0.$
4. For any $B \in \mathcal{S}$

$$x^*(B) = m^* \gamma (B \cap [s^*, \bar{s}]) + \int_{s^*}^{\bar{s}} \Gamma(s, B \cap [s^*, \bar{s}]) \, x^*(ds)$$

5. Market clearing:

$$Y^d(p^\star) = \int_{s^\star}^{\bar{s}} s \; f(n^\star(s; p^\star)) x^\star(ds)$$

- Average output of the firm is given by

$$\frac{Y}{N} = \frac{\int_{s^\star}^{\bar{s}} s \; f[n^*(s)] \; x^*(ds)}{\int_{s^\star}^{\bar{s}} x^*(ds)}$$

- Share of output produced by the top 1% of firms. Need to find $\tilde{s}$

$$\frac{\int_{\tilde{s}}^{\bar{s}} x^*(ds)}{\int_S x^*(ds)} = .01$$

$$\frac{\int_{\tilde{s}}^{\bar{s}} s \;\; f[n^*(s)] \;\; x^*(ds)}{\int_{s^\star}^{\bar{s}} s \;\; f[n^*(s)] \;\; x^*(ds)}$$

- Fraction of firms in the top 1% two periods in a row (s continuous)

$$\int_{s \geq \tilde{s}} \int_{s' \geq \tilde{s}} \Gamma_{ss'} x^*(ds)$$

- Gini coefficient.

Consider adjustment costs to labor $c\left(n^{-}, n\right)$ due to hiring n units of labor in t as

- *Convex Adjustment Costs*: if the firm wants to vary the units of labor, it has to pay $\alpha\left(n_t - n_{t-1}\right)^2$ units of the numeraire good. The cost here depends on the size of the adjustment.

- *Training Costs or Hiring Costs*: if the firm wants to increase labor, it has to pay $\alpha\left[n_t - (1-\delta)\, n_{t-1}\right]^2$ units of the numeraire good only if $n_t > n_{t-1}$. We can write this as

$$1_{\{n_t > n_{t-1}\}} \alpha\left[n_t - (1-\delta)\, n_{t-1}\right]^2,$$

  where 1 is the indicator function and δ measures the exogenous attrition of workers in each period.

- *Firing Costs*: the firm has to pay if it wants to reduce the number of workers.

$$V\left(s,n^{-};p\right)=\max\left\{0,\max_{n\geq 0}\ sf\left(n\right)-wn-c^{v}-c\left(n^{-},n\right)+\frac{1}{(1+r)}\int_{s'\in S}V\left(s',n,p\right)\ \Gamma(s,ds')\right\},$$

$c(\cdot,\cdot)$ is cost function (note limited liability: exit value is 0)

Note $n=g(s,n^{-};p)<\bar{N}$. Let $\mathcal{N}$ be a σ-algebra on $[0,\bar{N}]$.

$$x'\left(B^{S},B^{N}\right)=m\gamma\left(B^{S}\cap[s^{*},\bar{s}]\right)1_{\{0\in B^{N}\}}+\int_{s^{*}}^{\bar{s}}\int_{0}^{\bar{N}}1_{\{g(s,n_{-};p)\in B^{N}\}}\ \Gamma\left(s,B^{S}\cap[s^{*},\bar{s}]\right)x\left(ds,dn_{-}\right),$$

$$\forall\ B^{S}\in\mathcal{S},\forall\ B^{N}\in\mathcal{N}.$$

Exercise

Write the first order conditions.

Exercise

Define the recursive competitive equilibrium for this economy.

Exercise

Another example of labor adjustment costs is when the firm has to post vacancies to attract labor. As an example of such case, suppose the firm faces a firing cost according to function c. The firm also pays a cost κ to post vacancies and after posting vacancies, it takes one period for the workers to be hired. How can we write the problem of firms in this environment?

Exercise

Add Adjustment Costs to Capital

- So far *stationary industry equilibria* (invariant distribution of firms).
- If p were constant, the firm distribution would converge to the stationary equilibrium distribution x^*.
- What is an alternative?
- Prices are changing over time and so is the distribution of firms.
- There are two ways of modeling non-stationary equilibria
  - In Sequence Space (or stochastic process state)
  - Recursively
- What is best depends on the purpose. They should give the same answer. It is an issue of computation.
- We will look at both ways (for now deterministic).
- Given the convergence that we talked about we need a rationale for the non stationarity.
- Consider demand shifters z_t so that $D(P, z_t)$ where $z_{t+1} = \phi(z_t)$ so we can choose to represent it as a sequence or recursively.

- Note the need for an initial condition. Then objects are relatively simple.
- Given a path $\{z_t\}_{t=0}^{\infty}$ and an initial x_0, an equilibrium defined in term of sequences is: Sequences $\{p_t, m_t, s_t^*\}$ of numbers, a sequence of measures x_t, and sequences $\{V_t(s), n_t(s)\}_{t=0}^{\infty}$ of functions:

1. **Optimality**: Given $\{p_t\}$, $\{V_t, s_t^*, n_t\}$ sole

$$V_t(s) = \max \left\{ 0, \max \; p_t \; s \; f(n) - wn - c^v + \frac{\int_S V_{t+1}(s') \Gamma(s, ds')}{1+r} \right\}$$

2. **Free-entry**: $\int V_t(s) \; \gamma(ds) \leq c^e$, with strict equality if $m_t > 0$.
3. **Law of motion**: $x_{t+1}(B) = m_{t+1}\gamma \left(\cap [s_{t+1}^*, \bar{s}] \right) + \int_{s_t^*}^{\bar{s}} \Gamma \left(s, B \cap [s_{t+1}^*, \bar{s}] \right) \; x_t(ds)$, $\forall B \in \mathcal{S}$.
4. **Market clearing**: $D[p_t, z_t] = \int_{s_t^*}^{\bar{s}} p_t \; s \; f[n_t(s)] \; x_t(ds)$.

- Only from today to tomorrow: need objects that given the state today, $\{z, x\}$, give us the state tomorrow $\{\phi, G\}$.
- Given ϕ, an equil defined recursively is functions $G(z,x)$, $m(z,x)$, $p(z,x)$, values and decisions $\{V(s,z,x), n(s,z,x), s^*(s,z,x)\}$ s.t.

1. **Optimality**: $\{V(s,z,x), s^*(s,z,x), n(s,z,x)\}$ solve

$$V(s,z,x) = \max_n \left\{0, \max p(s,z,x) s\ f(n) - wn - c^v + \frac{1}{1+r}\int_S V[s', \phi(z), G(z,x)]\ \Gamma(s, ds')\right\}$$

2. **Free-entry**: $\int V(s,z,x)\ \gamma(ds) \leq c^e$, ($=$ if $m(z,x) > 0$).
3. **Law of motion**: $\forall B \in \mathcal{S}$, we have $G(z,x)(B) = m(z,x)\ \gamma(B \cap [s^*(s,z,x), \bar{s}]) + \int_{s^*(s,z,x)}^{\bar{s}} \Gamma(s, B \cap [s^*(s,z,x), \bar{s}]) x(ds)$,
4. **Market clearing**: $D(p(z,x), z) = \int_{s^*(s,z,x)}^{\bar{s}} p(z,x)\ s\ f[n(s,z,x)]\ x(ds)$.

- It is the same but in Stochastic Processes Language

- They extend the same for sequences and for the Recursive

- Obviously You have to add the Expectations to the terms of one period later.

- There is a new (Boppart, Mitman & Krusell (2017)) way of thinking of Stochastic Equilibria that is NOT recursive.

- It is based on a linear approximation to a completely unanticipated (MIT) shock.

- It requires to compute a transition as a Perfect Foresight Equilibrium

- Then do linear approximations in sequence space.

- Consider the social planner's problem (with full depreciation)

$$\begin{aligned} V(k_t) \quad &= \quad \max_{c_t, k_{t+1}} u(c_t) + \beta V(k_{t+1}) \\ & \quad\ \text{s.t. } c_t + k_{t+1} \leq f(k_t), \qquad \forall\ t \geq 0 \\ & \quad\ c_t, k_{t+1} \geq 0, \ \forall \qquad t \geq 0 \\ & \quad\ k_0 > 0 \text{ given.} \end{aligned}$$

- The solution $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ satisfies

$$u_c(c_t) = \beta\ u_c(c_{t+1})\ f_k(k_{t+1}), \forall\ t \geq 0$$

$$c_t + k_{t+1} = f(k_t), \quad \forall\ t \geq 0$$

$$\lim_{t \to \infty} \beta^t \ \ u_c(c_t) \ \ k_{t+1} = 0$$

- Derive the above equilibrium conditions.

- Look at the a steady state k^*
- Rewrite solution as

$$\psi(k_t, k_{t+1}, k_{t+2}) = u_c[f(k_t - k_{t+1})] - \beta\ u_c[f(k_{t+1} - k_{t+2})]\ f_k(k_{t+1}) = 0,$$

a second order difference equation with exactly two boundary conditions, k_0 and $k_\infty = k^*$.

- It is solvable:
  1. guess k_1, use k_0 and $\psi(k_t, k_{t+1}, k_{t+2}) = 0$ to get $k_2, k_3, \ldots$ forward up until some T, and solve $k_T^\psi(k_1) = k^*$.
  2. Or guess k_{T-1} solve backward using ψ to find $k_0^\psi(k_{t-1}) = k_0$
  3. Solve for the whole sequence as a system of equations (almost diagonal)
  4. Use dynare.
- Either way you get a numerical solution starting from any k_0

- We can compute any transition. Also one with time varying ψ.
- Consider this model with $c_t + k_{t+1} = e^{z_t}\ f(k_t),\ z_{t+1} = \rho z_t,\ z_0 = 1$.

$$\psi_t(k_t, k_{t+1}, k_{t+2}) = u_c[\rho^t f(k_t - k_{t+1})] - \beta u_c[\rho^{t+1} f(k_{t+1} - k_{t+2})] f_k(k_{t+1}),$$

- In this case we can look at an MIT shock or impulse response. Here $k_0 = k_\infty = k^*$, but $k_1 \neq k^*$
- We can again obtain the transition k_t.
- Let now $\widehat{k_t} = \log k_t - \log k^*$, (log st st deviation).
- This is in fact an impulse response function.

- We want now to simulate a response of the economy to shocks. Consider an AR(1) process for z_t : with $z_{t+1} = \rho^t z_t + \epsilon_{t+1}$.) where $\epsilon_t \sim \mathcal{N}(\int, \supset^{\in})$.
- **Want:** Solve for the solution by linearly approximating using $\{\widehat{k}_t\}_{t=0}^{\infty}$ the equilibrium given any sequence of innovations $\{\epsilon_t\}$.).
- Obtain $\widetilde{k}_t(k_0, \epsilon^{t-1})$ again in deviations from steady state. Note that the following linear approximation is what we want.

$$
\begin{aligned}
\widetilde{k}_1(k_0, \epsilon_0) &= \epsilon_0 \ \widehat{k}_1 \\
\widetilde{k}_2(k_0, \epsilon_0, \epsilon_1) &= \epsilon_0 \ \widehat{k}_2 + \epsilon_1 \ \widehat{k}_1, \\
&\vdots \\
\widetilde{k}_{t+1}(k_0, \epsilon^t) &= \sum_{\tau=0}^{t} \epsilon_t \ \widehat{k}_{t-\tau+1} \qquad \text{exact if } \epsilon_0 = 1, \ \epsilon_t = 0, \forall t \neq 0,
\end{aligned}
$$

- This can be done for all Economies.
- Including industry equilibria.
- For all Statistics of all Economies.
- The computational costs is linear not exponential in the number of shocks.
- We do not know how to use it for asymmetric shocks (e.g. downward rigid wages)

Exercise

1. *What happens if demand suddenly doubles starting from a stationary equilibrium?*

2. *Define Formally the stochastic counterparts (sequentially and recursivrly) to the ones that we wrote above?*

3. *Sketch an algorithm to find the equilibrium prices.*

4. *Describe a way to compute the evolution of the Gini Index or the Herfindahl Index of the industry over the first fifteen periods.*

5. *Imagine now that the industry is subject to demand shocks that follow an AR(1). Describe an algorithm to approximate it.*

Incomplete Market Models

- Consider the problem of a farmer with storage possibilities

$$V(s,a) = \max_{c,a' \geq 0} \quad u(c) + \beta \sum_{s'} \Gamma_{ss'} \; V(s', a') \qquad s.t.$$

$$c + qa' = a + s$$

a assets, c consumption, and $s \in \{s^1, \cdots, s^{N^s}\} = S$ has transition Γ. q units today yield 1 unit tomorrow. Only nonnegative storage.

- If s constant, then

$$V(a) = \max_{c,a' \geq 0} \left\{ u\left(a + s - qa'\right) + \beta V\left(a'\right) \right\}.$$

- with FOC $q\, u_c \geq \beta u_c'$
- With equality if $a' > 0$. Then
    - if $q > \beta$ (i.e. nature is more stingy, or the farmer is less patient),
        - Either $c' < c$ and the farmer dis-saves
        - Or $c = s$ and $a' = 0$.
    - If $q < \beta$, $c' > c$ and consumption grows without bound.
    - If $q = \beta$, $c' = c$ (with noise and $u_{ccc} > 0$ grows without bound).
- So we assume $\beta / q < 1$

- Assuming $\beta/q < 1$, allows us to bound asset holdings.
- They also save in best states when a is low.
- The FOC is

$$u_c\left[c\left(s,a\right)\right] \geq \frac{\beta}{q}\sum_{s'}\Gamma_{ss'}\ u_c\left(c\left[s',g\left(s,a\right)\right]\right),$$

with equality when $a' = g\left(s,a\right) > 0$

- Note: $a \gg g\left(s,a\right),\ \ \forall s$ for sufficiently large a. So $\exists\ \bar{a},\ s.t.\ a' \in A = [0,\bar{a}]$
- We can construct a prob distribution over states $S \times A$. Define $\mathcal{B}$ as all subsets of S times Borel-σ-algebra sets in A.
- For any such prob measure x its evolution is

$$x'\left(B\right) = \widetilde{T}(B,x;\Gamma,g) = \sum_{s}\int_{0}^{\bar{a}}\sum_{s'\in B_s}\Gamma_{ss'}\ 1_{\{g(s,a)\in B_a\}}\ x\left(s,da\right),\quad \forall B\in\mathcal{B}$$

where B_s and B_a are projections of B on S and A,

Theorem

With a well behaved Γ, there is a unique stationary probability x^*, so that:

$$x^*(B) = \widetilde{T}(B, x^*; \Gamma, g)(B), \quad \forall B \in \mathcal{B},$$
$$x^*(B) = \lim_{n\to\infty} \widetilde{T}^n(B, x_0; \Gamma, g)(B), \quad \forall B \in \mathcal{B},$$

for all initial probability measures X_0 on $(E, \mathcal{B})$.

We use compactness of $[0, \overline{A}]$.

1. Our ignorance of what is going on with the farmer or fisherman.

    - Even if we know at $t = 0$ s, a, no news lead us to x^*.

    - We can use x^* to compute the statistics of what happens to the fisherman: Average wealth is $\int_{S\times A} a \, dx^*$.

2. A description of a large number of fishermen (an archipelago). Notice how even if there is no contact between them. Stationarity arises (İmrohoroğlu (1989))

    - There is a unique distribution of wealth.

- How can $a < 0$? Because of borrowing.
- Consider now an economy with many farmers and NO storage.

$$\begin{aligned} V(s,a) = \max_{c \geq 0, a'} \quad & u(c) + \beta \sum_{s'} \Gamma_{ss'} V(s', a') \\ s.t. \quad & c + q\, a' = a + s \\ & a' \geq \underline{a}, \end{aligned}$$

where $\underline{a} < 0$ and $\beta / q < 1$. With solution $a' = g(s,a)$. Again

- One possibility for $\underline{a}$ is the natural borrowing limit: the agent can pay back his debt with certainty, no matter what:

$$a^n := -\frac{s_{\min}}{\left(\frac{1}{q} - 1\right)}. \tag{23}$$

- Or it could be tighter which makes it likely to bind $0 > \underline{a} > a^n$.

- To determine q in general equilibrium, consider this function of q:

$$\int_{A\times S} a\, dx^*(q) \qquad \text{Aggregate asset holdings}$$

- A Stationary Equilibrium requires two things

$$\begin{aligned} \int_{A\times S} a\, dx^*(q) &= 0, \\ x^*(q) &= \widetilde{T}^n(B, x^*(q); \Gamma, g)(B). \end{aligned}$$

- It exists in $q \in (\beta, \infty]$ (intermediate value thm). Need to ensure:
    1. $\int_{A\times S} a\, dX^*(q)$ is a continuous function of q;
    2. $\lim_{q\to\beta} \int_{A\times S} a\, dX^*(q) \to \infty$; (As $q \to \beta$, the interest rate $R = 1/q$ increases up to $1/\beta$, (steady state interest rate in deterministic Econ. With $u_{ccc} > 0$ we have precautionary savings
    3. $\lim_{q\to\infty} \int_{A\times S} a\, dX^*(q) < 0$. As $q \to \infty$, arbitrary large consumption is achievable by borrowing.

- Workhorse models of modern macroeconomics.
- An Environment like the ones before
- On top of a growth model with $f(K, L)$ that yield factor prices.

$$K = \int_{A \times S} a\, dx,$$

$$N = \int_{A \times S} s\, dx.$$

- s fluctuations in the employment status (either efficiency units of labor or actual employment).
- Now we need $\beta(1 + r) < 1$. We write

$$V(s, a) = \max_{c, a' \geq 0} \quad u(c) + \beta \int_{s'} V(s', a') \; \Gamma(s, d\, s') \qquad s.t.$$

$$c + a' = (1 + r)\, a + ws$$

where r is the return on savings and w is the wage rate.

- Factor prices depend on the capital-labor ratio: $x^{*}\left(\frac{K}{L}\right)$. Equilibrium requires

$$\frac{K^*}{L^*} = \frac{\int_{A\times S} a \;\; dX^*\left(\frac{K^*}{L^*}\right)}{\int_{A\times S} s \, dX^*\left(\frac{K^*}{L^*}\right)}.$$

Exercise

Show that aggregate capital is higher in the stationary equilibrium of the Aiyagari economy than it is the standard representative agent economy.

Exercise

Not necessarily so if leisure has value (Pijoan-Mas (2006))

Exercise

Rewrite the economy when households like leisure

- Let the Economy's parameters be summarized by $\theta = \{u, \beta, s, \Gamma, F\}$.
- $V(s, a; \theta)$ and $x^*(\theta)$ are functions of those parameters.
- Suppose an unexpected policy change that shifts θ to $\hat{\theta} = \{u, \beta, s, \hat{\Gamma}, F\}$.
- Consider $V\left(s, a; \hat{\theta}\right)$ and $x^*\left(\hat{\theta}\right)$.
- Define $\eta(s, a)$ by

$$V\left(s, a + \eta(s, a); \hat{\theta}\right) = V(s, a; \theta),$$

- Transfer necessary to make the (a, s) agent indifferent between living in the old environment and in the new.
- Total transfer needed to compensate all agents to live in $\hat{\theta}$ is

$$\int_{A \times S} \eta(s, a)\, dX^*(\theta).$$

- This is NOT a Welfare Comparison.
- This compares being parachuted in the stationary distribution of θ versus $\hat{\theta}$.
- Welfare computing the transition from the SAME initial conditions.
- Otherwise the best tax policy in the Rep agent (which is Pareto Optimal) would be to subsidize capital to maximize steady state consumption.

- What if aggregate shocks as in e.g. $z\ F\left(K,\bar{N}\right)$.
- Without leisure aggregate capital is a sufficient statistic for factor prices.
- Will aggregate capital be $K'=G\left(z,K\right)$ or $K'=G\left(z,x\right)$?
- The latter. Decision rules are not usually linear. But then $x'=G\left(z,x\right)$

$$
\begin{aligned}
V\left(z,X,s,a\right)=\max_{c,a'\geq 0}\quad & u\left(c\right)+\beta\sum_{z',s'}\Pi_{zz'}\Gamma_{ss'}^{z'}V\left(z',X',s',a'\right)\\
s.t.\quad & c+a'=a\ z\ f_k\left(K,\bar{N}\right)+s\ z\ f_n\left(K,\bar{N}\right)\\
& K=\int a\ dX\left(s,a\right)\\
& X'=G\left(z,X\right)
\end{aligned}
$$

(replaced factor prices with marginal productivities)

- Computationally, this problem is a beast! So, what then?

- They people believe tomorrow's capital depends only on K and not on x. This, obviously, is not an economy with rational expectations. The agent's problem in such a setting is

$$\widetilde{V}(z, K, s, a) = \max_{c, a'} \quad u(c) + \beta \sum_{z', s'} \Pi_{zz'} \Gamma_{ss'}^{z'} \widetilde{V}\left(z', K', s', a'\right)$$

$$s.t. \quad c + a' = a \; z \; f_k\left(K, \bar{N}\right) + szf_n\left(K, \bar{N}\right)$$

$$K' = \widetilde{G}(z, K)$$

- We could approximate the equilibrium in the computer by posing a linear approximation to $\widetilde{G}$. A pain but doable. Krusell Smith (1997).

- They found it works well in boring settings (things are pretty linear)

- We can use the same linear approx in sequences as before for any shocks:

  1. Find the steady state

  2. Obtain the the impulse response (the perfect foresight equilibium) given an MIT shock that is treated as an innovation.

  3. Use these responses to approximate the behavior of any aggregate.

- Valuable for SMALL shocks like all linear approximations.

- Consider an Aiyagari economy with an AR(1) TFP shock z.
  - Choose an initial size innovation $\overline{\epsilon}_0$ (does not have to be 1) and compute the perfect foresight Equilibria of this MIT shock.
    - This involves a fixed point in the space of sequence of capital labor ratios.
    - But can be done with some effort:
    - To evaluate it, given prices solve the household's problem backwards from the final steady state.
    - Then update the distribution forward from the initial steady state obtaining new prices.
    - We look for a fixed point of this (not necessarily iterating mechanically but as solution of a system of equations)

- We have now the sequence of x_t and any prices that we care for.
- Compute the sequence of all statistics $\{d_t\}_t^T$ of that economy that you care for.
- Get a random draw $\{\epsilon_t\}_{t=0}^T$.
- Linearly approximate those statistic like we did before the same way that we approximated

$$\begin{aligned}
\widetilde{d}_1(x_0, \epsilon_0) &= \frac{\epsilon_0}{\bar{\epsilon}_0}\ \widehat{d}_1 \\
\widetilde{d}_2(x_0, \epsilon_0, \epsilon_1) &= \frac{\epsilon_0}{\bar{\epsilon}_0}\ \widehat{d}_2 + \frac{\epsilon_1}{\bar{\epsilon}_0}\ \widehat{d}_1, \\
&\vdots \\
\widetilde{d}_{t+1}(x_0, \epsilon^t) &= \sum_{\tau=0}^{t} \frac{\epsilon_t}{\bar{\epsilon}_0}\ \widehat{d}_{t-\tau+1} \qquad \text{exact if } \epsilon_0 = \widetilde{\epsilon}_0,\ \epsilon_t = 0, \forall t \neq 0.
\end{aligned}$$

- Agents can either not work or work: $\varepsilon = \{0, 1\}$,
- Agents can exert painful effort h to search for a job increasing the probability $\phi(h)$ (with $\phi' > 0$) of finding it.
- An employed worker, does not search for a job so $h = 0$, but its job can be destroyed with some exogenous probability δ.
- s is Markovian (Γ) labor labor productivity. Then the unemployed

$$V(s,0,a) = \max_{c,h,a'\geq 0} u(c,h) + \beta \sum_{s'} \Gamma_{ss'} \left[\phi(h) V(s',1,a') + (1-\phi(h)) V(s',0,a')\right]$$

$$s.t. \quad c + a' = h + (1+r)a$$

the employed

$$V(s,1,a) = \max_{c,a'\geq 0} u(c) + \beta \sum_{s'} \Gamma_{ss'} \left[\delta V(s',0,a') + (1-\delta) V(s',1,a')\right]$$

$$s.t. \quad c + a' = sw + (1+r)a$$

- Suppose every period agents choose an occupation: entrepreneur or a worker.
- Entrepreneurs run their own business: manage a project that combines entrepreneurial ability (η), capital (k), and labor(n); while workers work for somebody else.
- If worker

$$V^w(s,\eta,a) = \max_{c,a'\geq 0, d\in\{0,1\}} u(c) + \beta \sum_{s',\eta'} \Gamma_{ss'}\Gamma_{\eta\eta'} \left[dV^w(s',\eta',a') + (1-d)V^e(s',\eta',a')\right]$$

$$s.t. \quad c + a' = ws + (1+r)a$$

- Similarly, the entrepreneur's problem can be formulated as follows

$$V^e(s,\eta,a) = \max_{c,a'\geq 0, d\in\{0,1\}} u(c) + \beta \sum_{s',\eta'} \Gamma_{ss'}\Gamma_{\eta\eta'} \left[d\ V^w(s',\eta',a') + (1-d)V^e(s',\eta',a')\right]$$

$$s.t. \quad c + a' = \pi(s,\eta,a)$$

- Income is from profits $\pi(a,s,\eta)$ not wages. Assume entrepreneurs have a DRS technology f. Profits are

$$\pi(s,\eta,a) = \max_{k,n} \eta f(k,n) + (1-\delta)k - (1+r)(k-a) - w\max\{n-s,0\}$$

$$s.t. \quad k - a \leq \phi a$$

- The constraint here reflects the fact that entrepreneurs can only make loans up to a fraction ϕ of his total wealth.

- Entrepreneurs never make an operating loss within a period, (can always choose $k = n = 0$ and earn the risk free rate on savings).
- Agents with high entrepreneurial ability η have access to an investment technology f that provides higher returns than workers so become richer.
- Even the prospects (high η) low wealth suffice to induce high savings? (Γ)
- Who becomes an entrepreneur in this economy? Without financial constraints, wealth will play no role. $\exists \eta^*$ above which it becomes an entrepreneur.
- With financial constraints wealth matters. Wealthy agents with high h will while the poor with low η will not.
- For the rest, it depends. If η is persistent, poor individuals with high entrepreneurial ability will save to one day become entrepreneurs, while rich agents with low entrepreneurial ability will lend their assets and become workers.

- The price of lending incorporates the possibility of default.

- Assume upon default punished to autarky forever after (no borrowing or lending)

- If no default the budget constraint is $c + q(a')a' = a + ws$,

$$V(s,a) = \max\left\{u(ws) + \beta \sum_{s'} \Gamma_{ss'} \bar{V}(s'),\right.$$
$$\left.\max_{c,a'} u[ws + a - q(a')\ a'] + \beta \sum_{s'} \Gamma_{ss'} V(s', a')\right\}$$

where $\bar{V}(s') = \frac{1}{1-\beta} u(ws')$ is the value of autarky.

- What determines $q(a')$? A zero profit on lenders: Probability of default

Monopolistic Competition

- Models with Nominal Prices.
- Price/Wage Rigidity.
- Firms are sufficiently "different" to set prices.
- Small in the Context of the Aggregate Economy. Hence Monopolistic Competition.

- Consumers have a taste for variety

    - The consumer's utility function has constant elasticity of substitution (CES)

$$u\left(\{c(i)\}_{i\in[0,n]}\right) = \left(\int_0^n c(i)^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{\sigma}{\sigma-1}}$$

where σ is the elasticity of substitution, and $c(i)$ is the quantity consumed of variety i. For ease of notation, we rename $c(i) = c_i$.

- Assume the agents receive exogenous *nominal* income I

- They are endowed with one unit of time.

$$\max_{\{c_i\}_{i\in[0,n]}} \left(\int_0^n c_i^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{\sigma}{\sigma-1}}$$

$$s.t. \quad \int_0^n p_i \; c_i \; di \leq I$$

- Deriving the FOC, and relating the demand for varieties i and j

$$c_j = c_i \left(\frac{p_j}{p_i}\right)^{-\sigma}$$

- Multiplying both sides by p_j and integrating over j, yields

$$c_i^* = \frac{I}{\int_0^n p_j^{1-\sigma} dj} \, p_i^{-\sigma}$$

- Here c_i^* depends on the price of i and an aggregate price

- Convenient to define the aggregate price index P as

$$P = \left(\int_0^n p_j^{1-\sigma} dj \right)^{\frac{1}{1-\sigma}}$$

- and thus

$$c_i^* = \frac{I}{P} \left(\frac{p_i}{P} \right)^{-\sigma}$$

real income times a measure of the relative price of i.

Exercise

Show the following within this monopolistic competition framework

1. *σ is the elasticity of substitution between varieties.*
2. *Price index P is the expenditure to purchase a unit-level utility.*
3. *Consumer utility is increasing in the number of varieties n.*
4. *Is there a missing n?*

- Assume linear production technology: $f(\ell_j) = \ell_j$.
- Nominal wage rate is given by W.
- The firm solves

$$\max_{p_j} \pi(p_j) = p_j \ c_j^*(p_j) - W \ c_j^*(p_j)$$
$$\text{s.t.} \qquad c_j^* = \frac{I}{P}\left(\frac{p_j}{P}\right)^{-\sigma}$$

- Firms do not affect P. Solve for the FOC:

$$p_j^* = \frac{\sigma}{\sigma - 1} W \qquad \forall j$$

- $\frac{\sigma}{\sigma - 1}$ is a constant mark-up over the marginal cost,
- When varieties are close substitutes ($\sigma \to \infty$), prices converge to W.

Set the wage as numeraire. An Eq is prices $\{p_i^*\}_{i\in[0,n]}$, the aggregate price index P, household's consumption, $\{c_i^*\}_{i\in[0,n]}$, income I, firm's labor demand $\{\ell_i^*\}_{i\in[0,n]}$ and profits $\{\pi_i^*\}_{i\in[0,n]}$, such that

1. Given prices, $\{c_i^*\}_{i\in[0,n]}$ solves the household's problem
2. Given P and I, p_i^* and π_i^* solve the firm's problem $\forall i \in [0, n]$
3. Price Aggregation

$$P = \left(\int_0^n (p_j^*)^{1-\sigma} \, dj \right)^{\frac{1}{1-\sigma}}$$

4. Markets clear

$$\int_0^n \ell_i^* \, di = 1$$

$$1 + \int \pi_i^* \, di = I$$

A symmetric equilibria: $c_i^* = \bar{c}$, $p_i^* = \bar{p}$, $\ell_i^* = \bar{\ell}$, $\pi_i^* = \bar{\pi}$ for all i.

- To study inflation, (meaningful interactions between output and inflation) needs
    1. A dynamic model
    2. Some source of nominal frictions so nominal variables (things measured in dollars) can affect real variables.
- Most popular friction is *price rigidity*. (firms cannot adjust their prices freely)
    1. *Rotemberg pricing* (menu costs)
    2. *Calvo pricing* (some (randomly set) firms can change prices, others cannot).

- Firms face adjustment cost $\phi(p_j, p_j^-)$ when changing their prices p_j each period.

- Let the Agg State be S, and let $I(S)$, $W(S)$, $P(S)$. Then firm's per period profit under Rotemberg pricing in a dynamic setup as follows:

$$\Omega(S, p_j^-) = \max_{p_j} \; p_j c_j^* - W(S) c_j^* - \phi(p_j, p_j^-)$$

$$+ E\{R^{-1}(G(S))\, \Omega(G(S), p_j)\}$$

$$\text{where} \quad c_j^* = \left(\frac{p_j}{P(S)}\right)^{-\sigma} \frac{I(S)}{P(S)}$$

- easy algebra when quadratic price adjustment cost.

- Without capital $S = P^-$ and Aggregate Shocks.

- Firms can adjust their prices each period with probability θ.

- A firm that can change its price

$$\Omega^1(S, p_j^-) = \max_{p_j} p_j c_j^* - W(S) c_j^* + (1-\theta) E\{R^{-1}(S')\, \Omega^0(S', p_j)\} + \theta\, E\{R^{-1}(S')\, \Omega^1(S', p_j)\}$$

$$\text{where} \quad c_j^* = \left(\frac{p_j}{P(S)}\right)^{-\sigma} \frac{I(S)}{P(S)} \quad \text{and} \quad S' = G(S)$$

- A firm that cannot

$$\Omega^0(S, p_j^-) = [p_j^- - W(S)] c_j^* + (1-\theta) E\{R^{-1}(S')\, \Omega^0(S', p_j^-)\} + \theta\, E\{R^{-1}(S')\, \Omega^1(S', p_j^-)\}$$

Exercise

Derive the following for the dynamic model with Calvo pricing

1. *Solve the firm's problem in sequence space and write the firm's equilibrium pricing $p_{j,t}$ as a function of present and future aggregate prices, wages, and endowments: $\{P_t, W_t, I_t\}_{t=0}^{\infty}$.*
2. *Show that under flexible pricing ($\theta = 1$), the firm's pricing strategy is identical to the static model.*
3. *Show that with price rigidity ($\theta < 1$), the firm's pricing strategy is identical to the static model in a steady state with zero inflation.*

- Rotenberg: pricing equilibrium is $P^{-} \in S$ and $P = p^{*}(S, P^{-})$
- Calvo: there is a $P^{-} \in S$ and recall that $P = \left(\int_0^n p_j^{1-\sigma} dj\right)^{\frac{1}{1-\sigma}}$.
  - Because we have θ firms adjusting and $1-\theta$ not, we have

$$P = \left[\theta \left(P^{-}\right)^{1-\sigma} + (1-\theta)\left(p^{*}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}$$

q for the optimally chosen p^{*}.

  - This is the magic of Calvo pricing: The distritribution of prices is NOT a state variable.
  - That turns out to satisfy (after using representative agent condition)

$$P^{*} = \frac{\sigma}{\sigma-1} \frac{E\left\{\sum_{\tau}(\theta\beta)^{\tau} u_c \; P_{\tau}^{\sigma-1} \; \varphi_{\tau} \; y_{\tau}\right\}}{E\left\{\sum_{\tau}(\theta\beta)^{\tau} u_c \; P_{\tau}^{\sigma-1} \; y_{\tau}\right\}}$$

where φ_{τ} is nominal marginal cost

  - Is this a nightmare? No. Log-linearization comes to help

- Let X
- Let $\overline{X}$ be the steady state.
- Sometimes we want to use

$$\widehat{x} = \log X - \log \overline{X}$$

- We say Log Deviations

- Products

$$Z = X^{\alpha}\ Y^{\beta} \quad \Longrightarrow \widehat{z} = \alpha\ \widehat{x} + \beta\ \widehat{y}$$

- Sums

$$\overline{Z}\ \widehat{z} = \alpha\ \overline{X}\ \widehat{x} + \beta\ \overline{Y}\ \widehat{y}$$

- Smooth Functions $Z = f(X, Y) \Longrightarrow$

$$\overline{Z} \simeq \widehat{z} = f_x(\overline{X}, \overline{Y})\ \ \overline{X}\ \widehat{x} + \beta\ f_y(\overline{X}, \overline{Y})\ \overline{Y}\ \widehat{y}$$

- Recall the Law of motion for the price level

$$P = \left[\theta\left(P^{-}\right)^{1-\sigma} + (1-\theta)\left(p^{*}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}$$

- Log-linearizing around the steady state

$$\frac{1}{1-\sigma}\overline{P}\widehat{p} \simeq \theta\frac{1}{1-\sigma}\overline{P}^{-} + (1-\theta)\frac{1}{1-\sigma}\overline{P}^{*}\widehat{p}^{*}$$

ignoring the consants which always cancels from both sides, noting that in St St $\overline{P} = \overline{P}^{*}$ we have $\widehat{p} = \theta\ \widehat{p^{-}} + (1-\theta)\ \widehat{p}^{*}$

- and because the steady state is common $\widehat{p} \simeq \theta\ \widehat{p^{-}} + (1-\theta)\ \widehat{p}^{*}$ so

$$p \simeq \theta\ p^{-} + (1-\theta)\ p^{*}$$

- Which implies for inflation that

$$\pi = p - p^{-} = (1-\theta)\ (\widehat{p}^{*} - \widehat{p}^{-})$$

- Price setting

$$P^* = \frac{\sigma}{\sigma - 1} \frac{E\left\{\sum_\tau (\theta\beta)^\tau u_c \; P_\tau^{\sigma-1} \; \varphi_\tau \; y_\tau\right\}}{E\left\{\sum_\tau (\theta\beta)^\tau u_c \; P_\tau^{\sigma-1} \; y_\tau\right\}}$$

or

$$E\left\{\sum_\tau (\theta\beta)^\tau u_c \; P_\tau^{\sigma-1} \; y_\tau\right\} \; P^* = \frac{\sigma}{\sigma-1} E\left\{\sum_\tau (\theta\beta)^\tau u_c \; P_\tau^{\sigma-1} \; \varphi_\tau \; y_\tau\right\}$$

- Approximating the left hand side gives the terms

$$E\left\{\sum_\tau \; (\theta\beta)^\tau \; \overline{U}_c \; \overline{P}^{\sigma-1} \; \overline{Y} \; \overline{P}^* \left[\widehat{u}_{c,\tau} + (\sigma-1)\widehat{p}_\tau + \widehat{y}_\tau + \widehat{p}^*\right]\right\}$$

Steady state values $\overline{U}_c$, $\overline{P}$ etc are common to all terms in the sum

- Approximating the rigth hand side yields

$$\frac{\sigma}{\sigma-1} E\left\{\sum_{\tau} (\theta\beta)^{\tau}\ \overline{U}_c\ \overline{P}^{\sigma-1}\ \overline{\varphi}\ \overline{Y}\ \overline{P}^{*}\left[\widehat{u}_{c,\tau}+(\sigma-1)\widehat{p}_{\tau}+\widehat{\varphi}_{\tau}+\widehat{y}_{\tau}+\right]\right\}$$

- Because in St St $\overline{P}^{*}=\frac{s}{s-1}\ \overline{\varphi}$ we can cancell all the common terms so

$$E\left\{\sum_{\tau} (\theta\beta)^{t}\ \widehat{p}^{*}\right\}=\widehat{p}^{*}\ E\sum_{\tau} (\theta\beta)^{t} \simeq E\left\{\sum_{\tau} (\theta\beta)^{\tau}\ \overline{\varphi}_{\tau}\right\}$$

- Calculating the sum yields $\widehat{p}^{*} \simeq (1-\theta\beta)\left\{\sum_{\tau} (\theta\beta)^{\tau}\ \overline{\varphi}_{\tau}\right\}$
- And Adding back in Steady State terms yield

$$\widehat{p}^{*}=\mu+(1-\theta\beta)\ E\left\{\sum_{\tau}(\theta\beta)^{\tau}\ \left[mc_{\tau}+p_{\tau}\right]\right\}$$

where log mark $\mu=\log\frac{\sigma}{\sigma-1}$ and where mv_{τ} is log real marginal cost

Extreme Value Shocks

The General Problem 1: Estimation of Discrete Choice

- Let an agent have I choices that yield utility.
- Let the cost (or something else) of each choice be z^i, with vector $z = \{z^i\}_{i=1}^I$.
- We want to make sense of
  1. Percentage of choices being $x^i(z)$
  2. For various vectors of prices z (so that we have a theory of changes of behavior). In particular to learn about elasticity.
- Let $u_i + v(c)$ be fundamental utility of choice i where c is other consumption.
- Let ϵ^i be an idyosincratic shock to each agent. then

$$\max_i \{u^i + \epsilon^i + v(y - z^i)\} = \max_i \{u^i + \beta z^i + \epsilon^i\}$$

- If ϵ^i extreme value Gumbel then the probability of i, p^i is logit

$$p^i = \frac{\exp^{u^i + \beta z^i}}{\sum_{j=1}^I \exp^{u^j + \beta z^j}}$$

- This is now estimated (ML). Estimation should include Variance of shocks.
- Problem of correlated choices (blue/red bus). A Solution is to nest.

- Savings (or Durables, retirement, quits, marriage and so on).
- In one period normally $\max\left\{u(y,0), u(y-q,1)\right\}$
- If separable and strictly concave, solution is to do 0 for $y < \overline{y}$ and 1 for $y \geq \overline{y}$, implying a drop in c.
- The problem is that discontinuities propagate in time. A solution is to pose Extreme Value Shocks e.g. (without adjustment costs)

$$
\begin{aligned}
V(s,a) = \max\ &\left\{V^0(a), V^1(a)\right\} = \\
&\max\left\{\max_{a'}\ u(aR+s-a',0)+\epsilon^0+E\ V(s',a'),\right. \\
&\qquad\left.\max_{a'}\ u(aR+s-a'-q,1)+\epsilon^1+E\ V(s',a')\right\}
\end{aligned}
$$

- This gets rid of kinks and discontinuities as both choices are always possible for any a. But can cause problems.

- If ϵ follows i.i.d. $G(\mu, \alpha)$, where the mode μ is non-zero, we have

$$V^1 = E\{\epsilon\} = \mu + \alpha\ \gamma$$

$$\text{Mode}\ \ \{\epsilon\} = \mu$$

$$\text{Median}\{\epsilon\} = \mu - \alpha\ \ \ln(\ln 2)$$

$$\text{Var}\{\epsilon\} = \frac{\pi^2\ \alpha^2}{6}$$

$$\text{cdf}\{\epsilon\} = e^{\left\{-e^{\left[-\frac{(\epsilon-\mu)}{\alpha}\right]}\right\}}$$

- Expected maximum of N Gumbel random variables $G(\mu, \alpha)$. Let

$$X^N = \max\left\{\epsilon^1, \epsilon^2, \cdots, \epsilon^N\right\}$$
$$V^N = \mathbb{E}\left[X^N\right]$$

- We have

$$X^N \sim G\left(\mu + \alpha \; \ln N, \alpha\right)$$

$$V^N = \mu + \alpha \; \ln N + \alpha \; \gamma$$

- To make V^N independent of the number of choices N, either

$$V^N = \bar{V} \Rightarrow \alpha(N) = \frac{\bar{V} - \mu}{\gamma + \ln N}$$

$$V^N = \bar{V} \Rightarrow \mu(N) = \bar{V} - \alpha \; \ln N - \alpha \; \gamma$$

- η^i follows $G(\mu, \alpha)$, let $\epsilon^i = \eta^i + \delta^i$, $\epsilon^i \sim G(\mu + \delta^i, \alpha)$.

$$X^N \sim G\left(\alpha \ln \sum_i e^{\frac{\mu^i}{\alpha}}, \alpha\right) = G\left(\mu + \alpha \ln \sum_i e^{\frac{\delta^i}{\alpha}}, \alpha\right)$$

$$V^N = \mu + \alpha \ln \sum_i e^{\frac{\delta^i}{\alpha}} + \alpha\, \gamma$$

- To make V^N independent of the number of choices, we can require

$$V^N = \bar{V} \Rightarrow \alpha(N) = \frac{\bar{V} - \mu}{\gamma + \ln \sum_i e^{\frac{\mu^i}{\alpha(N)}}}$$

$$V^N = \bar{V} \Rightarrow \mu(N) = \bar{V} - \alpha\left[\gamma + \ln \sum_i e^{\frac{\mu^i}{\alpha}}\right]$$

- No closed-form solution for $\alpha(N)$

The continuum

- Consider an interval $C = [0, \bar{c}]$, and an $\epsilon(c), \forall c \in C$. We want

$$V^C = E\left\{\max_{c \in C} \{\epsilon(c)\}\right\}, \quad \epsilon(c) \sim G(0, \alpha(C)), \qquad \text{for some} \quad V^C > 0.$$

- We proceed by instead letting N draws in an equal sized grid over C and associating to each $n \in \{1, 2, \cdots, N\}$ a Gumbel $\epsilon^n \sim G(0, \alpha(N))$.

- Let $X^N = \max_{n \in \{1,2,\cdots,N\}} \{\epsilon^n\}$ and $V^N = E\left\{X^N\right\}$.

- We choose $\alpha(V^C, N)$ so that $V^N = V^C$: $\qquad \alpha\left(V^C, N\right) = \frac{V^C}{\ln N + \gamma}$ for any N.

- As we have seen, V^N is increasing in N. So no good to set μ so that $V^1 = 0$. More choice gives more utility.

- Is this fundamental?

- It depends. But if it is, there is a form of precautionary savings: Agents want to save to have more choice (a larger choice set C) in the future.

- Violating the Euler equation by choice becomes a valuable privilege.

- If so we have to design algorithms that respect this feature.

- We have to think of V^C as a fundamental parameter that determines the size of the utility bonus for the richest agent (the one with the largest choice set).

How to choose for a poorer agent $\widetilde{c} < \overline{c}$

- Let $\widetilde{c} < \overline{c}$, $[0, \widetilde{c}]$ a smaller choice set.

- Let $N^{\widetilde{c}} = \max_{n \geq 0} \frac{n}{N^{\overline{c}}} < \frac{\widetilde{c}}{\overline{c}}$, the point to the left of an imagined grid of size $N^{\overline{c}} + 1$.

- Then we associate with choice set $C^{\widetilde{c}}$, a draw of $N^{\widetilde{c}}$ ϵ's with probability $\underline{p}(\widetilde{c}) = \frac{N^{\widetilde{c}}+1}{N^{\overline{c}}} - \frac{\widetilde{c}}{\overline{c}}$, and a draw of $N^{\widetilde{c}} + 1$ with probability $\overline{p}(\widetilde{c}) = \frac{\widetilde{c}}{\overline{c}} - \frac{N^{\widetilde{c}}}{N^{\overline{c}}}$.

- Drawing zero ϵ yields expected utility 0.

- Let $$V^{\widetilde{c}} = \underline{p}(\widetilde{c}) V^{N^{\widetilde{c}}} + \overline{p}(\widetilde{c}) V^{N^{\widetilde{c}}+1}.$$

- Where $$V^{n} = \alpha \left(V^{\overline{c}}, N^{\overline{c}} \right) (\ln n + \gamma), \ \text{ for } n = N^{\widetilde{c}}, N^{\widetilde{c}} + 1.$$

- Note that the utility bonus $V^{\overline{c}}$ is of the right size given V^{C}.

- When writing algorithms, we have to be aware that the density of grid points is not the same as the size of the choice set $[0, \overline{c}]$.
- So we have to adjust for both.
  1. Find $\widetilde{c}(j)$, the maximum consumption attainable in state j. It depends on the budget constraint and prices not on the details of the grid.
  2. Compute $V^{\widetilde{c}(j)}$ as explained above
  3. Find the appropriate $\alpha(V^{\widetilde{c}}, j)$. This requires
     - Find $M(x)$ this is the number of grid points accessible from x. This depends on the grid system but also on prices.
     - Solve for $\alpha(V^{\widetilde{c}}, x) = \frac{V^{\widetilde{c}(x)}}{\ln M(x) + \gamma}$.
- Now you can iterate on the value function that includes the utility bonus.

Agents in Aiyagari worlds with Extreme Value Shocks

- The fundamental problem

$$v(s,a) = \max_{a',c=sw+aR-a'} \left\{ \frac{c^{1-\sigma}-1}{1-\sigma} + \epsilon(c) + \sum_{s'} \Gamma_{s,s'}\ v(s',a') \right\}$$

- Fix N, a large integer, we approximate the problem by

$$v(s,a) = \max_{a^{n\prime}=sw+aR-c^n,c^n} \left\{ \frac{c^{1-\sigma}-1}{1-\sigma} + \epsilon^n + \sum_{s'} \Gamma_{s,s'}\ v(s',a^{n\prime}) \right\}$$

We have to impute the right probabilities

Macro and COVID-19

- Short Horizons (No investment)
- Choose what Issues to Worry About
    1. *Mitigation Policy and Heterogeneity Age/Sector*
- Choose wich Allocation Mechanism to Model (large externality)
    1. All Econ choices are Government choices

- All variables are shares of a measure 1 population
- Three health states, $j \in \{s, i, r\}$ susceptible, infected, recovered or dead, with associated population shares S, I, R. Initial conditions $S(0), I(0), R(0)$.
- Two parameters: β governs rate of infection, κ the rate of recovery (or death)
- System of differential Equations

$$\begin{aligned} \dot{S}(t) &= -\beta S(t)I(t) \\ \dot{I}(t) &= \beta S(t)I(t) - \kappa I(t) \\ \dot{R}(t) &= \kappa I(t) \end{aligned}$$

- Basic Reproduction Number: define $R_0 = \frac{\beta}{\kappa}$

- Growth rate of infections given by $\frac{\dot{I}(t)}{I(t)} = \beta S(t) - \kappa$
- Let $I(0) = \epsilon, S(0) = 1 - I(0)$, when $\epsilon > 0$ is very small, $S(0) \approx 1$.
- Since $\dot{S}(t) = -\beta S(t) I(t)$ and for t close to zero,

  $I(t) \approx 0, S(t) \approx 1$, then $\dot{I}(t)/I(t)$ is roughly constant and equal to

  $$\dot{S}(t) = -\beta S(0) I(0) \qquad \text{So}$$

  $$I(t) = I(0) e^{\kappa\left(\frac{\beta}{\kappa} S(0) - 1\right)} \approx I(0) e^{\kappa\left(\frac{\beta}{\kappa} - 1\right)}$$
- If $R_0 = \frac{\beta}{\kappa} > 1$ exponential growth early (if $I(0) > 0$).
- If $R_0 = \frac{\beta}{\kappa} < 1$ then infections fall to zero and epidemic disappears immediately.

- The Ratio of differential equations: $\frac{\dot{I}(t)}{\dot{S}(t)} = -1 + \frac{1}{R_0 S(t)}$
- Integrating yields $I(t) = -S(t) + \frac{\ln(S(t))}{R_0} + q$

  where q is a constant of integration that does not depend on time.
- Evaluating at $t = 0$ yields (using $R(0) = 0$, thus $S(0) + I(0) = 1$

$$q = 1 - \frac{\ln(S(0))}{R_0}$$

- What is $S(\infty) = S^{\star}$? share of the population never to get infected
- Evaluating at $t = \infty$ and using the fact that $I(\infty) = 0$ yields

$$S^{\star} = 1 + \frac{\ln\left[S^{\star}/S(0)\right]}{R_0}$$

- Steady state satisfies the trascendental equation:

$$S^{\star} = 1 + \frac{\ln\left[S^{\star}/S(0)\right]}{R_0}$$

and $R^{\star} = 1 - S^{\star}, I^{\star} = 0$.

- If $R_0 > 1$ and $S(0) < 1$, $\exists$ a unique long-run S^*.

  Strictly decreasing in R_0 and strictly increasing in $S(0)$.

- For $R_0 \approx 1$ (but > 1), $S^{\star} = \frac{1}{R_0}$ and $R^{\star} = \frac{R_0 - 1}{R_0}$

  This approximation (a first good rule of thumb) uses $S(0) \approx 1$ and

$$\ln(1/R_0) = -\ln(R_0) = -\ln(1 + R_0 - 1) \approx 1 - R_0.$$

- With **costly** transfers across agents
- To Assess combination of two policies
  - Shutdown / mitigation (less output but also less contagion)
  - Redistribution toward those whose jobs are shuttered
- Characterize optimal policy
- Key interaction:
  - Mitigation creates the need for more redistribution
  - But if redistribution is costly, want less mitigation
  - Need heterogeneous-agent model to analyze this

- Stage of the disease
  - **S**usceptible
  - Infected **A**symptomatic
  - Infected with **F**lu-like symptoms
  - Infected and needing **E**mergency hospital care
  - **R**ecovered (or dead)
- Worst case disease progression: $S \to A \to F \to E \to D$
- But *R*ecovery is possible at each stage
- Three infected types spread virus in different ways:
  - A at work, while consuming, at home
  - F at home
  - E to health-care workers

- Age $i \in \{y, o\}$
  - Only young work
  - Old have more adverse outcomes conditional on contagion
  - But young more prone to contagion (they work)
- Sector of production $\{b, \ell\}$
  - Basic (health care / food production etc.)
    - Will never want shut-downs in this sector
    - Workers in this sector care for the hospitalized
  - Luxury (restaurants, entertainment etc.)
    - Workers in this sector face shutdown unemployment risk
    - But they are less likely to get infected

- Mitigation
  - Reduces contagion
  - Reduces risk of hospital overload
  - Reduces average consumption
  - Increases inequality (more unemployment in shuttered sectors)
- Redistribution
  - Helps the unemployed $\Rightarrow$ makes mitigation more palatable
  - But redistribution is costly $\Rightarrow$ makes mitigation more expensive
- What policy time paths do different types prefer? When (and how much) to shut down, when to open up? Size of Coronavirus check?
- How does the utilitarian optimal policy vary with the cost of redistribution?

- Lifetime utility for *old*

$$E\left\{\int e^{-\rho_o t}\ \left[u^o(c_t^o)+\bar{u}+\widehat{u}_t^j\right]\ dt\right\}$$

  - ρ_o: time discount rate
  - $u^o(c_t^o)$ instantaneous utility from old age consumption c_t^o
  - $\bar{u}$: value of life
  - $\widehat{u}_t^j$: intrinsic utility from health status j (zero for $j \in \{s, a, r\}$)

- Similar lifetime utility for *young*.
- Differences in expected longevity through $\rho_y \neq \rho_o$ (no aging)

- Young permanently assigned to b or ℓ
- Linear production: output equals number of workers
- Only workers with $j \in \{s, a, r\}$ work
- Output in basic sector:

$$y^b = x^{ybs} + x^{yba} + x^{ybr}$$

- Output in luxury sector is

$$y^\ell = [1 - m] \left(x^{y\ell s} + x^{y\ell a} + x^{y\ell r} \right)$$

- Total output given by

$$y = y^b + y^\ell.$$

- Fixed amount of output $\eta\Theta$ spent on emergency health care
- Θ measures capacity of emergency health system, η its unit cost

- Types of transmission
  - Work: young S workers infected by A workers, prob $\beta_w(m)$
  - Consumption: young & old S infected by A, prob $\beta_c(m) \times y(m)$
  - Home: young & old S infected by A and F, prob β_h
  - ER: basic S workers infected by E, prob β_e
- Shutdowns (mitigation) help by:
  - Reducing workers $\Rightarrow$ less workplace transmission
  - Reducing output $y(m) \Rightarrow$ less consumption transmission
  - Reducing infection-generating rates $\beta_w(m)$ & $\beta_c(m)$

$$\beta_w(m) = \frac{y^b}{y(m)}\alpha_w + \frac{y^\ell(m)}{y(m)}\alpha_w(1-m)$$

- Similar for $\beta_c(m)$
- Micro-founded via sectoral heterogeneity in social contact rates
- Smart mitigation shutters most contact-intensive sub-sectors first

Flow into asymptomatic (out of susceptible)

$$
\begin{aligned}
\dot{x}^{ybs} &= -\beta_w(m)\left[x^{yba} + (1-m)x^{y\ell a}\right]x^{ybs} \\
&\quad - \left[\beta_c(m)x^a y(m) + \beta_h\left(x^a + x^f\right) + \beta_e x^e\right]x^{ybs} \\
\dot{x}^{y\ell s} &= -\left[\beta_w(m)\left[x^{yba} + (1-m)x^{y\ell a}\right](1-m)x^{y\ell s}\right] \\
&\quad - \left[\beta_c(m)x^a y(m) + \beta_h\left(x^a + x^f\right)\right]x^{y\ell s} \\
\dot{x}^{os} &= -\left[\beta_c(m)x^a y(m) + \beta_h\left(x^a + x^f\right)\right]x^{os}
\end{aligned}
$$

- For each type $j \in \{yb, y\ell, o\}$

$$\dot{x}^{ja} = -\dot{x}^{js} - \left(\sigma^{jaf} + \sigma^{jar}\right) x^{ja}$$
$$\dot{x}^{jf} = \sigma^{jaf} x^{ja} - \left(\sigma^{jfe} + \sigma^{jfr}\right) x^{jf}$$
$$\dot{x}^{je} = \sigma^{jfe} x^{jf} - \left(\sigma^{jed} + \sigma^{jer}\right) x^{je}$$
$$\dot{x}^{jr} = \sigma^{jar} x^{ja} + \sigma^{jfr} x^{jf} + (\sigma^{jer} - \varphi) x^{je}$$
$$\varphi = \lambda_o \max\{x^e - \Theta, 0\}.$$

- All flow rates σ vary by age

- $x^e - \Theta$ measures excess demand for emergency health care. Reduces flow of recovered (Increases flow into death)

- Costly transfers between workers, non-workers (old, sick, unemployed)
- Utilitarian planner (or taxes / transfers that cannot depend on age, sector, health)
  - $\Rightarrow$ Workers share common consumption level c^w
  - $\Rightarrow$ Non-workers share common consumption level c^n
- Define measures of non-working and working as

$$\begin{aligned} \mu^n &= x^{y\ell f} + x^{y\ell e} + x^{ybf} + x^{ybe} + m\left(x^{y\ell s} + x^{y\ell a} + x^{y\ell r}\right) + x^o \\ \mu^w &= x^{ybs} + x^{yba} + x^{ybr} + [1-m]\left(x^{y\ell s} + x^{y\ell a} + x^{y\ell r}\right) \\ \nu^w &= \frac{\mu^w}{\mu^w + \mu^n} \end{aligned}$$

- Aggregate resource constraint

$$\mu^w c^w + \mu^n c^n + \mu^n T(c^n) = \mu^w - \eta\Theta$$

where $T(c^n)$ is per-capita cost of transferring c^n to non-workers

- Consumption allocation does not affect disease dynamics $\Rightarrow$ optimal redistribution is a static problem
- With log-utility and equal weights, period social welfare given by

$$W(x,m) = \max_{c^n, c^w} [\mu^w \log(c^w) + \mu^n \log(c^n)] + (\mu^w + \mu^n)\bar{u} + \sum_{i,j \in \{f,e\}} x^{ij} \widehat{u}^j$$

- Maximization subject to resource constraint gives $\frac{c^w}{c^n} = 1 + T'(c^n)$.
- Period welfare

$$\begin{aligned} W(x,m) &= [\mu^w + \mu^n]\, w(x,m) \\ w(x,m) &= \log(c^n) + \nu \log(1 + T'(c^n)) + \bar{u} + \sum_{i,j \in \{f,e\}} \frac{x^{ij}}{\mu^w + \mu^w} \widehat{u}^j \end{aligned}$$

- Assume $\mu^n T(c^n) = \mu^w \frac{\tau}{2} \left(\frac{\mu^n c^n}{\mu^w}\right)^2$

- Optimal allocation

$$\begin{aligned} c^n &= \frac{\sqrt{1 + 2\tau \frac{1-\nu^2}{\nu} \tilde{y}} - 1}{\tau \frac{1-\nu^2}{\nu}} \\ c^w &= c^n(1 + T'(c^n))) = c^n \left(1 + \tau \frac{1-\nu}{\nu} c^n\right) \end{aligned}$$

  Where $\tilde{y} = \nu - \frac{\eta\Theta}{\mu^w + \mu^n}$.

- $\left(1 + \tau \frac{1-\nu}{\nu} c^n\right)$ is the effective marginal cost (MC) of transfers.

- It increases with c^n and τ, decreases with share of workers ν

- Higher mitigation m reduces ν, thus increases MC

- $\Rightarrow$ policy interaction between m, τ.

Endogenous Growth and R&D

- Exogenous Growth

$$F(K, N) = A\, K^{\theta_1}\, L^{\theta_2},$$

- Need $\theta_1 + \theta_2 \leq 1$ for consistency with the notion of competitive equilibrium. (Even $<$ is a bit problematic).
- Then the economy cannot grow in per capita terms.
- So it has to be A: Exogenous
- Still, empirically, the problem is NOT accounting for **growth rate** differences but for output **LEVEL** differences

- The AK model: technology is linear in reproducible capital (it can include human capital as long it is accumulated by using reproducible factors, i.e. schools not time.

- The existence of externalities in production. Consider a firm production function with an aggregate externality:

$$F(k,n) = A\ K^{1-\theta_1}\ k^{\theta_1}\ n^{\theta_2},$$

- An explicit accumulation of technology

- Three sectors in the economy.
  1. Final goods are competitive use labor and intermediate goods according to
  $$N_{1,t}^{\alpha}\int_0^{A_t} x_t\left(i\right)^{1-\alpha} di$$
  where $x\left(i\right)$ denotes the utilization of intermediate good of variety $i \in [0, A_t]$.
  2. Intermediate producers are monopolists. They have a differentiated technology of the form:
  $$x\left(i\right) = \frac{k\left(i\right)}{\eta}.$$
  Note: aggregate demand of capital is $\int_0^{A_t} \eta x\left(i\right) di$.
  3. R&D sector. A new good is a new variety of the intermediate good produced using labor:
  $$\frac{A_{t+1}}{A_t} = 1 + \xi N_{2,t}.$$
  we can write $\quad A_{t+1} - A_t = A_t \xi N_{2,t}, \quad$ so the flow of new intermediate goods is determined by the current stock of them in the economy (an externality).

     Right to produce new goods sold to new monopolists.

Remark

The reason we see A_t on the previous expression as an externality is that it is indeed used as an input in the process of R&D, while, it is not paid for. Thus, inventors, in a sense, do not pay the investors of the previous varieties, while using their inventions. They only pay for the labor they hire. Perhaps, the basic idea of this production function might be traced back to Isaac Newton's quote: "If I have seen further, it is only by standing on the shoulders of giants".

Exercise

If the price of all varieties are the same, what is the optimal choice of input vector for a producer?

Exercise

What if they do not have the same amount? Would a firm decide not to use a variety in the production?

- Preferences

$$\sum_{t=0}^{\infty} \beta^t u(c_t),$$

- Budget constraint

$$c_t + k_{t+1} \leq r_t k_t + w_t + (1-\delta) k_t + \pi_t.$$

Remark

In this economy, GDP is $Y_t = W_t + r_t K_t + \pi_t,$ *where* π_t *are profits.*
In terms of expenditures, GDP is $Y_t = C_t + K_{t+1} - (1-\delta) K_t + \pi_t,$ *where* $K_{t+1} - (1-\delta) K_t$ *is the investment in physical capital. In terms of value added, it is* $Y_t = N_t^{\alpha} \int_0^{A_t} x_t(i)^{1-\alpha} di + p_t (A_{t+1} - A_t)$.

- Not a model that maps well to the data, yet carefully crafted to convey ideas.

- Final good producer; it chooses $N_{1,t}$ and $x_t(i)$, $\forall i \in [0, A_t]$,

$$\max \; N_{1,t}^{\alpha} \int_0^{A_t} x_t(i)^{1-\alpha} \, di - w_t N_{1,t} - \int_0^{A_t} q_t(i) \, x_t(i) \, di,$$

where $q_t(i)$ is the price of variety i in period t. First order conditions are:

1. $N_{1,t}$: $\alpha N_{1,t}^{\alpha-1} \int_0^{A_t} x_t(i)^{1-\alpha} \, di = w_t$; and,
2. $x_t(i)$: $(1-\alpha) N_{1,t}^{\alpha} x_t(i)^{-\alpha} = q_t(i)$, for all $i \in [0, A_t]$.

- Note the monopolistic competition type of condition

$$x_t(i) = \left(\frac{(1-\alpha)}{q_t(i)} \right)^{\frac{1}{\alpha}} N_{1,t},$$

- which, given N_{1t}, is the *demand function* for variety i, by the final good producer.

- $$\pi_t(i) = \max_{\{q_t(i)\}} \quad q_t(i)\, x_t(q_t(i)) - r_t \eta x_t(q_t(i))$$

  $$s.t. \quad x_t(q_t(i)) = \left(\frac{(1-\alpha)}{q_t(i)}\right)^{\frac{1}{\alpha}} N_{1,t},$$

  we substituted for the technology of the monopolist, $x(i) = k(i)/\eta$.

- FOC wrt to $q_t(i)$, is $x_t(q_t(i)) + (q_t(i) - r_t\eta)\frac{\partial x_t(q_t(i))}{\partial q_t(i)} = 0$, which implies

  $$\frac{(1-\alpha)^{\frac{1}{\alpha}}}{q_t(i)^{\frac{1}{\alpha}}} N_{1,t} = \frac{(q_t(i) - r_t\eta)}{\alpha} \frac{(1-\alpha)^{\frac{1}{\alpha}}}{q_t(i)^{\frac{1+\alpha}{\alpha}}} N_{1,t}.$$

- Rearranging yields $q_t(i) = \frac{1}{(1-\alpha)} r_t \eta$ and substituting

  $$x_t(i) = \left[\frac{(1-\alpha)^2}{r_t\eta}\right]^{\frac{1}{\alpha}} N_{1,t},$$

  and the demand for capital services is simply $\eta x_t(i)$.

- In a symmetric equilibrium $\int_0^{A_t} x_t(i)\, di = A_t x_t = \frac{k_t}{\eta}$,
- Therefore $x_t = \frac{k_t}{\eta A_t}$.
- let Y_t be the production of the final good

$$Y_t = N_{1,t} A_t \left[\frac{(1-\alpha)^2}{r_t \eta} \right]^{\frac{1-\alpha}{\alpha}}.$$

- Hence the model displays constant returns to scale in $N_{1,t}$ and A_t.
- A representative competitive firm chooses $N_{2,t}$ to solve

$$\max_{N_{2,t}} \quad p_t A_t \xi N_{2,t} - w_t N_{2,t}.$$

- With FOC $p_t = \frac{w_t}{A_t \xi}$.

1. Intertemporal Euler equation:

$$u'(c_t) = \beta u'(c_{t+1}) [r_{t+1} + (1 - \delta)] .$$

2. allocates labor demand for R&D, and that for final good production. For determining the labor choices $N_{1,t}$ and $N_{2,t}$. Note that as long as there are profits in the intermediate good sector, new monopolists will enter yielding a zero profit condition:

$$p_t = \sum_{s=t}^{\infty} \left(\prod_{\tau=t}^{s} \frac{1}{1 + r_\tau - \delta} \right) \pi_s .$$

3. Output can grow at the same rate as A_t and as K_t.
4. Growth comes from the externality in the R&D sector. Without that, we cannot get sustained growth in this model.
5. This model neatly delivers balanced growth, with just enough structure.

Overlapping Generations

- Every period there is death and birth of agents.
- We want birth to have new agents be different than existing agents, e.g. poor.
- We want death to prevent certain things such as excessive wealth accumulation.
- We may also want an inefficient economy (the interest rate is too low) and OLG's are natural.
    - May also happen in Aiyagari type economies Aguiar, Amador, and Arellano (2021)
- We may just want to be realistic about the finite nature of the length of life.

- Agents live up to I period
- They own assets A_i,
    - $A_1 = A_{I+1} = 0, \quad \sum_i A_i \mu_i = K$. We may consider different cohort sized μ_i.
- Standard Recursive Representation with State $\{A_2, \cdots, A_i, A_I\}$.
- Many Bells and Whistles are possible.

- Simplest Case, Example Economy.
- $I = 2$, No Storage. Endowment $\{\omega^y, \omega^o\}, \omega^y > \omega^o$.
- $u(c^y, c^o) = \log c^y + \log c^o$
- What happens? Nobody to trade with. So autarky?
- Perhaps there is Money as a store of Value.
- Consider

$$m_t = \frac{\omega^y - c_t^y}{p_t}$$

$$c_{t+1}^o + = \frac{m_t}{p_{t+1} + m_t}$$

- Many Monetary Equilibria $M_t = 1$

- Solutions to a difference equation

$$\frac{\omega^o + \frac{1}{p_{t+1}}}{\omega^y - \frac{1}{p_t}} = \frac{p_{t+1}}{p_t}$$

- A stationary one is $\frac{1}{p^*} = \frac{\omega^y - \omega^o}{2}$.

- There are many more with $P_0 > P^*$, converging to ∞

- Still, Why accept Money from older agents? Who needs them?

Growth Model with Many Firms Suitable for Pandemic Times

- This is a growth model suitable to study business cycles.
- Emphasis on small business creation not on inequality so rep hholds.
- Creation and destruction of small firms both for technological and for financial reasons.
- Household cannot help its small businesses in distress.
- We have in mind that even though Pandemic affects both Supply (want less work) and Demand (Less consumption) there is a reduction in output sold per unit of good produced of $\phi(S)$.

- Two sectors as in Quadrini (2000): Corporate and non corporate sector.
- Corporate sector uses capital and labor via aggr prod fn $F(K,N)$
- Non corporate sector: type/size firms $i \in \{1,\cdots,I\}$, $f^i(n)$, $f^i_n > 0$, (provided the firm has the required number of managers, λ^i).
- A firm requires creation: It costs ξ^i to open a new firm of size i.
- Some Firms are destroyed.
  - Firms invest m in maintenance.
  - Probability that a firm survives is $q^i(m)$, $q^i(0)=0$, $q^i(\infty)<1$, $q^i_m > 0$.
- Aggregate measure of type i firms is X_i
- The law of motion of new firms is:

$$X_i' = q^i(M_i)\; X_i + B_i$$

- The Aggregate Feasibility Constraint is

$$C + [K' - (1-\delta)K] + \sum_i X_i \; M_i + \sum_i B_i \; \xi_i = \sum_i X_i \; f_i(N_i) + F(K,N).$$

- Household owns measure x_i of firms of type $i \in \{1, \cdots, \mathcal{I}\}$
- The household may be rationed in its workforce: i.e. it may not be in its static Euler equation.
- Households create b^i new firms of type i at cost ξ^i each,
- Managers choose maintenance and profits.
- In addition to its firms, households own a units of corporate capital which they can increase by savings.
- Households allocate its members to managers, workers or enjoyers of leisure:

$$n + \sum_i \lambda^i x^i + \ell = 1.$$

(implicitly we are guessing (to be verified) that all business are operated).

- Households have preferences over consumption c and leisure ℓ, using utility function $u(c, \ell)$ and discounts the future at rate β.

- Small firms cannot access financing once they are born.
- They can only give benefits to the household:

$$\Omega^i(S) = \max_{n \geq 0, m \leq \psi(S) f^i(n) - w\ n} \psi(S)\ f^i(n) - w\ n - m + \frac{q^i(m)}{R(S')}\ \Omega^i(S')$$

Here, S is the aggregate state and s in the individual state, $\Psi(S) < 1$ is capacity used which is demand determined and $R(S')$ is the rate of return used by the firm.

- Implictly assuming that there is no need to index $\Omega^i(S)$ by s.

Exercise

Get the FOC assuming first that m is unrestricted and then that $m \leq \psi(S) f^i(n) - w\ n$.

$$V(S, a, x_1, \cdots, x_I) = \max_{c,n,b_1,\cdots,b_I,a'} \; u(c,\; 1 - n - \sum_i \lambda^i \; x^i) + \beta \; V(S', a', x_1', \cdots, x_I') \qquad s.t.$$

$$c + \sum_i \; b_i \; \xi_i + a' = n \; w(S) + a \; R(S) + \sum_i \; \pi_i(S) \; x_i$$

$$x_i' = q^i(M_i) \; x_i + b_i \qquad i \in \{1, \cdots, I\}.$$

Exercise

Get the FOCs for b^i a' and n assuming first that $\lambda^i = 0$ and $\pi^i > 0$ and charaterize the solution (the relation between the FOC of b^i, m^i and a'). Then characterize the FOC when $\lambda^i > 0$.

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