# Macro 7210 Lectures

Preliminary

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# Introduction

- Study of Aggregates
  - You should not leave anyone behind
- Methods: General Equilibrium
- Dynamics
- Timely issues (Great Recession, COVID)

- A model is an artificial economy used to ask questions.
- The description of a model's environment includes specifying agents' preferences and endowments, technology available, information structure as well as property rights.
- The workhorse model in Macro is the Neoclassical Growth Model.
- It delivers some fundamental properties that are characteristics of industrialized economies. Kaldor (1957) summarizes six (plus one) stylized facts.

- A model requires an equilibrium concept.
- Equilibrium is a prediction of what will happen in the economy, i.e. a mapping from environments to outcomes (allocations, prices, etc.).
- One such equilibrium concept is Competitive Equilibrium (CE).
- Characterizing equilibrium usually involves finding solutions to a system of an infinite number of equations. Three ways around it

  1. To invoke the first welfare theorem to solve for the allocation and then find the equilibrium prices associated with it (not so general: market incompleteness, externalities, distortions, heterogeneity (Negishi)).
  2. Construct the equilibrium (not good to learn about the world)
  3. Recursive Competitive Equilibrium (RCE) directly.

1. Output per capita has grown at a roughly constant rate
2. The capital-output ratio has remained roughly constant (capital measured using the perpetual inventory method)
3. Consumption as a fraction of output has been roughly constant
4. The Wage rate has grown at roughly the same rate as output
5. The real interest rate has roughly constant over a long period of time
6. Labor income as a share of output has remained roughly constant
7. Hours worked per capita have been roughly constant.

1. Exogenous Technical Change (there is no systematic variation of growth rates that really calls for a theory of growth rates)
2. Cobb-Douglas Technology not other (ot at least aggregates to Cobb-Douglas)
3. Balanced growth Preferences
    - Cobb-Douglas:

$$u(c,\ell) = \frac{\left[c^{\theta}\ \ell^{1-\theta}\right]^{1-\sigma}}{1-\sigma}$$

    - Log plus Constant Frisch: :

$$u(c,1-\ell) = u(c,n) \log c + \chi \frac{n^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}}$$

# Recursive Equilibria without Distortions

- A natural extension of the dynamic programming problem.
- It requires the definition of state variables
  - Aggregate $K$
  - Individual $a$
- In addition to decision rules we need
  - Pricing Functions (of aggregate variables)
  - Laws of motion of aggregate states
  - Equilibrium Conditions/ Representative Agent Conditions

- Aggregate State $K$ with law of motion $K' = G(K)$
- Individual State $a$
- Equilibrium Prices $w(K), \quad R(K)$

$$
\begin{aligned}
V(K, a; G) = \max_{c, a'} \quad u(c) \quad &+ \quad \beta V(K', a'; G) \\
\text{s.t. } c + a' \quad &= \quad w(K) + R(K)a \\
K' \quad &= \quad G(K), \\
c \quad &\geq \quad 0
\end{aligned}
$$

- $c = c(K, a; G)$, $a' = g(K, a; G)$, $V(K, a; G)$ satisfy (use envelope)

$$
u_c\left[c(K, a; G)\right] = \beta V_{a'}\left[G(K), g(K, a; G); G\right]
$$

$$
V_a(K, a; G) = R(K)\, u_c\left[c(K, a; G)\right]
$$

- The Rep Agent Equilibrium Condition requires

$$G(K) = g(K, K; G)$$

- The most convenient is to summarize all conditions by successive substitution
- Yields a functional equation in $K$ (after using marginal productivities)

$$u_c\left[w(K) + R(K)K - G(K)\right] =$$

$$\beta \; u_{c'}\{w[G(K)] + R[G(K)]G(K) - G[G(K)]\} \; R[G(K)]$$

- In this case we can use the $G(K)$ that comes out of the social planner's dynamic programming problem as the candidate for RCE.

# Economies with Distortions and Heterogeneity

- Wedges: Externalities, Governments, Heterogeneity
- Just define Equilibria directly.
- Lump sum Taxes $T(K)$ levied for Parks. Government has a period by period balance budget constraint.

$$\begin{aligned} V(K, a; T, P, G) = \max_{c \geq 0, a'} \quad & u[c, P(K)] \quad + \quad \beta V(K', a'; T, P, G) \\ \text{s.t.} \quad c + a' \quad & = \quad w(K) + R(K)a - T(K) \\ K' \quad & = \quad G(K) \end{aligned}$$

with solution $a' = g(K, a; T, P, G)$.

- Equilibrium requires

$$\begin{aligned} G^*(K) &= g(K, K; T, P^*, G^*), \\ P^*(K) &= T(K). \end{aligned}$$

- If labor income tax, substitute $T(K)$ with $\tau(K)\ w(K)$.

$$
\begin{aligned}
V(K, a; \tau, G) = \max_{c \geq 0, a'} \quad & u(c, P) \quad + \quad \beta \, V(K', a'; \tau, G) \\
\text{s.t.} \quad c + a' \quad & = \quad w(K) + a\left[1 + r(K)\left(1 - \tau(K)\right)\right] \\
K' \quad & = \quad G(K) \\
P \quad & = \quad P(K).
\end{aligned}
$$

- Eq Cond: $P^*(K) = \tau(K) r^*(K) K$, and $R(K) = 1 + r(K)$ plus Rep Agent.
- The First Welfare Theorem fails and the RCE is not Pareto optimal. (if $\tau(K) > 0$ there will be a wedge, and the efficiency conditions will not be satisfied).

**Exercise**

*Derive the first order conditions in the above problem to see the wedge introduced by taxes.*

- Aggregate State is $K$ and $B$
- Government policy (for now assume it can):

$$\tau\left(K,B\right),\ P\left(K,B\right)\ \text{and}\ B'\left(K,B\right).$$

- The government budget constraint reduces the degrees of freedom

$$B+P(K,B)=\tau(K,B)R(K)K+q(K,B)B'(K,B)$$

- The household does not care about the composition of his portfolio as long as assets have the same rate of return, which is true because of the no arbitrage condition.
- So individual state is just $a$

- The household needs to know the evolution of capital and *debt*

$$
\begin{aligned}
V(K,B,a) &= \max_{c\geq 0,a'} && u(c,P(K,B)) + \beta V(K',B',a') \\
&\text{s.t.} \quad c+a' &&= w(K)+aR(K)(1-\tau(K,B)) \\
& K' &&= G(K,B) \\
& B' &&= H(K,B)
\end{aligned}
$$

with solution $g(K,B,a)$.

## Definition

A *Rational Expectations Recursive Competitive Equilibrium given* $P(K,B)$ *and* $\tau(K,B)$*, are functions* $V$*,* $g$*,* $G$*,* $H$*,* $w$*,* $q$*, and* $R$*, s.t.*

1. *Given* $w$ *and* $R$*,* $V$ *and* $g$ *solve the household's problem,*
2. *Factor prices are paid their marginal productivities:* $w(K) = F_2(K,1)$ *and* $R(K) = F_1(K,1)$*.*
3. *Rep agent condition*

$$g[K,B,K+q(K^-,B^-)B] = G(K,B) + q(K,B)\ H(K,B),$$

4. *No arbitrage*

$$\frac{1}{q(K,B)} = [1-\tau(G(K,B),H(K,B))]\ \ R(G(K))$$

5. *Gov b constr:* $B + P(K,B) = \tau(K,B)R(K)K + q(K,B)H(K,B)$
6. *Government debt is bounded:* $\exists$ *some* $\bar{B}$*, such that for all* $K \in \left[0,\tilde{k}\right)$ *and* $B \leq \bar{B}$*,* $H(K,B) \leq \bar{B}$*.*

1. *Habit formation:* $u\left(c, c^{-}\right)$, *increasing in* $c$, *decreasing in* $c^{-}$ *(e.g.* $u\left(c, c^{-}\right) = v\left(c\right) - \left(c - c^{-}\right)^{2}$*). Agg. state* $\{K, C^{-}\}$, *individual* $\{a, c^{-}\}$.

**Exercise**

*Define it. Is the equilibrium optimum in this case?*

2. *Catching up with the Jones* $u\left(c, C^{-}\right)$. *Externality from aggregate consumption. Aggregate state* $\{K, C^{-}\}$, *while* $c^{-}$ *is not a state.*

**Exercise**

*How does the agent know* $C$*? Is the equilibrium optimum?*

3. *Keeping up with the Jones* $u\left(c, C\right)$:

**Exercise**

*How does the agent know* $C$*? Is the equilibrium optimum?*

- Rep firms buy and install capital; own one unit of land used to produce $F(K, L)$.
- Firm's shares are publicly traded and bought by households.
- Agg State is $K$, ind state is shares $a$. The hhold solves

$$\begin{aligned} V(K, a) = \max_{c, a'} \quad u(c) \quad &+ \quad \beta V[G(K), a'] \\ \text{s.t.} \quad c + P(K)a' \quad &= \quad a\left[D(K) + P(K)\right] \end{aligned}$$

- $P(K)$ is shares price; $D(K)$ dividends per share. Soltn, $a' = h(K, a)$
- Firm solves

$$\begin{aligned} \Omega(K, k) = \max_{d, k'} \quad d \quad &+ \quad q[G(K)]\, \Omega[G(K), k'] \\ \text{s.t.} \quad F(k, 1) \quad &= \quad d + k' \end{aligned}$$

- $d$ dividends (solution $d(K, k)$), $q\left[G(K)\right]$ is price of good tomorrow.

A Rec Comp Eq are functions, $V$, $\Omega$, $h$, $g$, $d$, $q$, $D$, $P$, $G$ so that:

1. Given prices, $V$ and $h$ solve the household's problem,
2. $\Omega$, $g$, and $d$ solve the firm's problem,
3. Representative household holds all shares: $h(K,1) = 1$
4. Rep Firm

$$\begin{aligned} F(K,K) - d(K,K) &= G(K) \\ d(K,K) &= D(K) \end{aligned}$$

5. Value of a representative firm equals price plus dividends

$$\Omega(K,K) = D(K) + P(K),$$

## Exercise

*Find missing condition. [Hint: it relates $q(G(K))$ with the price and dividends ($P(K), P(G(K))$, and $D(G(K))$).]*

## Exercise

*Define the RCE if a were savings paying $R(K)$ instead of shares.*

- Two types of households differing only in wealth: $R$ (rich) and $P$ (poor) with measures $\mu$ and $1-\mu$. Otherwise identical.

$$V(K^R, K^P, a) = \max_{c,a'} \quad u(c) \quad + \quad \beta V(K^{R'}, K^{P'}, a')$$

$$\text{s.t.} \quad c + a' \quad = \quad w\left[(\mu K^R + (1-\mu)K^P\right] + aR\left[\mu K^R + (1-\mu)K^P\right]$$

$$K^{i'} \quad = \quad G^i(K^R, K^P) \qquad \text{for } i = R, P.$$

**Remark**

*Decision rules are not linear (even though they might be almost linear); therefore, we need two states, $K^1$ and $K^2$, not aggregate $K$.*

### Definition

A Rec Comp Equil are functions $V$, $g$, $w$, $R$, $G^1$, and $G^2$ such that:

1. Given prices, $V$ and $g$ solve the household's probl
2. $w$ and $R$ are the marginal products of labor and capital, respectively
3. Consistency: representative agent conditions are satisfied, i.e.

$$
\begin{aligned}
g(K^R, K^P, K^R) &= G^R(K^R, K^P) \\
g(K^R, K^P, K^P) &= G^P(K^R, K^P).
\end{aligned}
$$

### Remark

*Note that $G^R\left(K^R, K^P\right) = G^P\left(K^P, K^R\right)$ (look at the arguments carefully). Why? (How are rich and poor different?)*

- In steady state, the Euler equations for the two types simplify to

$$u'\left(c^{R^*}\right) = \beta R \; u'\left(c^{R^*}\right), \text{ and } u'\left(c^{P^*}\right) = \beta R \; u'\left(c^{P^*}\right).$$

$$\text{so } \beta R = 1, \text{ where } R = F_K\left(\mu K^{R^*} + (1-\mu)K^{P^*}, 1\right).$$

- Using hhold's budget constraint and $a^i = K^i$ because of the rep agent's condition

$$c^i + a^i = w + a^i R \quad \text{for } i = R, P$$

- We have three equations (2 budget constraints and Euler equation) and four unknowns ($a^{i^*}$ and $c^{i^*}$ for $i = R, P$).
- The theory is silent about the steady state distribution of wealth!
- If savings are linear in $a$ state (i.e. $g(K, a) = \mu^i(K) + \lambda(K)a$, and all have the same preferences, then aggregate capital can be expressed as the choice of a representative agent (with savings decision given by $g(K, K) = \bar{\mu}(K) + \lambda(K)K$).

- Type $i$ has labor skill $\epsilon_i$, $\mu^1 = \mu^2 = 1/2$. $\mu^1\epsilon_1 + \mu^2\epsilon_2 = 1$.

- The value functions are now indexed by type:

$$
\begin{aligned}
V^i(K^1, K^2, a) = \max_{c, a'} \quad & u(c) \quad + \quad \beta V^i(K^{1'}, K^{2'}, a') \\
\text{s.t.} \quad c + a' \quad & = \quad w\left(\frac{K^1 + K^2}{2}\right)\epsilon_i + aR\left(\frac{K^1 + K^2}{2}\right) \\
K^{i'} \quad & = \quad G^i(K^1, K^2) \quad \text{for } i = 1, 2.
\end{aligned}
$$

with solution $g^i(K^1, K^2, a)$.

**Exercise**

*Define the RCE.*

**Remark**

*We can also rewrite this problem as*

$$
\begin{aligned}
V^{i}(K,\lambda,a) = \max_{c,a'} \quad & \left\{u(c) + \beta V^{i}\left(K',\lambda',a'\right)\right\} \\
s.t. \quad & c + a' = R(K)\,a + W(K)\,\epsilon_i \\
& K = G(K,\lambda) \\
& \lambda' = H(K,\lambda),
\end{aligned}
$$

*where $K$ is aggregate capital, and $\lambda$ is the share of type 1.*

*Then the consistency conditions of the RCE must be:*

$$
\begin{aligned}
G(K,\lambda) &= \frac{1}{2}\left[g^{1}(K,\lambda,2\lambda K) + g^{2}(K,\lambda,2(1-\lambda)K)\right], \\
H(K,\lambda) &= \frac{g^{1}(K,\lambda,2\lambda K)}{2G(K,\lambda)}.
\end{aligned}
$$

- Have to define what is a country.
  - A Place?
  - A Technology?
  - A Policy?
  - A set of Trade Restrictions?
- Today, two countries, 1 and 2, labor is immobile, but capital markets perfect. Traded goods flow within the period. Different technology.
- Aggregate resource constraint is:

$$C^1 + C^2 + K^{1'} + K^{2'} = F^1(K^1, 1) + F^2(K^2, 1)$$

- There are mutual funds that own all firms countries. They choose labor and installs capital. Shares are traded in the world market.
- *What are the appropriate aggregate states in this world?*
  - Capital in each country.
  - Need also a variable for wealth distribution, say, shares in country 1.

- Hhold Probl. $A$ are shares held by country 1 hholds. $a$ are own shares.

$$V^i\left(K^1, K^2, A, a\right) = \max_{c, a'(z)} \quad u(c) + \beta V^i\left(K^{1'}, K^{2'}, A', a'\right)$$

$$s.t. \quad c + Q(K^1, K^2, A)a' = w^i\left(K^i\right) + a\Phi(K^1, K^2, A)$$

$$K^{i'} = G^i\left(K^1, K^2, , A\right), \quad \text{for } i = 1, 2$$

$$A' = H\left(K^1, K^2, A\right)$$

- Mutual Funds' problem (note wages are country specific)

$$\Phi(K^1, K^2, A, k^1, k^2) \quad = \quad \max_{k^{1'}, k^{2'}, n^1, n^2} \sum_i \left[F^i(k^i, n^i) - n^i w^i(K_i) - k^{i'}\right] + \frac{1}{R(K^{1'}, K^{2'}, A)} \Phi(K^{1'}, K^{2'}, A', k^{1'}, k^{2'})$$

$$s.t. \qquad K^{i'} = G^i\left(K^1, K^2, , A\right), \quad \text{for } i = 1, 2$$

$$A' = H\left(K^1, K^2, A\right)$$

## Definition

Rec Comp Equil: $\left\{V^i, h^i, g^i, n^i, w^i, G^i\right\}_{i=1,2}$, $\Phi$, $H$, $Q$, and $R$, S.t.:

1. Given prices and aggregate laws of motion, $V^i$ and $h^i$ solve hholds' probl
2. Samo: $\Phi$, $\left\{g^i, n^i\right\}_{i=1,2}$ solve mutual funds' probl,
3. Labor markets clear $n^i(K^1, K^2, A, K^1, K^2) = 1 \qquad \text{for } i = 1, 2,$
4. Consistency (MF)

$$g^i(K^1, K^2, A, K^1, K^2) = G^i(K^1, K^2, A) \qquad \text{for } i = 1, 2,$$

5. Consistency (Households)

$$\begin{aligned} h^1(K^1, K^2, A, A) &= H(K^1, K^2, A) \\ h^1(K^1, K^2, A, A) + h^2(K^1, K^2, A, 1-A) &= 1 \end{aligned}$$

6. No arbitrage $Q(K^1, K^2, A) = \frac{1}{R(K^{1'}, K^{2'}, A')} \Phi(K^{1'}, K^{2'}, A', K^{1'}, K^{2'})$

## Exercise

*Solve for the mutual fund's decision rules. Is next period capital in each country chosen by the mutual fund priced differently? What about labor?*

# Overlapping Generations

- Every period there is death and birth of agents.
- We want birth to have new agents be different than existing agents, e.g. poor.
- We want death to prevent certain things such as excessive wealth accumulation.
- We may also want an inefficient economy (the interest rate is too low) and OLG's are natural.
    - May also happen in Aiyagari type economies Aguiar, Amador, and Arellano (2021)
- We may just want to be realistic about the finite nature of the length of life.

- Agents live up to $I$ period
- They own assets $A_i$,
  - $A_1 = A_{I+1} = 0, \quad \sum_i A_i \mu_i = K$. We may consider different cohort sized $\mu_i$.
- Standard Recursive Representation with State $\{A_2, \cdots, A_i, A_I\}$.
- The Rep Agent condition takes the form $G^{i+1}(A) = g^i(A, A_i)$
- Many Bells and Whistles are possible.

1. Hormones (i.e. exogenous)
2. Learning by doing (working more today increases your wage tomorrow)
3. Learning by not doing (i.e. Ben-Porath (1967)). From the time measure to be working a fraction is devoted to learn. Such time is not productive today.
4. Wage-Ladder: People start with low wage and over their lives the run into better opportunities and they switch. Occasionally the move down.
5. Wealth helps because either
    5.1 They are in a better bargaining position because wealth makes quitting better
    5.2 Rich people can afford to search longer because of a better match.
6. Age shapes your bargaining power and wages ensues (only really used at the end of life)
7. Various combinations of the above.

- Simplest Case, Example Economy.
- $I = 2$, No Storage. Endowment $\{\omega^y, \omega^o\}, \omega^y > \omega^o$.
- $u(c^y, c^o) = \log c^y + \log c^o$
- What happens? Nobody to trade with. So autarky?
- Perhaps there is Money as a store of Value.
- Consider

$$\begin{aligned} m_t &= \frac{\omega^y - c_t^y}{p_t} \\ c_{t+1}^o &= \frac{m_t}{p_{t+1} + m_t} \end{aligned}$$

- Many Monetary Equilibria $M_t = 1$

- Solutions to a difference equation

$$\frac{\omega^o + \frac{1}{p_{t+1}}}{\omega^y - \frac{1}{p_t}} = \frac{p_{t+1}}{p_t}$$

- A stationary one is $\frac{1}{p^*} = \frac{\omega^y - \omega^o}{2}$.

- There are many more with $P_0 > P^*$, converging to $\infty$

- Still, Why accept Money from older agents? Who needs them?

# The Lucas Tree

- The Purpose: To Price Assets so they do the right thing
- The Environment:
  - Goods: A measure one of trees that give fruit, $z$, that follows a Markov Process with transition matrix $\Gamma_{zz'}$.
  - Preferences: $E \; \sum_t \beta^t \; u(c_t)$.
  - Markets: Hholds buy shares $s'$ of trees in stock markets at price $p(z)$, and consume fruit. They receive dividends $d(z)$ and have shares.
- State Variables
  - Aggregate $z$
  - Individual $s$

$$V(z,s) = \max_{c,s'} \quad u(c) + \beta \sum_{z'} \Gamma_{zz'} V(z', s')$$

$$s.t. \quad c + p(z)\, s' = s\,[p(z) + d(z)]\,,$$

**Definition**

A Rational Expectations Recursive Competitive Equilibrium is a set of functions, $V$, $g$, $d$, and $p$, such that

1. $V$ and $g$ solves the household's problem given prices,
2. $d(z) = z$, and,
3. $g(z,1) = 1$, for all $z$.

- Recall

$$u_c\left(c\left(z,s\right)\right) = \beta \sum_{z'} \Gamma_{zz'} \left[\frac{p\left(z'\right) + d\left(z'\right)}{p\left(z\right)}\right] \; u_c\left(c\left(z',s'\right)\right).$$

- In equilibrium $s = 1$ and $c(z,1) = z$ so we have $u_c\left(z\right) := u_c\left(c\left(z,1\right)\right)$. The

$$p\left(z\right) u_c\left(z\right) = \beta \sum_{z'} \Gamma_{zz'} u_c\left(z'\right) \left[p\left(z'\right) + z'\right] \qquad \forall z.$$

- A system of $n_z$ equations. Denote $\mathrm{p} := \left[p\left(z_1\right) \vdots p\left(z_n\right)\right]_{(n_z \times 1)}$ and

$$\mathrm{u}_c := \begin{bmatrix} u_c\left(z_1\right) & & 0 \\ & \ddots & \\ 0 & & u_c\left(z_n\right) \end{bmatrix}_{(n_z \times n_z)} .$$

- Then

$$\mathrm{u}_c.\mathrm{p} = \begin{bmatrix} p(z_1)\, u_c(z_1) \\ \vdots \\ p(z_n)\, u_c(z_n) \end{bmatrix}_{(n_z \times 1)},$$

- Now, rewrite the system above as

$$\mathrm{u}_c\mathrm{p} = \beta\Gamma\mathrm{u}_c\mathrm{z} + \beta\Gamma\mathrm{u}_c\mathrm{p},$$

- where $\Gamma$ is the transition matrix for $z$, as before. Hence, share prices are

$$\left(\mathrm{I}_{\mathbf{n_z}} - \beta\Gamma\right)\mathrm{u}_c\mathrm{p} = \beta\Gamma\mathrm{u}_c\mathrm{z},$$

- or

$$\mathrm{p} = \left(\left[\mathrm{I}_{\mathbf{n_z}} - \beta\Gamma\right]\mathrm{u}_c\right)^{-1}\beta\Gamma\mathrm{u}_c\mathrm{z},$$

- An asset is "a claim to a chunk of fruit, sometime in the future."

- An asset that promises $m_t\left(z^t\right)$ after history $z^t = (z_0, z_1, \ldots, z_t) \in H^t$. The price of such an asset is the price of what it entitles its owner to.

- This follows from a no-arbitrage argument.

$$p^m\left(z_0\right) = \sum_t \sum_{z^t \in H^t} q_t^0\left(z^t\right)\ a_t\left(z^t\right),$$

$q_t^0\left(z^t\right)$ is the price of one unit of fruit after $z^t$ in time zero's goods.

- Given the $\left\{q_t^0(z^t)\right\}$, we can *replicate any possible asset by a set of state-contingent claims* and use this formula to price that asset.

- To find those $q^0$ consider a world where agents solve

$$\max_{c_t(z^t)} \quad \sum_{t=0}^{\infty} \beta^t \sum_{z^t} \pi_t\left(z^t\right) u\left(c_t\left(z^t\right)\right)$$

$$s.t. \quad \sum_{t=0}^{\infty} \sum_{z^t} q_t^0\left(z^t\right) c_t\left(z^t\right) \leq \sum_{t=0}^{\infty} \sum_{h^t} q_t^0\left(z^t\right) z_t.$$

- The $\pi(z^t)$ are the prob and can be constructed recursively with $\Gamma$.
- Note that this is the familiar Arrow-Debreu market structure, where the household owns a tree, and the tree yields $z \in Z$ amount of fruit in each period). The FOC for this problem (taking the ratio of the first period FOC and that of the history $z^t$) and imposing the equilibrium condition $c(z^t) = z_t(z^t)$ imply:

$$q_t^0\left(z^t\right) = \beta^t \pi_t\left(z^t\right) \frac{u_c\left(z_t\right)}{u_c\left(z_0\right)}.$$

- This enables us to price the good in each history of the world and price any asset accordingly.

- Hhold Probl

$$V(z,s,b) = \max_{c,s',b'(z')} \quad u(c) + \beta \sum_{z'} \Gamma_{zz'} V\left(z', s', b'\left(z'\right)\right)$$

$$s.t. \quad c + p(z)\, s' + \sum_{z'} q\left(z, z'\right) b'\left(z'\right) = s\left[p(z) + z\right] + b.$$

- A characterization of $q$ can be obtained by the FOC, evaluated at the equilibrium, and thus written as:

$$q\left(z, z'\right) u_c(z) = \beta \Gamma_{zz'} u_c\left(z'\right).$$

- We can thus price *all types* of securities using $p$ and $q$ in this economy.

- To sell the tree tomorrow at price $P$

$$\widehat{q}(z,P)=\sum_{z'} q\left(z,z'\right)\max\left\{P-p\left(z'\right),0\right\},$$

- The (American) option to sell either tomorrow or the day after

$$\tilde{q}(z,P)=\sum_{z'} q\left(z,z'\right)\ \max\left\{P-p\left(z'\right),\widehat{q}\left(z',P\right)\right\}.$$

- The European option to buy the day after tomorrow is

$$\bar{q}(z,P)=\sum_{z'}\sum_{z''}\max\left\{p\left(z''\right)-P,0\right\}q\left(z',z''\right)q\left(z,z'\right).$$

- If today's shock is $z$, the gross risk free rate

$$R(z) = \left[\sum_{z'} q\left(z, z'\right)\right]^{-1}$$

- The unconditional gross risk free rate is

$$R^f = \sum_{z} \mu_z^* R(z)$$

where $\mu^*$ is the steady-state distribution of the shocks.

- The average gross rate of return on the stock market is

$$\sum_{z} \mu_z^* \sum_{z'} \Gamma_{zz'} \left[ \frac{p(z') + z'}{p(z)} \right]$$

- The Risk Premia is

$$\sum_{z} \mu_z^* \left( \sum_{z'} \Gamma_{zz'} \left[ \frac{p(z') + z'}{p(z)} \right] - R(z) \right).$$

- Use the expressions for p and q and the properties of the utility function to show that risk premium is positive.

- The fruit is constant over time (normalized to 1)
- The agent is subject to preference shocks for the fruit each period given by $\theta \in \Theta$ with transition $\Gamma^{\theta}$.

$$V(\theta, s) = \max_{c, s'} \quad \theta u(c) + \beta \sum_{\theta'} \Gamma_{\theta\theta'} V(\theta', s')$$
$$s.t. \quad c + p(\theta) s' = s[p(\theta) + d(\theta)].$$

- The equilibrium is defined as before.
- In Eq $d(\theta) = 1$
- Discussion of Demand vs Supply Shocks and what RBC vs Lucas trees are.

# An Introduction to Search with a Particular Application: Endogenous Productivity in a Product Search Model

- Most of Economics posts Supply and Demand
- This is NOT the only wat to think.
- There are Trades all the time (houses jobs). What does it mean to clear the market?
- Search theory models decentralised exchanges: Trades require pairwise meetings of buyers and sellers (workers, firms, prospective couples) which do not happen automatically:
- Difficulties in meeting partners.
- After meeting, trades may happen or not.

# A Twist on the Lucas Tree Model

- So far
  - Hholds own the tree
  - Purchase Shares
  - To access the fruit they JUST have to Purchase it.
- Now They also have to FIND the fruit

- There is matching function $M(T, D)$: Trees and Search Effort.
  - Constant Returns to Scale, e.g. $D^{\varphi} T^{1-\varphi}$. Let $\frac{1}{Q} := \frac{D}{T}$, i.e. the ratio of shoppers per trees, *the market tightness*.
  - Other more natural matching functions $D \leq M(T, D) \leq D$.
- The probability that a unit of shopping effort finds a tree is

$$= \Psi^{h}(Q) := \frac{M(T, D)}{D} = Q^{1-\varphi}$$

- The probability that a tree finds a shopper is

$$\Psi^{f}(Q) := \frac{M(T, D)}{T} = Q^{-\varphi}$$

- Here $T = 1$. The number of trees is constant.

- A hunger (demand) shock $\theta$ with transition matrix $\Gamma_{\theta\theta'}$
- A Productvity (TFP, supply) shock $z$ with transition matrix $\Gamma_{zz'}$
- We look for a Lucas tree *type* Equilibrium
- State Variables
  - Aggregate $\theta, z$
  - Individual $s$

$$V(\theta, z, s) = \max_{c,d,s'} \quad u(c, d, \theta) + \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} \ V(\theta', z', s')$$

$$s.t. \quad c + P(\theta, z) s' = P(\theta, z) \left[ s \left( 1 + \widehat{R}(\theta, z) \right) \right]$$

$$c = d \ \Psi^h (Q(\theta, z)) z$$

- $P(\theta, z)$ is the price of the tree relative to that of consumption
- $\widehat{R}(\theta, z)$ is the dividend income (in units of the tree).
- $Q(\theta, z)$ is market tightness.

- Substitute the constraints into the objective, solve for $d$ and get the Euler equation for the household.

- Using THEN the market clearing condition in equilibrium, the problem is reduced to one equation and two unknowns, $P(\theta, z)$ and $Q(\theta, z)$

- Still need another functional equation.

- We need to specify the search protocol (how it happens).

**Exercise**

*Derive the Euler equation of the household from the problem defined above.*

- It is a particular search protocol of what is called non-random (or directed) search.
- Ex-ante Commitment to the terms of trade (in other search protocols it is not the case)
- Consider a world consisting of a large number of islands. Each island has a sign that displays two numbers, $P(\theta, z)$ and $Q(\theta, z)$. (price and market tightness)
- Searchers and (trees and household effort) choose which island to go to. They have different trade-offs of price versus tightness.
- Equilibrium determines which island (Optimal so unique).

Firms maximize returns by choosing market, $Q, P$. It helps to use trees as numeraire, so $\widehat{P}(Q) = 1/P$ is the price of consumption. We want to characterize the set of available markets for firms, $\widehat{P}(Q)$ by looking at the implications for firms that face it:

$$\widehat{R}(\theta, z) = \max_{Q} \ \widehat{P}(Q) \ \Psi^{f}(Q) \ z = \frac{z \ \Psi^{f}(Q)}{P(Q)}$$

with FOC

$$\widehat{P}'(Q) \Psi^{f}(Q) + \widehat{P}(Q) \Psi^{f'}(Q) = 0,$$

The set of pairs $P$ a that satisfies FOC yields a relation of indifference between the firms the pairs $\{P, Q\}$ for the firms that implicitly determines $\widehat{P}(Q)$ as

$$\frac{\widehat{P}'(Q)}{\widehat{P}(Q)} = -\frac{\Psi^{f'}(Q)}{\Psi^{f}(Q)} = \frac{P(Q)}{P'(Q)}.$$

$$V(\theta, z, s) = \max_{c,d,s',P,Q} \quad u(\theta \; c, d) + \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} V(\theta', z', s') \tag{1}$$

$$s.t. \quad c + Ps' = P\left[s\left(1 + \widehat{R}(\theta, z)\right)\right], \tag{2}$$

$$c = d \; \Psi^h(Q) \, z \tag{3}$$

$$\frac{z\Psi^f(Q)}{P} \geq \widehat{R}(\theta, z) \tag{4}$$

- The last constraint states that for a market to exist firms have to be guaranteed $\widehat{R}(\theta, z)$.

Plug the first two constraints into the objective function ( $c$ and $s'$ as functions of $d$) and (recall that $\Psi^h = Q^{1-\varphi}$) :

$$\theta Q^{1-\varphi} z u_c(\theta d Q^{1-\varphi} z, d) + u_d(\theta d Q^{1-\varphi} z, d) = \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} V_3 \left(\theta', z', s(1+\widehat{R}(\theta, z)) - \frac{d Q^{1-\varphi} z}{P}\right) \frac{Q^{1-\varphi} z}{P} \quad (5)$$

Get rid of $V_3$ using original problem and use the envelope theorem

$$V_3(\theta, z, s) = \left[\theta u_c(\theta d Q^{1-\varphi} z, d) + \frac{u_d(\theta d Q^{1-\varphi} z, d)}{Q^{1-\varphi} z}\right] P(1+\widehat{R}(\theta, z))$$

Combining these two gives the Euler equation:

$$\theta u_c(\theta d Q^{1-\varphi} z, d) + \frac{u_d(\theta d Q^{1-\varphi} z, d)}{Q^{1-\varphi} z} = \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} \frac{P'(1+\widehat{R}(\theta', z'))}{P} \left[\theta' u_c(\theta' d' Q'^{1-\varphi} z', d') + \frac{u_d(\theta' d' Q'^{1-\varphi} z', d')}{Q'^{1-\varphi} z'}\right] \quad (6)$$

# FOC with respect to $Q$ and $P$.

$\lambda$: Lagrange multiplier on the firm's participation constraint, then

$$\theta d(1-\varphi)Q^{-\varphi} z u_c(\theta d Q^{1-\varphi} z, d) =$$
$$\beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} V_3 \left(\theta', z', s(1+\widehat{R}(\theta, z)) - \frac{dQ^{1-\varphi} z}{P}\right)$$
$$\frac{d(1-\varphi)Q^{-\varphi} z}{P} - \lambda \frac{\varphi Q^{-\varphi-1} z}{P} \quad (7)$$

and

$$\beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} V_3 \left(\theta', z', s(1+\widehat{R}(\theta, z)) - \frac{dQ^{1-\varphi} z}{P}\right) dQ = -\lambda \quad (8)$$

Combining these two equation gives us:

$$\theta u_c(\theta d Q^{1-\varphi} z, d) = \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'}$$

$$V_3\left(\theta', z', s(1+\widehat{R}(\theta, z)) - \frac{dQ^{1-\varphi} z}{P}\right)\left[\frac{1}{(1-\varphi)P}\right] \quad (9)$$

Recall $V_3(\cdot,\cdot,\cdot)$ so

$$(1-\varphi)\theta u_c(\theta d Q^{1-\varphi} z, d) = \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'}$$

$$\frac{P'(1+\widehat{R}(\theta', z'))}{P}\left[\theta' u_c(\theta' d' {Q'}^{1-\varphi} z', d') + \frac{u_d(\theta' d' {Q'}^{1-\varphi} z', d')}{{Q'}^{1-\varphi} z'}\right] \quad (10)$$

**Definition**

An Eq with competitive search is functions $\{V, c, d, s', P, Q, \widehat{R}\}$ that:

1. Household's budget constraint, (condition 2)
2. Household's shopping constraint, (condition 3)
3. Household's Euler equation, (condition 6)
4. Market condition, (condition 10)
5. Firm's participation constraint, (condition 4), which gives us that the dividend payment is the profit of the firm, $\widehat{R}(\theta, z) = \frac{zQ^{-\varphi}}{P}$,
6. Market clearing, i.e. $s' = 1$ and $Q = 1/d$.

# Measure Theory

Measure theory is a tool that helps us aggregate.

**Definition**

For a set $S$, $\mathcal{S}$ is a family of subsets of $S$, if $B \in \mathcal{S}$ implies $B \subseteq S$ (but not the other way around).

**Remark**

*Note that in this section we will assume the following convention*

1. *small letters (e.g. $s$) are for elements,*
2. *capital letters (e.g. $S$) are for sets, and*
3. *fancy letters (e.g. $\mathcal{S}$) are for a set of subsets (or families of subsets).*

## Definition

A family of subsets of $S$, $\mathcal{S}$, is called a $\sigma$-algebra in $S$ if

1. $S, \emptyset \in \mathcal{S}$;
2. if $A \in \mathcal{S} \Rightarrow A^c \in \mathcal{S}$ (i.e. $\mathcal{S}$ is closed with respect to complements and $A^c = S \backslash A$); and,
3. for $\{B_i\}_{i \in \mathbb{N}}$, if $B_i \in \mathcal{S}$ for all $i \Rightarrow \bigcap_{i \in \mathbb{N}} B_i \in \mathcal{S}$ (i.e. $\mathcal{S}$ is closed with respect to countable intersections.

## Example

1. The power set of $S$ and $\{\emptyset, S\}$ are $\sigma$-algebras in $S$.
2. $\{\emptyset, S, S_{1/2}, S_{2/2}\}$, where $S_{1/2}$ means the lower half of $S$ (imagine $S$ as an closed interval in $\mathbb{R}$), is a $\sigma$-algebra in $S$.
3. If $S = [0, 1]$, then $\mathcal{S} = \left\{\emptyset, \left[0, \frac{1}{2}\right), \left\{\frac{1}{2}\right\}, \left[\frac{1}{2}, 1\right], S\right\}$ is *not* a $\sigma$-algebra in $S$. But $\mathcal{S} = \left\{\emptyset, \left\{\frac{1}{2}\right\}, \left\{\left[0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right]\right\}, S\right\}$ is.

It allows us to define sets where things happen and we can *weigh* those sets (avoiding math troubles)

**Definition**

Suppose $\mathcal{S}$ is a $\sigma$-algebra in $S$. A measure is a real-valued function $x : \mathcal{S} \to \mathbb{R}_+$, that satisfies

1. $x(\emptyset) = 0$;
2. if $B_1, B_2 \in \mathcal{S}$ and $B_1 \cap B_2 = \emptyset \Rightarrow x(B_1 \cup B_2) = x(B_1) + x(B_2)$ (additivity); and,
3. if $\{B_i\}_{i \in \mathbb{N}} \in \mathcal{S}$ and $B_i \cap B_j = \emptyset$ for all $i \neq j \Rightarrow x(\cup_i B_i) = \sum_i x(B_i)$ (countable additivity).

A set $S$, a $\sigma$-algebra in it ($\mathcal{S}$), and a measure on $\mathcal{S}$ $x$, define a measurable space, $(S, \mathcal{S}, x)$.

**Definition**

A Borel $\sigma$-algebra is a $\sigma$-algebra generated by the family of all open sets $\mathfrak{B}$ (generated by a topology). A Borel set is any set in $\mathfrak{B}$.

A Borel $\sigma$-algebra corresponds to complete information.

**Definition**

A probability measure $x$ is a measure where $x(S) = 1$. $(S, \mathcal{S}, x)$ is a probability space. The probab of an event is then given by $x(A)$, where $A \in \mathcal{S}$.

**Definition**

Given a m'able space $(S, \mathcal{S}, x)$, a real-valued function $f : S \to \mathbb{R}$ is m'able (with respect to the m'able space) if, for all $a \in \mathbb{R}$, we have

$$\{b \in S \mid f(b) \leq a\} \in \mathcal{S}.$$

Interpret $\sigma$-algebras as describing available information.

Similarly, a function is m'able wrt a $\sigma$-algebra $\mathcal{S}$, if it can be evaluated

**Example**

Suppose $S = \{1, 2, 3, 4, 5, 6\}$. Consider a function $f$ that maps the element 6 to the number 1 (i.e. $f(6) = 1$) and any other elements to -100. Then $f$ is NOT measurable with respect to $\mathcal{S} = \{\emptyset, \{1, 2, 3\}, \{4, 5, 6\}, S\}$. Why? Consider $a = 0$, then $\{b \in S \mid f(b) \leq a\} = \{1, 2, 3, 4, 5\}$. But this set is not in $\mathcal{S}$.

Extend the notion of Markov stuff to any measurable space

**Definition**

Given a measurable space $(S, \mathcal{S}, x)$, a function $Q : S \times \mathcal{S} \to [0, 1]$ is a transition probability if

1. $Q(s, \cdot)$ is a probability measure for all $s \in S$; and,
2. $Q(\cdot, B)$ is a measurable function for all $B \in \mathcal{S}$.

Intuitively, for $B \in \mathcal{S}$ and $s \in S$, $Q(s, B)$ gives the probability of being in set $B$ tomorrow, given that the state is $s$ today.

1. A *Markov chain* with transition matrix given by

$$\Gamma = \begin{bmatrix} 0.2 & 0.2 & 0.6 \\ 0.1 & 0.1 & 0.8 \\ 0.3 & 0.5 & 0.2 \end{bmatrix},$$

on $S = \{1, 2, 3\}$, with the the power set being the $\sigma$-algebra of $S$).

$$Q(3, \{1, 2\}) = \Gamma_{31} + \Gamma_{32} = 0.3 + 0.5 \ .$$

2. Consider a measure $x$ on $\mathcal{S}$. $x_i$ is the fraction of type $i$. Then

$$\begin{aligned} x_1' &= x_1\Gamma_{11} + x_2\Gamma_{21} + x_3\Gamma_{31}, \\ x_2' &= x_1\Gamma_{12} + x_2\Gamma_{22} + x_3\Gamma_{32}, \\ x_3' &= x_1\Gamma_{13} + x_2\Gamma_{23} + x_3\Gamma_{33}. \end{aligned}$$

In other words: $\mathrm{x}' = \Gamma^T \mathrm{x}$, where $\mathrm{x}^T = (x_1, x_2, x_3)$.

From a measure $x$ today to one tomorrow $x'$

$$\begin{aligned} x'(B) &= T(x, Q)(B) \\ &= \int_S Q(s, B)\, x(ds), \quad \forall B \in \mathcal{S}, \end{aligned}$$

we integrated over all $s \in S$ to get the prob of $B$ tomorrow.

A stationary distribution is a fixed point of $T$, that is $x^*$ such that

$$x^*(B) = T(x^*, Q)(B), \quad \forall B \in \mathcal{S}.$$

**Theorem**

*If $Q$ has nice properties (American Dream and Nightmare) then $\exists$ a unique stationary distribution $x^*$ and*

$$x^* = \lim_{n \to \infty} T^n(x_0, Q), \qquad \textit{for any } x_0.$$

**Exercise**

*Consider unemployment in a very simple economy (in which the transition matrix is exogenous). There are two states of the world: being employed and being unemployed. The transition matrix is given by*

$$\Gamma = \begin{pmatrix} 0.95 & 0.05 \\ 0.50 & 0.50 \end{pmatrix}.$$

*Compute the stationary distribution corresponding to this Markov transition matrix.*

# Industry Equilibrium

- Study the dynamics of the distribution of firms in partial equilibrium
- A single firm produces a good using labor:
- Output is $sf(n)$ ( $f$ increasing, strictly concave, $f(0) = 0$, $s$ is productivity.
- Markets are competitive, ($p$ and $w = 1$) as given.
- A firm solves

$$\pi(s,p) = \max_{n \geq 0} \{psf(n) - wn\}. \tag{11}$$

- With FOC

$$psf_n(n^*) = 1. \tag{12}$$

  Solution is $n^*(s,p)$.

- $n^*$ is an increasing function of both arguments. Prove it.

- A mass of firms in the industry, indexed by $s \in S \subset \mathbb{R}_+$, $S := [\underline{s}, \bar{s}]$.
- $\mathcal{S}$ is a $\sigma$-algebra on $S$ (a Borel $\sigma$-algebra, for instance).
- $x$ is a measure on $(S, \mathcal{S})$, which describes the cross-sectional distribution of productivity among firms.
- Use $x$ to define statistics of the industry: Since individual supply is $sf(n^*(s,p))$, then the aggregate supply

$$Y^S(p) = \int_S sf(n^*(s,p))\, x(ds)\,. \tag{13}$$

$Y^S$ is a function of the price $p$ only.

- Let Demand $Y^D(p)$. Then $p^*$ clears the market:

$$Y^D(p^*) = Y^S(p^*)\,. \tag{14}$$

Where does $x$ come from?

- Price $p$ and output $Y$ are constant over time.
- Firms face the problem above every period and discount profits at exogenous $r$.
- Each firm faces a probability $1-\delta$ of disappearing in each period.
- The choice is static. The value of an $s$ firm is

$$V(s;p) = \sum_{t=0}^{\infty} \left(\frac{\delta}{1+r}\right)^{t} \pi(s,p) = \qquad \left(\frac{1+r}{1+r-\delta}\right) \pi(s,p)$$

- Every period a mass of firms die. To achieve a stationary equilibrium we need firms entry: assume that there is a constant flow of firms entering the economy in each as well, so that entry equals exit.
- $x$ is the measure of firms. Firms that die are $(1-\delta)x(S)$.
- Entrants draw $s$ from probability measure $\gamma$ over $(S, \mathcal{S})$.

- What keeps other firms out of the market in the first place?
- (if $\pi(s;p) = psf(n^*(s;p)) - wn^*(s;p) > 0$, then any firm with $s \in S$ would enter.
- Assume a fixed entry cost, $c^E$ before learning $s$. Value of an entrant

$$V^E(p) = \int_S V(s;p)\,\gamma(ds) - c^E. \tag{15}$$

If $V^E > 0$ there will be entry.
- Equilibrium requires $V^E = 0$

- $x_t$ : cross-sectional distribution of firms. For any $B \subset S$, fraction $1-\delta$ of firms with $s \in B$ die and mass $m$ of newcomers enter. Next period's measure of firms on set $B$ is

$$x_{t+1}(B) = \delta x_t(B) + m\gamma(B). \tag{16}$$

- Mass $m$ of firms would enter $t+1$, with fraction $\gamma(B)$ having $s \in B$, $\forall B \in \mathcal{S}$.
- Cross-sectional distribution of firms completely determined by $\gamma$.
- Consider an updating operator $T$

$$Tx(B) = \delta x(B) + m\gamma(B), \quad \forall B \in \mathcal{S}, \tag{17}$$

a stationary dbon is a fixed point, i.e. $x^*$ such that $Tx^* = x^*$, so

$$x^*(B; m) = \frac{m}{1-\delta}\gamma(B), \quad \forall B \in \mathcal{S}. \tag{18}$$

- Demand and supply condition in equation (14) is

$$Y^D\left(p^*\left(m\right)\right) = \int_S s\ f\left[n^*\left(s;p\right)\right]\ dx^*\left(s;m\right), \tag{19}$$

whose solution $p^*\left(m\right)$ is a continuous function

- We have two equations, (15) and (19), and two unknowns, $p$ and $m$.

**Definition**

A stationary distribution for this environment consists of functions $V$, $\pi^*$, $n^*$, $p^*$, $x^*$, and $m^*$, that satisfy:

1. Given prices, $V$, $\pi^*$, and $n^*$ solve the incumbent firm's problem;
2. $Y^D\left(p^*\left(m\right)\right) = \int_S s\ f\left[n^*\left(s;p\right)\right]\ dx^*\left(s;m\right)$;
3. $\int_s V\left(s;p\right)\gamma\left(ds\right) - c^E = 0$; and,
4. $x^*\left(B\right) = \delta x^*\left(B\right) + m^*\gamma\left(B\right), \quad \forall B \in \mathcal{S}$.

- Assume $s$ follows a Markov process with transition $\Gamma$. This would change the mapping $T$ in Equation (17) to

$$Tx(B) = \delta \int_S \Gamma(s, B)\, x(ds) + m\gamma(B), \quad \forall B \in \mathcal{S}. \tag{20}$$

But no firm exits ($c^E$ is a sunk cost). Still not much Econ.

- Suppose now an operating cost $c^v$ each period.
    - when $s$ is low, firm's profits maybe negative and firm exits
    - But it is not enough. Assume $\Gamma$ satisfies stochastic dominance: $s^1 > s^2$ implies $\sum_{s'=1}^{\widehat{s}} \Gamma_{s^1,s'} < \sum_{s'=1}^{\widehat{s}} \Gamma_{s^2,s'}$.
    - Then $\exists$ a threshold, $s^* \in S$, below which firms exit and above stay.

$$V(s;p) = \max \left\{0, \pi(s;p) + \frac{1}{(1+r)} \int_S V(s';p)\, \Gamma(s, ds') - c^v \right\}. \tag{21}$$

- Updating operator becomes

$$x'(B) = \int_{s^*}^{\bar{s}} \Gamma(s, B \cap [s^*, \bar{s}])\, x(ds) + m\gamma(B \cap [s^*, \bar{s}]), \quad \forall B \in \mathcal{S}. \tag{22}$$

  A stationary distribution of the firms in this economy, $x^*$, is the fixed point of this equation.

- With $x^*$ we get all class of statistics:
  - Threshold for being in top 10% by size? Solve for $\widehat{s}$

$$\frac{\int_{\hat{s}}^{\bar{s}} x^*(ds)}{\int_{s^*}^{\bar{s}} x^*(ds)} = 0.1,$$

  - Fraction of workers in largest top 10% of firms

$$\frac{\int_{\hat{s}}^{\bar{s}} n^*(s,p)\, x^*(ds)}{\int_{s^*}^{\bar{s}} n^*(s,p)\, x^*(ds)}.$$

**Exercise**

*Compute the average growth rate of the smallest one third of the firms.*

**Exercise**

*What would be the fraction of firms in the top 10% largest firms in the economy that remain in the top 10% in next period?*

**Exercise**

*What is the fraction of firms younger than five years?*

**Definition**

$\pi^*, n^*, d^*, s^*, V$, a price $p^*$, a measure $x^*$, and mass $m^*$ such that

1. Given $p^*$, the functions $V, \pi^*, n^*, d^*$ solve the firm's
2. The reservation productivity $s^*$ satisfies $d^*(s; p^*) = \begin{cases} 1 & \text{if } s \geq s^* \\ 0 & \text{otherwise} \end{cases}$.
3. Free-entry condition: $V^E(p^*) = 0$.
4. For any $B \in \mathcal{S}$

$$x^*(B) = m^* \gamma(B \cap [s^*, \bar{s}]) + \int_{s^*}^{\bar{s}} \Gamma(s, B \cap [s^*, \bar{s}]) x^*(ds)$$

5. Market clearing:

$$Y^d(p^\star) = \int_{s^\star}^{\bar{s}} s\ f(n^\star(s; p^\star)) x^\star(ds)$$

- Average output of the firm is given by

$$\frac{Y}{N} = \frac{\int_{s^\star}^{\bar{s}} s\ f[n^*(s)]\ x^*(ds)}{\int_{s^\star}^{\bar{s}} x^*(ds)}$$

- Share of output produced by the top 1% of firms. Need to find $\tilde{s}$

$$\frac{\int_{\tilde{s}}^{\bar{s}} x^*(ds)}{\int_S x^*(ds)} = .01$$

$$\frac{\int_{\tilde{s}}^{\bar{s}} s\ \ f[n^*(s)]\ \ x^*(ds)}{\int_{s^\star}^{\bar{s}} s\ \ f[n^*(s)]\ \ x^*(ds)}$$

- Fraction of firms in the top 1% two periods in a row ($s$ continuous)

$$\int_{s \geq \tilde{s}} \int_{s' \geq \tilde{s}} \Gamma_{ss'} x^*(ds)$$

- Gini coefficient.

Consider adjustment costs to labor $c\left(n^{-}, n\right)$ due to hiring $n$ units of labor in $t$ as

- *Convex Adjustment Costs*: if the firm wants to vary the units of labor, it has to pay $\alpha\left(n_t - n_{t-1}\right)^2$ units of the numeraire good. The cost here depends on the size of the adjustment.

- *Training Costs or Hiring Costs*: if the firm wants to increase labor, it has to pay $\alpha\left[n_t - (1-\delta)\, n_{t-1}\right]^2$ units of the numeraire good only if $n_t > n_{t-1}$. We can write this as

$$1_{\{n_t > n_{t-1}\}}\alpha\left[n_t - (1-\delta)\, n_{t-1}\right]^2,$$

  where 1 is the indicator function and $\delta$ measures the exogenous attrition of workers in each period.

- *Firing Costs*: the firm has to pay if it wants to reduce the number of workers.

$$V\left(s,n^{-};p\right)=\max\left\{0,\max_{n\geq 0}\ sf\left(n\right)-wn-c^{v}-c\left(n^{-},n\right)+\frac{1}{(1+r)}\int_{s'\in S}V\left(s',n,p\right)\ \Gamma(s,ds')\right\},$$

$c(\cdot,\cdot)$ is cost function (note limited liability: exit value is 0)

Note $n=g(s,n^{-};p)<\bar{N}$. Let $\mathcal{N}$ be a $\sigma$-algebra on $[0,\bar{N}]$.

$$x'\left(B^{S},B^{N}\right)=m\gamma\left(B^{S}\cap[s^{*},\bar{s}]\right)1_{\{0\in B^{N}\}}+\int_{s^{*}}^{\bar{s}}\int_{0}^{\bar{N}}1_{\{g(s,n_{-};p)\in B^{N}\}}\ \Gamma\left(s,B^{S}\cap[s^{*},\bar{s}]\right)x\left(ds,dn_{-}\right),$$

$$\forall\ B^{S}\in\mathcal{S},\forall\ B^{N}\in\mathcal{N}.$$

**Exercise**

*Write the first order conditions.*

**Exercise**

*Define the recursive competitive equilibrium for this economy.*

**Exercise**

*Another example of labor adjustment costs is when the firm has to post vacancies to attract labor. As an example of such case, suppose the firm faces a firing cost according to function $c$. The firm also pays a cost $\kappa$ to post vacancies and after posting vacancies, it takes one period for the workers to be hired. How can we write the problem of firms in this environment?*

**Exercise**

*Add Adjustment Costs to Capital*

**Exercise**

*Add R& D*

- So far *stationary industry equilibria* (invariant distribution of firms).
- If $p$ were constant, the firm distribution would converge to the stationary equilibrium distribution $x^*$.
- What is an alternative?
- Prices are changing over time and so is the distribution of firms.
- There are two ways of modeling non-stationary equilibria
    - In Sequence Space (or stochastic process state)
    - Recursively
- What is best depends on the purpose. They should give the same answer. It is an issue of computation.
- We will look at both ways (for now deterministic).
- Given the convergence that we talked about we need a rationale for the non stationarity.
- Consider demand shifters $z_t$ so that $D(P, z_t)$ where $z_{t+1} = \phi(z_t)$ so we can choose to represent it as a sequence or recursively.

- Note the need for an initial condition. Then objects are relatively simple.
- Given a path $\{z_t\}_{t=0}^{\infty}$ and an initial $x_0$, an equilibrium defined in term of sequences is: Sequences $\{p_t, m_t, s_t^*\}$ of numbers, a sequence of measures $x_t$, and sequences $\{V_t(s), n_t(s)\}_{t=0}^{\infty}$ of functions:

1. **Optimality**: Given $\{p_t\}$, $\{V_t, s_t^*, n_t\}$ sole

$$V_t(s) = \max \left\{0, \max \ p_t \ s \ f(n) - wn - c^v + \frac{\int_S V_{t+1}(s')\Gamma(s, ds')}{1+r}\right\}$$

2. **Free-entry**: $\int V_t(s) \ \gamma(ds) \leq c^e$, with strict equality if $m_t > 0$.
3. **Law of motion**: $x_{t+1}(B) = m_{t+1}\gamma\left(\cap[s_{t+1}^*, \bar{s}]\right) + \int_{s_t^*}^{\bar{s}} \Gamma\left(s, B \cap [s_{t+1}^*, \bar{s}]\right) \ x_t(ds)$, $\forall B \in \mathcal{S}$.
4. **Market clearing**: $D[p_t, z_t) = \int_{s_t^*}^{\bar{s}} p_t \ s \ f[n_t(s)] \ x_t(ds)$.

- Only from today to tomorrow: need objects that given the state today, $\{z, x\}$, give us the state tomorrow $\{\phi, G\}$.
- Given $\phi$, an equil defined recursively is functions $G(z,x)$, $m(z,x)$, $p(z,x)$, values and decisions $\{V(s,z,x), n(s,z,x), s^*(z,x)\}$ s.t.

1. **Optimality**: $\{V(s,z,x), s^*(z,x), n(s,z,x)\}$ solve

$$V(s,z,x) = \max_n \left\{0, \max p(z,x)s\ f(n) - wn - c^v + \frac{1}{1+r}\int_S V[s', \phi(z), G(z,x)]\ \Gamma(s, ds')\right\}$$

2. **Free-entry**: $\int V(s,z,x)\ \gamma(ds) \leq c^e$, $(=$ if $m(z,x) > 0)$.

3. **Law of motion**: $\forall B \in \mathcal{S}$, we have
$G(z,x)(B) = m(z,x)\ \gamma(B \cap [s^*(z,x), \bar{s}]) + \int_{s^*(z,x)}^{\bar{s}} \Gamma(s, B \cap [s^*(z,x), \bar{s}])x(ds)$,

4. **Market clearing**: $D(p(z,x), z) = \int_{s^*(z,x)}^{\bar{s}} p(z,x)\ s\ f[n(s,z,x)]\ x(ds)$.

- It is the same but in Stochastic Processes Language

- They extend the same for sequences and for the Recursive

- Obviously You have to add the Expectations to the terms of one period later.

# Numerical Approximations I: Using an Exact Transition

- A Recursive (or sequence) equilibrium entails finding an infinite dimensional function $x = g(s)$ (or sequence $\{x_t\}_{t=1}^{\infty}$).
- This cannot be done as such. So we substitute.
    - General Functions with functions belonging to a class that can be represented by a finite number of real numbers (or its computer representation) if the equilibrium is recursive
    - Finite number of periods and then some assumption is made if Equilibrium is made up of sequences.
- So a computational method for recusive equilibria consists of
    1. Choosing a function $\Phi(s, \theta)$, where $s$ is the state and $\theta \in R^M$ is a vector (eg J-piece cubic splines). We will use $\Phi$ instead of $g$ for some suitable chosen $\theta^*$.
    2. Choosing a metric $\|g, \Phi(., \theta)\|$ and a criterion (a maximum distance) that we will tolerate to justify $\theta^*$ (say $10^{-6}$ a the maximum the error in the Euler equation when evaluated in some prespecified grid).
    3. Specify some tricks or procedures to effectively compute $\theta^*$ (say iterate backward from the future to the present using successive approximations).

- Boppart, Krusell, and Mitman (2018) think of Stochastic Equilibria in a way that is NOT recursive.
- It is based on a linear approximation to a completely unanticipated (MIT) shock.
- It requires to compute a transition as a Perfect Foresight Equilibrium. This amounts to the same complexity that the computation of a non-deterministic non-stationary equilibrium that happens to coverge to a stationary allocation.
- Then do linear approximations in sequence space.
- We will see how it works via a simple example.

- Consider the social planner's problem (with full depreciation)

$$
\begin{aligned}
V(k_t) \quad = \quad & \max_{c_t, k_{t+1}} u(c_t) + \beta V(k_{t+1}) \\
& \text{s.t. } c_t + k_{t+1} \leq f(k_t), \qquad \forall\, t \geq 0 \\
& c_t, k_{t+1} \geq 0, \ \forall \quad t \geq 0 \\
& k_0 > 0 \text{ given.}
\end{aligned}
$$

- The solution $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ satisfies

$$u_c(c_t) = \beta\ u_c(c_{t+1})\ f_k(k_{t+1}), \forall\ t \geq 0$$

$$c_t + k_{t+1} = f(k_t), \quad \forall\ t \geq 0$$

$$\lim_{t \to \infty} \beta^t \ \ u_c(c_t) \ \ k_{t+1} = 0$$

- Derive the above equilibrium conditions.

- Look at the steady state $k^*$
- Rewrite solution as

$$\psi(k_t, k_{t+1}, k_{t+2}) = u_c[f(k_t) - k_{t+1}] - \beta\ u_c[f(k_{t+1}) - k_{t+2}]\ f_k(k_{t+1}) = 0,$$

a second order diff equation with exactly two boundary conditions, $k_0$, $k_\infty = k^*$.

- It is solvable by various alternative strategies
  1. guess $k_1$, use $k_0$ and $\psi(k_t, k_{t+1}, k_{t+2}) = 0$ to get $k_2, k_3, \dots$ forward up until some $T$, and solve $k_T^\psi(k_1) = k^*$.
  2. Or guess $k_{T-1}$ solve backward using $\psi$ to find $k_0^\psi(k_{t-1}) = k_0$
  3. Solve for the whole sequence as a system of equations (almost diagonal)
  4. Use dynare.
- Either way you get a numerical solution starting from any $k_0$. This is something that you should be able to compute always.
- It is the only way to do policy analysis (as the evaluation of the equilibrium under alternative policies **GIVEN** initial conditions.

- We can compute any transition. Also one with time varying $\psi$.

- Consider this model with $c_t + k_{t+1} = e^{z_t}\ f(k_t),\ z_{t+1} = \rho z_t,\ z_0 = 1$.

$$\psi_t(k_t, k_{t+1}, k_{t+2}) = u_c[\rho^t f(k_t) - k_{t+1}] - \beta\ u_c[\rho^{t+1} f(k_{t+1}) - k_{t+2}]\ f_k(k_{t+1}),$$

- In this case we can look at an MIT shock or impulse response. Here $k_0 = k_\infty = k^*$, but $k_1 \neq k^*$

- We can again obtain the transition $k_t$.

- Let now $\widehat{k_t} = \log k_t - \log k^*$, (log st st deviation).

- This is in fact an impulse response function.

- We want now to simulate a response of the economy to shocks. Consider an AR(1) process for $z_t$ : with $z_{t+1} = \rho^t z_t + \epsilon_{t+1}$.) where $\epsilon_t \sim \mathcal{N}(0, \sigma^2)$.
- **Want:** Solve for the solution by linearly approximating using $\{\widehat{k}_t\}_{t=0}^{\infty}$ the equilibrium given any sequence of innovations $\{\epsilon_t\}$.).
- Obtain $\widetilde{k}_t(k_0, \epsilon^{t-1})$ again in deviations from steady state. Note that the following linear approximation is what we want.

$$
\begin{aligned}
\widetilde{k}_1(k_0, \epsilon_0) &= \epsilon_0 \ \widehat{k}_1 \\
\widetilde{k}_2(k_0, \epsilon_0, \epsilon_1) &= \epsilon_0 \ \widehat{k}_2 + \epsilon_1 \ \widehat{k}_1, \\
&\vdots \\
\widetilde{k}_{t+1}(k_0, \epsilon^t) &= \sum_{\tau=0}^{t} \epsilon_t \ \widehat{k}_{t-\tau+1} \qquad \text{exact if } \epsilon_0 = 1, \ \epsilon_t = 0, \forall t \neq 0,
\end{aligned}
$$

- It is a $MA(\infty)$ process.

- This can be done for all Economies.
- Including industry equilibria.
- For all Statistics of all Economies.
- The computational costs are linear not exponential in the number of shocks.
- We do not know how to use it for asymmetric shocks (e.g. downward rigid wages)

**Exercise**

1. *What happens if demand suddenly doubles starting from a stationary equilibrium?*
2. *Define Formally the stochastic counterparts (sequentially and recursivrly) to the ones that we wrote above?*
3. *Sketch an algorithm to find the equilibrium prices.*
4. *Describe a way to compute the evolution of the Gini Index or the Herfindahl Index of the industry over the first fifteen periods.*
5. *Imagine now that the industry is subject to demand shocks that follow an AR(1). Describe an algorithm to approximate it.*

# Incomplete Market Models

- Consider the problem of a farmer with storage possibilities

$$V(s,a) = \max_{c,a' \geq 0} \quad u(c) + \beta \sum_{s'} \Gamma_{ss'} \; V\left(s', a'\right) \qquad s.t.$$

$$c + qa' = a + s$$

$a$ assets, $c$ consumption,

$s \in \{s^1, \cdots, s^{N^s}\} = S$ has transition $\Gamma$,
$q$ units today yield 1 unit tomorrow. Only nonnegative storage.

- If $s$ constant, then

$$V(a) = \max_{c,a' \geq 0} \left\{ u\left(a + s - qa'\right) + \beta V\left(a'\right) \right\}.$$

- with FOC $\quad q\, u_c \geq \beta\, u_c'$
- With equality if $a' > 0$. Then
  - if $q > \beta$ (i.e. nature is more stingy, or the farmer is less patient),
    - Either $c' < c$ and the farmer dis-saves
    - Or $c = s$ and $a' = 0$.
  - If $q < \beta$, $c' > c$ and consumption grows without bound.
  - If $q = \beta$, $c' = c$ (with noise and $u_{ccc} > 0$ grows without bound).
- So we assume $\beta/q < 1$

- Assuming $\beta/q < 1$, allows us to bound asset holdings.
- They also save in best states when $a$ is low.
- The FOC is

$$u_c\left[c\left(s,a\right)\right] \;\geq \frac{\beta}{q} \sum_{s'} \Gamma_{ss'}\; u_c\left(c\left[s', g\left(s,a\right)\right]\right),$$

with equality when $a' = g\left(s,a\right) > 0$

- Note: $a \gg g\left(s,a\right), \;\; \forall s$ for sufficiently large $a$. So $\exists\, \bar{a},\;\; s.t.\; a' \in A = [0, \bar{a}]$
- We can construct a prob distribution over states $S \times A$. Define $\mathcal{B}$ as all subsets of $S$ times $\mathcal{A}$, the Borel-$\sigma$-algebra sets in $A$.
- Using $\Gamma$ and $g$ we get a transition over $\{S \times A, \mathcal{S} \times \mathcal{A}\}$: for any such prob measure $x$ its evolution is

$$x'\left(B\right) = \widetilde{T}(B, x; \Gamma, g) = \sum_{s} \int_0^{\bar{a}} \sum_{s' \in B_s} \Gamma_{ss'}\; 1_{\{g(s,a) \in B_a\}}\; x\left(s, da\right), \quad \forall B \in \mathcal{B}$$

where $B_s$ and $B_a$ are projections of $B$ on $S$ and $A$,

**Theorem**

*With a well behaved $\Gamma$, there is a unique stationary probability $x^*$, so that:*

$$\begin{aligned} x^*(B) &= \widetilde{T}(B, x^*; \Gamma, g)(B), \quad \forall B \in \mathcal{B}, \\ x^*(B) &= \lim_{n \to \infty} \widetilde{T}^n(B, x_0; \Gamma, g)(B), \quad \forall B \in \mathcal{B}, \end{aligned}$$

*for all initial probability measures $X_0$ on $(E, \mathcal{B})$.*

We use compactness of $[0, \overline{A}]$.

1. Our ignorance of what is going on with the farmer or fisherman.

   - Even if we know at $t = 0$ $s, a$, no news lead us to $x^*$.

   - We can use $x^*$ to compute the statistics of what happens to the fisherman: Average wealth is $\int_{S \times A} a \, dx^*$.

2. A description of a large number of fishermen (an archipelago). Notice how even if there is no contact between them. Stationarity arises (İmrohoroğlu (1989))

   - There is a unique distribution of wealth.

- How can $a < 0$? Because of borrowing.
- Consider now an economy with many farmers and NO storage.

$$V(s,a) = \max_{c \geq 0, a'} \quad u(c) + \beta \sum_{s'} \Gamma_{ss'} V(s', a')$$

$$s.t. \quad c + q\, a' = a + s$$

$$a' \geq \underline{a},$$

where $\underline{a} < 0$ and $\beta/q < 1$. With solution $a' = g(s,a)$. Again

- One possibility for $\underline{a}$ is the natural borrowing limit: the agent can pay back his debt with certainty, no matter what:

$$a^n := -\frac{s_{\min}}{\left(\frac{1}{q} - 1\right)}. \tag{23}$$

- Or it could be tighter which makes it likely to bind $0 > \underline{a} > a^n$.

- To determine $q$ in general equilibrium, consider this function of $q$:

$$\int_{A\times S} a\, dx^{*}(q) \qquad \text{Aggregate asset holdings}$$

- A Stationary Equilibrium requires two things

$$\begin{aligned} \int_{A\times S} a\, dx^{*}(q) &= 0, \\ x^{*}(q) &= \widetilde{T}^{n}\left(B, x^{*}(q); \Gamma, g\right)(B). \end{aligned}$$

- It exists in $q \in (\beta, \infty]$ (intermediate value thm). Need to ensure:
  1. $\int_{A\times S} a\, dX^{*}(q)$ is a continuous function of $q$;
  2. $\lim_{q\to\beta} \int_{A\times S} a\, dX^{*}(q) \to \infty$; (As $q \to \beta$, the interest rate $R = 1/q$ increases up to $1/\beta$, (steady state interest rate in deterministic Econ. With $u_{ccc} > 0$ we have precautionary savings
  3. $\lim_{q\to\infty} \int_{A\times S} a\, dX^{*}(q) < 0$. As $q \to \infty$, arbitrary large consumption is achievable by borrowing.

- Workhorse models of modern macroeconomics.
- An Environment like the ones before
- On top of a growth model with $f(K, L)$ that yield factor prices.

$$K = \int_{A \times S} a \, dx,$$
$$N = \int_{A \times S} s \, dx.$$

- $s$ fluctuations in the employment status (either efficiency units of labor or actual employment).
- Now we need $\beta(1+r) < 1$. We write

$$V(s, a) = \max_{c, a' \geq 0} \quad u(c) + \beta \int_{s'} V(s', a') \; \Gamma(s, d\, s') \qquad s.t.$$
$$c + a' = (1+r)\, a + ws$$

where $r$ is the return on savings and $w$ is the wage rate.

- Factor prices depend on the capital-labor ratio: $x^*\left(\frac{K}{L}\right)$. Equilibrium requires

$$\frac{K^*}{L^*} = \frac{\int_{A\times S} a \; dX^*\left(\frac{K^*}{L^*}\right)}{\int_{A\times S} s \, dX^*\left(\frac{K^*}{L^*}\right)}.$$

**Exercise**

*Show that aggregate capital is higher in the stationary equilibrium of the Aiyagari economy than it is the standard representative agent economy.*

**Exercise**

*Not necessarily so if leisure has value (Pijoan-Mas (2006))*

**Exercise**

*Rewrite the economy when households like leisure*

- Let the Economy's parameters be summarized by $\theta = \{u, \beta, s, \Gamma, F\}$.
- $V(s, a; \theta)$ and $x^*(\theta)$ are functions of those parameters.
- Suppose an unexpected policy change that shifts $\theta$ to $\hat{\theta} = \{u, \beta, s, \hat{\Gamma}, F\}$.
- Consider $V\left(s, a; \hat{\theta}\right)$ and $x^*\left(\hat{\theta}\right)$.
- Define $\eta(s, a)$ by

$$V\left(s, a + \eta(s, a); \hat{\theta}\right) = V(s, a; \theta),$$

- Transfer necessary to make the $(a, s)$ agent indifferent between living in the old environment and in the new.
- Total transfer needed to compensate all agents to live in $\hat{\theta}$ is

$$\int_{A\times S} \eta(s, a)\, dX^*(\theta).$$

- This is NOT a Welfare Comparison.
- This compares being parachuted in the stationary distribution of $\theta$ versus $\hat{\theta}$.
- Welfare computing the transition from the SAME initial conditions.
- Otherwise the best tax policy in the Rep agent (which is Pareto Optimal) would be to subsidize capital to maximize steady state consumption.
- To analyze the policy we need to compute the whole transition as depicted in a previous lecture. It is a NON-stationary equilibrium.

# Business Cycles with Economies with Measures of Agent

- What if aggregate shocks as in e.g. $z\ F\left(K, \bar{N}\right)$.
- Without leisure aggregate capital is a sufficient statistic for factor prices.
- Will aggregate capital be $K' = G\left(z, K\right)$ or $K' = G\left(z, x\right)$ ?
- The latter. Decision rules are not usually linear. But then $x' = G\left(z, x\right)$

$$
\begin{aligned}
V\left(z, X, s, a\right) = \max_{c, a' \geq 0} \quad & u\left(c\right) + \beta \sum_{z', s'} \Pi_{zz'} \Gamma_{ss'}^{z'} V\left(z', X', s', a'\right) \\
s.t. \quad & c + a' = a\ z\ f_k\left(K, \bar{N}\right) + s\ z\ f_n\left(K, \bar{N}\right) \\
& K = \int a\ dX\left(s, a\right) \\
& X' = G\left(z, X\right)
\end{aligned}
$$

(replaced factor prices with marginal productivities)

- Computationally, this problem is a beast! So, what then?

- They people believe tomorrow's capital depends only on $K$ and not on $x$. This, obviously, is not an economy with rational expectations. The agent's problem in such a setting is

$$\begin{aligned} \widetilde{V}(z, K, s, a) = \max_{c, a'} \quad & u(c) + \beta \sum_{z', s'} \Pi_{zz'} \Gamma_{ss'}^{z'} \widetilde{V}\left(z', K', s', a'\right) \\ s.t. \quad & c + a' = a\ z\ f_k\left(K, \bar{N}\right) + szf_n\left(K, \bar{N}\right) \\ & K' = \widetilde{G}(z, K) \end{aligned}$$

- We could approximate the equilibrium in the computer by posing a linear approximation to $\widetilde{G}$. A pain but doable. Krusell Smith (1997).

- They found it works well in boring settings (things are pretty linear)

- We can use the same linear approx in sequences as before for any shocks:

    1. Find the steady state

    2. Obtain the the impulse response (the perfect foresight equilibium) given an MIT shock that is treated as an innovation.

    3. Use these responses to approximate the behavior of any aggregate.

- Valuable for SMALL shocks like all linear approximations.

- Consider an Aiyagari economy with an AR(1) TFP shock $z$.
  - Choose an initial size innovation $\bar{\epsilon}_0$ (does not have to be 1) and compute the perfect foresight Equilibria of this MIT shock.
    - This involves a fixed point in the space of sequence of capital labor ratios.
    - But can be done with some effort:
    - To evaluate it, given prices solve the household's problem backwards from the final steady state.
    - Then update the distribution forward from the initial steady state obtaining new prices.
    - We look for a fixed point of this (not necessarily iterating mechanically but as solution of a system of equations)

- We have now the sequence of $x_t$ and any prices that we care for.
- Compute the sequence of all statistics $\{d_t\}_t^T$ of that economy that you care for.
- Get a random draw $\{\epsilon_t\}_{t=0}^T$.
- Linearly approximate those statistic like we did before the same way that we approximated

$$\begin{aligned}
\widetilde{d}_1(x_0, \epsilon_0) &= \frac{\epsilon_0}{\bar{\epsilon}_0} \ \widehat{d}_1 \\
\widetilde{d}_2(x_0, \epsilon_0, \epsilon_1) &= \frac{\epsilon_0}{\bar{\epsilon}_0} \ \widehat{d}_2 + \frac{\epsilon_1}{\bar{\epsilon}_0} \ \widehat{d}_1, \\
&\vdots \\
\widetilde{d}_{t+1}(x_0, \epsilon^t) &= \sum_{\tau=0}^{t} \frac{\epsilon_t}{\bar{\epsilon}_0} \ \widehat{d}_{t-\tau+1} \qquad \text{exact if } \epsilon_0 = \widetilde{\epsilon}_0, \ \epsilon_t = 0, \forall t \neq 0.
\end{aligned}$$

# Numerical Approximations II: Using a Linear Approximation to Get the Transition

**Taken mostly from a May 20 Remote Brown Bag on Computational Economics and Finance by Auclert, Bardóczy, Rognlie, and Straub**

Auclert, Bardóczy, Rognlie, and Straub (2021)

- Directly solve linear system in the sequence space: same, but faster!
- Steps of Method:

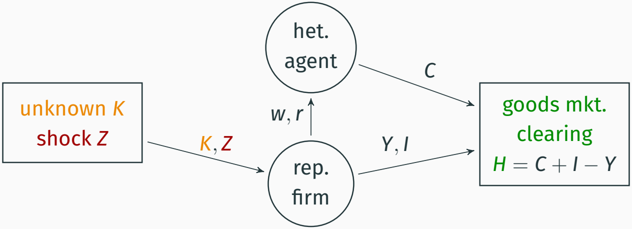
1. Write Heterog A model as a collection of blocks along a directed acyclic graph (DAG)
2. Compute the Jacobian of each block: key "sufficient statistic" for General Equilibrium interactions
3. Use Jacobians to get Impulse Response Functions

1. Fast: for state-of-the-art, two-asset HANK model,
    - Impulse response takes about 5s (vs about 100s with leading alternative methods)
    - Additional impulse responses take about 100ms (vs 100s) by alternative methods) reusing Jacobians
    - This makes model estimation possible
2. Accurate: no "model reduction" necessary, only error is from truncation
3. Modular: easy to build complex models by stitching blocks together
4. Intuitive: block Jacobians often have simple interpretation [eg MPCs]
5. Accessible: key steps automated in publicly available code [in Python]
    - Most ideas are also easily implemented in Julia or even Matlab

- Block: Mapping from sequence of inputs to sequence of outputs. Examples
  1. Heterogeneous household block $\{r_t, w_t\}$ß$\{C_t\}$
  2. Representative firm block with $L = 1\{K_t, Z_t\} \rightarrow \{Y_t, I_t, r_t, w_t\}$
  3. Goods market clearing block $\{Y_t, C_t, I_t\} \rightarrow \{H_t C_t + I_t Y_t\}$
- Model: Set of blocks, arranged along a directed acyclic graph (DAG)
  - Some inputs are exogenous shocks, e.g. $\{Z_t\}$
  - Some inputs are endogenous unknowns, e.g. $\{K_t\}$
  - Some outputs are target sequences that must equal zero in GE, e.g. $\{H_t\}$ [must have as many targets as unknowns]
- Many models can be written in this way.
- Key restriction: agents interact via limited set of aggregate variables

# Krusell-Smith model DAG

Capital market clearing

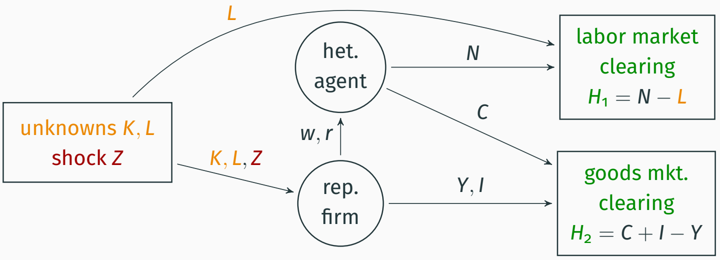



- DAG can be collapsed into mapping

$$H_t(\{K_s\}, \{Z_s\}) = C_t + I_t - Y_t$$

- GE path of $\{K_s\}$ achieves $H_t(\{K_s\}, \{Z_s\}) = 0$

## Dealing with endogenous labor: add an unknown and a target



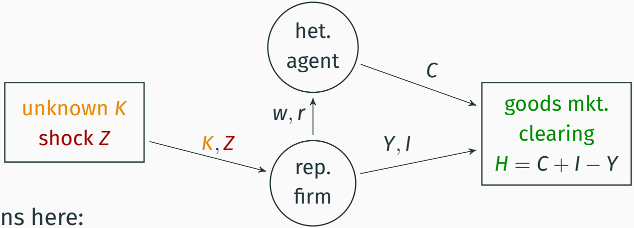

- DAG can be collapsed into mapping

$$\mathbf{H}_t\left(\{K_s, L_s\}, \{Z_s\}\right) = \{C_t + I_t - Y_t, N_s - L_s\}$$

- GE path of $\{K_s, L_s\}$ achieves $\mathbf{H}_t\left(\{K_s, L_s\}, \{Z_s\}\right) = 0$

- Suppose we have set the directed acyclic graph (DAG) and initial conditions (typically the steady state)
- Define a block Jacobian as the derivatives of its outputs wrt its inputs
- e.g. household block in the Heterogenous Agent Model that takes as inputs wages $w$ and rates of return $r$ and as output aggregate Consumption $C$ has two Jacobians
  - $\mathcal{J}^{C,w}_{t,s} \equiv \frac{\partial C_t}{\partial w_s}$
  - $\mathcal{J}^{C,r}_{t,s} \equiv \frac{\partial C_t}{\partial r_s}$
- Next: block Jacobians are sufficient to compute GE impulse responses

## Krusell-Smith model Jacobians




- Jacobians here:
  - het. agent: $\left\{\frac{\partial C_t}{\partial w_s}\right\}, \left\{\frac{\partial C_t}{\partial r_s}\right\} \rightsquigarrow$ denote $\mathcal{J}^{C,w}, \mathcal{J}^{C,r}$
  - rep. firm: $\left\{\frac{\partial w_t}{\partial K_s}\right\}, \left\{\frac{\partial w_t}{\partial Z_s}\right\}, \left\{\frac{\partial r_t}{\partial K_s}\right\}, \left\{\frac{\partial r_t}{\partial Z_s}\right\}, \ldots \rightsquigarrow$ denote $\mathcal{J}^{w,K}, \mathcal{J}^{w,Z}, \mathcal{J}^{r,K}, \mathcal{J}^{r,Z}, \ldots$

- To Get the Jacobians of the Markets Clearing BLOCK

$$\frac{\partial H}{\partial K} = \mathcal{J}^{C,r}\mathcal{J}^{r,K} + \mathcal{J}^{C,w}\mathcal{J}^{w,K} + \mathcal{J}^{I,K} - \mathcal{J}^{Y,K}$$

$$\frac{\partial H}{\partial Z} = \mathcal{J}^{C,r}\mathcal{J}^{r,Z} + \mathcal{J}^{C,w}\mathcal{J}^{w,Z} + \mathcal{J}^{I,Z} - \mathcal{J}^{Y,Z}$$

- Once the Jacobians are chained to get $\frac{\partial H}{\partial K}$ and $\frac{\partial H}{\partial Z}$ we are done

- To see this suppose we have $dZ = \{dZ_t\}$ [with $dZ_t = 0,\ t \geq T_0$] and we want to get the impulse response.
  1. Note that we need to get $H(K, Z) = 0$ as a result of the shock. We need to solve for $dK$:
  $$dK = -\left(\frac{\partial H}{\partial K}\right)^{-1} \frac{\partial H}{\partial Z}\, dZ$$
  2. Use Jacobians to back out all Impulse Response Functions of interest, e.g. that of output is
  $$dY = \mathcal{J}^{Y,K}\, dK + \mathcal{J}^{Y,Z}\, dZ +$$
  3. This allows us to obtain everything including the moments of the distribution $x$.

**Exercise**

*Show how to get those distribution moments (call them $m_x$) by properly constructing $\mathcal{J}^{m_x \cdot}$.*

- Certainty equivalence implies that $dK$ is also the $MA(\infty)$ in models with aggregate shocks:
- Suppose $\{d\widehat{Z}_t\}$ is $MA(\infty)$ in iid structural innovation vectors $\{\epsilon_t\}$:

$$d\widehat{Z}_t = \sum_{s=0}^{\infty} dZ_s \ \epsilon_{t-s}$$

then

$$d\widehat{K}_t = \sum_{s=0}^{\infty} dK_s \ \epsilon_{t-s}$$

- Applications:
    1. Simulation method (immediate)
    2. Analytical second moments for any $X$, $Y$:

$$Cov(d\widehat{X}_t, d\widehat{Y}_\tau) = \sigma_\epsilon^2 \sum_{s=0}^{T-(\tau-t)} dX_s \ dY_{s+\tau-t}$$

    3. Even Estimation

- Let $\Phi(\theta)$ be the covariance matrix for a bunch of relevant variables given a vector $\theta$ of parameters to be estimated

- With normal (Gaussian) innovations, log-likelyhood of data $Y$ given $\theta$:

$$\mathcal{L}(Y;\theta) = -\frac{1}{2} \log\det V(\theta) - \frac{1}{2} Y^T V(\theta) Y$$

- No Need for Kalman filter

- Estimating the parameters of the shocks is almost free (we use the same Jacobians for any $dZ$

- For other parameters estimation is still very fast if we do not need to recalculate the heterogeneous agents blocks (capital adjustment costs for instance) because we can use the same Jacobians $\mathcal{J}^{C,w}, \mathcal{J}^{C,r}$.

1. In practice, this method involves the inversion of an $nT \times nT$ matrix $\frac{\partial H}{\partial K}$ where $n = \#$ unknowns and $T$ is the truncation horizon [typically T= 300-500]
   - very fast as long as DAG doesn't have too many unknowns
   - key benefit of DAGs: reduce $n$ without any loss in accuracy [typically n  3]
   - in practice, choice of $T$ depends on persistence of exogenous variables
2. This matrix is invertible if the model is locally determinate
3. Jacobians are also useful to get the nonlinear perfect-foresight solution
   - Solve $H(K, Z) = 0$ using Newton's method because $\frac{\partial H}{\partial K}$ is the steady state Jacobian.

# Computing heterogeneous-agent Jacobians

**So far:** DAG + Jacobians $\Rightarrow$ IRFs, determinacy, estimation, nonlinear transitions

But how do we get the block **Jacobians?**

- *simple blocks*: (e.g. representative firms) simple, sparse matrix
- *HA blocks?* $\rightarrow$ **next**

## Jacobian of consumption with respect to wage

- Want to know $\mathcal{J}_{t,s} \equiv \frac{\partial C_t}{\partial w_s}$ for $s, t \in \{0, \ldots, T-1\}$ [intertemporal MPCs]
  - Assume initial condition is s.s., with $r_t = r$, $w_t = w$, $D_0(e, k_-) = D(e, k_-)$

- **Direct algorithm**: perturb $w_s \equiv w + \epsilon$
  1. iterate backwards to get perturbed policies: $\mathbf{c}_t^s(e, k_-), \mathbf{k}_t^s(e, k_-)$
  2. iterate forward to get perturbed distributions $D_t^s(e, k_-)$
  3. put together to get perturbed aggregate consumption: $C_t^s = \int \mathbf{c}_t^s(e, k_-) D_t^s(e, dk_-)$
  4. compute $\mathcal{J}$ from $\mathcal{J}_{t,s} \equiv (C_t^s - C)/\epsilon$

- This is **slow**, since 1–4 needs to be done $T$ times, once for each $s$

- Paper proposes **fake news algorithm** that is $T$ times faster:
  - requires **single** backward iteration & **single** forward iteration
  - key idea: exploit **time symmetries** around the steady-state

## (0) The fake news matrix

- We can think of $\mathcal{J} \equiv \left(\frac{\partial C_t}{\partial w_s}\right)$ as a **news matrix**
  - column $s$ = response to news that shock hits in period $s$
- Define a new auxiliary matrix:

$$\mathcal{F}_{t,s} \equiv \begin{cases} \frac{\partial C_t}{\partial w_s} & s = 0 \text{ or } t = 0 \\ \frac{\partial C_t}{\partial w_s} - \frac{\partial C_{t-1}}{\partial w_{s-1}} & s, t > 0 \end{cases}$$

- Can think of this as **fake news matrix:**
  - at $t = 0$: news shock that period $s$ shock hits $\rightarrow \frac{\partial C_0}{\partial w_s}$
  - at $t = 1$: news shock that there won't be a shock at $s \rightarrow \frac{\partial C_1}{\partial w_s} - \frac{\partial C_0}{\partial w_{s-1}}$
  - useful: starting in $t = 1$, **agents' policy functions are unchanged** by fake news shock
- Can recover $\mathcal{J}$ from $\mathcal{F}$: news shock = sequence of fake news shocks

## (o) The fake news matrix

$$\mathcal{J} = \begin{pmatrix} \mathcal{J}_{00} & \mathcal{J}_{01} & \mathcal{J}_{02} & \cdots \\ \mathcal{J}_{10} & \mathcal{J}_{11} & \mathcal{J}_{12} & \cdots \\ \mathcal{J}_{20} & \mathcal{J}_{12} & \mathcal{J}_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad \mathcal{F} = \begin{pmatrix} \mathcal{J}_{00} & \mathcal{J}_{01} & \mathcal{J}_{02} & \cdots \\ \mathcal{J}_{10} & \mathcal{J}_{11} - \mathcal{J}_{00} & \mathcal{J}_{12} - \mathcal{J}_{01} & \cdots \\ \mathcal{J}_{20} & \mathcal{J}_{12} - \mathcal{J}_{10} & \mathcal{J}_{22} - \mathcal{J}_{11} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

- Can recover $\mathcal{J}$ from $\mathcal{F}$ by adding elements from top left diagonal

## (1) Single backward iteration

- **Claim:** Single backward iteration is enough to recover $\mathbf{c}_t^s(e, k_-), \mathbf{k}_t^s(e, k_-)$
- Why? only the **time** $s - t$ **until the perturbation matters**

$$\mathbf{c}_t^s(e, k_-) = \begin{cases} \mathbf{c}(e, k_-) & s < t \\ \mathbf{c}_{T-1-(s-t)}^{T-1}(e, k_-) & s \geq t \end{cases}$$

- Thus, only need a single backward iteration with $s = T - 1$ to get all the $\mathbf{c}_t^s$
- From these we get:
  - $C_0^s = \int \mathbf{c}_0^s(e, k_-) D(e, dk_-)$, so first row of Jacobian $\mathcal{J}_{0s} = \frac{\partial C_0}{\partial w_s} = \mathcal{F}_{0s}$
  - $D_1^s(e, dk_-)$, distributions at date 1 implied by new policy $\mathbf{c}_0^s$ at date 0

## (2) Single forward iteration

- Let's iterate those distributions forward using **s.s. policies**

$$D_1^s(e, dk_-) \mapsto D_2^s(e, dk_-) \mapsto D_3^s(e, dk_-) \mapsto \dots$$

  - this is just a **linear map**: $\mathbf{D}_t^s = (\Lambda')^{t-1}\,\mathbf{D}_1^s$ where $\Lambda$ is s.s. transition matrix

- Now construct aggregate consumption using **s.s. policies c**

$$C_t^s \equiv \int \mathbf{c}(e, k_-)\, D_t^s(e, dk_-) \quad \Rightarrow \quad C_t^s = \mathbf{c}'\left(\Lambda'\right)^{t-1}\mathbf{D}_1^s$$

  - this only requires computing $\mathbf{c}'$, $\mathbf{c}'\Lambda'$, $\mathbf{c}'\left(\Lambda'\right)^2$, $\dots \to$ like a **single** forward iteration!

- This is exactly the **fake news matrix**

$$\mathcal{F}_{t,s} = (C_t^s - C)/\epsilon$$

# Extensions of the Aiyagari Economy

- Agents can either not work or work: $\varepsilon = \{0, 1\}$,
- Agents can exert painful effort $h$ to search for a job increasing the probability $\phi(h)$ (with $\phi' > 0$) of finding it.
- An employed worker, does not search for a job so $h = 0$, but its job can be destroyed with some exogenous probability $\delta$.
- $s$ is Markovian ($\Gamma$) labor labor productivity. Then the unemployed

$$V(s,0,a) = \max_{c,h,a' \geq 0} u(c,h) + \beta \sum_{s'} \Gamma_{ss'} \left[\phi(h)V(s',1,a') + (1-\phi(h))V(s',0,a')\right]$$

$$s.t. \quad c + a' = b + (1+r)a$$

The employed

$$V(s,1,a) = \max_{c,a' \geq 0} u(c) + \beta \sum_{s'} \Gamma_{ss'} \left[\delta V(s',0,a') + (1-\delta)V(s',1,a')\right]$$

$$s.t. \quad c + a' = s\ w + (1+r)a$$

- Consider a Matching Function $M(H, T)$ where $H$ is aggregate household search effort and $T$ is the number of vacancies created. There is no way to separate workers by type.
- Creating a vacancy requires a machine of size $\kappa$ and the cost off positing a vacancy $c^{\kappa}$.
- The household size looks similar but no the firm size.
- Firms are machines that may or may not be matched with a worker. They get destroyed at rate $\delta$.
- If they are matched there is a need to split output.
- Need to specify protocol: Bargaining, wage posting, wage as a function of $s$.
- We can mix this with Industry Equilibria, but today the $s$ would be that of the worker.
- Define Stationary Equilibrium

- Suppose every period agents choose an occupation: entrepreneur or a worker.
- Entrepreneurs run their own business: manage a project that combines entrepreneurial ability ($\eta$), capital ($k$), and labor($n$); while workers work for somebody else.
- If worker

$$V^w(s,\eta,a) = \max_{c,a'\geq 0, d\in\{0,1\}} u(c) + \beta \sum_{s',\eta'} \Gamma_{ss'}\Gamma_{\eta\eta'}\left[dV^w(s',\eta',a') + (1-d)V^e(s',\eta',a')\right]$$

$$s.t. \quad c + a' = w\ s + (1+r)a$$

- Similarly, the entrepreneur's problem can be formulated as follows

$$V^e(s,\eta,a) = \max_{c,a'\geq 0, d\in\{0,1\}} u(c) + \beta \sum_{s',\eta'} \Gamma_{ss'}\Gamma_{\eta\eta'} \left[d \; V^w(s',\eta',a') + (1-d)V^e(s',\eta',a')\right]$$

$$s.t. \quad c + a' = \pi(s,\eta,a)$$

- Income is from profits $\pi(a,s,\eta)$ not wages. Assume entrepreneurs have a Decreasing Returns to Scale technology $f$. Profits are

$$\pi(s,\eta,a) = \max_{k,n} \eta f(k,n) + (1-\delta)k - (1+r)(k-a) - w\max\{n-s,0\}$$

$$s.t. \quad k - a \leq \phi a$$

- The constraint here reflects the fact that entrepreneurs can only make loans up to a fraction $\phi$ of his total wealth.

- Entrepreneurs never make an operating loss within a period, (can always choose $k = n = 0$ and earn the risk free rate on savings).
- Agents with high entrepreneurial ability $\eta$ have access to an investment technology $f$ that provides higher returns than workers so become richer.
- Even the prospects (high $\eta$) low wealth suffice to induce high savings? ($\Gamma$)
- Who becomes an entrepreneur in this economy? Without financial constraints, wealth will play no role. $\exists \eta^*$ above which it becomes an entrepreneur.
- With financial constraints wealth matters. Wealthy agents with high $h$ will while the poor with low $\eta$ will not.
- For the rest, it depends. If $\eta$ is persistent, poor individuals with high entrepreneurial ability will save to one day become entrepreneurs, while rich agents with low entrepreneurial ability will lend their assets and become workers.

- The price of lending incorporates the possibility of default.
- Assume upon default punished to autarky forever after (no borrowing or lending)
- If no default the budget constraint is $c + q(a')a' = a + ws$,

$$V(s,a) = \max \left\{ u(ws) + \beta \sum_{s'} \Gamma_{ss'} \bar{V}(s'), \right.$$
$$\left. \max_{c,a'} u[ws + a - q(a')\ a'] + \beta \sum_{s'} \Gamma_{ss'} V(s',a') \right\}$$

where $\bar{V}(s') = \frac{1}{1-\beta} u(ws')$ is the value of autarky.

- What determines $q(a')$? A zero profit on lenders: Probability of default
- Note that this problem displays time inconsistency

# Monopolistic Competition and New Keynesian Models

**Read McKay and Ravn Chapter 16 of Book**

- Models with Nominal Prices.
- Price/Wage Rigidity.
- Firms are sufficiently "different" to set prices.
- Small in the Context of the Aggregate Economy. Hence Monopolistic Competition.

- Consumers have a taste for variety
    - The consumer's utility function has constant elasticity of substitution (CES)

$$u\left(\{c(i)\}_{i\in[0,n]}\right) = \left(\int_0^n c(i)^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{\sigma}{\sigma-1}}$$

where $\sigma$ is the elasticity of substitution, and $c(i)$ is the quantity consumed of variety $i$. For ease of notation, we rename $c(i) = c_i$.

- Assume the agents receive exogenous *nominal* income $I$
- They are endowed with one unit of time.

$$\max_{\{c_i\}_{i\in[0,n]}} \left(\int_0^n c_i^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{\sigma}{\sigma-1}}$$

$$s.t. \quad \int_0^n p_i \; c_i \; di \leq I$$

- Deriving the FOC, and relating the demand for varieties $i$ and $j$

$$c_j = c_i \left(\frac{p_j}{p_i}\right)^{-\sigma}$$

- Multiplying both sides by $p_j$ and integrating over $j$, yields

$$c_i^* = \frac{I}{\int_0^n p_j^{1-\sigma} dj} \, p_i^{-\sigma}$$

- Here $c_i^*$ depends on the price of $i$ and an aggregate price

- Convenient to define the aggregate price index $P$ as

$$P = \left( \int_0^n p_j^{1-\sigma} dj \right)^{\frac{1}{1-\sigma}}$$

- and thus

$$c_i^* = \frac{I}{P} \left( \frac{p_i}{P} \right)^{-\sigma}$$

real income times a measure of the relative price of $i$.

**Exercise**

*Show the following within this monopolistic competition framework*

1. *$\sigma$ is the elasticity of substitution between varieties.*
2. *Price index $P$ is the expenditure to purchase a unit-level utility.*
3. *Consumer utility is increasing in the number of varieties $n$.*
4. *Is there a missing $n$?*

- Assume linear production technology: $f(\ell_j) = Z\ \ell_j$. (When in steady state $Z = 1$)
- Nominal wage rate is given by $W$.
- The firm solves

$$\max_{p_j} \pi(p_j) = p_j\ c_j^*(p_j) - W\ c_j^*(p_j)$$

$$\text{s.t.} \qquad c_j^* = \frac{I}{P}\left(\frac{p_j}{P}\right)^{-\sigma}$$

- Firms do not affect $P$. Solve for the FOC:

$$p_j^* = \frac{\sigma}{\sigma - 1} W \qquad \forall j$$

- $\frac{\sigma}{\sigma-1}$ is a constant mark-up over the marginal cost,
- When varieties are close substitutes ($\sigma \to \infty$), prices converge to $W$.

Set the wage as numeraire. An Eq is prices $\{p_i^*\}_{i\in[0,n]}$, the aggregate price index $P$, household's consumption, $\{c_i^*\}_{i\in[0,n]}$, income $I$, firm's labor demand $\{\ell_i^*\}_{i\in[0,n]}$ and profits $\{\pi_i^*\}_{i\in[0,n]}$, such that

1. Given prices, $\{c_i^*\}_{i\in[0,n]}$ solves the household's problem
2. Given $P$ and $I$, $p_i^*$ and $\pi_i^*$ solve the firm's problem $\forall i \in [0, n]$
3. Price Aggregation

$$P = \left(\int_0^n (p_j^*)^{1-\sigma}\, dj\right)^{\frac{1}{1-\sigma}}$$

4. Markets clear

$$\begin{aligned}\int_0^n \ell_i^*\, di &= 1 \\ 1 + \int \pi_i^*\, di &= I\end{aligned}$$

A symmetric equilibria: $c_i^* = \bar{c}$, $p_i^* = \bar{p}$, $\ell_i^* = \bar{\ell}$, $\pi_i^* = \bar{\pi}$ for all $i$.

- To study inflation, (meaningful interactions between output and inflation) needs
    1. A dynamic model
    2. Some source of nominal frictions so nominal variables (things measured in dollars) can affect real variables.
- Most popular friction is *price rigidity* (firms cannot adjust their prices freely)
    1. *Rotemberg pricing* (menu costs)
    2. *Calvo pricing* (some (randomly set) firms can change prices, others cannot).

- Firms face adjustment cost $\phi(p_j, p_j^-)$ when changing their prices $p_j$ each period.
- Let the Agg State be $S$, and let $I(S)$, $W(S)$, $P(S)$. Then firm's per period profit under Rotemberg pricing in a dynamic setup as follows:

$$\Omega(S, p_j^-) = \max_{p_j} \; p_j c_j^* - W(S) c_j^* - \phi(p_j, p_j^-)$$

$$+ E\{R^{-1}(G(S))\, \Omega(G(S), p_j)\}$$

$$\text{where} \quad c_j^* = \left(\frac{p_j}{P(S)}\right)^{-\sigma} \frac{I(S)}{P(S)}$$

- easy algebra when quadratic price adjustment cost.

**Exercise**

*Derive the First Order Condition of the Firm's Problem*

- Without capital $S = P^-$ and Aggregate Shocks.

- Firms can adjust their prices each period with probability $\theta$.
- A firm that can change its price

$$\Omega^1(S, p_j^-) = \max_{p_j} p_j c_j^* - W(S)c_j^* + (1-\theta)E\{R^{-1}(S')\,\Omega^0(S', p_j)\}$$
$$+ \qquad \theta\, E\{R^{-1}(S')\,\Omega^1(S', p_j)\}$$

$$\text{where} \quad c_j^* = \left(\frac{p_j}{P(S)}\right)^{-\sigma} \frac{I(S)}{P(S)} \quad \text{and} \quad S' = G(S)$$

- A firm that cannot

$$\Omega^0(S, p_j^-) = [p_j^- - W(S)]c_j^* +$$
$$(1-\theta)E\{R^{-1}(S')\,\Omega^0(S', p_j^-)\} +$$
$$\theta\, E\{R^{-1}(S')\,\Omega^1(S', p_j^-)\}$$

- We can also write $\Omega(S, p)$ generically and when choosing price $p(S) = \Omega(S, p)$.

**Exercise**

*Derive the following for the dynamic model with Calvo pricing*

1. *Solve the firm's problem in sequence space and write the firm's equilibrium pricing* $p_{j,t}$ *as a function of present and future aggregate prices, wages, and endowments:* $\{P_t, W_t, I_t\}_{t=0}^{\infty}$.
2. *Show that under flexible pricing* ($\theta = 1$)*, the firm's pricing strategy is identical to the static model.*
3. *Show that with price rigidity* ($\theta < 1$)*, the firm's pricing strategy is identical to the static model in a steady state with zero inflation.*

- Rotenberg: pricing equilibrium is $P^{-} \in S$ and $P = p^{*}(S, P^{-})$
- Calvo: there is a $P^{-} \in S$ and recall that $P = \left(\int_0^n p_j^{1-\sigma} dj\right)^{\frac{1}{1-\sigma}}$.
  - Because we have $\theta$ firms adjusting and $1-\theta$ not, we have

$$P = \left[\theta \left(P^{-}\right)^{1-\sigma} + (1-\theta)\left(p^{*}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}$$

    for the optimally chosen $p^{*}$.
  - This is the magic of Calvo pricing: The distritribution of prices is NOT a state variable.
  - That turns out to satisfy (after using representative agent condition)

$$P^{*} = \frac{\sigma}{\sigma-1} \frac{E\left\{\sum_{\tau}(\theta\beta)^{\tau} u_c \; P_{\tau}^{\sigma-1} \; \varphi_{\tau} \; y_{\tau}\right\}}{E\left\{\sum_{\tau}(\theta\beta)^{\tau} u_c \; P_{\tau}^{\sigma-1} \; y_{\tau}\right\}}$$

    where $\varphi_{\tau}$ is nominal marginal cost
  - Is this a nightmare? No. Log-linearization comes to help

- Let $X$

- Let $\overline{X}$ be the steady state.

- Sometimes we want to use

$$\widehat{x} = \log X - \log \overline{X}$$

- We say Log Deviations

- Products

$$Z = X^{\alpha}\ Y^{\beta} \quad \Longrightarrow\ \widehat{z} = \alpha\ \widehat{x} + \beta\ \widehat{y}$$

- Sums

$$\overline{Z}\ \widehat{z} = \alpha\ \overline{X}\ \widehat{x} + \beta\ \overline{Y}\ \widehat{y}$$

- Smooth Functions $Z = f(X, Y) \Longrightarrow$

$$\overline{Z} \simeq \widehat{z} = f_x(\overline{X}, \overline{Y})\ \ \overline{X}\ \widehat{x} + \beta\ f_y(\overline{X}, \overline{Y})\ \overline{Y}\ \widehat{y}$$

- Recall the Law of motion for the price level

$$P = \left[\theta \left(P^{-}\right)^{1-\sigma} + (1-\theta)\left(p^{*}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}$$

- Log-linearizing around the steady state

$$\frac{1}{1-\sigma}\overline{P}\widehat{p} \simeq \theta \frac{1}{1-\sigma}\overline{P}\widehat{p^{-}} + (1-\theta)\frac{1}{1-\sigma}\overline{P}^{*}\widehat{p}^{*}$$

ignoring the constants (they cancel from both sides), and noting that in St St $\overline{P} = \overline{P}^{*}$ we have $\widehat{p} = \theta\ \widehat{p^{-}} + (1-\theta)\ \widehat{p}^{*}$

- and because the steady state is common $\widehat{p} \simeq \theta\ \widehat{p^{-}} + (1-\theta)\ \widehat{p}^{*}$ so

$$p \simeq \theta\ p^{-} + (1-\theta)\ p^{*}$$

- Which implies for inflation that

$$\pi = p - p^{-} = (1-\theta)\ \left(\widehat{p}^{*} - \widehat{p^{-}}\right)$$

- Price setting

$$P^* = \frac{\sigma}{\sigma - 1} \frac{E\left\{\sum_\tau (\theta\beta)^\tau u_c \; P_\tau^{\sigma-1} \; \varphi_\tau \; y_\tau\right\}}{E\left\{\sum_\tau (\theta\beta)^\tau u_c \; P_\tau^{\sigma-1} \; y_\tau\right\}}$$

or

$$E\left\{\sum_\tau (\theta\beta)^\tau u_c \; P_\tau^{\sigma-1} \; y_\tau\right\} \; P^* = \frac{\sigma}{\sigma - 1} E\left\{\sum_\tau (\theta\beta)^\tau u_c \; P_\tau^{\sigma-1} \; \varphi_\tau \; y_\tau\right\}$$

- Approximating the left hand side gives the terms

$$E\left\{\sum_\tau \, (\theta\beta)^\tau \; \overline{U}_c \; \overline{P}^{\sigma-1} \; \overline{Y} \; \overline{P}^* \, [\widehat{u}_{c,\tau} + (\sigma - 1)\widehat{p}_\tau + \widehat{y}_\tau + \widehat{p}^*]\right\}$$

Steady state values $\overline{U}_c$, $\overline{P}$ etc are common to all terms in the sum

- Approximating the right hand side yields

$$\frac{\sigma}{\sigma-1} E\left\{\sum_{\tau} (\theta\beta)^{\tau}\ \overline{U}_c\ \overline{P}^{\sigma-1}\ \overline{\varphi}\ \overline{Y}\ \overline{P}^{*} \left[\widehat{u}_{c,\tau} + (\sigma-1)\widehat{p}_{\tau} + \widehat{\varphi}_{\tau} + \widehat{y}_{\tau} + \right]\right\}$$

- Because in St St $\overline{P}^{*} = \frac{s}{s-1}\ \overline{\varphi}$ we can cancel all the common terms so

$$E\left\{\sum_{\tau} (\theta\beta)^{t}\ \widehat{p}^{*}\right\} = \widehat{p}^{*}\ E\sum_{\tau} (\theta\beta)^{t} \simeq E\left\{\sum_{\tau} (\theta\beta)^{\tau}\ \overline{\varphi}_{\tau}\right\}$$

- Calculating the sum yields $\widehat{p}^{*} \simeq (1-\theta\beta)\ E\left\{\sum_{\tau} (\theta\beta)^{\tau}\ \overline{\varphi}_{\tau}\right\}$
- And Adding back in Steady State terms yield

$$\widehat{p}^{*} = \mu + (1-\theta\beta)\ E\left\{\sum_{\tau}(\theta\beta)^{\tau}\ \left[mc_{\tau} + p_{\tau}\right]\right\}$$

where $\mu = \log\frac{\sigma}{\sigma-1}$ and where $mc_{\tau}$ is log real marginal cost

- It solves

$$v(S,b) = \max_{c,\ell} u(c,\ell) + \beta \; E\{v(S',b') \qquad \text{s.t.}$$

$$b' + P(S)\; c = P(S)\; w\; \ell + [1+i^{-}(S)]b + P(S) \int m_j(S)\; dj.$$

here $b$ are nominal bonds $m_j$ are the profits of firm $j$ and $i$ the nominal interest rate,

- Dividing the Budget Constraint by Prices to get it in real terms

$$\frac{b'}{P(S)} + c = w\; \ell + \frac{1+i^{-}(S)}{1+\pi(S)}\; \frac{b}{P(S)} + \int \pi_j(S)\; dj.$$

with $\pi$ being the inflation rate.

- We get the Euler equation and the intratemporal condition

$$u_c[C(S,b),\ell(S,b)] = \beta\; E\left\{\frac{1+i(S)}{1+\pi(S')}\; u_c[C(S',b'),\ell(S',b')]\right\}$$

$$u_c[C(S,b),\ell(S,b)]\; w = u_\ell[C(S,b),\ell(S,b)].$$

- We have the law of motion of the price level (backward looking)

$$P(S) = \left[\theta \left(P^{-}\right)^{1-\sigma} + (1-\theta)\left(p(S)\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}$$

- But inflation is forward looking. Recall that the resetting price satisfies

$$p(S) = \frac{\left[\frac{P^{1-\sigma}-\theta(P^{-})}{1-\theta}\right]^{\frac{1}{1-\sigma}}}{P} = \left[\frac{1-\theta(1+\pi)^{\sigma-1}}{1-\theta}\right]^{\frac{1}{1-\sigma}}$$

- Rearranging

$$1+\pi = \left[\frac{1-(1-\theta)P(S)^{1-\sigma}}{\theta}\right]^{\frac{1}{1-\sigma}}$$

- Price dispersion is source of inefficiency (output varieties' dispersion):

$$Y = \left(\int_0^1 y_j^{\frac{\sigma-1}{\sigma}} \, dj\right)^{\frac{1}{1-\sigma}} = \frac{A}{D} L, \qquad \text{where } D = \int \left(\frac{p_j}{P}\right)^{-\sigma}$$

- Note also that $D = (1-\theta)P(S)^{\sigma} + \theta(1+\pi)^{\sigma} \; D^{-}$ .

- The state variables are the shocks which may include
  - TFP Shocks, $Z$
  - Mark-ups Shocks to $\sigma$
  - Patience Shocks to $\beta$
  - Shocks to the Mood of the Central Bank $\omega$
- The Degree of Price Dispersion $D^{-}$

- There is still a missing piece: Nominal determination.
- Sets nominal interest rates
- Often represented through an interest rate rule that specifies the interest rate as an explicit function of macroeconomic variables such as the rate of inflation.

$$i = \widehat{i} + \phi_\pi\ (\pi - \pi^*) + \phi_x\ x + \omega$$

there is an inflation target $\pi^*$ and $\widehat{i}$ is the nominal interest rate consistent in steady state with $\pi^*$. So $\widehat{i} = \beta^{-1}(1 + \pi^*) - 1$. This is a Taylor rule ($\phi_\pi = 1.5$ and $\phi_x = .5$. The monetary shocks are $\omega$.

**Exercise**

*Define Equilibrium recursively*

1. The IS Curve (log-linearized Euler equation)

$$x = E\{x'\} - \frac{1}{\sigma} \left( \log(1+i) - E\{\pi'\} + \log\beta - \frac{\sigma(1+\psi)}{\sigma\ \psi}\ (Z - E\{Z'\}) \right)$$

   Where we use CRRA and constant Frisch $\psi$. Here $x$ is the output gap, and the term after the inflation one is the real natural rate of interest.

2. The New Keynesian Phillips curve that comes from using the firms FOC to yield

$$\pi = \kappa\ x + \beta\ E\{\pi'\}$$

3. The Taylor (or other) monetary policy Rule

$$i = \widehat{i} + \phi_\pi\ (\pi - \pi^*) + \phi_x\ x + \omega$$

- Similar tools and logic as with Sticky Prices.
- Instead of using "raw" undifferentiated labor, firms use an aggregate of many labor varieties:

$$L_j = \left( \int_{i=0}^{1} \ell_{ji}^{\frac{\xi-1}{\xi}} \right)^{\frac{\xi}{\xi-1}}$$

  they pay $\int_{i=0}^{1} \ell_{ji} \, w_i$.
- Each Household has a measure one of workers, encompassing all varieties. Each member of the household is asked to work $\ell_i$ hours.
- Each Labor variety $i$ has a union that only wants to maximize the revenue of its members (not the household's wellbeing). It has monopsonist power for a given aggregate production, so its sets the wage.

**Exercise**

*Pose the problem of the union as a function of aggregate output (when prices are symmetric)*

- This can be posed dynamically with wage rigidity a la Calvo o a la Rotemberg.

# Extreme Value Shocks

- Let an agent have $I$ choices that yield utility.
- Let the cost (or something else) of each choice be $z^i$, with vector $z = \{z^i\}_{i=1}^I$.
- We want to make sense of
  1. Percentage of choices being $x^i(z)$
  2. For various vectors of prices $z$ (so that we have a theory of changes of behavior). In particular to learn about elasticity.
- Let $u_i + v(c)$ be fundamental utility of choice $i$ where $c$ is other consumption.
- Let $\epsilon^i$ be an idyosincratic shock to each agent. then

$$\max_i \{u^i + \epsilon^i + v(y - z^i)\} = \max_i \{u^i + \beta z^i + \epsilon^i\}$$

- If $\epsilon^i$ extreme value Gumbel then the probability of $i$, $p^i$ is logit

$$p^i = \frac{\exp^{u^i + \beta z^i}}{\sum_{j=1}^I \exp^{u^j + \beta z^j}}$$

- This is now estimated (ML). Estimation should include Variance of shocks.
- Problem of correlated choices (blue/red bus). A Solution is to nest.

- Savings (or Durables, retirement, quits, marriage and so on).
- In one period normally $\max\{u(y,0), u(y-q,1)\}$
- If separable and strictly concave, solution is to do 0 for $y < \overline{y}$ and 1 for $y \geq \overline{y}$, implying a drop in $c$.
- The problem is that discontinuities propagate in time. A solution is to pose Extreme Value Shocks e.g. (without adjustment costs)

$$V(s,a) = \max\ \{V^0(a), V^1(a)\} =$$
$$\max\left\{\max_{a'}\ u(aR+s-a',0)+\epsilon^0+E\ V(s',a'),\right.$$
$$\left.\max_{a'}\ u(aR+s-a'-q,1)+\epsilon^1+E\ V(s',a')\right\}$$

- This gets rid of kinks and discontinuities as both choices are always possible for any $a$. But can cause problems.

- If $\epsilon$ follows i.i.d. $G(\mu, \alpha)$, where the mode $\mu$ is non-zero, we have

$$V^1 = E\{\epsilon\} = \mu + \alpha\ \gamma$$

$\gamma \simeq .57721$ is the Euler Mascheroni constant

$$\text{Mode}\ \ \{\epsilon\} = \mu$$

$$\text{Median}\{\epsilon\} = \mu - \alpha\ \ \ln(\ln 2)$$

$$\text{Var}\{\epsilon\} = \frac{\pi^2\ \alpha^2}{6}$$

$$\text{cdf}\{\epsilon\} = e^{\left\{-e^{\left[-\frac{(\epsilon-\mu)}{\alpha}\right]}\right\}}$$

- Expected maximum of $N$ Gumbel random variables $G(\mu, \alpha)$. Let

$$X^N = \max\left\{\epsilon^1, \epsilon^2, \cdots, \epsilon^N\right\}$$

- We have

$$X^N \sim G\left(\mu + \alpha \ \ln N, \alpha\right)$$

$$\mathbb{E}\left[X^N\right] = \mu + \alpha \ \ln N + \alpha \ \gamma$$

- To make $\mathbb{E}\left[X^N\right]$ independent of the number of choices $N$, either

$$\mathbb{E}\left[X^N\right] = \bar{V} \Rightarrow \alpha(N) = \frac{\bar{V} - \mu}{\gamma + \ln N}$$

$$\mathbb{E}\left[X^N\right] = \bar{V} \Rightarrow \mu(N) = \bar{V} - \alpha \ \ln N - \alpha \ \gamma$$

better the latter so that they are all Gumbel

- $\eta^i$ follows $G(\mu, \alpha)$, let $\epsilon^i = \eta^i + \delta^i$, $\epsilon^i \sim G(\mu + \delta^i, \alpha)$.

$$X^N \sim G\left(\alpha \ln \sum_i e^{\frac{\mu^i}{\alpha}}, \alpha\right) = G\left(\mu + \alpha \ln \sum_i e^{\frac{\delta^i}{\alpha}}, \alpha\right)$$

$$\mathbb{E}\left[X^N\right] = \mu + \alpha \ln \sum_i e^{\frac{\delta^i}{\alpha}} + \alpha\,\gamma$$

- To make $\mathbb{E}\left[X^N\right]$ independent of the number of choices, we can require

$$\mathbb{E}\left[X^N\right] = \bar{V} \Rightarrow \alpha(N) = \frac{\bar{V} - \mu}{\gamma + \ln \sum_i e^{\frac{\mu^i}{\alpha(N)}}}$$

$$\mathbb{E}\left[X^N\right] = \bar{V} \Rightarrow \mu(N) = \bar{V} - \alpha\left[\gamma + \ln \sum_i e^{\frac{\mu^i}{\alpha}}\right]$$

- No closed-form solution for $\alpha(N)$

# The continuum

- Consider an interval $C = [0, \bar{c}]$, and an $\epsilon(c), \forall c \in C$ . We want

$$V^C = E\left\{\max_{c \in C} \{\epsilon(c)\}\right\}, \quad \epsilon(c) \sim G(0, \alpha(C)), \qquad \text{for some} \quad V^C > 0.$$

- We proceed by instead letting $N$ draws in an equal sized grid over $C$ and associating to each $n \in \{1, 2, \cdots, N\}$ a Gumbel $\epsilon^n \sim G(0, \alpha(N))$.

- Let $X^N = \max_{n \in \{1,2,\cdots,N\}} \{\epsilon^n\}$ and $V^N = E\left\{X^N\right\}$.

- We choose $\alpha(V^C, N)$ so that $V^N = V^C$ : $\qquad \alpha\left(V^C, N\right) = \frac{V^C}{\ln N + \gamma}$ for any $N$.

- As we have seen, $V^N$ is increasing in $N$. So no good to set $\mu$ so that $V^1 = 0$. More choice gives more utility.

- Is this fundamental?

- It depends. But if it is, there is a form of precautionary savings: Agents want to save to have more choice (a larger choice set $C$) in the future.

- Violating the Euler equation by choice becomes a valuable privilege.

- If so we have to design algorithms that respect this feature.

- We have to think of $V^C$ as a fundamental parameter that determines the size of the utility bonus for the richest agent (the one with the largest choice set).

- Let $\widetilde{c} < \overline{c}$, $[0, \widetilde{c}]$ a smaller choice set.
- Let $N^{\widetilde{c}} = \max_{n \geq 0} \frac{n}{N^{\overline{c}}} < \frac{\widetilde{c}}{\overline{c}}$, the point to the left of an imagined grid of size $N^{\overline{c}} + 1$.
- Then we associate with choice set $C^{\widetilde{c}}$, a draw of $N^{\widetilde{c}}$ $\epsilon'$s with probability $\underline{p}(\widetilde{c}) = \frac{N^{\widetilde{c}}+1}{N^{\overline{c}}} - \frac{\widetilde{c}}{\overline{c}}$, and a draw of $N^{\widetilde{c}} + 1$ with probability $\overline{p}(\widetilde{c}) = \frac{\widetilde{c}}{\overline{c}} - \frac{N^{\widetilde{c}}}{N^{\overline{c}}}$.
- Drawing zero $\epsilon$ yields expected utility 0.
- Let $$V^{\widetilde{c}} = \underline{p}(\widetilde{c}) V^{N^{\widetilde{c}}} + \overline{p}(\widetilde{c}) V^{N^{\widetilde{c}}+1}.$$
- Where $$V^n = \alpha\left(V^{\overline{c}}, N^{\overline{c}}\right)(\ln n + \gamma), \text{ for } n = N^{\widetilde{c}}, N^{\widetilde{c}} + 1.$$
- Note that the utility bonus $V^{\overline{c}}$ is of the right size given $V^C$.

- When writing algorithms, we have to be aware that the density of grid points is not the same as the size of the choice set $[0, \overline{c}]$.

- So we have to adjust for both.

  1. Find $\widetilde{c}(j)$, the maximum consumption attainable in state $j$. It depends on the budget constraint and prices not on the details of the grid.

  2. Compute $V^{\widetilde{c}(j)}$ as explained above

  3. Find the appropriate $\alpha(V^{\widetilde{c}}, j)$. This requires

     - Find $M(x)$ this is the number of grid points accessible from $x$. This depends on the grid system but also on prices.

     - Solve for $\alpha(V^{\widetilde{c}}, x) = \frac{V^{\widetilde{c}(x)}}{\ln M(x)+\gamma}$.

- Now you can iterate on the value function that includes the utility bonus.

# Agents in Aiyagari worlds with Extreme Value Shocks

- The fundamental problem

$$v(s,a) = \max_{a',c=sw+aR-a'} \left\{ \frac{c^{1-\sigma}-1}{1-\sigma} + \epsilon(c) + \sum_{s'} \Gamma_{s,s'} \; v(s',a') \right\}$$

- Fix $N$, a large integer, we approximate the problem by

$$v(s,a) = \max_{a^{n\prime}=sw+aR-c^n,c^n} \left\{ \frac{c^{1-\sigma}-1}{1-\sigma} + \epsilon^n + \sum_{s'} \Gamma_{s,s'} \; v(s',a^{n\prime}) \right\}$$

We have to impute the right probabilities

# Endogenous Growth and R&D

- Exogenous Growth

$$F(K, N) = A\ K^{\theta_1}\ L^{\theta_2},$$

- Need $\theta_1 + \theta_2 \leq 1$ for consistency with the notion of competitive equilibrium. (Even $<$ is a bit problematic).

- Then the economy cannot grow in per capita terms.

- So it has to be $A$: Exogenous

- Still, empirically, the problem is NOT accounting for **growth rate** differences but for output **LEVEL** differences

- The $AK$ model: technology is linear in reproducible capital (it can include human capital as long it is accumulated by using reproducible factors, i.e. schools not time.

- The existence of externalities in production. Consider a firm production function with an aggregate externality:

$$F(k,n) = A\, K^{1-\theta_1}\, k^{\theta_1}\, n^{\theta_2},$$

- An explicit accumulation of technology

- Three sectors in the economy.
  1. Final goods are competitive use labor and intermediate goods according to

     $$N_{1,t}^{\alpha} \int_0^{A_t} x_t(i)^{1-\alpha} \, di$$

     where $x(i)$ denotes the utilization of intermediate good of variety $i \in [0, A_t]$.

  2. Intermediate producers are monopolists. They have a differentiated technology of the form:

     $$x(i) = \frac{k(i)}{\eta}.$$

     Note: aggregate demand of capital is $\int_0^{A_t} \eta x(i) \, di$.

  3. R&D sector. A new good is a new variety of the intermediate good produced using labor:

     $$\frac{A_{t+1}}{A_t} = 1 + \xi N_{2,t}.$$

     we can write $A_{t+1} - A_t = A_t \xi N_{2,t}$, so the flow of new intermediate goods is determined by the current stock of them in the economy (an externality).

     Right to produce new goods sold to new monopolists.

**Remark**

*The reason we see $A_t$ on the previous expression as an externality is that it is indeed used as an input in the process of R&D, while, it is not paid for. Thus, inventors, in a sense, do not pay the investors of the previous varieties, while using their inventions. They only pay for the labor they hire. Perhaps, the basic idea of this production function might be traced back to Isaac Newton's quote: "If I have seen further, it is only by standing on the shoulders of giants".*

**Exercise**

*If the price of all varieties are the same, what is the optimal choice of input vector for a producer?*

**Exercise**

*What if they do not have the same amount? Would a firm decide not to use a variety in the production?*

- Preferences

$$\sum_{t=0}^{\infty} \beta^t u(c_t),$$

- Budget constraint

$$c_t + k_{t+1} \leq r_t k_t + w_t + (1-\delta) k_t + \pi_t.$$

**Remark**

*In this economy, GDP is* $Y_t = W_t + r_t K_t + \pi_t$, *where* $\pi_t$ *are profits.* *In terms of expenditures, GDP is* $Y_t = C_t + K_{t+1} - (1-\delta) K_t + \pi_t$, *where* $K_{t+1} - (1-\delta) K_t$ *is the investment in physical capital. In terms of value added, it is* $Y_t = N_t^{\alpha} \int_0^{A_t} x_t(i)^{1-\alpha} di + p_t (A_{t+1} - A_t)$.

- Not a model that maps well to the data, yet carefully crafted to convey ideas.

- Final good producer; it chooses $N_{1,t}$ and $x_t(i)$, $\forall i \in [0, A_t]$,

$$\max \; N_{1,t}^{\alpha} \int_0^{A_t} x_t(i)^{1-\alpha} \, di - w_t N_{1,t} - \int_0^{A_t} q_t(i) \, x_t(i) \, di,$$

where $q_t(i)$ is the price of variety $i$ in period $t$. First order conditions are:

1. $N_{1,t}$: $\alpha N_{1,t}^{\alpha-1} \int_0^{A_t} x_t(i)^{1-\alpha} \, di = w_t$; and,
2. $x_t(i)$: $(1-\alpha) N_{1,t}^{\alpha} x_t(i)^{-\alpha} = q_t(i)$, for all $i \in [0, A_t]$.

- Note the monopolistic competition type of condition

$$x_t(i) = \left( \frac{(1-\alpha)}{q_t(i)} \right)^{\frac{1}{\alpha}} N_{1,t},$$

- which, given $N_{1t}$, is the *demand function* for variety $i$, by the final good producer.

- $$\pi_t(i) = \max_{\{q_t(i)\}} \quad q_t(i)\, x_t(q_t(i)) - r_t \eta x_t(q_t(i))$$

$$s.t. \quad x_t(q_t(i)) = \left(\frac{(1-\alpha)}{q_t(i)}\right)^{\frac{1}{\alpha}} N_{1,t},$$

  we substituted for the technology of the monopolist, $x(i) = k(i)/\eta$.

- FOC wrt to $q_t(i)$, is $x_t(q_t(i)) + (q_t(i) - r_t\eta)\frac{\partial x_t(q_t(i))}{\partial q_t(i)} = 0$, which implies

$$\frac{(1-\alpha)^{\frac{1}{\alpha}}}{q_t(i)^{\frac{1}{\alpha}}} N_{1,t} = \frac{(q_t(i) - r_t\eta)}{\alpha} \frac{(1-\alpha)^{\frac{1}{\alpha}}}{q_t(i)^{\frac{1+\alpha}{\alpha}}} N_{1,t}.$$

- Rearranging yields $q_t(i) = \frac{1}{(1-\alpha)} r_t \eta$ and substituting

$$x_t(i) = \left[\frac{(1-\alpha)^2}{r_t\eta}\right]^{\frac{1}{\alpha}} N_{1,t},$$

  and the demand for capital services is simply $\eta x_t(i)$.

- In a symmetric equilibrium $\int_0^{A_t} x_t(i)\, di = A_t x_t = \frac{k_t}{\eta}$,
- Therefore $x_t = \frac{k_t}{\eta A_t}$.
- let $Y_t$ be the production of the final good

$$Y_t = N_{1,t} A_t \left[ \frac{(1-\alpha)^2}{r_t \eta} \right]^{\frac{1-\alpha}{\alpha}}.$$

- Hence the model displays constant returns to scale in $N_{1,t}$ and $A_t$.
- A representative competitive firm chooses $N_{2,t}$ to solve

$$\max_{N_{2,t}} \quad p_t A_t \xi N_{2,t} - w_t N_{2,t}.$$

- With FOC $p_t = \frac{w_t}{A_t \xi}$.

1. Intertemporal Euler equation:

$$u'(c_t) = \beta u'(c_{t+1})\left[r_{t+1} + (1-\delta)\right].$$

2. allocates labor demand for R&D, and that for final good production. For determining the labor choices $N_{1,t}$ and $N_{2,t}$. Note that as long as there are profits in the intermediate good sector, new monopolists will enter yielding a zero profit condition:

$$p_t = \sum_{s=t}^{\infty}\left(\prod_{\tau=t}^{s}\frac{1}{1+r_\tau-\delta}\right)\pi_s.$$

3. Output can grow at the same rate as $A_t$ and as $K_t$.
4. Growth comes from the externality in the R&D sector. Without that, we cannot get sustained growth in this model.
5. This model neatly delivers balanced growth, with just enough structure.

# Growth Model with Many Firms Suitable for Pandemic Times

- This is a growth model suitable to study business cycles.
- Emphasis on small business creation not on inequality so rep hholds.
- Creation and destruction of small firms both for technological and for financial reasons.
- Household cannot help its small businesses in distress.
- We have in mind that even though Pandemic affects both Supply (want less work) and Demand (Less consumption) there is a reduction in output sold per unit of good produced of $\phi(S)$.

- Two sectors as in Quadrini (2000): Corporate and non corporate sector.
- Corporate sector uses capital and labor via aggr prod fn $F(K, N)$
- Non corporate sector: type/size firms $i \in \{1, \cdots, I\}$, $f^i(n)$, $f^i_n > 0$, (provided the firm has the required number of managers, $\lambda^i$).
- A firm requires creation: It costs $\xi^i$ to open a new firm of size $i$.
- Some Firms are destroyed.
  - Firms invest $m$ in maintenance.
  - Probability that a firm survives is $q^i(m)$, $q^i(0) = 0$, $q^i(\infty) < 1$, $q^i_m > 0$ .
- Aggregate measure of type $i$ firms is $X_i$
- The law of motion of new firms is:

$$X_i' = q^i(M_i)\ X_i + B_i$$

- The Aggregate Feasibility Constraint is

$$C + [K' - (1-\delta)K] + \sum_i X_i\ M_i + \sum_i B_i\ \xi_i = \sum_i X_i\ f_i(N_i) + F(K, N).$$

- Household owns measure $x_i$ of firms of type $i \in \{1, \cdots, \mathcal{I}\}$
- The household may be rationed in its workforce: i.e. it may not be in its static Euler equation.
- Households create $b^i$ new firms of type $i$ at cost $\xi^i$ each,
- Managers choose maintenance and profits.
- In addition to its firms, households own $a$ units of corporate capital which they can increase by savings.
- Households allocate its members to managers, workers or enjoyers of leisure:

$$n + \sum_i \lambda^i x^i + \ell = 1.$$

(implicitly we are guessing (to be verified) that all business are operated).

- Households have preferences over consumption $c$ and leisure $\ell$, using utility function $u(c, \ell)$ and discounts the future at rate $\beta$.

- Small firms cannot access financing once they are born.
- They can only give benefits to the household:

$$\Omega^i(S) = \max_{n\geq 0, m\leq \psi(S)f^i(n)-w\ n} \psi(S)\ f^i(n) - w\ n - m + \frac{q^i(m)}{R(S')}\ \Omega^i(S')$$

Here, $S$ is the aggregate state and $s$ in the individual state, $\Psi(S) < 1$ is capacity used which is demand determined and $R(S')$ is the rate of return used by the firm.

- Implictly assuming that there is no need to index $\Omega^i(S)$ by $s$.

**Exercise**

*Get the FOC assuming first that $m$ is unrestricted and then that $m \leq \psi(S)f^i(n) - w\ n$.*

$$V(S, a, x_1, \cdots, x_I) = \max_{c, n, b_1, \cdots, b_I, a'} u(c,\ 1 - n - \sum_i \lambda^i\ x^i) + \beta\ V(S', a', x_1', \cdots, x_I') \qquad s.t.$$

$$c + \sum_i\ b_i\ \xi_i + a' = n\ w(S) + a\ R(S) + \sum_i\ \pi_i(S)\ x_i$$

$$x_i' = q^i(M_i)\ x_i + b_i \qquad i \in \{1, \cdots, I\}.$$

**Exercise**

*Get the FOCs for* $b^i$ $a'$ *and* $n$ *assuming first that* $\lambda^i = 0$ *and* $\pi^i > 0$ *and charaterize the solution (the relation between the FOC of* $b^i$, $m^i$ *and* $a'$*). Then characterize the FOC when* $\lambda^i > 0$.

# An Integraded Analysis Model of Climate Change

- Consider a world with a global externality: using fossil fuel for energy creates carbon dioxide.

- Energy is a required input for the production technology.

- Goal: Derive the optimal policy —here a tax on carbon— so that the externality is internalized.

- Higher levels of carbon dioxide in the atmosphere contributes to global warming, which in turn causes damages like production shortfalls, poor health or deaths, capital destruction and much more.

- Map carbon concentration to climate, and then map climate to damages.

- **Expected sum of future damage elasticities:** the percentage change in output resulting from a percentage change in the amount of carbon in the atmosphere, caused by emitting a unit of carbon today.

- Discounted because of time preferences and because of carbon depreciating.

- Carbon circulation system: carbon is exchanged through various reservoirs such as the atmosphere, the terrestrial biosphere, and different layers of the ocean. A unit of Carbon will remain in the atmosphore $s$ periods after emmited according to

$$\phi_L + (1 - \phi_L)\phi_0(1 - \phi)^s$$

  - $\phi_L$: the share of carbon that stays in the atmosphere forever
  - $(1 - \phi_0)$: of the carbon that does not stay in the atmosphere forever, this is the share that exits the atmosphere into the biosphere or ocean within a decade
  - the remaining carbon in the atmosphere, $(1 - \phi_L)\phi_0$, decays at a geometric rate $\phi$
- We then have a non-linear function $T_{t+1} = \mathcal{T}(T_t, S_t)$ with a steady state like

$$T(f) = \frac{\eta}{(\kappa_{Planck} - \kappa_{other} - \kappa_{refl})} \frac{1}{\ln 2} \ln\left(\frac{S}{\bar{\bar{S}}}\right)$$

- Surprisingly, non-linearities in the relation between $CO_2$ and Temperature seem to cancel each other in most advanced climate models. The global mean temperature thus becomes approximately linear in cumulative emissions. $T_t = \sigma_{CCR} \sum_{s=0}^{t} Emm_s$
- According to the latest (6th) IPCC report, $\sigma_{CCR}$ is "likely" (2/3 confidence interval) between 1.0 and 2.3 degrees Celsius per 1000 GtC (corresponding to 0.27-0.63$^\circ / Tt$CO$_2$). This constant is called CCR (Carbon Climate Response, sometimes CRE or TCRE).
- Here, we postulate a Damage Function: Carbon reduces output proportionally so what we have left is $[1 - D_t(S_t)]$
- Nordhaus summarizes various studies of effects:
- Positive effects if initial temperature is below 11.5 degrees. Suggests quadratic damage $D(T) = \alpha_{ag}^1 \left(T + T_0^j\right) + \alpha_{ag}^2 \left(T + T_0^j\right)^2 + \alpha_{ag}^j$.

1. A joint model of the climate and the economy.
2. Production Process (GDP) affected by Climate Change
3. Households with preferences (needed to evaluate outcomes)
4. Explicit use of energy that both contributes to GDP and emits $CO_2$
5. Inclusion of Exhaustible Resources that induces savvy economic behavior.

- Technology $Y_t = F_t(K_t, N_t, E_t, S_t)$
- There are many types of energy inputs $E_{j,t}$, $j = 1, \cdots, J$
- The first $J_g - 1$ sectors are "dirty" and the last one is "clean" energy
- For the dirty energy firms, $E_{j,t}$ is normalized so that one unit of $E_{j,t}$ produces one unit of carbon. Emissions are $\sum_{j=1}^{J_g-1} E_j$
- $E_t = \sum_{j=1}^{J} E_{j,t}\alpha^j$, Actual amount of energy used
- Some energy resources have a finite stock, which is accounted for by the constraint $R_{j,t+1} = R_{j,t} - E^j_{j,t} \geq 0$
- Dirty energy has cost constant cost $\xi_j$. Clean energy has convex cost $\xi_J(E_{J,T})$.

- The climate variable $S_t$ is the amount of carbon in the atmosphere.
- Depends on past emissions as in the reduced form way
- Define a function $\tilde{S}_t$ that maps the history of man made pollution into the current level of carbon dioxide.

$$S_t = \tilde{S}_t \left( \sum_{j=1}^{J_g - 1} E_{j,-T}, \sum_{j=1}^{J_g - 1} E_{j,-T+1}, ..., \sum_{j=1}^{J_g - 1} E_{j,t} \right)$$

- Here, $-T$ is defined as the start of industrialization.

1. $U(C) = \log(C)$

2. $F_t(K_t, N_t, E_t, S_t) = [1 - D_t(S_t)]\, \widetilde{F}_t(K_t, N_t, E_t)$

   (has already subtracted the costs $\xi_j$ of producing energy source $j$)

3. Damages: $[1 - D_t(S_t)] = \exp\{-\gamma_t(S_t - \bar{S})\}$

4. The function $\tilde{S}_t$ is linear and has the depreciation structure:

$$S_t - \bar{S} = \sum_{s=0}^{t+T} \sum_{j=1}^{J_g - 1} E_{j,t-s}$$

- It is found by solving a social planner's problem
  - Representative household of the world
  - Technological, Climate and Exhaustability Constraints
- After that we worry about implementation

$$\max_{\substack{\{C_t, N_t, K_{t+1}, R_{j,t+1}, \\ E_{j,t}, S_t\}_{t=0}^{\infty} \geq 0}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(C_t) \qquad \text{s.t.}$$

$$C_t + K_{t+1} = F_t(K_t, N_t, E_t, S_t) + (1-\delta)K_t \qquad \text{FB}$$

$$E_t = \sum_j E_{j,t}\ \alpha^j \qquad \text{AGE}$$

$$R_{j,t+1} = R_{j,t} - E_{j,t} \geq 0 \qquad \text{for all } j \qquad ExE$$

$$S_t = \tilde{S}_t \left( \sum_{j=1}^{J_g - 1} E_{j,-T}, \sum_{j=1}^{J_g - 1} E_{j,-T+1}, \ldots, \sum_{j=1}^{J_g - 1} E_{j,t} \right) \qquad CC$$

- $E_{j,t}$ is output of Energy of Sector (type) $j$ measured in units of carbon emitted.

- $\alpha^j$ Conversion of units of energy of type $j$ from being in terms of carbon emissions to units of energy.

- The marginal externality damage is the same for all $j$:

$$\Lambda_t^s = \mathbb{E}\sum_{i=0}^{\infty}\beta^i \frac{U'(C_{t+i})}{U'(C_t)}\frac{\partial F_{t+i}}{\partial S_{t+i}}\frac{\partial S_{t+i}}{\partial E_{j,t}}$$

- Under our specific assumptions, this expression simplifies to:

$$\Lambda_t^s = \mathbb{E}\sum_{i=0}^{\infty}\beta^i C_t \frac{Y_{t+i}}{C_{t+i}}\gamma_{t+i}(1-d_i)$$

- Further, if the planner's problem implies a constant savings rate, then the expression can be written as:

$$\Lambda_t^s = Y_t\left[\mathbb{E}\sum_{i=0}^{\infty}\beta^i\gamma_{t+i}(1-d_i)\right]$$

- The FOC of the planner says

$$\alpha_j \frac{\partial F_t}{\partial E_t} - \xi_j - \Lambda_t^s = 0$$

$$\max_{\{C_t, N_t, K_{t+1}\}_{t=0}^{\infty}} \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(C_t)$$

$$\text{subject to} \qquad \mathbb{E}_0 \sum_{t=0}^{\infty} q_t (C_t + K_{t+1})$$

$$= \mathbb{E}_0 \sum_{t=0}^{\infty} q_t ((1 + r_t - \delta) K_t + w_t N_t + T_t) + \Pi_t .$$

$$\Pi_0 = \max_{\{K_t, N_t, E_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} q_t \left[ F_t(K_t, N_t, E_t, S_t) \right.$$

$$\left. - r_t K_t - w_t N_t - \sum_{j=1}^{J} p_{j,t} E_{j,t} \right]$$

- $\tau_{j,t} = \Lambda_t^s$ for the "dirty" energy firms, and $\tau_{j,t} = 0$ for the "clean" energy firms.

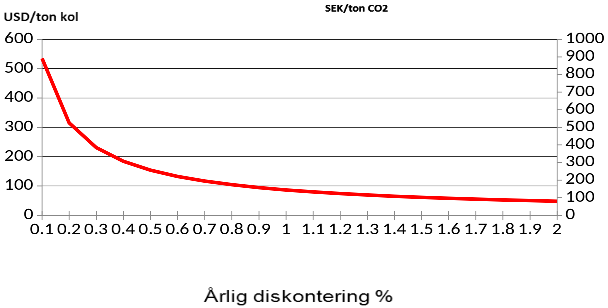
- **This is the optimal first best tax on carbon emissions.**

- If there are multiple externalities (for instance an R&D component to the model) then a separate Pigouvian tax is required for each externality.

To understand the magnitude of the optimal tax rates given by this model, they can be compared with estimates from other models, and also with tax rates that are currently being used around the world.

- Nordhaus (2008) uses a discount rate of 1.5% and gets a tax of $30 per ton of coal. With the same discount rate, this paper gives a tax of $56.9 per ton of coal.
- Stern (2007) uses a discount rate of 0.1% and gets a tax of $250 per ton of coal. With the same discount rate, this paper gives a tax of $500 per ton of coal.
- In Sweden, the current tax on private consumption of carbon exceeds $600 per ton of carbon, which is larger than the estimates for the optimal tax in this paper. However, these taxes are significantly higher than many other countries, for instance the EU has a tax of around $77 per ton of carbon.

# Sum damages over time => ”optimal” tax!




Sweden has carbon tax ~ 600 USD/tC!

- Let's use a recent (natural science-based) approximation of the effects on global temperature of fossil-fuel emissions.
- "Carbon Climate Response (CCR): for each 1,000GtC in cumulative historic emissions, global temperature rises by 1-2.1 degrees Celsius (1.8-3.8F).
- We have emitted about 550GtC so far (since industrial revolution).
- Remaining (conventional) oil+gas: about 300GtC. Limited warming if we use it up!
- Remaining coal: much more, possibly over 3,000GtC.
- => Coal is the main threat!

- Wouldn't affect (conventional) oil and gas use.
  - A tax on oil and gas makes little difference: these fuels are so cheap to produce that markets will keep using them despite the tax.
  - It is indeed efficient from an economic perspective to use them up!
- A different story for coal:
  - Coal doesn't give a big profit per unit so a tax would make us stop using most of the coal.
  - Taking the climate damage into account, using coal simply isn't worth it.
- So: bad for the coal industry (the world over), no big deal otherwise

## How costly is the optimal tax for us?

- Suppose we use "very cautious" discounting of 0.1%, implying a tax of $600/tC.
- Turns out Sweden has had that tax for over a decade. They did better than average during the Great Recession, no noticeable "leakage" of firms abroad.
- Significant scope for
  - Energy saving
  - Alternative technology

- Baseline recommendation:
  - Tax carbon, world-wide
  - Required rate will not be a big blow to our global economy, but will (must) shake up coal industries
- What about alternatives, like cap-and-trade?
  - If managed so that the emission rights are as expensive as the carbon tax, ok!
  - In Europe, this is not the case —low world demand and high caps culprits.
- Do we need green subsidies?
  - Under an optimal carbon tax, maybe not; otherwise, yes.
- Should all countries mainly reduce emissions at home?
  - No: reduce them where they are least needed/least efficient (e.g., buy emission rights in EU trading system, pay to keep forests, ...)

## Broad concluions so far

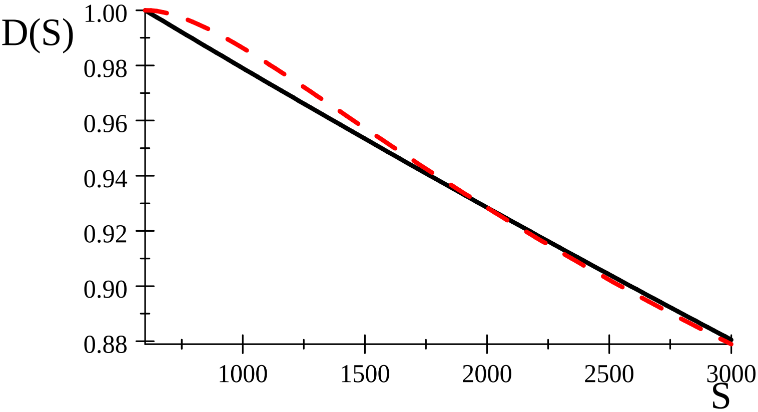
- climate change likely leads to non-negligible global damages
- very uneven effects across regions of world
- for world as a whole, costs likely not catastrophically large
- a robust result (in Golosov, et al., 2013): optimal policy involves rather modest tax on CO 2 and would not pose threat to economic well-being
- some elements of analysis subject to substantial uncertainty

- The burning of fossil fuel (oil, coal, natural gas) increases the $CO_2$ concentration in the atmosphere.
- $CO_2$ in the atmosphere is a greenhouse gas: it lets solar radiation pass through but blocks heat radiation.
- This leads to global warming. The logic is undisputed among scientists.
- The direct warming effect is significant, but not catastrophic.
- There are, however, feedback effects: creation of water vapor, melting of ice caps lowering solar reflection, cloud formation, ....
- The quantitative magnitudes of feedback are disputed. The "average" view seems to be that feedbacks strengthen the direct warming effect considerably, but there is much uncertainty.

- Global warming affects economic activity; in many places, the effect is to cause damages (to agriculture, human health, and so on).
- This is an externality: those emitting carbon into the atmosphere are not charged for the costs.
- Thus, in classical economic terms, we have a failure of markets. The prescription is government intervention: we need to artificially raise the cost of emissions to its proper societal value.
- Main recipe: use a tax. Well-known since Pigou (1920).
- The tax must be global: the externality is global.
- What is the appropriate level of the tax? For this, we use standard cost-benefit analysis.

- Nordhaus's aggregate damage function maps temperature into damages.
- Now consider collapsing the two steps, i.e.,
  1. the one from increased $CO_2$ concentration ($S$) to the change in global mean temperature ($T$)
  2. the one from $T$ to damages, into one: from $S$ to damages directly.
- For the first step use Arrhenius $T(S) = \frac{3}{\ln 2} \ln\left(\frac{S+600}{600}\right)$ where $S$ is GtC over the pre-industrial level (600 GtC).
- For the second let $D(T)$ be Nordhaus's global damage function.
- Together, the two steps are $D(T(S))$ mapping additional atmospheric carbon to damages. Let's examine the mapping.

- It turns out that $1 - D(T(S))$, i.e., how much is left after damages as a function of $S$, is well approximated by the function $e^{-\gamma S}$: for $\gamma = 5.3 * 10^{-5}$ (black), it is quite close to $1 - D(T(S))$ (red dashed), as seen in the figure.



- Define $Y_{net}$ as output net of damages and $Y$ as gross output, implying $Y_{net} = (1 - D(T(S)))Y$.
- Using the approximation $(1 - D(T(S))) \approx e^{-\gamma S}$, we have $Y_{net} = e^{-\gamma S}Y$.
- Then, $\frac{\partial Y_{net}}{\partial S}\frac{1}{Y_{net}}$ is the marginal loss of net output from additional GtC in the atmosphere expressed as a share of net output.
- Using our approximation, we have $\frac{\partial Y_{net}}{\partial S}\frac{1}{Y_{net}} = \frac{\partial\left(e^{-\gamma S}Y\right)}{\partial S}\frac{1}{e^{-\gamma S}Y} = -\gamma$. I.e., the marginal losses are a constant proportion of GDP!
- This "elasticity" is thus independent of GDP and $CO_2$ concentration.
- With $\gamma = 5.3 * 10^{-5}$, one GtC extra in the atmosphere gives extra damages at 0.0053%. Recall the rate of accumulation of $S_t$.
- Robust?

# 2024 Update: Granular Model with various Suboptimal Policies

# Climate Policy in the Wide World

**John Hassler, Per Krusell, and Conny Olovsson**

**from the Simpson Lecture in Princeton University April 15, 2024**

- Focus is on “the wide world”.
    - This framework is a neoclassical growth model with carbon-cycle and climate blocks that builds on Nordhaus, but these blocks are updated to reflect the latest climate-science insights.
- We include very high geographic resolution: $1^\circ \times 1^\circ$ latitude-longitude cells, with each cell assigned to a country.
- Despite this complexity, our model is also highly accessible to others, i.e., no need for advanced numerical toolboxes.

- They conduct three kinds of suboptimal policy options that have been discussed or already implemented.

  1. How far can a modest uniform carbon tax go to limit global warming and damages around the world?

  2. How costly is it to deviate from tax uniformity and allow poor countries to not tax carbon?

  3. How successful is a policy that refrains from carbon taxation and instead focuses on promoting green energy (reminiscent of the Inflation Reduction Act)?

- The first policy is *successful* in mitigating global warming, the second is very *costly*, and the third is both *costly* and *unsuccessful*.

- The model consists of all countries in the world.
- Each country is divided into regions (grid cells) that use capital, labor, and energy to produce a final good that is identical across regions and countries.
- Most regions cannot produce oil: those that can only export oil in exchange for consumption goods (no further international trade).
  - Additional energy sources (coal, green etc.) are produced within each country.
- Capital, labor, and energy services can move freely within countries but not at all across.
  - Hence, factor prices are required to be the same within each country.
- They make assumptions such that saving rates are easy to solve for separately.

- Each country $j$ contains a large number of identical consumers with preferences given by

$$\sum_{t=0}^{\infty} N_{j,t}\ \beta^t\ \log(c_t),$$

  – $N_j$ is total population that follows an exogenous path; we define $x_{j,t+1} \equiv N_{j,t+1}/N_{j,t}$

- Consumers in country $j$ are initially endowed with $k_{j0}$ units of capital/capita and they can save by investing in capital in their own country. (No international capital markets!)

- Consumers in oil-producing regions extract and sell oil from a finite reserve, $R_j$, at zero cost (as in Golosov et al., 2014).

- Governments in each country tax emissions and rebate all the proceeds to consumers in the country.

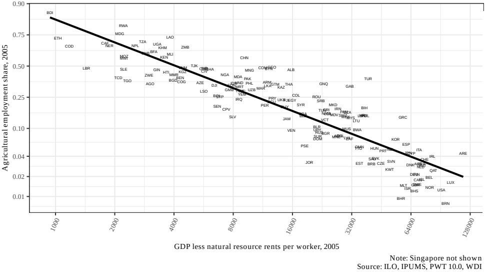
- One sector features firms that produce final output.
- The other sectors are the energy input producers.
- The production function is the same for these activities but the TFP components may differ.
    - The relative TFP factors will constitute the relative prices of the different energy inputs.
- Final-good producing firms in region $i$ and country $j$ solve

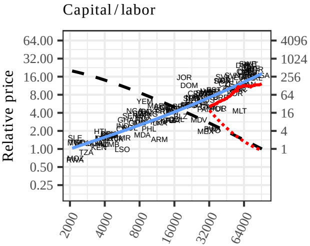
$$\pi_{ijt} = \max_{k_{ijt}, n_{ijt}, e_{ijt}} A_{ijt} \left( k_{ijt}^{\alpha} n_{ijt}^{1-\alpha-\nu_j} e_{ijt}^{\nu_j} \right)^{\varphi_j} - r_{jt} k_{ijt} - n_{jt} w_{ijt} - p_{jt} e_{ijt}.$$

- Implicit in the above expression is a fixed factor that can be thought of as land.

Poor countries are about agriculture, and capital/labor shares in agriculture increase with development.

Left: ag. empl. shares on GDP/worker; right: $k/l$, $r/w$ (dashed).




Call for a richer production structure, or at the very least $\alpha_j$.

They use log, one sector, and Cobb-Douglas to get easy-to-solve-for saving rates. An alternative is elasticity $\epsilon$ and CRRA $\epsilon$.

- All energy inputs (except oil) can be produced within each country with $p_\kappa$ units of the final good are required to produce $e_\kappa$ units of energy source, $\kappa \in \{c, g, f\}$.

- A region that has access to the fracking input produces an "oil composite" $o$ as

$$o = (\lambda e_o^{\rho_o} + (1 - \lambda) e_f^{\rho_o})$$

$1\rho_o - \rho_o$ determines the EOS between the $e_o$ and $e_f$.

- The supply of energy services is then a CES aggregate

$$e = \left(\lambda_o o^\rho + \lambda_c e_c^\rho + \lambda_g e_g^\rho\right)$$

$1\rho - \rho$ determines the EOS between the energy inputs.

- "$A$" indicates TFP and it has several components. Formally:

$$
\begin{aligned}
A_{ijt} &= \exp(z_{ijt}) D_{ijt} \\
z_{ijt} &= z_{ij} + \sum_{s=0}^{\infty} g_{js}
\end{aligned}
$$

  - $g$ is the exogenous growth rate. For $t > T$, $g_j = g$.

- Note that TFP
  - has one *region-specific* component that is constant over time ($z_{ij}$),
  - one *country-specific* component that is changing over time ($g_{jt}$), and
  - one *endogenous* component that responds to climate change ($D_{ijt}$).

- TFP damages are described by a U-shape in local temperature:

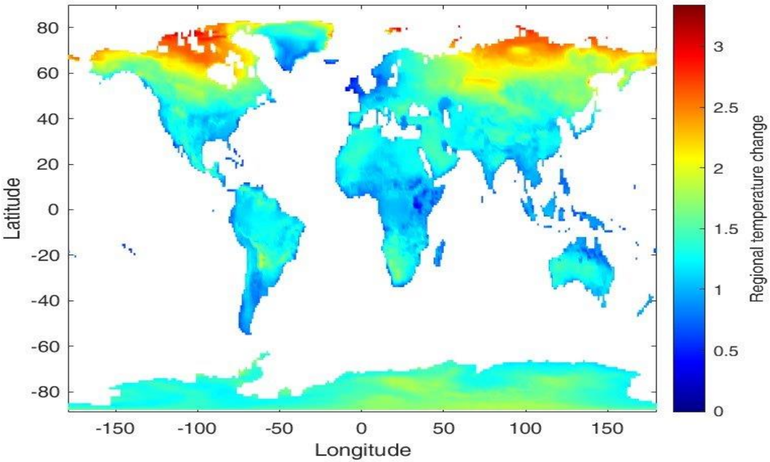
$$D_{ijt} = D(T_{ijt}) = \exp(\kappa_1 T_{ijt})(1 + \exp(\kappa_2(T_{ijt} - \kappa_3)))$$

-$\kappa_4\kappa_2$,

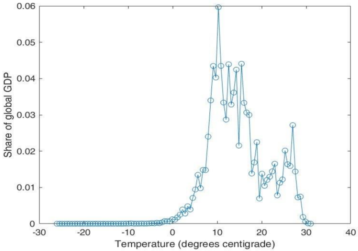
– $T_{ijt}$ is the local temperature in region $i$ and country $j$ at time $t$ (inspired by Krusell and Smith (2022), Burke, Hsiang, and Miguel (2015) and Cruz and Rossi-Hansberg (2023).

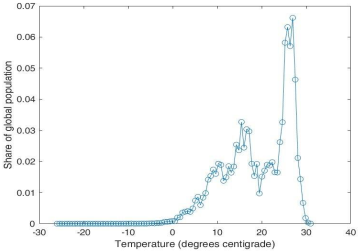
- Compute $T_{ijt}$ with "statistical downscaling" where the global temperature is a sufficient statistic for the temperature in each region:

$$T_{ijt} = \hat{T}ij + \gamma_{ij}(T_t - T_0).$$

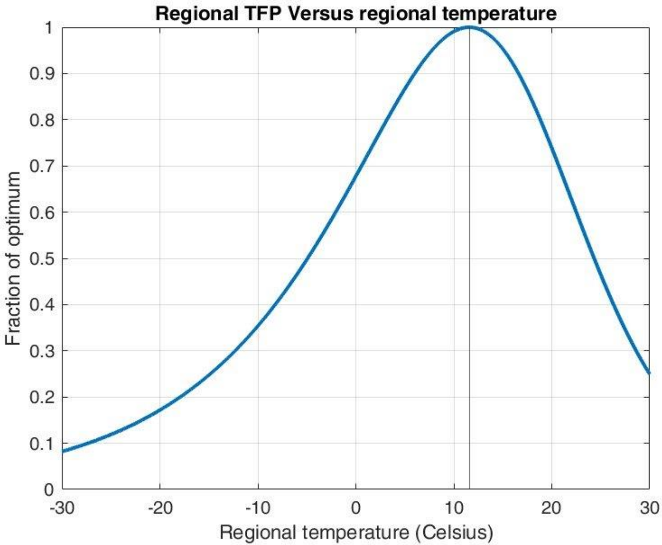


Latitude
80
60
40
20
0
-20
-40
-60
-80
-150
-100
-50
0
50
100
150
Longitude
Regional temperature change
3
2.5
2
1.5
1
0.5
0




- Most of the output are produced in regions where $\mathbb{E}[T] \approx 11.6^\circ$C.
- A lot of people live where $\mathbb{E}[T] > 11.6^\circ$C.


Regional TFP Versus regional temperature
Fraction of optimum
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
-30
-20
-10
0
10
20
30
Regional temperature (Celsius)

- The carbon cycle and temperature dynamics, respectively:

$$\begin{aligned} S_t - S_{t-1} &= \phi_{12} S_{t-1} + \phi_{21} S^U_{t-1} + E_{t-1} \\ S^U_t - S^U_{t-1} &= \phi_{12} S_{t-1} - (\phi_{21} + \phi_{23}) S^U_{t-1} + \phi_{32} S^L_{t-1} \\ S^L_t - S^L_{t-1} &= \phi_{23} S^U_{t-1} + \phi_{32} S^L_{t-1}, \end{aligned}$$

- $E$: global (sum over $(i, j)$) emissions; $S$, $S^U$, and $S^L$: $CO_2$ stocks in the atmosphere, surface oceans and biosphere, and deep oceans.

$$\begin{aligned} T_t &= T_{t-1} + \sigma_1 \left( F_t - \kappa T_{t-1} - \sigma_2 \left( T_{t-1} - T^L_{t-1} \right) \right) \\ T^L_t &= T^L_{t-1} + \sigma_3 \left( T_{t-1} - T^L_{t-1} \right) \end{aligned}$$

- $F_t = \chi \frac{\eta}{\ln 2} \ln \left( \frac{S_t}{S_0} \right)$ ($\chi > 1$ captures non-CO2 forcing) and $T^L$ is ocean temperature. System replicates temperature graphs above.

- For given policy sequences $\tau_{ijt}{}_{t=0}^{\infty}$ across the world, the model can be solved forward.
  - The only forward-looking decisions involve the consumers' problems that delivers solutions for saving rates that only depend on exogenous parameters.
  - The saving rates themselves are time-dependent and forward-looking, but satisfy a simple recursion.
- Given $(k_j, n_j)$, one can compute country $y_j$ and input

  demands, assuming a value for the world price of oil. TFP levels are endogenous but predetermined at each point in time.
- Use a simple fixed-point algorithm for finding the oil price that clears the world markets period by period.

$$
\begin{aligned}
p_j &= \left( \lambda_o^{\frac{1}{1-\rho}} \hat{p}_{oj}^{\frac{\rho}{\rho-1}} + \lambda_c^{\frac{1}{1-\rho}} p_{cj}^{\frac{\rho}{\rho-1}} + \lambda_g^{\frac{1}{1-\rho}} p_{gj}^{\frac{\rho}{\rho-1}} \right)^{\frac{\rho-1}{\rho}} \\
\hat{p}_{oj} &= \left( \lambda^{\frac{1}{1-\rho_h}} p_o^{\frac{\rho_h}{\rho_h-1}} + (1-\lambda)^{\frac{1}{1-\rho_h}} p_{fj}^{\frac{\rho_h}{\rho_h-1}} \right)^{\frac{\rho_h-1}{\rho_h}} \\
e_{oij} &= e_{ij} \left( \frac{\lambda_o p_j}{\hat{p}_{oj}} \right)^{\frac{1}{1-\rho}} \left( \frac{\lambda \hat{p}_{oj}}{p_o} \right)^{\frac{1}{1-\rho_h}} \\
e_{fij} &= e_{ij} \left( \frac{\lambda_o p_j}{\hat{p}_{oj}} \right)^{\frac{1}{1-\rho}} \left( \frac{(1-\lambda) \hat{p}_{oj}}{p_f} \right)^{\frac{1}{1-\rho_h}} \\
e_{mij} &= e_{ij} \left( \frac{\lambda_m p_j}{p_{mj}} \right)^{\frac{1}{1-\rho}}, \; m = c, g.
\end{aligned}
$$

All underlying prices are exogenously given except $p_o$,

- The production function for a regional firms (omitting time subscripts) can be written as

$$y_i = \left(\frac{\nu_j \varphi_j}{p_j}\right)$$

$\nu_j \varphi_j 1 - \nu_j - \varphi_j A_{ij}$

$11 \text{ - } \nu_j - \varphi_j (k_i^{\alpha} n_i^{1-\alpha-\nu_j})^{\frac{\varphi_j}{1-\nu_j\varphi_j}}.$

- Summing over regions: $y_j = \sum_{i=1}^{I} y_i,\ k_j = \sum_{i=1}^{I} k_i,\ n_j = \sum_{i=1}^{I} n_i$ we get per-capita output in a country $j$

$$y_j = \left(\frac{\nu_j \varphi_j}{p_j}\right)$$

$\nu_j \varphi_j 1 - \nu_j - \varphi_j \left(\sum_{i=1}^{I} A_{ij}\right.$

$1(1 \text{ - } \nu_j\varphi_j)(1-\varphi_j)^{1-\varphi_j} k_j^{\alpha_k \varphi_j} n_j^{\alpha_n \varphi_j}.$

- The only remaining endogenous variable is $p_j$, which is determined on the world market.

Above, we have an expression for $e_{oijt}$. Summing over regions and countries we get, after manipulation, oil demand

$$\text{Oil demand}_t = \sum_{ij} \Pi_{jt}(p_{ot})\ \nu_j\ \varphi_j\ y_{ijt}\ N_{jt},$$

where $\Pi_{jt}$ is a known function of $p_{ot}$.

Turning to supply, the oil producer's maximization problem delivers

$$R_{t+1} = \beta \frac{1 - s_t}{1 - s_{t+1}} R_t\ ,\ s_t = \frac{\beta x_{t+1}}{1 - s_{t+1} + \beta x_{t+1}}.$$

Given an exogenous sequence $\{x_{t+1}\}_{t=0}^{\infty}$, we can solve backwards:

$$\text{Oil supply}_t = \beta \sum_j \frac{1 - s_{jt}}{1 - s_{jt+1}} N_{jt} R_{jt}.$$

Solve for $p_{ot}$ by setting Oil supply$_t$ = Oil demand$_t$.

They can also derive a forward-looking equation in saving rates in oil-consuming regions, and write per-capita savings of country $j$ as

$$k_{j,t+1} = \frac{s_{jt}(1+\hat{\tau}_{jt})}{x_{j,t+1}} A_{jt} k_{jt}^{\frac{\alpha\varphi_j}{1-\nu_j\varphi_j}},$$

with

$$s_{jt} = \frac{\frac{\alpha\beta\varphi}{1-\nu\varphi} x_{j,t+1}}{1 - s_{j,t+1} + \frac{\alpha\beta\varphi}{1-\nu\varphi} x_{j,t+1}}.$$

The heterogeneity across economies appear in multiple places:

- saving rates
- population growth rates
- taxes, $\varphi_j$, $\nu_j$
- TFP, costs of producing energy services.

1. Solve for the saving rates $\forall j$ (no endogenous variables).

2. Compute the equilibrium forward, starting at time 0. The endogenous state variables at $t = 0$ are $K_j$, $T_j$, oil resources by $j$; state variables in the carbon cycle and climate system.

    - Compute all TFP levels around the world and solve for the oil price in the period, which requires a numerical solution but only involves one equation in one unknown.
    - $p_{o,0}$ and $\tau_j$ gives the demand for all fuels and thus total emissions, so temperatures can be updated to next-period values.
    - The government BC is used to compute the carbon-revenue transfer rates $\hat{\tau}_j$.
    - Update the capital stocks and oil resources to their next-period values.

3. This completes the procedure for going from period 0 to period 1. Proceed to all future periods.

- They make use of the G-Econ database, version 4.0, which provides data on GDP and population ($N$) for every $1^\circ \times 1^\circ$ cell that contains land for the model's base year, 2005.
  - The database contains GDP and $N$ data for 16,443 cells in 2005, and these cells that comprise the basic unit of analysis in the model.
- Estimated and projected $N$ growth rates from 1990 to 2100 by country taken from the United Nations. Between 2100 and 2200, assume a linear progression from the 2100 rate to 0.
- The exogenous part of TFP grows at a rate of 1.5%/year, but developing countries are allowed to catch up.
- MPK:s are equalized in period 0 (following Caselli & Feyrer, 2007), which can be used to pin down the $\varphi$'s (are found in the range $0.8 \leq \varphi_j \leq 1$).

- The elasticity of substitution between oil, coal, and the green energy source is set to 2 (for all countries).

- The elasticity of substitution between conventional oil and fracking is set to 10 (only the U.S. is assumed to have fracking).

- $E[p_o]_{2005-2009}$=\$70 per barrel or \$606.5/ton of carbon.

- $E[p_c]_{2005-2009}$=\$74/ton or \$103.35/ton of carbon.

- They set $p_g$ based on the current relative price between green energy and oil.

- With these prices and observed quantities, the $\lambda$s in the energy aggregator can be computed (Golosov, Hassler, Krusell, and Tsyvinski (2014)).

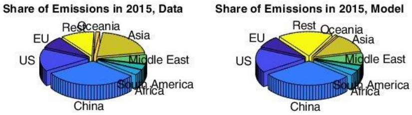
They incorporate heterogeneity in the energy income share with data on national energy use from the World Bank by computing

$$\nu_i = \frac{e_i^{int}}{\hat{e}_i^{int}} \nu,$$

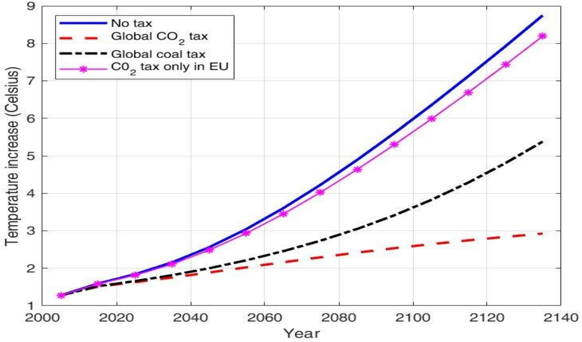
where $e_i^{int}$ is national energy intensity (energy use in oil equivalents divided by PPP-adjusted GDP in year 2000), $\hat{e}_i^{int}$ is average energy intensity, and $\nu$=0.035.

(Relies on the price of energy being equal across countries.)

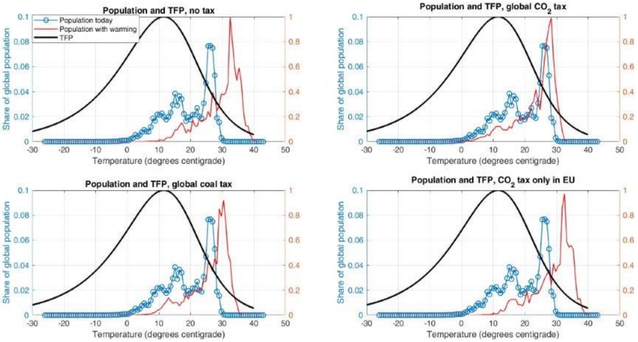
Note that the model is quite successful in matching observed $CO_2$ emissions, even though these were not directly targeted (China subsidizes fossil fuel use.).



Consider a tax of USD \$20/ton $CO_2$ at the initial date; it then grows at the rate of world GDP.

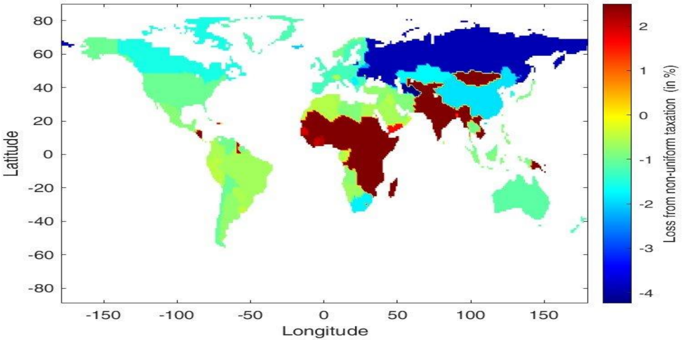



The difference between no tax and the modest tax is striking: about 6°C by 2140!




- Distributional consequences: populations being moved significantly to the right for the no-tax policy.
- As if hit by several Great Depressions at once for the some of the poorest regions.

- The Pigou principle: the tax should equal the negative externality caused. Since the negative externality is global, the tax should be the same everywhere: it should be uniform.
- However, we often hear arguments that for the sake of fairness, some poor regions should be "let off the hook".
- They here quantify exactly how costly deviations from a uniform taxation policy are in dollar terms.
- They again start with a $\tau$ of about US $20/ton.
    - Compare the results to a setting where the poorest countries, defined to be below 25% of global GDP/capita, have a zero/very low tax;
    - The tax in the ROW is then increased so that the increase in $T$ is the same as with uniform taxation (3.1°C) at $t = 15$ .
    - They consider a $\tau$ that is $20 \times \tau_{modest}$ in the participating countries, and $0.06 \times \tau_{modest}$ for the poor.


80
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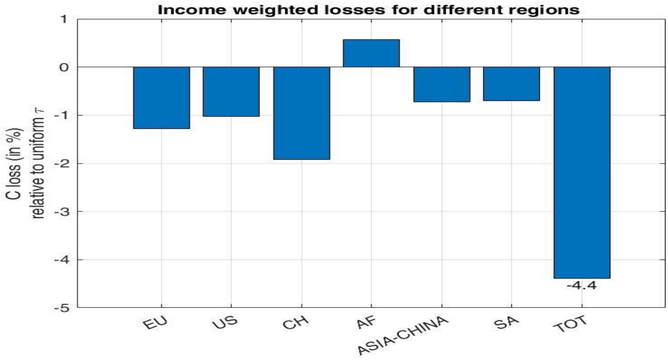
Loss from non-uniform taxation (in %)

# Aggregate effects of non-uniform taxation



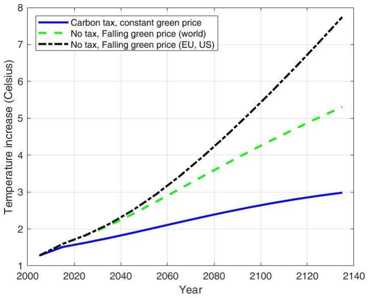
The climate-change aspects of the IRA boil down to the idea that cheap green technology will compete out fossil fuel.

They evaluate this idea by considering a scenario where the relative price of green energy falls by 2%/year (at zero cost), while the relative price of fossil fuel production is unchanged.

Two cases:

1. The U.S.-like policy generates fast green technology growth everywhere in the whole world.

2. The sped-up green technology growth will occur only in the U.S. and the EU.




- With green growth only in the EU and the U.S., global warming becomes almost as high as with no policy at all. Also insufficient when the technology spreads around the whole world.
- Key issue: green technology increases overall energy consumption but is ineffective in competing out fossil fuel.

- They offer a model of economics and climate change with very high regional resolution.
  - The model rests on standard microeconomic foundations that allows for cost-benefit analysis.
  - The spatial dispersion of the welfare effects of global warming are found to swamp the average effects.
  - Proof of concept: lots of room for improvements regarding heterogeneity in energy supplies and technologies, production structures.
- They also find that
  - Even a modest, globally uniform carbon tax would be extremely valuable.
  - A non-uniform tax on carbon is quite inefficient.
  - Relying on a push for green technical change only tax appears like a risky policy.

# Macro and COVID-19

- Short Horizons (No investment)
- Choose what Issues to Worry About
    1. *Mitigation Policy and Heterogeneity Age/Sector*
- Choose wich Allocation Mechanism to Model (large externality)
    1. All Econ choices are Government choices

- All variables are shares of a measure 1 population
- Three health states, $j \in \{s, i, r\}$ susceptible, infected, recovered or dead, with associated population shares $S, I, R$. Initial conditions $S(0), I(0), R(0)$.
- Two parameters: $\beta$ governs rate of infection, $\kappa$ the rate of recovery (or death)
- System of differential Equations

$$\begin{aligned}\dot{S}(t) &= -\beta S(t)I(t)\\ \dot{I}(t) &= \beta S(t)I(t) - \kappa I(t)\\ \dot{R}(t) &= \kappa I(t)\end{aligned}$$

- Basic Reproduction Number: define $R_0 = \frac{\beta}{\kappa}$

- Growth rate of infections given by $\frac{\dot{I}(t)}{I(t)} = \beta S(t) - \kappa$
- Let $I(0) = \epsilon, S(0) = 1 - I(0)$, when $\epsilon > 0$ is very small, $S(0) \approx 1$.
- Since $\dot{S}(t) = -\beta S(t) I(t)$ and for $t$ close to zero,

  $I(t) \approx 0, S(t) \approx 1$, then $\dot{I}(t)/I(t)$ is roughly constant and equal to

$$\dot{S}(t) = -\beta S(0) I(0) \qquad \text{So}$$

$$I(t) = I(0) e^{\kappa\left(\frac{\beta}{\kappa} S(0) - 1\right)} \approx I(0) e^{\kappa\left(\frac{\beta}{\kappa} - 1\right)}$$

- If $R_0 = \frac{\beta}{\kappa} > 1$ exponential growth early (if $I(0) > 0$).
- If $R_0 = \frac{\beta}{\kappa} < 1$ then infections fall to zero and epidemic disappears immediately.

- The Ratio of differential equations: $\frac{\dot{I}(t)}{\dot{S}(t)} = -1 + \frac{1}{R_0 S(t)}$
- Integrating yields $I(t) = -S(t) + \frac{\ln(S(t))}{R_0} + q$

  where $q$ is a constant of integration that does not depend on time.
- Evaluating at $t = 0$ yields (using $R(0) = 0$, thus $S(0) + I(0) = 1$

$$q = 1 - \frac{\ln(S(0))}{R_0}$$

- What is $S(\infty) = S^\star$? share of the population never to get infected
- Evaluating at $t = \infty$ and using the fact that $I(\infty) = 0$ yields

$$S^\star = 1 + \frac{\ln\left[S^\star / S(0)\right]}{R_0}$$

- Steady state satisfies the trascendental equation:

$$S^\star = 1 + \frac{\ln\left[S^\star/S(0)\right]}{R_0}$$

  and $R^\star = 1 - S^\star, I^\star = 0$.

- If $R_0 > 1$ and $S(0) < 1$, $\exists$ a unique long-run $S^*$.

  Strictly decreasing in $R_0$ and strictly increasing in $S(0)$.

- For $R_0 \approx 1$ (but $> 1$), $S^\star = \frac{1}{R_0}$ and $R^\star = \frac{R_0 - 1}{R_0}$

  This approximation (a first good rule of thumb) uses $S(0) \approx 1$ and

$$\ln(1/R_0) = -\ln(R_0) = -\ln(1 + R_0 - 1) \approx 1 - R_0.$$

- With **costly** transfers across agents
- To Assess combination of two policies
  - Shutdown / mitigation (less output but also less contagion)
  - Redistribution toward those whose jobs are shuttered
- Characterize optimal policy
- Key interaction:
  - Mitigation creates the need for more redistribution
  - But if redistribution is costly, want less mitigation
  - Need heterogeneous-agent model to analyze this

# The SAFER SIR Model

- Stage of the disease
  - **S**usceptible
  - Infected **A**symptomatic
  - Infected with **F**lu-like symptoms
  - Infected and needing **E**mergency hospital care
  - **R**ecovered (or dead)
- Worst case disease progression: $S \rightarrow A \rightarrow F \rightarrow E \rightarrow D$
- But *R*ecovery is possible at each stage
- Three infected types spread virus in different ways:
  - $A$ at work, while consuming, at home
  - $F$ at home
  - $E$ to health-care workers

# Heterogeneity by Age and Sector

- Age $i \in \{y, o\}$
  - Only young work
  - Old have more adverse outcomes conditional on contagion
  - But young more prone to contagion (they work)
- Sector of production $\{b, \ell\}$
  - Basic (health care / food production etc.)
    - Will never want shut-downs in this sector
    - Workers in this sector care for the hospitalized
  - Luxury (restaurants, entertainment etc.)
    - Workers in this sector face shutdown unemployment risk
    - But they are less likely to get infected

- Mitigation
  - Reduces contagion
  - Reduces risk of hospital overload
  - Reduces average consumption
  - Increases inequality (more unemployment in shuttered sectors)
- Redistribution
  - Helps the unemployed $\Rightarrow$ makes mitigation more palatable
  - But redistribution is costly $\Rightarrow$ makes mitigation more expensive
- What policy time paths do different types prefer? When (and how much) to shut down, when to open up? Size of Coronavirus check?
- How does the utilitarian optimal policy vary with the cost of redistribution?

- Lifetime utility for *o*ld

$$E\left\{\int e^{-\rho_o t}\ \left[u^o(c_t^o)+\bar{u}+\widehat{u}_t^j\right]\ dt\right\}$$

  - $\rho_o$: time discount rate
  - $u^o(c_t^o)$ instantaneous utility from old age consumption $c_t^o$
  - $\bar{u}$: value of life
  - $\widehat{u}_t^j$: intrinsic utility from health status $j$ (zero for $j \in \{s, a, r\}$)
- Similar lifetime utility for *y*oung.
- Differences in expected longevity through $\rho_y \neq \rho_o$ (no aging)

- Young permanently assigned to $b$ or $\ell$
- Linear production: output equals number of workers
- Only workers with $j \in \{s, a, r\}$ work
- Output in basic sector:

$$y^b = x^{ybs} + x^{yba} + x^{ybr}$$

- Output in luxury sector is

$$y^\ell = [1 - m] \left( x^{y\ell s} + x^{y\ell a} + x^{y\ell r} \right)$$

- Total output given by

$$y = y^b + y^\ell.$$

- Fixed amount of output $\eta\Theta$ spent on emergency health care
- $\Theta$ measures capacity of emergency health system, $\eta$ its unit cost

- Types of transmission
  - Work: young $S$ workers infected by $A$ workers, prob $\beta_w(m)$
  - Consumption: young & old $S$ infected by $A$, prob $\beta_c(m) \times y(m)$
  - Home: young & old $S$ infected by $A$ and $F$, prob $\beta_h$
  - ER: basic $S$ workers infected by $E$, prob $\beta_e$
- Shutdowns (mitigation) help by:
  - Reducing workers $\Rightarrow$ less workplace transmission
  - Reducing output $y(m) \Rightarrow$ less consumption transmission
  - Reducing infection-generating rates $\beta_w(m)$ & $\beta_c(m)$

$$\beta_w(m) = \frac{y^b}{y(m)}\alpha_w + \frac{y^\ell(m)}{y(m)}\alpha_w(1-m)$$

  - Similar for $\beta_c(m)$
  - Micro-founded via sectoral heterogeneity in social contact rates
  - Smart mitigation shutters most contact-intensive sub-sectors first

$$
\begin{aligned}
\dot{x}^{ybs} &= -\beta_w(m)\left[x^{yba} + (1-m)x^{y\ell a}\right]x^{ybs} \\
&\quad - \left[\beta_c(m)x^a y(m) + \beta_h\left(x^a + x^f\right) + \beta_e x^e\right]x^{ybs} \\
\dot{x}^{y\ell s} &= -\left[\beta_w(m)\left[x^{yba} + (1-m)x^{y\ell a}\right](1-m)x^{y\ell s}\right] \\
&\quad - \left[\beta_c(m)x^a y(m) + \beta_h\left(x^a + x^f\right)\right]x^{y\ell s} \\
\dot{x}^{os} &= -\left[\beta_c(m)x^a y(m) + \beta_h\left(x^a + x^f\right)\right]x^{os}
\end{aligned}
$$

- For each type $j \in \{yb, y\ell, o\}$

$$\dot{x}^{ja} = -\,\dot{x}^{js} - \left(\sigma^{jaf} + \sigma^{jar}\right)\; x^{ja}$$
$$\dot{x}^{jf} = \sigma^{jaf}\; x^{ja} - \left(\sigma^{jfe} + \sigma^{jfr}\right)\; x^{jf}$$
$$\dot{x}^{je} = \sigma^{jfe}\; x^{jf} - \left(\sigma^{jed} + \sigma^{jer}\right)\; x^{je}$$
$$\dot{x}^{jr} = \sigma^{jar} x^{ja} + \sigma^{jfr} x^{jf} + (\sigma^{jer} - \varphi) x^{je}$$
$$\varphi = \lambda_o \max\{x^e - \Theta, 0\}.$$

- All flow rates $\sigma$ vary by age

- $x^e - \Theta$ measures excess demand for emergency health care. Reduces flow of recovered (Increases flow into death)

- Costly transfers between workers, non-workers (old, sick, unemployed)
- Utilitarian planner (or taxes / transfers that cannot depend on age, sector, health)
  - $\Rightarrow$ Workers share common consumption level $c^w$
  - $\Rightarrow$ Non-workers share common consumption level $c^n$
- Define measures of non-working and working as

$$\begin{aligned}
\mu^n &= x^{y\ell f} + x^{y\ell e} + x^{ybf} + x^{ybe} + m\left(x^{y\ell s} + x^{y\ell a} + x^{y\ell r}\right) + x^o \\
\mu^w &= x^{ybs} + x^{yba} + x^{ybr} + [1-m]\left(x^{y\ell s} + x^{y\ell a} + x^{y\ell r}\right) \\
\nu^w &= \frac{\mu^w}{\mu^w + \mu^n}
\end{aligned}$$

- Aggregate resource constraint

$$\mu^w c^w + \mu^n c^n + \mu^n T(c^n) = \mu^w - \eta\Theta$$

where $T(c^n)$ is per-capita cost of transferring $c^n$ to non-workers

- Consumption allocation does not affect disease dynamics $\Rightarrow$ optimal redistribution is a static problem
- With log-utility and equal weights, period social welfare given by

$$W(x,m) = \max_{c^n, c^w} [\mu^w \log(c^w) + \mu^n \log(c^n)] + (\mu^w + \mu^n)\bar{u} + \sum_{i,j \in \{f,e\}} x^{ij} \widehat{u}^j$$

- Maximization subject to resource constraint gives $\frac{c^w}{c^n} = 1 + T'(c^n)$.
- Period welfare

$$\begin{aligned} W(x,m) &= [\mu^w + \mu^n]\, w(x,m) \\ w(x,m) &= \log(c^n) + \nu \log(1 + T'(c^n)) + \bar{u} + \sum_{i,j \in \{f,e\}} \frac{x^{ij}}{\mu^w + \mu^w} \widehat{u}^j \end{aligned}$$

- Assume $\mu^n T(c^n) = \mu^w \frac{\tau}{2} \left( \frac{\mu^n c^n}{\mu^w} \right)^2$
- Optimal allocation

$$\begin{aligned} c^n &= \frac{\sqrt{1 + 2\tau \frac{1-\nu^2}{\nu} \tilde{y}} - 1}{\tau \frac{1-\nu^2}{\nu}} \\ c^w &= c^n(1 + T'(c^n)) = c^n \left( 1 + \tau \frac{1-\nu}{\nu} c^n \right) \end{aligned}$$

Where $\tilde{y} = \nu - \frac{\eta\Theta}{\mu^w + \mu^n}$.

- $\left(1 + \tau \frac{1-\nu}{\nu} c^n\right)$ is the effective marginal cost (MC) of transfers.
- It increases with $c^n$ and $\tau$, decreases with share of workers $\nu$
- Higher mitigation $m$ reduces $\nu$, thus increases MC
- $\Rightarrow$ policy interaction between $m, \tau$.

# References


Aguiar, M., M. Amador, and C. Arellano (2021): "Micro Risks and Pareto Improving Policies," Mimeo, University of Minnesota.

Aiyagari, S. R. (1994): "Uninsured Idiosyncratic Risk and Aggregate Saving," *Quarterly Journal of Economics*, 109(3), 659–684.

Auclert, A., B. Bardóczy, M. Rognlie, and L. Straub (2021): "Using the Sequence-Space Jacobian to Solve and Estimate Heterogeneous-Agent Models," *Econometrica*, 89(5), 2375–2408.

Ben-Porath, Y. (1967): "The Production of Human Capital and the Life Cycle of Earnings," *Journal of Political Economy*, 75(4), 352–365.

Boppart, T., P. Krusell, and K. Mitman (2018): "Exploiting MIT shocks in heterogeneous-agent economies: the impulse response as a numerical derivative," *Journal of Economic Dynamics and Control*, 89(C), 68–92.

Burke, M., S. M. Hsiang, and E. Miguel (2015): "Global non-linear effect of temperature on economic production," *Nature*, 527, 235–239.

Cruz, J.-L., and E. Rossi-Hansberg (2023): "The Economic Geography of Global Warming," *The Review of Economic Studies*, 91(2), 899–939.

Golosov, M., J. Hassler, P. Krusell, and A. Tsyvinski (2014): "Optimal Taxes on Fossil Fuel in General Equilibrium," *Econometrica*, 82(1), 41–88.

Huggett, M. (1993): "The Risk-Free Rate in Heterogeneous-Agent, Incomplete-Insurance Economies," *Journal of Economic Dynamics and Control*, 17(5), 953–969.

İmrohoroğlu, A. (1989): "Cost of Business Cycles with Indivisibilities and Liquidity Constraints," *Journal of Political Economy*, 97(6), 1364–1383.

Kaldor, N. (1957): "A Model of Economic Growth," *The Economic Journal*, 67(268), 591–624.

Krusell, P., and J. Smith, Anthony A (2022): "Climate Change Around the World," Working Paper 30338, National Bureau of Economic Research.

Pijoan-Mas, J. (2006): "Precautionary Savings or Working Longer Hours?," *Review of Economic Dynamics*, 9, 326–352.

Quadrini, V. (2000): "Entrepreneurship, Saving, and Social Mobility," *Review of Economic Dynamics*, 3(1), 1–40.

Romer, P. M. (1990): "Endogenous Technological Change," *Journal of Political Economy*, 98, S71–S102.