Macro 7210 Lectures

Preliminary

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Introduction

What is Macro About?

- Study of Aggregates

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- Methods: General Equilibrium
- Dynamics
- Timely issues (Great Recession, COVID)

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- The description of a model's environment includes specifying agents' preferences and endowments, technology available, information structure as well as property rights.
- The workhorse model in Macro is the Neoclassical Growth Model.
- It delivers some fundamental properties that are characteristics of industrialized economies. Kaldor (1957) summarizes six (plus one) stylized facts.

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  3. Recursive Competitive Equilibrium (RCE) directly.

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Neoclassical Growth Model satisfies those facts provided

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   - Log plus Constant Frisch: :

$$u(c, 1-\ell) = u(c,n) \log c + \chi \frac{n^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}}$$

Recursive Equilibria without Distortions

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Ingredients

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  - Equilibrium Conditions/ Representative Agent Conditions

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- $c = c(K,a;G)$, $a' = g(K,a;G)$, $V(K,a;G)$ satisfy (use envelope)

$$u_c\left[c(K,a;G)\right] \quad = \quad \beta V_{a'}\left[G(K), g(K,a;G); G\right]$$

$$V_a\left(K,a;G\right) \quad = \quad R(K)\, u_c\left[c(K,a;G)\right]$$

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- In this case we can use the $G(K)$ that comes out of the social planner's dynamic programming problem as the candidate for RCE.

Economies with Distortions and Heterogeneity

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- If labor income tax, substitute $T(K)$ with $\tau(K)\ w(K)$.

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Exercise

Derive the first order conditions in the above problem to see the wedge introduced by taxes.

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- So individual state is just a

- The household needs to know the evolution of capital and *debt*

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with solution $g\left(K, B, a\right)$.

Definition

A *Rational Expectations Recursive Competitive Equilibrium given* $P(K, B)$ *and* $\tau(K, B)$*, are functions* V*,* g*,* G*,* H*,* w*,* q*, and* R*, s.t.*

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5. *Gov b constr:* $B + P(K,B) = \tau(K,B)R(K)K + q(K,B)H(K,B)$
6. *Government debt is bounded:*
   $\exists$ *some* $\bar{B}$*, such that for all* $K \in \left[0, \tilde{k}\right)$ *and* $B \leq \bar{B}$*,* $H(K,B) \leq \bar{B}$*.*

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- $P(K)$ is shares price; $D(K)$ dividends per share. Soltn, $a' = h(K, a)$

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- d dividends (solution $d(K, k)$), $q\,[G(K)]$ is price of good tomorrow.

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Exercise

Find missing condition. [Hint: it relates $q(G(K))$ with the price and dividends ($P(K), P(G(K))$, and $D(G(K))$).]

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5. Value of a representative firm equals price plus dividends

$$\Omega(K, K) = D(K) + P(K),$$

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Exercise

Define the RCE if a were savings paying $R(K)$ instead of shares.

- Two types of households differing only in wealth: R (rich) and P (poor) with measures μ and $1-\mu$. Otherwise identical.

$$V(K^R, K^P, a) = \max_{c,a'} \quad u(c) \quad + \quad \beta V(K^{R'}, K^{P'}, a')$$

$$\text{s.t.} \quad c + a' \quad = \quad w\left[(\mu K^R + (1-\mu)K^P\right] + aR\left[\mu K^R + (1-\mu)K^P\right]$$

$$K^{i'} \quad = \quad G^i(K^R, K^P) \qquad \text{for } i = R, P.$$

Remark

Decision rules are not linear (even though they might be almost linear); therefore, we need two states, K^1 and K^2, not aggregate K.

Definition

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Remark

Note that $G^R\left(K^R, K^P\right) = G^P\left(K^P, K^R\right)$ (look at the arguments carefully). Why? (How are rich and poor different?)

- In steady state, the Euler equations for the two types simplify to

$$u'\left(c^{R^*}\right) = \beta R\ u'\left(c^{R^*}\right), \text{ and } u'\left(c^{P^*}\right) = \beta R\ u'\left(c^{P^*}\right).$$

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- We have three equations (2 budget constraints and Euler equation) and four unknowns (a^{i^*} and c^{i^*} for $i = R, P$).
- The theory is silent about the steady state distribution of wealth!
- If savings are linear in a state (i.e. $g(K,a) = \mu^i(K) + \lambda(K)a$, and all have the same preferences, then aggregate capital can be expressed as the choice of a representative agent (with savings decision given by $g(K,K) = \bar{\mu}(K) + \lambda(K)K$).

- Type i has labor skill ϵ_i, $\mu^1 = \mu^2 = 1/2$. $\mu^1 \epsilon_1 + \mu^2 \epsilon_2 = 1$.

Exercise

Define the RCE.

- Type i has labor skill ϵ_i, $\mu^1 = \mu^2 = 1/2$. $\mu^1\epsilon_1 + \mu^2\epsilon_2 = 1$.
- The value functions are now indexed by type:

$$V^i(K^1, K^2, a) = \max_{c,a'} \quad u(c) \quad + \quad \beta V^i(K^{1'}, K^{2'}, a')$$

$$\text{s.t.} \quad c + a' = w\left(\frac{K^1 + K^2}{2}\right)\epsilon_i + aR\left(\frac{K^1 + K^2}{2}\right)$$

$$K^{i'} = G^i(K^1, K^2) \quad \text{for } i = 1, 2.$$

with solution $g^i(K^1, K^2, a)$.

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Define the RCE.

Remark

We can also rewrite this problem as

$$\begin{aligned} V^{i}\left(K,\lambda,a\right) = \max_{c,a'} \quad & \left\{u\left(c\right)+\beta V^{i}\left(K',\lambda',a'\right)\right\} \\ s.t. \quad & c + a' = R\left(K\right)a + W\left(K\right)\epsilon_i \\ & K = G\left(K,\lambda\right) \\ & \lambda' = H\left(K,\lambda\right), \end{aligned}$$

where K is aggregate capital, and λ is the share of type 1.

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where K is aggregate capital, and λ is the share of type 1.

Then the consistency conditions of the RCE must be:

$$\begin{aligned} G(K,\lambda) &= \frac{1}{2}\left[g^1(K,\lambda,2\lambda K) + g^2(K,\lambda,2(1-\lambda)K)\right], \\ H(K,\lambda) &= \frac{g^1(K,\lambda,2\lambda K)}{2G(K,\lambda)}. \end{aligned}$$

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  - Need also a variable for wealth distribution, say, shares in country 1.

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s.t. \quad & c + Q(K^1, K^2, A)a' = w^i\left(K^i\right) + a\Phi(K^1, K^2, A) \\
& K^{i'} = G^i\left(K^1, K^2, , A\right), \quad \text{for } i = 1, 2 \\
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$$K^{i'}=G^i\left(K^1,K^2,,A\right),\quad \text{for } i=1,2$$

$$A'=H\left(K^1,K^2,A\right)$$

- Mutual Funds' problem (note wages are country specific)

$$\Phi(K^1,K^2,A,k^1,k^2) \quad = \quad \max_{k^{1'},k^{2'},n^1,n^2}\sum_i\left[F^i(k^i,n^i)-n^iw^i(K_i)-k^{i'}\right]+ \frac{1}{R(K^{1'},K^{2'},A)}\Phi(K^{1'},K^{2'},A',k^{1'},k^{2'})$$

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Exercise

Solve for the mutual fund's decision rules. Is next period capital in each country chosen by the mutual fund priced differently? What about labor?

Overlapping Generations

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- We may just want to be realistic about the finite nature of the length of life.

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- Many Bells and Whistles are possible.

1. Hormones (i.e. exogenous)
2. Learning by doing (working more today increases your wage tomorrow)
3. Learning by not doing (i.e. Ben-Porath (1967)). From the time measure to be working a fraction is devoted to learn. Such time is not productive today.
4. Wage-Ladder: People start with low wage and over their lives the run into better opportunities and they switch. Occasionally the move down.
5. Wealth helps because either
    5.1 They are in a better bargaining position because wealth makes quitting better
    5.2 Rich people can afford to search longer because of a better match.
6. Age shapes your bargaining power and wages ensues (only really used at the end of life)
7. Various combinations of the above.

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- Consider

$$\begin{aligned} m_t &= \frac{\omega^y - c_t^y}{p_t} \\ c_{t+1}^o &= \frac{m_t}{p_{t+1} + m_t} \end{aligned}$$

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- Still, Why accept Money from older agents? Who needs them?

The Lucas Tree

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Definition

A Rational Expectations Recursive Competitive Equilibrium is a set of functions, V, g, d, and p, such that

1. V and g solves the household's problem given prices,
2. $d(z) = z$, and,
3. $g(z, 1) = 1$, for all z.

- Recall

$$u_c\left(c\left(z,s\right)\right) = \beta \sum_{z'} \Gamma_{zz'} \left[\frac{p\left(z'\right) + d\left(z'\right)}{p\left(z\right)}\right] \; u_c\left(c\left(z',s'\right)\right).$$

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- In equilibrium $s=1$ and $c(z,1)=z$ so we have $u_c\left(z\right):=u_c\left(c\left(z,1\right)\right)$. The

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- A system of n_z equations. Denote $\mathrm{p}:=\left[p\left(z_1\right)\vdots p\left(z_n\right)\right]_{(n_z\times 1)}$ and

$$\mathrm{u}_c:=\begin{bmatrix} u_c\left(z_1\right) & & 0 \\ & \ddots & \\ 0 & & u_c\left(z_n\right)\end{bmatrix}_{(n_z\times n_z)}.$$

- Then

$$\mathrm{u}_c.\mathrm{p} = \begin{bmatrix} p(z_1)\, u_c(z_1) \\ \vdots \\ p(z_n)\, u_c(z_n) \end{bmatrix}_{(n_z \times 1)},$$

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- This follows from a no-arbitrage argument.

$$p^m\left(z_0\right) = \sum_t \sum_{z^t \in H^t} q_t^0\left(z^t\right) \; a_t\left(z^t\right),$$

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- Given the $\left\{q_t^0(z^t)\right\}$, we can *replicate any possible asset by a set of state-contingent claims* and use this formula to price that asset.

- To find those q^0 consider a world where agents solve

$$\max_{c_t(z^t)} \quad \sum_{t=0}^{\infty} \beta^t \sum_{z^t} \pi_t\left(z^t\right) u\left(c_t\left(z^t\right)\right)$$

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- Note that this is the familiar Arrow-Debreu market structure, where the household owns a tree, and the tree yields $z \in Z$ amount of fruit in each period). The FOC for this problem (taking the ratio of the first period FOC and that of the history z^t) and imposing the equilibrium condition $c(z^t) = z_t(z^t)$ imply:

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- This enables us to price the good in each history of the world and price any asset accordingly.

- Hhold Probl

$$
\begin{aligned}
V(z,s,b) = \max_{c,s',b'(z')} \quad & u(c) + \beta \sum_{z'} \Gamma_{zz'} V\left(z', s', b'\left(z'\right)\right) \\
s.t. \quad & c + p(z)\, s' + \sum_{z'} q\left(z, z'\right) b'\left(z'\right) = s\left[p(z) + z\right] + b.
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- We can thus price *all types* of securities using p and q in this economy.

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- The European option to buy the day after tomorrow is

$$\bar{q}(z,P)=\sum_{z'}\sum_{z''}\max\left\{p\left(z''\right)-P,0\right\}q\left(z',z''\right)q\left(z,z'\right).$$

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- The unconditional gross risk free rate is

$$R^f = \sum_{z} \mu_z^* R(z)$$

where μ^* is the steady-state distribution of the shocks.

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- Use the expressions for p and q and the properties of the utility function to show that risk premium is positive.

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- The equilibrium is defined as before.
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- Discussion of Demand vs Supply Shocks and what RBC vs Lucas trees are.

An Introduction to Search with a Particular Application: Endogenous Productivity in a Product Search Model

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- After meeting, trades may happen or not.

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- Now They also have to FIND the fruit

- There is matching function $M(T, D)$: Trees and Search Effort.

A Slightly Different Environment

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- Here $T = 1$. The number of trees is constant.

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$$V(\theta, z, s) = \max_{c,d,s'} \quad u(c, d, \theta) + \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} \; V(\theta', z', s')$$

$$s.t. \quad c + P(\theta, z) s' = P(\theta, z) \left[s \left(1 + \widehat{R}(\theta, z) \right) \right]$$

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Exercise

Derive the Euler equation of the household from the problem defined above.

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- Searchers and (trees and household effort) choose which island to go to. They have different trade-offs of price versus tightness.
- Equilibrium determines which island (Optimal so unique).

Firms maximize returns by choosing market, Q, P. It helps to use trees as numeraire, so $\widehat{P}(Q) = 1/P$ is the price of consumption. We want to characterize the set of available markets for firms, $\widehat{P}(Q)$ by looking at the implications for firms that face it:

$$\widehat{R}(\theta, z) = \max_{Q} \; \widehat{P}(Q) \; \Psi^{f}(Q) \; z = \frac{z \; \Psi^{f}(Q)}{P(Q)}$$

with FOC

$$\widehat{P}'(Q) \Psi^{f}(Q) + \widehat{P}(Q) \Psi^{f'}(Q) = 0,$$

The set of pairs P a that satisfies FOC yields a relation of indifference between the firms the pairs $\{P, Q\}$ for the firms that implicitly determines $\widehat{P}(Q)$ as

$$\frac{\widehat{P}'(Q)}{\widehat{P}(Q)} = -\frac{\Psi^{f'}(Q)}{\Psi^{f}(Q)} = \frac{P(Q)}{P'(Q)}.$$

$$V(\theta, z, s) = \max_{c,d,s',P,Q} \quad u(\theta \;\, c, d) + \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} V\left(\theta', z', s'\right) \tag{1}$$

$$s.t. \quad c + Ps' = P\left[s\left(1 + \widehat{R}(\theta, z)\right)\right], \tag{2}$$

$$c = d\; \Psi^h(Q)\, z \tag{3}$$

$$\frac{z\Psi^f(Q)}{P} \geq \widehat{R}(\theta, z) \tag{4}$$

- The last constraint states that for a market to exist firms have to be guaranteed $\widehat{R}(\theta, z)$.

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Plug the first two constraints into the objective function (c and s' as functions of d) and (recall that $\Psi^h = Q^{1-\varphi}$) :

$$\theta Q^{1-\varphi} z u_c(\theta d Q^{1-\varphi} z, d) + u_d(\theta d Q^{1-\varphi} z, d) = \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} V_3 \left(\theta', z', s(1+\widehat{R}(\theta, z)) - \frac{dQ^{1-\varphi} z}{P} \right) \frac{Q^{1-\varphi} z}{P} \quad (5)$$

Get rid of V_3 using original problem and use the envelope theorem

$$V_3(\theta, z, s) = \left[\theta u_c(\theta d Q^{1-\varphi} z, d) + \frac{u_d(\theta d Q^{1-\varphi} z, d)}{Q^{1-\varphi} z} \right] P(1+\widehat{R}(\theta, z))$$

Combining these two gives the Euler equation:

$$\theta u_c(\theta d Q^{1-\varphi} z, d) + \frac{u_d(\theta d Q^{1-\varphi} z, d)}{Q^{1-\varphi} z} = \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} \frac{P'(1+\widehat{R}(\theta', z'))}{P} \left[\theta' u_c(\theta' d' Q'^{1-\varphi} z', d') + \frac{u_d(\theta' d' Q'^{1-\varphi} z', d')}{Q'^{1-\varphi} z'} \right] \quad (6)$$

λ: Lagrange multiplier on the firm's participation constraint, then

$$\theta d(1-\varphi)Q^{-\varphi} z u_c(\theta d Q^{1-\varphi} z, d) = \\ \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} V_3 \left(\theta', z', s(1+\widehat{R}(\theta, z)) - \frac{dQ^{1-\varphi} z}{P}\right) \\ \frac{d(1-\varphi)Q^{-\varphi} z}{P} - \lambda \frac{\varphi Q^{-\varphi-1} z}{P} \tag{7}$$

and

$$\beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} V_3 \left(\theta', z', s(1+\widehat{R}(\theta, z)) - \frac{dQ^{1-\varphi} z}{P}\right) dQ = -\lambda \tag{8}$$

Combining these two equation gives us:

$$\theta u_c(\theta d Q^{1-\varphi} z, d) = \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'}$$

$$V_3\left(\theta', z', s(1+\widehat{R}(\theta, z)) - \frac{dQ^{1-\varphi} z}{P}\right)\left[\frac{1}{(1-\varphi)P}\right] \quad (9)$$

Recall $V_3(\cdot, \cdot, \cdot)$ so

$$(1-\varphi)\theta u_c(\theta d Q^{1-\varphi} z, d) = \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'}$$

$$\frac{P'(1+\widehat{R}(\theta', z'))}{P}\left[\theta' u_c(\theta' d' {Q'}^{1-\varphi} z', d') + \frac{u_d(\theta' d' {Q'}^{1-\varphi} z', d')}{{Q'}^{1-\varphi} z'}\right] \quad (10)$$

Definition

An Eq with competitive search is functions $\{V, c, d, s', P, Q, \widehat{R}\}$ that:

1. Household's budget constraint, (condition 2)
2. Household's shopping constraint, (condition 3)
3. Household's Euler equation, (condition 6)
4. Market condition, (condition 10)
5. Firm's participation constraint, (condition 4), which gives us that the dividend payment is the profit of the firm, $\widehat{R}(\theta, z) = \frac{zQ^{-\varphi}}{P}$,
6. Market clearing, i.e. $s' = 1$ and $Q = 1/d$.

Measure Theory

Measure theory is a tool that helps us aggregate.

Definition

For a set S, $\mathcal{S}$ is a family of subsets of S, if $B \in \mathcal{S}$ implies $B \subseteq S$ (but not the other way around).

Remark

Note that in this section we will assume the following convention

1. *small letters (e.g. s) are for elements,*
2. *capital letters (e.g. S) are for sets, and*
3. *fancy letters (e.g. $\mathcal{S}$) are for a set of subsets (or families of subsets).*

Definition

A family of subsets of S, $\mathcal{S}$, is called a σ-algebra in S if

1. $S, \emptyset \in \mathcal{S}$;
2. if $A \in \mathcal{S} \Rightarrow A^c \in \mathcal{S}$ (i.e. $\mathcal{S}$ is closed with respect to complements and $A^c = S \backslash A$); and,
3. for $\{B_i\}_{i \in \mathbb{N}}$, if $B_i \in \mathcal{S}$ for all $i \Rightarrow \bigcap_{i \in \mathbb{N}} B_i \in \mathcal{S}$ (i.e. $\mathcal{S}$ is closed with respect to countable intersections.

Example

1. The power set of S and $\{\emptyset, S\}$ are σ-algebras in S.
2. $\{\emptyset, S, S_{1/2}, S_{2/2}\}$, where $S_{1/2}$ means the lower half of S (imagine S as an closed interval in $\mathbb{R}$), is a σ-algebra in S.
3. If $S = [0, 1]$, then $\mathcal{S} = \left\{\emptyset, \left[0, \frac{1}{2}\right), \left\{\frac{1}{2}\right\}, \left[\frac{1}{2}, 1\right], S\right\}$ is *not* a σ-algebra in S. But $\mathcal{S} = \left\{\emptyset, \left\{\frac{1}{2}\right\}, \left\{\left[0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right]\right\}, S\right\}$ is.

It allows us to define sets where things happen and we can *weigh* those sets (avoiding math troubles)

Definition

Suppose $\mathcal{S}$ is a σ-algebra in S. A measure is a real-valued function $x : \mathcal{S} \to \mathbb{R}_+$, that satisfies

1. $x(\emptyset) = 0$;
2. if $B_1, B_2 \in \mathcal{S}$ and $B_1 \cap B_2 = \emptyset \Rightarrow x(B_1 \cup B_2) = x(B_1) + x(B_2)$ (additivity); and,
3. if $\{B_i\}_{i \in \mathbb{N}} \in \mathcal{S}$ and $B_i \cap B_j = \emptyset$ for all $i \neq j \Rightarrow x(\cup_i B_i) = \sum_i x(B_i)$ (countable additivity).

A set S, a σ-algebra in it ($\mathcal{S}$), and a measure on $\mathcal{S}$ x, define a measurable space, $(S, \mathcal{S}, x)$.

Definition

A Borel σ-algebra is a σ-algebra generated by the family of all open sets $\mathfrak{B}$ (generated by a topology). A Borel set is any set in $\mathfrak{B}$.

A Borel σ-algebra corresponds to complete information.

Definition

A probability measure x is a measure where $x(S) = 1$. $(S, \mathcal{S}, x)$ is a probability space. The probab of an event is then given by $x(A)$, where $A \in \mathcal{S}$.

Definition

Given a m'able space $(S, \mathcal{S}, x)$, a real-valued function $f : S \to \mathbb{R}$ is m'able (with respect to the m'able space) if, for all $a \in \mathbb{R}$, we have

$$\{b \in S \mid f(b) \leq a\} \in \mathcal{S}.$$

Interpret σ-algebras as describing available information.

Similarly, a function is m'able wrt a σ-algebra $\mathcal{S}$, if it can be evaluated

Example

Suppose $S = \{1, 2, 3, 4, 5, 6\}$. Consider a function f that maps the element 6 to the number 1 (i.e. $f(6) = 1$) and any other elements to -100. Then f is NOT measurable with respect to $\mathcal{S} = \{\emptyset, \{1, 2, 3\}, \{4, 5, 6\}, S\}$. Why? Consider $a = 0$, then $\{b \in S \mid f(b) \leq a\} = \{1, 2, 3, 4, 5\}$. But this set is not in $\mathcal{S}$.

Extend the notion of Markov stuff to any measurable space

Definition

Given a measurable space $(S, \mathcal{S}, x)$, a function $Q : S \times \mathcal{S} \to [0, 1]$ is a transition probability if

1. $Q(s, \cdot)$ is a probability measure for all $s \in S$; and,
2. $Q(\cdot, B)$ is a measurable function for all $B \in \mathcal{S}$.

Intuitively, for $B \in \mathcal{S}$ and $s \in S$, $Q(s, B)$ gives the probability of being in set B tomorrow, given that the state is s today.

1. A *Markov chain* with transition matrix given by

$$\Gamma = \begin{bmatrix} 0.2 & 0.2 & 0.6 \\ 0.1 & 0.1 & 0.8 \\ 0.3 & 0.5 & 0.2 \end{bmatrix},$$

on $S = \{1, 2, 3\}$, with the the power set being the σ-algebra of S).

$$Q\left(3, \{1, 2\}\right) = \Gamma_{31} + \Gamma_{32} = 0.3 + 0.5 \ .$$

2. Consider a measure x on $\mathcal{S}$. x_i is the fraction of type i. Then

$$\begin{aligned} x_1' &= x_1\Gamma_{11} + x_2\Gamma_{21} + x_3\Gamma_{31}, \\ x_2' &= x_1\Gamma_{12} + x_2\Gamma_{22} + x_3\Gamma_{32}, \\ x_3' &= x_1\Gamma_{13} + x_2\Gamma_{23} + x_3\Gamma_{33}. \end{aligned}$$

In other words: $\mathrm{x}' = \Gamma^T \mathrm{x}$, where $\mathrm{x}^T = (x_1, x_2, x_3)$.

From a measure x today to one tomorrow x'

$$\begin{aligned} x'(B) &= T(x, Q)(B) \\ &= \int_S Q(s, B)\, x(ds), \quad \forall B \in \mathcal{S}, \end{aligned}$$

we integrated over all $s \in S$ to get the prob of B tomorrow.

A stationary distribution is a fixed point of T, that is x^* such that

$$x^*(B) = T(x^*, Q)(B), \quad \forall B \in \mathcal{S}.$$

Theorem

If Q has nice properties (American Dream and Nightmare) then $\exists$ a unique stationary distribution x^* and

$$x^* = \lim_{n \to \infty} T^n(x_0, Q), \qquad \textit{for any } x_0.$$

Exercise

Consider unemployment in a very simple economy (in which the transition matrix is exogenous). There are two states of the world: being employed and being unemployed. The transition matrix is given by

$$\Gamma = \begin{pmatrix} 0.95 & 0.05 \\ 0.50 & 0.50 \end{pmatrix}.$$

Compute the stationary distribution corresponding to this Markov transition matrix.

Industry Equilibrium

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- n^* is an increasing function of both arguments. Prove it.

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- Use x to define statistics of the industry: Since individual supply is $sf\left(n^*\left(s,p\right)\right)$, then the aggregate supply

$$Y^S\left(p\right) = \int_S sf\left(n^*\left(s,p\right)\right) x\left(ds\right). \tag{13}$$

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- Let Demand $Y^D\left(p\right)$. Then p^* clears the market:

$$Y^D\left(p^*\right) = Y^S\left(p^*\right). \tag{14}$$

Where does x come from?

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- The choice is static. The value of an s firm is

$$V(s;p) = \sum_{t=0}^{\infty} \left(\frac{\delta}{1+r}\right)^t \pi(s,p) = \qquad \left(\frac{1+r}{1+r-\delta}\right)\pi(s,p)$$

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- x is the measure of firms. Firms that die are $(1-\delta)x(S)$.
- Entrants draw s from probability measure γ over $(S, \mathcal{S})$.

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- Equilibrium requires $V^E = 0$

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$$Tx(B) = \delta x(B) + m\gamma(B), \quad \forall B \in \mathcal{S}, \tag{17}$$

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$$x^*(B; m) = \frac{m}{1-\delta}\gamma(B), \quad \forall B \in \mathcal{S}. \tag{18}$$

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$$Y^D\left(p^*\left(m\right)\right) = \int_S s\ f\left[n^*\left(s;p\right)\right]\ dx^*\left(s;m\right), \tag{19}$$

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  - Then $\exists$ a threshold, $s^* \in S$, below which firms exit and above stay.

$$V(s;p) = \max \left\{0, \pi(s;p) + \frac{1}{(1+r)} \int_S V(s';p)\, \Gamma(s, ds') - c^v \right\}. \tag{21}$$

- Updating operator becomes

$$x'(B) = \int_{s^*}^{\bar{s}} \Gamma(s, B \cap [s^*, \bar{s}]) \, x(ds) + m\gamma(B \cap [s^*, \bar{s}]), \quad \forall B \in \mathcal{S}. \tag{22}$$

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  - Fraction of workers in largest top 10% of firms

$$\frac{\int_{\hat{s}}^{\bar{s}} n^*(s,p)\, x^*(ds)}{\int_{s^*}^{\bar{s}} n^*(s,p)\, x^*(ds)}.$$

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Exercise

Compute the average growth rate of the smallest one third of the firms.

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- Gini coefficient.

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$$\forall\ B^{S} \in \mathcal{S}, \forall\ B^{N} \in \mathcal{N}.$$

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- Consider demand shifters z_t so that $D(P, z_t)$ where $z_{t+1} = \phi(z_t)$ so we can choose to represent it as a sequence or recursively.

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- Obviously You have to add the Expectations to the terms of one period later.

Numerical Approximations I: Using an Exact Transition

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    3. Specify some tricks or procedures to effectively compute θ^* (say iterate backward from the future to the present using successive approximations).

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- Then do linear approximations in sequence space.
- We will see how it works via a simple example.

- Consider the social planner's problem (with full depreciation)

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\begin{aligned}
V(k_t) \quad = \quad & \max_{c_t, k_{t+1}} u(c_t) + \beta V(k_{t+1}) \\
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- Derive the above equilibrium conditions.

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- It is the only way to do policy analysis (as the evaluation of the equilibrium under alternative policies **GIVEN** initial conditions.

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- This is in fact an impulse response function.

- We want now to simulate a response of the economy to shocks. Consider an AR(1) process for z_t : with $z_{t+1} = \rho^t z_t + \epsilon_{t+1}$.) where $\epsilon_t \sim \mathcal{N}(0, \sigma^2)$.

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- **Want:** Solve for the solution by linearly approximating using $\{\widehat{k}_t\}_{t=0}^{\infty}$ the equilibrium given any sequence of innovations $\{\epsilon_t\}$.).

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&\vdots \\
\widetilde{k}_{t+1}(k_0, \epsilon^t) &= \sum_{\tau=0}^{t} \epsilon_t \ \widehat{k}_{t-\tau+1} \qquad \text{exact if } \epsilon_0 = 1, \ \epsilon_t = 0, \forall t \neq 0,
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5. *Imagine now that the industry is subject to demand shocks that follow an AR(1). Describe an algorithm to approximate it.*

Incomplete Market Models

- Consider the problem of a farmer with storage possibilities

$$V\left(s,a\right)=\max_{c,a'\geq 0}\quad u\left(c\right)+\beta\sum_{s'}\Gamma_{ss'}\;V\left(s',a'\right)\qquad s.t.$$

$$c+qa'=a+s$$

a assets, c consumption,

$s\in\{s^1,\cdots,s^{N^s}\}=S$ has transition Γ,
q units today yield 1 unit tomorrow. Only nonnegative storage.

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- We can construct a prob distribution over states $S \times A$. Define $\mathcal{B}$ as all subsets of S times $\mathcal{A}$, the Borel-σ-algebra sets in A.
- Using Γ and g we get a transition over $\{S \times A, \mathcal{S} \times \mathcal{A}\}$: for any such prob measure x its evolution is

$$x'\left(B\right) = \widetilde{T}(B, x; \Gamma, g) = \sum_{s}\int_0^{\bar{a}}\sum_{s' \in B_s}\Gamma_{ss'}\; 1_{\{g(s,a)\in B_a\}}\; x\left(s, da\right), \quad \forall B \in \mathcal{B}$$

where B_s and B_a are projections of B on S and A,

Theorem

With a well behaved Γ, there is a unique stationary probability x^*, so that:

$$\begin{aligned} x^*(B) &= \widetilde{T}(B, x^*; \Gamma, g)(B), \quad \forall B \in \mathcal{B}, \\ x^*(B) &= \lim_{n\to\infty} \widetilde{T}^n(B, x_0; \Gamma, g)(B), \quad \forall B \in \mathcal{B}, \end{aligned}$$

for all initial probability measures X_0 on $(E, \mathcal{B})$.

We use compactness of $[0, \overline{A}]$.

Two Interpretations of x

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where $\underline{a} < 0$ and $\beta / q < 1$. With solution $a' = g(s,a)$. Again

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- Or it could be tighter which makes it likely to bind $0 > \underline{a} > a^n$.

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  3. $\lim_{q\to\infty} \int_{A\times S} a\, dX^*(q) < 0$. As $q \to \infty$, arbitrary large consumption is achievable by borrowing.

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Aiyagari (1994) Economy

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- Now we need $\beta(1+r) < 1$. We write

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$$c + a' = (1+r)\,a + ws$$

where r is the return on savings and w is the wage rate.

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Rewrite the economy when households like leisure

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- Define $\eta(s, a)$ by

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- Transfer necessary to make the (a, s) agent indifferent between living in the old environment and in the new.

- Let the Economy's parameters be summarized by $\theta = \{u, \beta, s, \Gamma, F\}$.
- $V(s, a; \theta)$ and $x^*(\theta)$ are functions of those parameters.
- Suppose an unexpected policy change that shifts θ to $\hat{\theta} = \{u, \beta, s, \hat{\Gamma}, F\}$.
- Consider $V\left(s, a; \hat{\theta}\right)$ and $x^*\left(\hat{\theta}\right)$.
- Define $\eta(s, a)$ by

$$V\left(s, a + \eta(s, a); \hat{\theta}\right) = V(s, a; \theta),$$

- Transfer necessary to make the (a, s) agent indifferent between living in the old environment and in the new.
- Total transfer needed to compensate all agents to live in $\hat{\theta}$ is

$$\int_{A \times S} \eta(s, a)\, dX^*(\theta).$$

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- Welfare computing the transition from the SAME initial conditions.
- Otherwise the best tax policy in the Rep agent (which is Pareto Optimal) would be to subsidize capital to maximize steady state consumption.
- To analyze the policy we need to compute the whole transition as depicted in a previous lecture. It is a NON-stationary equilibrium.

Business Cycles with Economies with Measures of Agent

- What if aggregate shocks as in e.g. $z\, F\left(K, \bar{N}\right)$.

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\begin{aligned}
V\left(z, X, s, a\right) = \max_{c, a' \geq 0} \quad & u\left(c\right) + \beta \sum_{z', s'} \Pi_{zz'} \Gamma_{ss'}^{z'} V\left(z', X', s', a'\right) \\
s.t. \quad & c + a' = a\ z\ f_k\left(K, \bar{N}\right) + s\ z\ f_n\left(K, \bar{N}\right) \\
& K = \int a\ dX\left(s, a\right) \\
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(replaced factor prices with marginal productivities)

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- Computationally, this problem is a beast! So, what then?

- They people believe tomorrow's capital depends only on K and not on x. This, obviously, is not an economy with rational expectations. The agent's problem in such a setting is

$$\widetilde{V}(z,K,s,a) = \max_{c,a'} \quad u(c) + \beta \sum_{z',s'} \Pi_{zz'} \Gamma_{ss'}^{z'} \widetilde{V}\left(z',K',s',a'\right)$$

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- They found it works well in boring settings (things are pretty linear)

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- Valuable for SMALL shocks like all linear approximations.

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        - We look for a fixed point of this (not necessarily iterating mechanically but as solution of a system of equations)

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$$\begin{aligned}\widetilde{d}_1(x_0,\epsilon_0) &= \frac{\epsilon_0}{\bar{\epsilon}_0}\ \widehat{d}_1 \\ \widetilde{d}_2(x_0,\epsilon_0,\epsilon_1) &= \frac{\epsilon_0}{\bar{\epsilon}_0}\ \widehat{d}_2 + \frac{\epsilon_1}{\bar{\epsilon}_0}\ \widehat{d}_1,\end{aligned}$$

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&\vdots \\
\widetilde{d}_{t+1}(x_0, \epsilon^t) &= \sum_{\tau=0}^{t} \frac{\epsilon_t}{\bar{\epsilon}_0} \; \widehat{d}_{t-\tau+1} \qquad \text{exact if } \epsilon_0 = \widetilde{\epsilon}_0, \; \epsilon_t = 0, \forall t \neq 0.
\end{aligned}
$$

Numerical Approximations II: Using a Linear Approximation to Get the Transition

Taken mostly from a May 20 Remote Brown Bag on Computational Economics and Finance by Auclert, Bardóczy, Rognlie, and Straub

Auclert, Bardóczy, Rognlie, and Straub (2021)

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Sequence-Space Jacobian to Solve and Estimate Heterogeneous-A Models

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  3. Use Jacobians to get Impulse Response Functions

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    - Most ideas are also easily implemented in Julia or even Matlab

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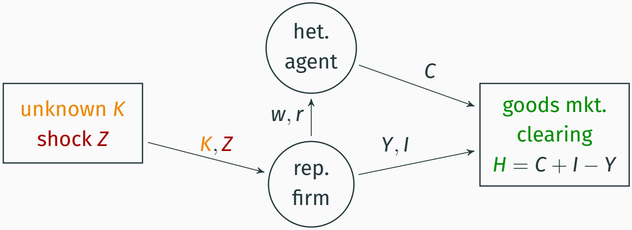
- Many models can be written in this way.
- Key restriction: agents interact via limited set of aggregate variables

Krusell-Smith model DAG

Capital market clearing

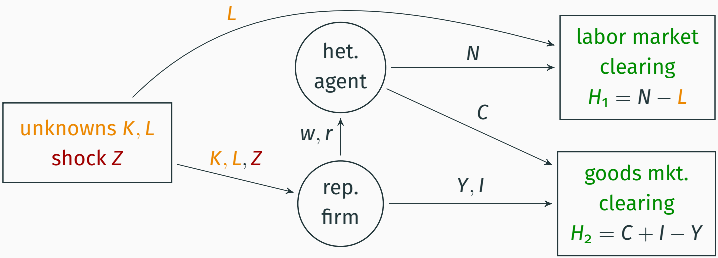


- DAG can be collapsed into mapping

$$H_t(\{K_s\}, \{Z_s\}) = C_t + I_t - Y_t$$

- GE path of $\{K_s\}$ achieves $H_t(\{K_s\}, \{Z_s\}) = 0$

Dealing with endogenous labor: add an unknown and a target



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$$\mathbf{H}_t\left(\{K_s, L_s\}, \{Z_s\}\right) = \{C_t + I_t - Y_t, N_s - L_s\}$$

- GE path of $\{K_s, L_s\}$ achieves $\mathbf{H}_t\left(\{K_s, L_s\}, \{Z_s\}\right) = 0$

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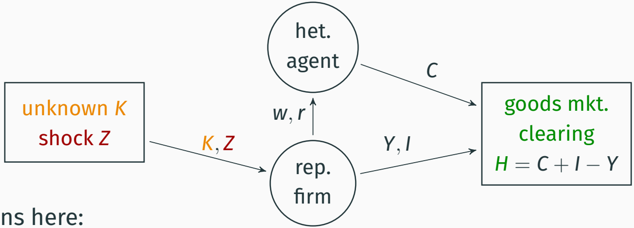
- Next: block Jacobians are sufficient to compute GE impulse responses

Krusell-Smith model Jacobians



- Jacobians here:
  - het. agent: $\left\{\frac{\partial C_t}{\partial w_s}\right\}, \left\{\frac{\partial C_t}{\partial r_s}\right\} \rightsquigarrow$ denote $\mathcal{J}^{C,w}, \mathcal{J}^{C,r}$
  - rep. firm: $\left\{\frac{\partial w_t}{\partial K_s}\right\}, \left\{\frac{\partial w_t}{\partial Z_s}\right\}, \left\{\frac{\partial r_t}{\partial K_s}\right\}, \left\{\frac{\partial r_t}{\partial Z_s}\right\}, \ldots \rightsquigarrow$ denote $\mathcal{J}^{w,K}, \mathcal{J}^{w,Z}, \mathcal{J}^{r,K}, \mathcal{J}^{r,Z}, \ldots$

- To Get the Jacobians of the Markets Clearing BLOCK

$$\frac{\partial H}{\partial K} = \mathcal{J}^{C,r}\mathcal{J}^{r,K} + \mathcal{J}^{C,w}\mathcal{J}^{w,K} + \mathcal{J}^{I,K} - \mathcal{J}^{Y,K}$$

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- To Get the Jacobians of the Markets Clearing BLOCK

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- Once the Jacobians are chained to get $\frac{\partial H}{\partial K}$ and $\frac{\partial H}{\partial Z}$ we are done

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Show how to get those distribution moments (call them m_x) by properly constructing $\mathcal{J}^{m_x \cdot}$.

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Aggregate shocks and the $MA(\infty)$ Representation

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- For other parameters estimation is still very fast if we do not need to recalculate the heterogeneous agents blocks (capital adjustment costs for instance) because we can use the same Jacobians $\mathcal{J}^{C,w}, \mathcal{J}^{C,r}$.

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3. Jacobians are also useful to get the nonlinear perfect-foresight solution
    - Solve $H(K, Z) = 0$ using Newton's method because $\frac{\partial H}{\partial K}$ is the steady state Jacobian.

Computing heterogeneous-agent Jacobians

So far: DAG + Jacobians $\Rightarrow$ IRFs, determinacy, estimation, nonlinear transitions

But how do we get the block **Jacobians?**

- *simple blocks*: (e.g. representative firms) simple, sparse matrix
- *HA blocks?* $\rightarrow$ **next**

Jacobian of consumption with respect to wage

- Want to know $\mathcal{J}_{t,s} \equiv \frac{\partial C_t}{\partial w_s}$ for $s, t \in \{0, \ldots, T-1\}$ [intertemporal MPCs]
  - Assume initial condition is s.s., with $r_t = r$, $w_t = w$, $D_0(e, k_-) = D(e, k_-)$

- **Direct algorithm**: perturb $w_s \equiv w + \epsilon$
  1. iterate backwards to get perturbed policies: $\mathbf{c}_t^s(e, k_-), \mathbf{k}_t^s(e, k_-)$
  2. iterate forward to get perturbed distributions $D_t^s(e, k_-)$
  3. put together to get perturbed aggregate consumption: $C_t^s = \int \mathbf{c}_t^s(e, k_-) D_t^s(e, dk_-)$
  4. compute $\mathcal{J}$ from $\mathcal{J}_{t,s} \equiv (C_t^s - C)/\epsilon$

- This is **slow**, since 1–4 needs to be done T times, once for each s

- Paper proposes **fake news algorithm** that is T times faster:
  - requires **single** backward iteration & **single** forward iteration
  - key idea: exploit **time symmetries** around the steady-state

(0) The fake news matrix

- We can think of $\mathcal{J} \equiv \left(\frac{\partial C_t}{\partial w_s}\right)$ as a **news matrix**
  - column s = response to news that shock hits in period s
- Define a new auxiliary matrix:

$$\mathcal{F}_{t,s} \equiv \begin{cases} \frac{\partial C_t}{\partial w_s} & s = 0 \text{ or } t = 0 \\ \frac{\partial C_t}{\partial w_s} - \frac{\partial C_{t-1}}{\partial w_{s-1}} & s, t > 0 \end{cases}$$

- Can think of this as **fake news matrix:**
  - at $t = 0$: news shock that period s shock hits $\rightarrow \frac{\partial C_0}{\partial w_s}$
  - at $t = 1$: news shock that there won't be a shock at $s \rightarrow \frac{\partial C_1}{\partial w_s} - \frac{\partial C_0}{\partial w_{s-1}}$
  - useful: starting in $t = 1$, **agents' policy functions are unchanged** by fake news shock
- Can recover $\mathcal{J}$ from $\mathcal{F}$: news shock = sequence of fake news shocks

(0) The fake news matrix

$$\mathcal{J} = \begin{pmatrix} \mathcal{J}_{00} & \mathcal{J}_{01} & \mathcal{J}_{02} & \cdots \\ \mathcal{J}_{10} & \mathcal{J}_{11} & \mathcal{J}_{12} & \cdots \\ \mathcal{J}_{20} & \mathcal{J}_{12} & \mathcal{J}_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad \mathcal{F} = \begin{pmatrix} \mathcal{J}_{00} & \mathcal{J}_{01} & \mathcal{J}_{02} & \cdots \\ \mathcal{J}_{10} & \mathcal{J}_{11} - \mathcal{J}_{00} & \mathcal{J}_{12} - \mathcal{J}_{01} & \cdots \\ \mathcal{J}_{20} & \mathcal{J}_{12} - \mathcal{J}_{10} & \mathcal{J}_{22} - \mathcal{J}_{11} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

- Can recover $\mathcal{J}$ from $\mathcal{F}$ by adding elements from top left diagonal

(1) Single backward iteration

- **Claim:** Single backward iteration is enough to recover $\mathbf{c}_t^s(e, k_-), \mathbf{k}_t^s(e, k_-)$
- Why? only the **time** $s - t$ **until the perturbation matters**

$$\mathbf{c}_t^s(e, k_-) = \begin{cases} \mathbf{c}(e, k_-) & s < t \\ \mathbf{c}_{T-1-(s-t)}^{T-1}(e, k_-) & s \geq t \end{cases}$$

- Thus, only need a single backward iteration with $s = T - 1$ to get all the $\mathbf{c}_t^s$
- From these we get:
    - $C_0^s = \int \mathbf{c}_0^s(e, k_-)D(e, dk_-)$, so first row of Jacobian $\mathcal{J}_{0s} = \frac{\partial C_0}{\partial w_s} = \mathcal{F}_{0s}$
    - $D_1^s(e, dk_-)$, distributions at date 1 implied by new policy $\mathbf{c}_0^s$ at date 0

(2) Single forward iteration

- Let's iterate those distributions forward using **s.s. policies**

$$D_1^s(e, dk_-) \mapsto D_2^s(e, dk_-) \mapsto D_3^s(e, dk_-) \mapsto \dots$$

  - this is just a **linear map**: $\mathbf{D}_t^s = (\Lambda')^{t-1}\, \mathbf{D}_1^s$ where Λ is s.s. transition matrix

- Now construct aggregate consumption using **s.s. policies c**

$$C_t^s \equiv \int \mathbf{c}(e, k_-)\, D_t^s(e, dk_-) \quad \Rightarrow \quad C_t^s = \mathbf{c}' \left(\Lambda'\right)^{t-1} \mathbf{D}_1^s$$

  - this only requires computing $\mathbf{c}'$, $\mathbf{c}'\Lambda'$, $\mathbf{c}'\,(\Lambda')^2$, $\dots \rightarrow$ like a **single** forward iteration!

- This is exactly the **fake news matrix**

$$\mathcal{F}_{t,s} = (C_t^s - C)/\epsilon$$

Extensions of the Aiyagari Economy

- Agents can either not work or work: $\varepsilon = \{0, 1\}$,

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- s is Markovian (Γ) labor labor productivity. Then the unemployed

$$V(s, 0, a) = \max_{c,h,a' \geq 0} u(c, h) + \beta \sum_{s'} \Gamma_{ss'} \left[\phi(h) V(s', 1, a') + (1 - \phi(h)) V(s', 0, a')\right]$$

$$s.t. \quad c + a' = b + (1 + r)a$$

The employed

$$V(s, 1, a) = \max_{c,a' \geq 0} u(c) + \beta \sum_{s'} \Gamma_{ss'} \left[\delta V(s', 0, a') + (1 - \delta) V(s', 1, a')\right]$$

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- Define Stationary Equilibrium

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- If worker

$$V^w(s,\eta,a) = \max_{c,a' \geq 0, d \in \{0,1\}} u(c) + \beta \sum_{s',\eta'} \Gamma_{ss'} \Gamma_{\eta\eta'} \left[dV^w(s',\eta',a') + (1-d)V^e(s',\eta',a') \right]$$

$$s.t. \quad c + a' = w\ s + (1+r)a$$

- Similarly, the entrepreneur's problem can be formulated as follows

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- Income is from profits $\pi(a,s,\eta)$ not wages. Assume entrepreneurs have a Decreasing Returns to Scale technology f. Profits are

$$\pi(s,\eta,a) = \max_{k,n} \eta f(k,n) + (1-\delta)k - (1+r)(k-a) - w \max\{n-s,0\}$$

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- The constraint here reflects the fact that entrepreneurs can only make loans up to a fraction ϕ of his total wealth.

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- Who becomes an entrepreneur in this economy? Without financial constraints, wealth will play no role. $\exists \eta^*$ above which it becomes an entrepreneur.
- With financial constraints wealth matters. Wealthy agents with high h will while the poor with low η will not.
- For the rest, it depends. If η is persistent, poor individuals with high entrepreneurial ability will save to one day become entrepreneurs, while rich agents with low entrepreneurial ability will lend their assets and become workers.

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- If no default the budget constraint is $c + q(a')a' = a + ws$,

$$V(s,a) = \max \left\{ u(ws) + \beta \sum_{s'} \Gamma_{ss'} \bar{V}(s'), \right.$$
$$\left. \max_{c,a'} u[ws + a - q(a')\ a'] + \beta \sum_{s'} \Gamma_{ss'} V(s',a') \right\}$$

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- What determines $q(a')$? A zero profit on lenders: Probability of default

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where $\bar{V}(s') = \frac{1}{1-\beta} u(ws')$ is the value of autarky.

- What determines $q(a')$? A zero profit on lenders: Probability of default
- Note that this problem displays time inconsistency

Monopolistic Competition and New Keynesian Models

Read McKay and Ravn Chapter 16 of Book

- Models with Nominal Prices.

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- Small in the Context of the Aggregate Economy. Hence Monopolistic Competition.

- Consumers have a taste for variety

$$u\left(\{c(i)\}_{i\in[0,n]}\right) = \left(\int_0^n c(i)^{\frac{\sigma-1}{\sigma}}\,di\right)^{\frac{\sigma}{\sigma-1}}$$

where σ is the elasticity of substitution, and $c(i)$ is the quantity consumed of variety i. For ease of notation, we rename $c(i) = c_i$.

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- Assume the agents receive exogenous *nominal* income I

- They are endowed with one unit of time.

$$\max_{\{c_i\}_{i\in[0,n]}} \left(\int_0^n c_i^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{\sigma}{\sigma-1}}$$

$$s.t. \quad \int_0^n p_i \; c_i \; di \leq I$$

- Deriving the FOC, and relating the demand for varieties i and j

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$$c_i^* = \frac{I}{\int_0^n p_j^{1-\sigma} dj} \, p_i^{-\sigma}$$

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- Here c_i^* depends on the price of i and an aggregate price

- Convenient to define the aggregate price index P as

$$P = \left(\int_0^n p_j^{1-\sigma} dj \right)^{\frac{1}{1-\sigma}}$$

Exercise

Show the following within this monopolistic competition framework

1. *σ is the elasticity of substitution between varieties.*
2. *Price index P is the expenditure to purchase a unit-level utility.*
3. *Consumer utility is increasing in the number of varieties n.*
4. *Is there a missing n?*

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- and thus

$$c_i^* = \frac{I}{P} \left(\frac{p_i}{P} \right)^{-\sigma}$$

real income times a measure of the relative price of i.

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- When varieties are close substitutes ($\sigma \to \infty$), prices converge to W.

Set the wage as numeraire. An Eq is prices $\{p_i^*\}_{i\in[0,n]}$, the aggregate price index P, household's consumption, $\{c_i^*\}_{i\in[0,n]}$, income I, firm's labor demand $\{\ell_i^*\}_{i\in[0,n]}$ and profits $\{\pi_i^*\}_{i\in[0,n]}$, such that

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4. Markets clear

$$\begin{aligned} \int_0^n \ell_i^*\, di &= 1 \\ 1 + \int \pi_i^*\, di &= I \end{aligned}$$

A symmetric equilibria: $c_i^* = \bar{c}$, $p_i^* = \bar{p}$, $\ell_i^* = \bar{\ell}$, $\pi_i^* = \bar{\pi}$ for all i.

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  1. *Rotemberg pricing* (menu costs)
  2. *Calvo pricing* (some (randomly set) firms can change prices, others cannot).

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$$\Omega(S, p_j^-) = \max_{p_j} \; p_j c_j^* - W(S) c_j^* - \phi(p_j, p_j^-)$$

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Exercise

Derive the First Order Condition of the Firm's Problem

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- Without capital $S = P^-$ and Aggregate Shocks.

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- We can also write $\Omega(S, p)$ generically and when choosing price $p(S) = \Omega(S, p)$.

Exercise

Derive the following for the dynamic model with Calvo pricing

1. *Solve the firm's problem in sequence space and write the firm's equilibrium pricing* $p_{j,t}$ *as a function of present and future aggregate prices, wages, and endowments:* $\{P_t, W_t, I_t\}_{t=0}^{\infty}$.
2. *Show that under flexible pricing* ($\theta = 1$)*, the firm's pricing strategy is identical to the static model.*
3. *Show that with price rigidity* ($\theta < 1$)*, the firm's pricing strategy is identical to the static model in a steady state with zero inflation.*

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  - Because we have θ firms adjusting and $1-\theta$ not, we have

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  - That turns out to satisfy (after using representative agent condition)

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    - Is this a nightmare? No. Log-linearization comes to help

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- We say Log Deviations

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- Smooth Functions $Z = f(X, Y) \Longrightarrow$

$$\overline{Z} \simeq \widehat{z} = f_x(\overline{X}, \overline{Y}) \;\; \overline{X} \; \widehat{x} + \beta \; f_y(\overline{X}, \overline{Y}) \; \overline{Y} \; \widehat{y}$$

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- Which implies for inflation that

$$\pi = p - p^{-} = (1-\theta)\ \left(\widehat{p}^{*} - \widehat{p^{-}}\right)$$

- Price setting

$$P^* = \frac{\sigma}{\sigma - 1} \frac{E\left\{\sum_\tau (\theta\beta)^\tau u_c \; P_\tau^{\sigma-1} \; \varphi_\tau \; y_\tau\right\}}{E\left\{\sum_\tau (\theta\beta)^\tau u_c \; P_\tau^{\sigma-1} \; y_\tau\right\}}$$

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- Approximating the left hand side gives the terms

$$E\left\{\sum_\tau \ (\theta\beta)^\tau \ \overline{U}_c \ \overline{P}^{\sigma-1} \ \overline{Y} \ \overline{P}^* \left[\widehat{u}_{c,\tau} + (\sigma-1)\widehat{p}_\tau + \widehat{y}_\tau + \widehat{p}^*\right]\right\}$$

Steady state values $\overline{U}_c, \ \overline{P}$ etc are common to all terms in the sum

- Approximating the right hand side yields

$$\frac{\sigma}{\sigma-1}E\left\{\sum_{\tau}(\theta\beta)^{\tau}\ \overline{U}_c\ \overline{P}^{\sigma-1}\ \overline{\varphi}\ \overline{Y}\ \overline{P}^{*}\left[\widehat{u}_{c,\tau}+(\sigma-1)\widehat{p}_{\tau}+\widehat{\varphi}_{\tau}+\widehat{y}_{\tau}+\right]\right\}$$

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- And Adding back in Steady State terms yield

$$\widehat{p}^{*} = \mu + (1-\theta\beta)\ E\left\{\sum_{\tau}(\theta\beta)^{\tau}\ \left[mc_{\tau} + p_{\tau}\right]\right\}$$

where $\mu = \log\frac{\sigma}{\sigma-1}$ and where mc_{τ} is log real marginal cost

- It solves

$$v(S,b) \quad = \quad \max_{c,\ell} u(c,\ell) + \beta \; E\{v(S',b') \qquad \text{s.t.}$$

$$b' + P(S)\; c = P(S)\; w\; \ell + [1 + i^{-}(S)]b + P(S) \int m_j(S)\; dj.$$

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- We get the Euler equation and the intratemporal condition

$$u_c[C(S,b),\ell(S,b)] = \beta\ E\left\{\frac{1 + i(S)}{1 + \pi(S')}\ u_c[C(S',b'),\ell(S',b')]\right\}$$
$$u_c[C(S,b),\ell(S,b)]\ w = u_\ell[C(S,b),\ell(S,b)].$$

- We have the law of motion of the price level (backward looking)

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- Note also that $D = (1-\theta)P(S)^{\sigma} + \theta(1+\pi)^{\sigma}\; D^{-}$.

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there is an inflation target π^* and $\widehat{i}$ is the nominal interest rate consistent in steady state with π^*. So $\widehat{i} = \beta^{-1}(1 + \pi^*) - 1$. This is a Taylor rule ($\phi_\pi = 1.5$ and $\phi_x = .5$. The monetary shocks are ω.

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Exercise

Define Equilibrium recursively

1. The IS Curve (log-linearized Euler equation)

$$x = E\{x'\} - \frac{1}{\sigma} \left(\log(1+i) - E\{\pi'\} + \log\beta - \frac{\sigma(1+\psi)}{\sigma\ \psi} \ (Z - E\{Z'\}) \right)$$

Where we use CRRA and constant Frisch ψ. Here x is the output gap, and the term after the inflation one is the real natural rate of interest.

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3. The Taylor (or other) monetary policy Rule

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- Instead of using "raw" undifferentiated labor, firms use an aggregate of many labor varieties:

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- This can be posed dynamically with wage rigidity a la Calvo o a la Rotemberg.

Extreme Value Shocks

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- Problem of correlated choices (blue/red bus). A Solution is to nest.

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- The problem is that discontinuities propagate in time. A solution is to pose Extreme Value Shocks e.g. (without adjustment costs)

$$V(s,a) = \max \ \{V^0(a), V^1(a)\} =$$

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- This gets rid of kinks and discontinuities as both choices are always possible for any a. But can cause problems.

- If ϵ follows i.i.d. $G(\mu, \alpha)$, where the mode μ is non-zero, we have

$$V^1 = E\{\epsilon\} = \mu + \alpha\ \gamma$$

$\gamma \simeq .57721$ is the Euler Mascheroni constant

$$\text{Mode}\ \ \{\epsilon\} = \mu$$

$$\text{Median}\{\epsilon\} = \mu - \alpha\ \ \ln(\ln 2)$$

$$\text{Var}\{\epsilon\} = \frac{\pi^2\ \alpha^2}{6}$$

$$\text{cdf}\{\epsilon\} = e^{\left\{-e^{\left[-\frac{(\epsilon-\mu)}{\alpha}\right]}\right\}}$$

- Expected maximum of N Gumbel random variables $G(\mu, \alpha)$. Let

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- To make $\mathbb{E}\left[X^N\right]$ independent of the number of choices N, either

$$\mathbb{E}\left[X^N\right] = \bar{V} \Rightarrow \alpha(N) = \frac{\bar{V} - \mu}{\gamma + \ln N}$$

$$\mathbb{E}\left[X^N\right] = \bar{V} \Rightarrow \mu(N) = \bar{V} - \alpha \ \ln N - \alpha \ \gamma$$

better the latter so that they are all Gumbel

- η^i follows $G(\mu, \alpha)$, let $\epsilon^i = \eta^i + \delta^i$, $\epsilon^i \sim G(\mu + \delta^i, \alpha)$.

$$X^N \sim G\left(\alpha \ln \sum_i e^{\frac{\mu^i}{\alpha}}, \alpha\right) = G\left(\mu + \alpha \ln \sum_i e^{\frac{\delta^i}{\alpha}}, \alpha\right)$$

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- To make $\mathbb{E}\left[X^N\right]$ independent of the number of choices, we can require

$$\mathbb{E}\left[X^N\right] = \bar{V} \Rightarrow \alpha(N) = \frac{\bar{V} - \mu}{\gamma + \ln \sum_i e^{\frac{\mu^i}{\alpha(N)}}}$$

$$\mathbb{E}\left[X^N\right] = \bar{V} \Rightarrow \mu(N) = \bar{V} - \alpha\left[\gamma + \ln \sum_i e^{\frac{\mu^i}{\alpha}}\right]$$

- η^i follows $G(\mu, \alpha)$, let $\epsilon^i = \eta^i + \delta^i$, $\epsilon^i \sim G(\mu + \delta^i, \alpha)$.

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- No closed-form solution for $\alpha(N)$

The continuum

- Consider an interval $C = [0, \overline{c}]$, and an $\epsilon(c), \forall c \in C$. We want

$$V^C = E\left\{\max_{c \in C} \{\epsilon(c)\}\right\}, \quad \epsilon(c) \sim G(0, \alpha(C)), \qquad \text{for some} \quad V^C > 0.$$

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- We choose $\alpha(V^C, N)$ so that $V^N = V^C$: $\qquad \alpha\left(V^C, N\right) = \frac{V^C}{\ln N + \gamma}$ for any N.

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- Violating the Euler equation by choice becomes a valuable privilege.
- If so we have to design algorithms that respect this feature.
- We have to think of V^C as a fundamental parameter that determines the size of the utility bonus for the richest agent (the one with the largest choice set).

- Let $\widetilde{c} < \overline{c}$, $[0, \widetilde{c}]$ a smaller choice set.

How to choose for a poorer agent $\widetilde{c} < \overline{c}$

- Let $\widetilde{c} < \overline{c}$, $[0, \widetilde{c}]$ a smaller choice set.
- Let $N^{\widetilde{c}} = \max_{n \geq 0} \frac{n}{N^{\overline{c}}} < \frac{\widetilde{c}}{\overline{c}}$, the point to the left of an imagined grid of size $N^{\overline{c}} + 1$.

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- Then we associate with choice set $C^{\widetilde{c}}$, a draw of $N^{\widetilde{c}}$ ϵ's with probability $\underline{p}(\widetilde{c}) = \frac{N^{\widetilde{c}}+1}{N^{\overline{c}}} - \frac{\widetilde{c}}{\overline{c}}$, and a draw of $N^{\widetilde{c}} + 1$ with probability $\overline{p}(\widetilde{c}) = \frac{\widetilde{c}}{\overline{c}} - \frac{N^{\widetilde{c}}}{N^{\overline{c}}}$.

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- Now you can iterate on the value function that includes the utility bonus.

Agents in Aiyagari worlds with Extreme Value Shocks

- The fundamental problem

$$v(s,a) = \max_{a', c = sw + aR - a'} \left\{ \frac{c^{1-\sigma} - 1}{1-\sigma} + \epsilon(c) + \sum_{s'} \Gamma_{s,s'} \; v(s', a') \right\}$$

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- Fix N, a large integer, we approximate the problem by

$$v(s,a) = \max_{a^{n\prime}=sw+aR-c^n,c^n} \left\{ \frac{c^{1-\sigma}-1}{1-\sigma} + \epsilon^n + \sum_{s'} \Gamma_{s,s'} \; v(s',a^{n\prime}) \right\}$$

We have to impute the right probabilities

Endogenous Growth and R&D

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$$F(K,N) = A\ K^{\theta_1}\ L^{\theta_2},$$

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- Still, empirically, the problem is NOT accounting for **growth rate** differences but for output **LEVEL** differences

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- An explicit accumulation of technology

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  1. Final goods are competitive use labor and intermediate goods according to

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  3. R&D sector. A new good is a new variety of the intermediate good produced using labor:

$$\frac{A_{t+1}}{A_t} = 1 + \xi N_{2,t}.$$

     we can write $A_{t+1} - A_t = A_t \xi N_{2,t},$ so the flow of new intermediate goods is determined by the current stock of them in the economy (an externality).

     Right to produce new goods sold to new monopolists.

Remark

The reason we see A_t on the previous expression as an externality is that it is indeed used as an input in the process of R&D, while, it is not paid for. Thus, inventors, in a sense, do not pay the investors of the previous varieties, while using their inventions. They only pay for the labor they hire. Perhaps, the basic idea of this production function might be traced back to Isaac Newton's quote: "If I have seen further, it is only by standing on the shoulders of giants".

Exercise

If the price of all varieties are the same, what is the optimal choice of input vector for a producer?

Exercise

What if they do not have the same amount? Would a firm decide not to use a variety in the production?

- Preferences

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Remark

In this economy, GDP is $Y_t = W_t + r_t K_t + \pi_t,$ *where* π_t *are profits.*
In terms of expenditures, GDP is $Y_t = C_t + K_{t+1} - (1-\delta) K_t + \pi_t,$ *where* $K_{t+1} - (1-\delta) K_t$ *is the investment in physical capital. In terms of value added, it is* $Y_t = N_t^{\alpha} \int_0^{A_t} x_t(i)^{1-\alpha} di + p_t (A_{t+1} - A_t).$

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- Not a model that maps well to the data, yet carefully crafted to convey ideas.

- Final good producer; it chooses $N_{1,t}$ and $x_t(i)$, $\forall i \in [0, A_t]$,

$$\max \; N_{1,t}^{\alpha} \int_0^{A_t} x_t(i)^{1-\alpha} \, di - w_t N_{1,t} - \int_0^{A_t} q_t(i) \, x_t(i) \, di,$$

where $q_t(i)$ is the price of variety i in period t. First order conditions are:

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- which, given N_{1t}, is the *demand function* for variety i, by the final good producer.

-

$$\pi_t(i) = \max_{\{q_t(i)\}} \quad q_t(i)\, x_t(q_t(i)) - r_t \eta x_t(q_t(i))$$

$$s.t. \quad x_t(q_t(i)) = \left(\frac{(1-\alpha)}{q_t(i)}\right)^{\frac{1}{\alpha}} N_{1,t},$$

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- FOC wrt to $q_t(i)$, is $x_t(q_t(i)) + (q_t(i) - r_t\eta)\frac{\partial x_t(q_t(i))}{\partial q_t(i)} = 0$, which implies

$$\frac{(1-\alpha)^{\frac{1}{\alpha}}}{q_t(i)^{\frac{1}{\alpha}}} N_{1,t} = \frac{(q_t(i) - r_t\eta)}{\alpha} \frac{(1-\alpha)^{\frac{1}{\alpha}}}{q_t(i)^{\frac{1+\alpha}{\alpha}}} N_{1,t}.$$

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$$s.t. \quad x_t(q_t(i)) = \left(\frac{(1-\alpha)}{q_t(i)}\right)^{\frac{1}{\alpha}} N_{1,t},$$

  we substituted for the technology of the monopolist, $x(i) = k(i)/\eta$.

- FOC wrt to $q_t(i)$, is $x_t(q_t(i)) + (q_t(i) - r_t\eta)\frac{\partial x_t(q_t(i))}{\partial q_t(i)} = 0,$ which implies

$$\frac{(1-\alpha)^{\frac{1}{\alpha}}}{q_t(i)^{\frac{1}{\alpha}}} N_{1,t} = \frac{(q_t(i) - r_t\eta)}{\alpha} \frac{(1-\alpha)^{\frac{1}{\alpha}}}{q_t(i)^{\frac{1+\alpha}{\alpha}}} N_{1,t}.$$

- Rearranging yields $q_t(i) = \frac{1}{(1-\alpha)} r_t \eta$ and substituting

$$x_t(i) = \left[\frac{(1-\alpha)^2}{r_t\eta}\right]^{\frac{1}{\alpha}} N_{1,t},$$

  and the demand for capital services is simply $\eta x_t(i)$.

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- With FOC $p_t = \frac{w_t}{A_t \xi}$.

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2. allocates labor demand for R&D, and that for final good production. For determining the labor choices $N_{1,t}$ and $N_{2,t}$. Note that as long as there are profits in the intermediate good sector, new monopolists will enter yielding a zero profit condition:

$$p_t=\sum_{s=t}^{\infty}\left(\prod_{\tau=t}^{s}\frac{1}{1+r_\tau-\delta}\right)\pi_s.$$

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4. Growth comes from the externality in the R&D sector. Without that, we cannot get sustained growth in this model.
5. This model neatly delivers balanced growth, with just enough structure.

Growth Model with Many Firms Suitable for Pandemic Times

- This is a growth model suitable to study business cycles.
- Emphasis on small business creation not on inequality so rep hholds.
- Creation and destruction of small firms both for technological and for financial reasons.
- Household cannot help its small businesses in distress.
- We have in mind that even though Pandemic affects both Supply (want less work) and Demand (Less consumption) there is a reduction in output sold per unit of good produced of $\phi(S)$.

- Two sectors as in Quadrini (2000): Corporate and non corporate sector.
- Corporate sector uses capital and labor via aggr prod fn $F(K, N)$
- Non corporate sector: type/size firms $i \in \{1, \cdots, I\}$, $f^i(n)$, $f^i_n > 0$, (provided the firm has the required number of managers, λ^i).
- A firm requires creation: It costs ξ^i to open a new firm of size i.
- Some Firms are destroyed.
    - Firms invest m in maintenance.
    - Probability that a firm survives is $q^i(m)$, $q^i(0) = 0$, $q^i(\infty) < 1$, $q^i_m > 0$.
- Aggregate measure of type i firms is X_i
- The law of motion of new firms is:

$$X_i' = q^i(M_i)\ X_i + B_i$$

- The Aggregate Feasibility Constraint is

$$C + [K' - (1-\delta)K] + \sum_i X_i\ M_i + \sum_i B_i\ \xi_i = \sum_i X_i\ f_i(N_i) + F(K, N).$$

- Household owns measure x_i of firms of type $i \in \{1, \cdots, \mathcal{I}\}$
- The household may be rationed in its workforce: i.e. it may not be in its static Euler equation.
- Households create b^i new firms of type i at cost ξ^i each,
- Managers choose maintenance and profits.
- In addition to its firms, households own a units of corporate capital which they can increase by savings.
- Households allocate its members to managers, workers or enjoyers of leisure:

$$n + \sum_i \lambda^i x^i + \ell = 1.$$

(implicitly we are guessing (to be verified) that all business are operated).

- Households have preferences over consumption c and leisure ℓ, using utility function $u(c, \ell)$ and discounts the future at rate β.

- Small firms cannot access financing once they are born.
- They can only give benefits to the household:

$$\Omega^i(S) = \max_{n \geq 0, m \leq \psi(S) f^i(n) - w \ n} \psi(S) \ f^i(n) - w \ n - m + \frac{q^i(m)}{R(S')} \ \Omega^i(S')$$

Here, S is the aggregate state and s in the individual state, $\Psi(S) < 1$ is capacity used which is demand determined and $R(S')$ is the rate of return used by the firm.

- Implictly assuming that there is no need to index $\Omega^i(S)$ by s.

Exercise

Get the FOC assuming first that m is unrestricted and then that $m \leq \psi(S) f^i(n) - w \ n$.

$$V(S, a, x_1, \cdots, x_I) = \max_{c,n,b_1,\cdots,b_I,a'} \; u(c,\; 1 - n - \sum_i \lambda^i\; x^i) + \beta\; V(S', a', x'_1, \cdots, x'_I) \qquad s.t.$$

$$c + \sum_i \; b_i\; \xi_i + a' = n\; w(S) + a\; R(S) + \sum_i \; \pi_i(S)\; x_i$$

$$x'_i = q^i(M_i)\; x_i + b_i \qquad i \in \{1, \cdots, I\}.$$

Exercise

Get the FOCs for b^i a' *and* n *assuming first that* $\lambda^i = 0$ *and* $\pi^i > 0$ *and charaterize the solution (the relation between the FOC of* b^i, m^i *and* a'*). Then characterize the FOC when* $\lambda^i > 0$.

An Integraded Analysis Model of Climate Change

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- Goal: Derive the optimal policy —here a tax on carbon— so that the externality is internalized.

- Higher levels of carbon dioxide in the atmosphere contributes to global warming, which in turn causes damages like production shortfalls, poor health or deaths, capital destruction and much more.

- Map carbon concentration to climate, and then map climate to damages.

- **Expected sum of future damage elasticities:** the percentage change in output resulting from a percentage change in the amount of carbon in the atmosphere, caused by emitting a unit of carbon today.

- Discounted because of time preferences and because of carbon depreciating.

- Carbon circulation system: carbon is exchanged through various reservoirs such as the atmosphere, the terrestrial biosphere, and different layers of the ocean. A unit of Carbon will remain in the atmosphere s periods after emmited according to

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- We then have a non-linear function $T_{t+1} = \mathcal{T}(T_t, S_t)$ with a steady state like

$$T(f) = \frac{\eta}{(\kappa_{Planck} - \kappa_{other} - \kappa_{refl})} \frac{1}{\ln 2} \ln\left(\frac{S}{\bar{\bar{S}}}\right)$$

- Surprisingly, non-linearities in the relation between CO_2 and Temperature seem to cancel each other in most advanced climate models. The global mean temperature thus becomes approximately linear in cumulative emissions. $T_t = \sigma_{CCR} \sum_{s=0}^{t} Emm_s$

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- According to the latest (6th) IPCC report, σ_{CCR} is "likely" (2/3 confidence interval) between 1.0 and 2.3 degrees Celsius per 1000 GtC (corresponding to 0.27-0.63$^\circ / Tt\text{CO}_2$). This constant is called CCR (Carbon Climate Response, sometimes CRE or TCRE).

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- Nordhaus summarizes various studies of effects:

- Positive effects if initial temperature is below 11.5 degrees. Suggests quadratic damage $D(T) = \alpha_{ag}^1 \left(T + T_0^j\right) + \alpha_{ag}^2 \left(T + T_0^j\right)^2 + \alpha_{ag}^j$.

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5. Inclusion of Exhaustible Resources that induces savvy economic behavior.

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- Dirty energy has cost constant cost ξ_j. Clean energy has convex cost $\xi_J(E_{J,T})$.

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$$S_t = \tilde{S}_t \left(\sum_{j=1}^{J_g - 1} E_{j,-T}, \sum_{j=1}^{J_g - 1} E_{j,-T+1}, ..., \sum_{j=1}^{J_g - 1} E_{j,t} \right)$$

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- Here, $-T$ is defined as the start of industrialization.

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4. The function $\tilde{S}_t$ is linear and has the depreciation structure:

$$S_t - \bar{S} = \sum_{s=0}^{t+T} \sum_{j=1}^{J_g - 1} E_{j,t-s}$$

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What is the best that can be done?

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  - Representative household of the world
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- After that we worry about implementation

$$\max_{\substack{\{C_t, N_t, K_{t+1}, R_{j,t+1}, \\ E_{j,t}, S_t\}_{t=0}^{\infty} \geq 0}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(C_t) \qquad \text{s.t.}$$

$$C_t + K_{t+1} = F_t(K_t, N_t, E_t, S_t) + (1-\delta)K_t \qquad \text{FB}$$

$$E_t = \sum_j E_{j,t}\ \alpha^j \qquad \text{AGE}$$

$$R_{j,t+1} = R_{j,t} - E_{j,t} \geq 0 \qquad \text{for all } j \qquad \mathit{ExE}$$

$$S_t = \tilde{S}_t \left(\sum_{j=1}^{J_g - 1} E_{j,-T}, \sum_{j=1}^{J_g - 1} E_{j,-T+1}, \ldots, \sum_{j=1}^{J_g - 1} E_{j,t} \right) \qquad \mathit{CC}$$

- $E_{j,t}$ is output of Energy of Sector (type) j measured in units of carbon emitted.

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- α^j Conversion of units of energy of type j from being in terms of carbon emissions to units of energy.

- The marginal externality damage is the same for all j:

$$\Lambda_t^s = \mathbb{E} \sum_{i=0}^{\infty} \beta^i \frac{U'(C_{t+i})}{U'(C_t)} \frac{\partial F_{t+i}}{\partial S_{t+i}} \frac{\partial S_{t+i}}{\partial E_{j,t}}$$

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- Under our specific assumptions, this expression simplifies to:

$$\Lambda_t^s = \mathbb{E} \sum_{i=0}^{\infty} \beta^i C_t \frac{Y_{t+i}}{C_{t+i}} \gamma_{t+i}(1 - d_i)$$

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- Further, if the planner's problem implies a constant savings rate, then the expression can be written as:

$$\Lambda_t^s = Y_t\left[\mathbb{E}\sum_{i=0}^{\infty}\beta^i\gamma_{t+i}(1-d_i)\right]$$

- The FOC of the planner says

$$\alpha_j \frac{\partial F_t}{\partial E_t} - \xi_j - \Lambda_t^s = 0$$

$$\max_{\{C_t, N_t, K_{t+1}\}_{t=0}^{\infty}} \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(C_t)$$

$$\text{subject to} \qquad \mathbb{E}_0 \sum_{t=0}^{\infty} q_t(C_t + K_{t+1})$$

$$= \mathbb{E}_0 \sum_{t=0}^{\infty} q_t((1 + r_t - \delta)K_t + w_t N_t + T_t) + \Pi_t.$$

$$\Pi_0 = \max_{\{K_t, N_t, E_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} q_t \left[F_t(K_t, N_t, E_t, S_t) \right.$$

$$\left. - r_t K_t - w_t N_t - \sum_{j=1}^{J} p_{j,t} E_{j,t} \right]$$

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- If there are multiple externalities (for instance an R&D component to the model) then a separate Pigouvian tax is required for each externality.

To understand the magnitude of the optimal tax rates given by this model, they can be compared with estimates from other models, and also with tax rates that are currently being used around the world.

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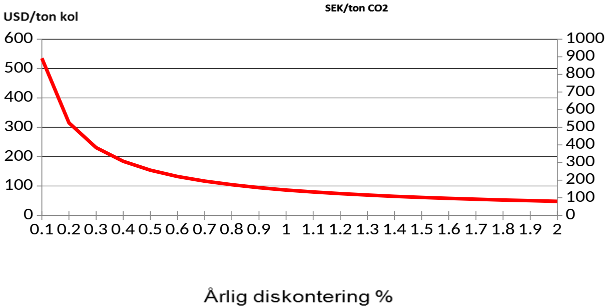
- In Sweden, the current tax on private consumption of carbon exceeds $600 per ton of carbon, which is larger than the estimates for the optimal tax in this paper. However, these taxes are significantly higher than many other countries, for instance the EU has a tax of around $77 per ton of carbon.

Sum damages over time => ”optimal” tax!



Sweden has carbon tax ~ 600 USD/tC!

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- => Coal is the main threat!

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- So: bad for the coal industry (the world over), no big deal otherwise

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  - No: reduce them where they are least needed/least efficient (e.g., buy emission rights in EU trading system, pay to keep forests, ...)

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- some elements of analysis subject to substantial uncertainty

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- The quantitative magnitudes of feedback are disputed. The "average" view seems to be that feedbacks strengthen the direct warming effect considerably, but there is much uncertainty.

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- What is the appropriate level of the tax? For this, we use standard cost-benefit analysis.

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- Together, the two steps are $D(T(S))$ mapping additional atmospheric carbon to damages. Let's examine the mapping.

- It turns out that $1 - D(T(S))$, i.e., how much is left after damages as a function of S, is well approximated by the function $e^{-\gamma S}$: for $\gamma = 5.3 * 10^{-5}$ (black), it is quite close to $1 - D(T(S))$ (red dashed), as seen in the figure.

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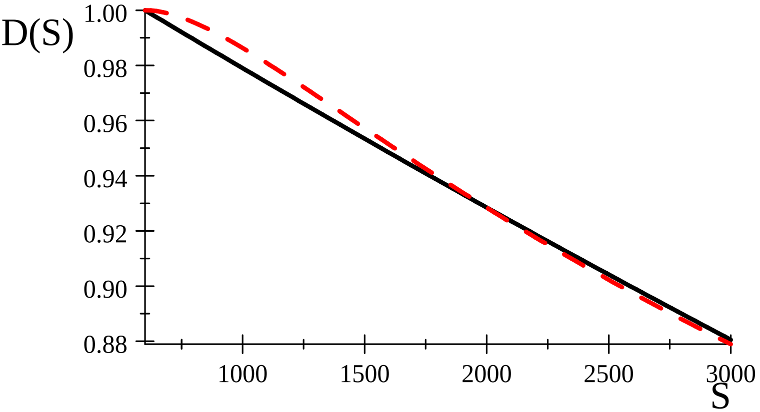

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- Robust?

2024 Update: Granular Model with various Suboptimal Policies

Climate Policy in the Wide World

John Hassler, Per Krusell, and Conny Olovsson

from the Simpson Lecture in Princeton University April 15, 2024

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- We include very high geographic resolution: $1^\circ \times 1^\circ$ latitude-longitude cells, with each cell assigned to a country.
- Despite this complexity, our model is also highly accessible to others, i.e., no need for advanced numerical toolboxes.

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  3. How successful is a policy that refrains from carbon taxation and instead focuses on promoting green energy (reminiscent of the Inflation Reduction Act)?

- The first policy is *successful* in mitigating global warming, the second is very *costly*, and the third is both *costly* and *unsuccessful*.

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- They make assumptions such that saving rates are easy to solve for separately.

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  – N_j is total population that follows an exogenous path; we define $x_{j,t+1} \equiv N_{j,t+1}/N_{j,t}$

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- Governments in each country tax emissions and rebate all the proceeds to consumers in the country.

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$$\pi_{ijt} = \max_{k_{ijt}, n_{ijt}, e_{ijt}} A_{ijt} \left(k_{ijt}^{\alpha} n_{ijt}^{1-\alpha-\nu_j} e_{ijt}^{\nu_j} \right)^{\varphi_j} - r_{jt} k_{ijt} - n_{jt} w_{ijt} - p_{jt} e_{ijt}.$$

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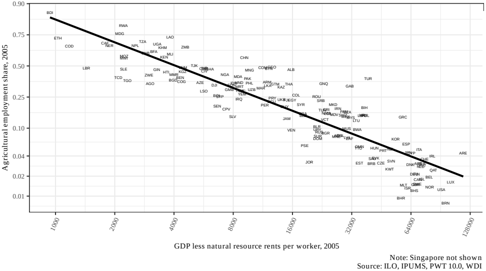
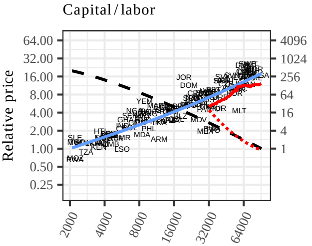
- Implicit in the above expression is a fixed factor that can be thought of as land.

Poor countries are about agriculture, and capital/labor shares in agriculture increase with development.

Left: ag. empl. shares on GDP/worker; right: k/l, r/w (dashed).





Call for a richer production structure, or at the very least α_j.

They use log, one sector, and Cobb-Douglas to get easy-to-solve-for saving rates. An alternative is elasticity ϵ and CRRA ϵ.

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$$o = (\lambda e_o^{\rho_o} + (1 - \lambda) e_f^{\rho_o})$$

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$1\rho_o - \rho_o$ determines the EOS between the e_o and e_f.

- The supply of energy services is then a CES aggregate

$$e = \left(\lambda_o o^\rho + \lambda_c e_c^\rho + \lambda_g e_g^\rho\right)$$

$1\rho - \rho$ determines the EOS between the energy inputs.

- "A" indicates TFP and it has several components. Formally:

$$\begin{aligned} A_{ijt} &= \exp(z_{ijt}) D_{ijt} \\ z_{ijt} &= z_{ij} + \sum_{s=0}^{\infty} g_{js} \end{aligned}$$

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    - has one *region-specific* component that is constant over time (z_{ij}),
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    - one *endogenous* component that responds to climate change (D_{ijt}).

- TFP damages are described by a U-shape in local temperature:

$$D_{ijt} = D(T_{ijt}) = \exp(\kappa_1 T_{ijt})(1 + \exp(\kappa_2 (T_{ijt} - \kappa_3)))$$

-$\kappa_4 \kappa_2$,

– T_{ijt} is the local temperature in region i and country j at time t (inspired by Krusell and Smith (2022), Burke, Hsiang, and Miguel (2015) and Cruz and Rossi-Hansberg (2023).

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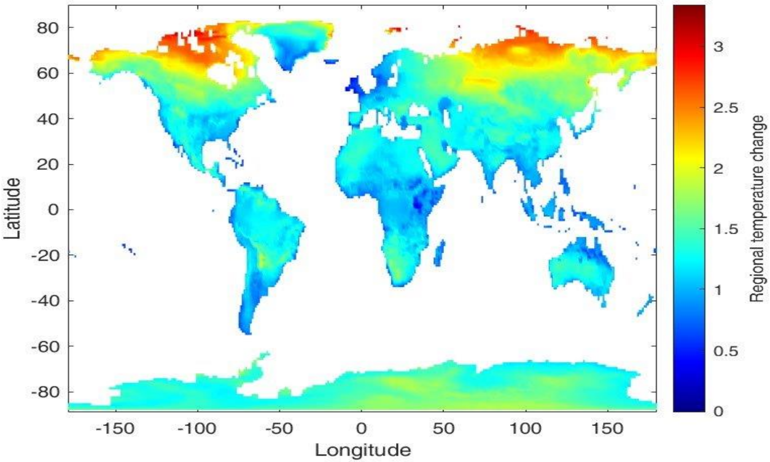
- Compute T_{ijt} with "statistical downscaling" where the global temperature is a sufficient statistic for the temperature in each region:

$$T_{ijt} = \hat{T}ij + \gamma_{ij}(T_t - T_0).$$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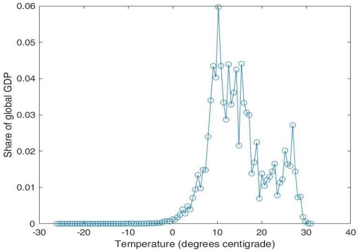
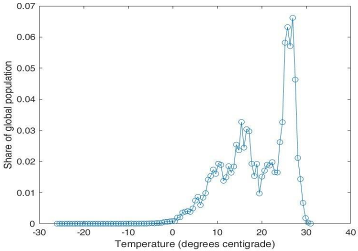
Regional temperature change

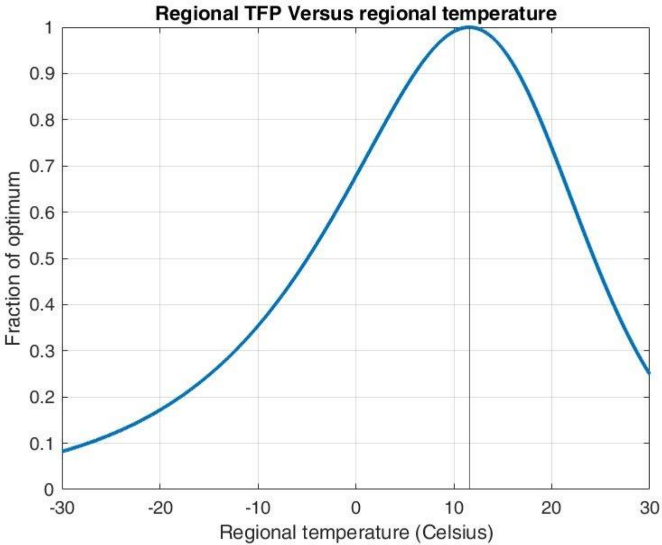
Temperatures and GDP and population shares (2005)





- Most of the output are produced in regions where $\mathbb{E}[T] \approx 11.6°\text{C}$.
- A lot of people live where $\mathbb{E}[T] > 11.6°\text{C}$.

The implied (inverse) U-shape for TFP



- The carbon cycle and temperature dynamics, respectively:

$$\begin{aligned} S_t - S_{t-1} &= \phi_{12} S_{t-1} + \phi_{21} S^U_{t-1} + E_{t-1} \\ S^U_t - S^U_{t-1} &= \phi_{12} S_{t-1} - (\phi_{21} + \phi_{23}) S^U_{t-1} + \phi_{32} S^L_{t-1} \\ S^L_t - S^L_{t-1} &= \phi_{23} S^U_{t-1} + \phi_{32} S^L_{t-1}, \end{aligned}$$

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$$\begin{aligned} S_t - S_{t-1} &= \phi_{12} S_{t-1} + \phi_{21} S_{t-1}^U + E_{t-1} \\ S_t^U - S_{t-1}^U &= \phi_{12} S_{t-1} - (\phi_{21} + \phi_{23}) S_{t-1}^U + \phi_{32} S_{t-1}^L \\ S_t^L - S_{t-1}^L &= \phi_{23} S_{t-1}^U + \phi_{32} S_{t-1}^L, \end{aligned}$$

- E: global (sum over (i,j)) emissions; S, S^U, and S^L: CO_2 stocks in the atmosphere, surface oceans and biosphere, and deep oceans.

$$\begin{aligned} T_t &= T_{t-1} + \sigma_1 \left(F_t - \kappa T_{t-1} - \sigma_2 \left(T_{t-1} - T_{t-1}^L \right) \right) \\ T_t^L &= T_{t-1}^L + \sigma_3 \left(T_{t-1} - T_{t-1}^L \right) \end{aligned}$$

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- $F_t = \chi \frac{\eta}{\ln 2} \ln \left(\frac{S_t}{S_0} \right)$ ($\chi > 1$ captures non-CO2 forcing) and T^L is ocean temperature. System replicates temperature graphs above.

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- Use a simple fixed-point algorithm for finding the oil price that clears the world markets period by period.

$$
\begin{aligned}
p_j &= \left(\lambda_o^{\frac{1}{1-\rho}} \hat{p}_{oj}^{\frac{\rho}{\rho-1}} + \lambda_c^{\frac{1}{1-\rho}} p_{cj}^{\frac{\rho}{\rho-1}} + \lambda_g^{\frac{1}{1-\rho}} p_{gj}^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}} \\
\hat{p}_{oj} &= \left(\lambda^{\frac{1}{1-\rho_h}} p_o^{\frac{\rho_h}{\rho_h-1}} + (1-\lambda)^{\frac{1}{1-\rho_h}} p_{fj}^{\frac{\rho_h}{\rho_h-1}}\right)^{\frac{\rho_h-1}{\rho_h}} \\
e_{oij} &= e_{ij}\left(\frac{\lambda_o p_j}{\hat{p}_{oj}}\right)^{\frac{1}{1-\rho}} \left(\frac{\lambda \hat{p}_{oj}}{p_o}\right)^{\frac{1}{1-\rho_h}} \\
e_{fij} &= e_{ij}\left(\frac{\lambda_o p_j}{\hat{p}_{oj}}\right)^{\frac{1}{1-\rho}} \left(\frac{(1-\lambda) \hat{p}_{oj}}{p_f}\right)^{\frac{1}{1-\rho_h}} \\
e_{mij} &= e_{ij}\left(\frac{\lambda_m p_j}{p_{mj}}\right)^{\frac{1}{1-\rho}}, \; m = c, g.
\end{aligned}
$$

All underlying prices are exogenously given except p_o,

- The production function for a regional firms (omitting time subscripts) can be written as

$$y_i = \left(\frac{\nu_j \varphi_j}{p_j} \right)$$

$\nu_j \varphi_j 1 - \nu_j - \varphi_j A_{ij}$

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- Summing over regions: $y_j = \sum_{i=1}^{I} y_i$, $k_j = \sum_{i=1}^{I} k_i$, $n_j = \sum_{i=1}^{I} n_i$ we get per-capita output in a country j

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- The only remaining endogenous variable is p_j, which is determined on the world market.

Above, we have an expression for e_{oijt}. Summing over regions and countries we get, after manipulation, oil demand

$$\text{Oil demand}_t = \sum_{ij} \Pi_{jt}(p_{ot})\ \nu_j\ \varphi_j\ y_{ijt}\ N_{jt},$$

where Π_{jt} is a known function of p_{ot}.

Turning to supply, the oil producer's maximization problem delivers

$$R_{t+1} = \beta \frac{1-s_t}{1-s_{t+1}} R_t\ ,\ s_t = \frac{\beta x_{t+1}}{1-s_{t+1}+\beta x_{t+1}}.$$

Given an exogenous sequence $\{x_{t+1}\}_{t=0}^{\infty}$, we can solve backwards:

$$\text{Oil supply}_t = \beta \sum_j \frac{1-s_{jt}}{1-s_{jt+1}} N_{jt} R_{jt}.$$

Solve for p_{ot} by setting Oil supply$_t$ = Oil demand$_t$.

They can also derive a forward-looking equation in saving rates in oil-consuming regions, and write per-capita savings of country j as

$$k_{j,t+1} = \frac{s_{jt}(1+\hat{\tau}_{jt})}{x_{j,t+1}} A_{jt} k_{jt}^{\frac{\alpha\varphi_j}{1-\nu_j\varphi_j}},$$

with

$$s_{jt} = \frac{\frac{\alpha\beta\varphi}{1-\nu\varphi} x_{j,t+1}}{1 - s_{j,t+1} + \frac{\alpha\beta\varphi}{1-\nu\varphi} x_{j,t+1}}.$$

The heterogeneity across economies appear in multiple places:

- saving rates
- population growth rates
- taxes, φ_j, ν_j
- TFP, costs of producing energy services.

1. Solve for the saving rates $\forall j$ (no endogenous variables).

2. Compute the equilibrium forward, starting at time 0. The endogenous state variables at $t = 0$ are K_j, T_j, oil resources by j; state variables in the carbon cycle and climate system.

    - Compute all TFP levels around the world and solve for the oil price in the period, which requires a numerical solution but only involves one equation in one unknown.
    - $p_{o,0}$ and τ_j gives the demand for all fuels and thus total emissions, so temperatures can be updated to next-period values.
    - The government BC is used to compute the carbon-revenue transfer rates $\hat{\tau}_j$.
    - Update the capital stocks and oil resources to their next-period values.

3. This completes the procedure for going from period 0 to period 1. Proceed to all future periods.

- They make use of the G-Econ database, version 4.0, which provides data on GDP and population (N) for every $1^\circ \times 1^\circ$ cell that contains land for the model's base year, 2005.
  - The database contains GDP and N data for 16,443 cells in 2005, and these cells that comprise the basic unit of analysis in the model.
- Estimated and projected N growth rates from 1990 to 2100 by country taken from the United Nations. Between 2100 and 2200, assume a linear progression from the 2100 rate to 0.
- The exogenous part of TFP grows at a rate of 1.5%/year, but developing countries are allowed to catch up.
- MPK:s are equalized in period 0 (following Caselli & Feyrer, 2007), which can be used to pin down the φ's (are found in the range $0.8 \leq \varphi_j \leq 1$).

- The elasticity of substitution between oil, coal, and the green energy source is set to 2 (for all countries).

- The elasticity of substitution between conventional oil and fracking is set to 10 (only the U.S. is assumed to have fracking).

- $E[p_o]_{2005-2009}$=\$70 per barrel or \$606.5/ton of carbon.

- $E[p_c]_{2005-2009}$=\$74/ton or \$103.35/ton of carbon.

- They set p_g based on the current relative price between green energy and oil.

- With these prices and observed quantities, the λs in the energy aggregator can be computed (Golosov, Hassler, Krusell, and Tsyvinski (2014)).

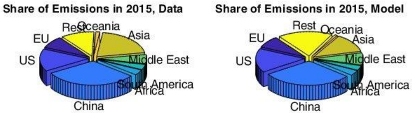
They incorporate heterogeneity in the energy income share with data on national energy use from the World Bank by computing

$$\nu_i = \frac{e_i^{int}}{\hat{e}_i^{int}}\nu,$$

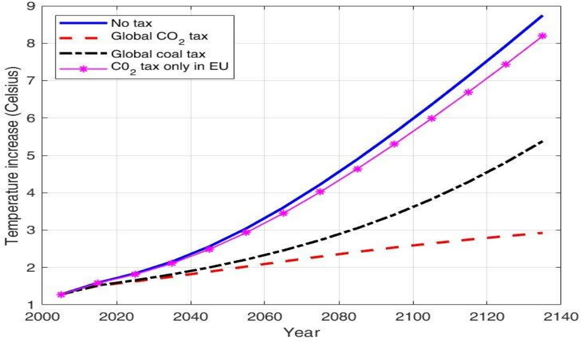
where e_i^{int} is national energy intensity (energy use in oil equivalents divided by PPP-adjusted GDP in year 2000), $\hat{e}_i^{int}$ is average energy intensity, and ν=0.035.

(Relies on the price of energy being equal across countries.)

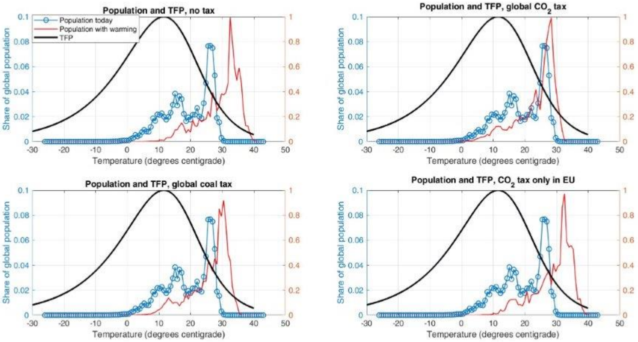
Note that the model is quite successful in matching observed CO_2 emissions, even though these were not directly targeted (China subsidizes fossil fuel use.).



Consider a tax of USD \$20/ton CO_2 at the initial date; it then grows at the rate of world GDP.

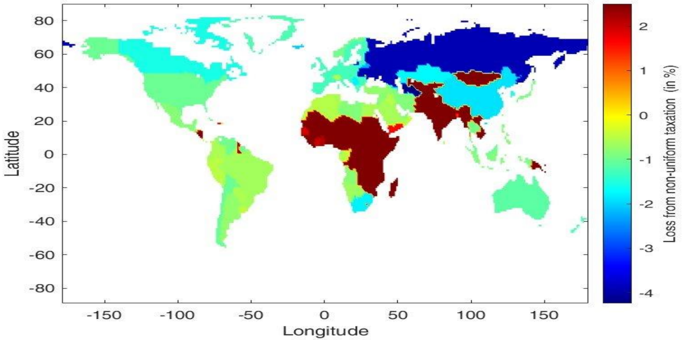


The difference between no tax and the modest tax is striking: about 6°C by 2140!



- Distributional consequences: populations being moved significantly to the right for the no-tax policy.
- As if hit by several Great Depressions at once for the some of the poorest regions.

- The Pigou principle: the tax should equal the negative externality caused. Since the negative externality is global, the tax should be the same everywhere: it should be uniform.
- However, we often hear arguments that for the sake of fairness, some poor regions should be "let off the hook".
- They here quantify exactly how costly deviations from a uniform taxation policy are in dollar terms.
- They again start with a τ of about US $20/ton.
    - Compare the results to a setting where the poorest countries, defined to be below 25% of global GDP/capita, have a zero/very low tax;
    - The tax in the ROW is then increased so that the increase in T is the same as with uniform taxation (3.1°C) at $t = 15$.
    - They consider a τ that is $20 \times \tau_{modest}$ in the participating countries, and $0.06 \times \tau_{modest}$ for the poor.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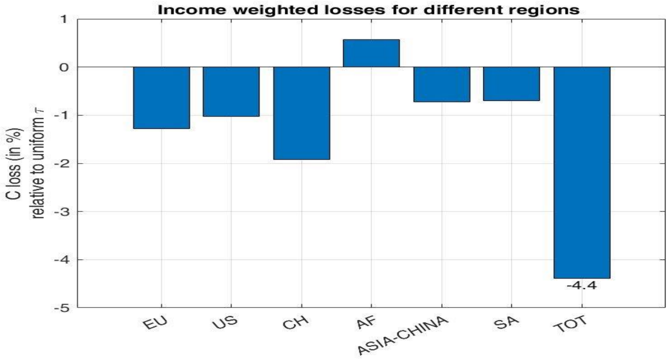
Loss from non-uniform taxation (in %)


Income weighted losses for different regions
C loss (in %)
relative to uniform τ
1
0
-1
-2
-3
-4
-5
-4.4
EU
US
CH
AF
ASIA-CHINA
SA
TOT

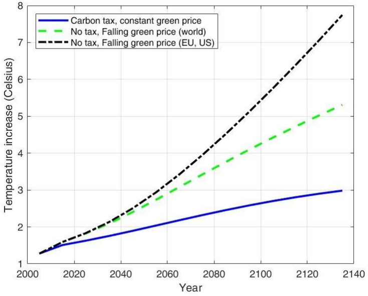
The climate-change aspects of the IRA boil down to the idea that cheap green technology will compete out fossil fuel.

They evaluate this idea by considering a scenario where the relative price of green energy falls by 2%/year (at zero cost), while the relative price of fossil fuel production is unchanged.

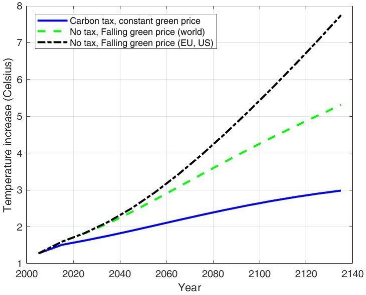
Two cases:

1. The U.S.-like policy generates fast green technology growth everywhere in the whole world.

2. The sped-up green technology growth will occur only in the U.S. and the EU.



- With green growth only in the EU and the U.S., global warming becomes almost as high as with no policy at all. Also insufficient when the technology spreads around the whole world.



- With green growth only in the EU and the U.S., global warming becomes almost as high as with no policy at all. Also insufficient when the technology spreads around the whole world.
- Key issue: green technology increases overall energy consumption but is ineffective in competing out fossil fuel.

- They offer a model of economics and climate change with very high regional resolution.
  - The model rests on standard microeconomic foundations that allows for cost-benefit analysis.
  - The spatial dispersion of the welfare effects of global warming are found to swamp the average effects.
  - Proof of concept: lots of room for improvements regarding heterogeneity in energy supplies and technologies, production structures.
- They also find that
  - Even a modest, globally uniform carbon tax would be extremely valuable.
  - A non-uniform tax on carbon is quite inefficient.
  - Relying on a push for green technical change only tax appears like a risky policy.

Macro and COVID-19

- Short Horizons (No investment)
- Choose what Issues to Worry About
  1. *Mitigation Policy and Heterogeneity Age/Sector*
- Choose wich Allocation Mechanism to Model (large externality)
  1. All Econ choices are Government choices

- All variables are shares of a measure 1 population
- Three health states, $j \in \{s, i, r\}$ susceptible, infected, recovered or dead, with associated population shares S, I, R. Initial conditions $S(0), I(0), R(0)$.
- Two parameters: β governs rate of infection, κ the rate of recovery (or death)
- System of differential Equations

$$\begin{aligned}\dot{S}(t) &= -\beta S(t)I(t) \\ \dot{I}(t) &= \beta S(t)I(t) - \kappa I(t) \\ \dot{R}(t) &= \kappa I(t)\end{aligned}$$

- Basic Reproduction Number: define $R_0 = \frac{\beta}{\kappa}$

- Growth rate of infections given by $\frac{\dot{I}(t)}{I(t)} = \beta S(t) - \kappa$
- Let $I(0) = \epsilon, S(0) = 1 - I(0)$, when $\epsilon > 0$ is very small, $S(0) \approx 1$.
- Since $\dot{S}(t) = -\beta S(t) I(t)$ and for t close to zero,

  $I(t) \approx 0, S(t) \approx 1$, then $\dot{I}(t)/I(t)$ is roughly constant and equal to

  $$\dot{S}(t) = -\beta S(0) I(0) \qquad \text{So}$$

  $$I(t) = I(0) e^{\kappa\left(\frac{\beta}{\kappa} S(0) - 1\right)} \approx I(0) e^{\kappa\left(\frac{\beta}{\kappa} - 1\right)}$$
- If $R_0 = \frac{\beta}{\kappa} > 1$ exponential growth early (if $I(0) > 0$).
- If $R_0 = \frac{\beta}{\kappa} < 1$ then infections fall to zero and epidemic disappears immediately.

- The Ratio of differential equations: $\frac{\dot{I}(t)}{\dot{S}(t)} = -1 + \frac{1}{R_{\mathbf{0}} S(t)}$
- Integrating yields $I(t) = -S(t) + \frac{\ln(S(t))}{R_{\mathbf{0}}} + q$

  where q is a constant of integration that does not depend on time.
- Evaluating at $t = 0$ yields (using $R(0) = 0$, thus $S(0) + I(0) = 1$

$$q = 1 - \frac{\ln(S(0))}{R_0}$$

- What is $S(\infty) = S^{\star}$? share of the population never to get infected
- Evaluating at $t = \infty$ and using the fact that $I(\infty) = 0$ yields

$$S^{\star} = 1 + \frac{\ln\left[S^{\star}/S(0)\right]}{R_0}$$

- Steady state satisfies the trascendental equation:

$$S^{\star} = 1 + \frac{\ln\left[S^{\star}/S(0)\right]}{R_0}$$

and $R^{\star} = 1 - S^{\star}, I^{\star} = 0$.

- If $R_0 > 1$ and $S(0) < 1$, $\exists$ a unique long-run S^{*}.

  Strictly decreasing in R_0 and strictly increasing in $S(0)$.

- For $R_0 \approx 1$ (but > 1), $S^{\star} = \frac{1}{R_0}$ and $R^{\star} = \frac{R_0 - 1}{R_0}$

  This approximation (a first good rule of thumb) uses $S(0) \approx 1$ and

$$\ln(1/R_0) = -\ln(R_0) = -\ln(1 + R_0 - 1) \approx 1 - R_0.$$

- With **costly** transfers across agents
- To Assess combination of two policies
  - Shutdown / mitigation (less output but also less contagion)
  - Redistribution toward those whose jobs are shuttered
- Characterize optimal policy
- Key interaction:
  - Mitigation creates the need for more redistribution
  - But if redistribution is costly, want less mitigation
  - Need heterogeneous-agent model to analyze this

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The SAFER SIR Model

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  - F at home
  - E to health-care workers

- Age $i \in \{y, o\}$
  - Only young work
  - Old have more adverse outcomes conditional on contagion
  - But young more prone to contagion (they work)
- Sector of production $\{b, \ell\}$
  - Basic (health care / food production etc.)
    - Will never want shut-downs in this sector
    - Workers in this sector care for the hospitalized
  - Luxury (restaurants, entertainment etc.)
    - Workers in this sector face shutdown unemployment risk
    - But they are less likely to get infected

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- How does the utilitarian optimal policy vary with the cost of redistribution?

- Lifetime utility for *old*

$$E\left\{\int e^{-\rho_o t}\ \left[u^o(c_t^o)+\bar{u}+\widehat{u}_t^j\right]\ dt\right\}$$

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- Differences in expected longevity through $\rho_y \neq \rho_o$ (no aging)

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- Θ measures capacity of emergency health system, η its unit cost

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- Micro-founded via sectoral heterogeneity in social contact rates
- Smart mitigation shutters most contact-intensive sub-sectors first

$$
\begin{aligned}
\dot{x}^{ybs} &= -\beta_w(m)\left[x^{yba} + (1-m)x^{y\ell a}\right] x^{ybs} \\
&\quad - \left[\beta_c(m)x^a y(m) + \beta_h\left(x^a + x^f\right) + \beta_e x^e\right] x^{ybs} \\
\dot{x}^{y\ell s} &= -\left[\beta_w(m)\left[x^{yba} + (1-m)x^{y\ell a}\right](1-m)x^{y\ell s}\right] \\
&\quad - \left[\beta_c(m)x^a y(m) + \beta_h\left(x^a + x^f\right)\right] x^{y\ell s} \\
\dot{x}^{os} &= -\left[\beta_c(m)x^a y(m) + \beta_h\left(x^a + x^f\right)\right] x^{os}
\end{aligned}
$$

- For each type $j \in \{yb, y\ell, o\}$

$$\begin{aligned}
\dot{x}^{ja} &= -\dot{x}^{js} - \left(\sigma^{jaf} + \sigma^{jar}\right) x^{ja} \\
\dot{x}^{jf} &= \sigma^{jaf} x^{ja} - \left(\sigma^{jfe} + \sigma^{jfr}\right) x^{jf} \\
\dot{x}^{je} &= \sigma^{jfe} x^{jf} - \left(\sigma^{jed} + \sigma^{jer}\right) x^{je} \\
\dot{x}^{jr} &= \sigma^{jar} x^{ja} + \sigma^{jfr} x^{jf} + (\sigma^{jer} - \varphi) x^{je} \\
\varphi &= \lambda_o \max\{x^e - \Theta, 0\}.
\end{aligned}$$

- All flow rates σ vary by age

- $x^e - \Theta$ measures excess demand for emergency health care. Reduces flow of recovered (Increases flow into death)

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- Define measures of non-working and working as

$$\begin{aligned}
\mu^n &= x^{y\ell f} + x^{y\ell e} + x^{ybf} + x^{ybe} + m\left(x^{y\ell s} + x^{y\ell a} + x^{y\ell r}\right) + x^o \\
\mu^w &= x^{ybs} + x^{yba} + x^{ybr} + [1-m]\ \left(x^{y\ell s} + x^{y\ell a} + x^{y\ell r}\right) \\
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- Aggregate resource constraint

$$\mu^w c^w + \mu^n c^n + \mu^n T(c^n) = \mu^w - \eta\Theta$$

where $T(c^n)$ is per-capita cost of transferring c^n to non-workers

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- Maximization subject to resource constraint gives $\frac{c^w}{c^n} = 1 + T'(c^n)$.

- Consumption allocation does not affect disease dynamics $\Rightarrow$ optimal redistribution is a static problem
- With log-utility and equal weights, period social welfare given by

$$W(x, m) = \max_{c^n, c^w} \left[\mu^w \log(c^w) + \mu^n \log(c^n)\right] + (\mu^w + \mu^n)\bar{u} + \sum_{i,j \in \{f,e\}} x^{ij} \widehat{u}^j$$

- Maximization subject to resource constraint gives $\frac{c^w}{c^n} = 1 + T'(c^n)$.
- Period welfare

$$\begin{aligned} W(x, m) &= \left[\mu^w + \mu^n\right] w(x, m) \\ w(x, m) &= \log(c^n) + \nu \log(1 + T'(c^n)) + \bar{u} + \sum_{i,j \in \{f,e\}} \frac{x^{ij}}{\mu^w + \mu^w} \widehat{u}^j \end{aligned}$$

- Assume $\mu^n T(c^n) = \mu^w \frac{\tau}{2}\left(\frac{\mu^n c^n}{\mu^w}\right)^2$
- Optimal allocation

$$
\begin{aligned}
c^n &= \frac{\sqrt{1 + 2\tau\frac{1-\nu^2}{\nu}\tilde{y}} - 1}{\tau\frac{1-\nu^2}{\nu}} \\
c^w &= c^n(1 + T'(c^n)) = c^n\left(1 + \tau\frac{1-\nu}{\nu}c^n\right)
\end{aligned}
$$

Where $\tilde{y} = \nu - \frac{\eta\Theta}{\mu^w + \mu^n}$.

- $\left(1 + \tau\frac{1-\nu}{\nu}c^n\right)$ is the effective marginal cost (MC) of transfers.
- It increases with c^n and τ, decreases with share of workers ν
- Higher mitigation m reduces ν, thus increases MC
- $\Rightarrow$ policy interaction between m, τ.

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