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Equilibria of Heterogeneous Economies with a Continuum of Agents

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EQUILIBRIA OF HETEROGENEOUS ECONOMIES WITH A CONTINUUM OF AGENTS

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The dynamic heterogeneous economies studied are described by a collection of heterogenous individuals, their individual states and an aggregate state, such that the individuals' actions are given by the policy obtained from an optimization program and the aggregate law of motion is given by the aggregation of the individuals' actions. These economies have been used in computer simulations, however the analytical information about the equilibria of such economies is scarce and the classical approach of Stokey and Lucas with Prescott (1989) does not apply. This paper defines the relevant concepts of equilibria and proves the existence of such equilibria using the Schauder Fixed Point Theorem. In order to apply Schauder's theorem, a metric for the space of operators between measures is provided, and the compactness of a specific operator is proved. Moreover, the existence of a steady state for the aggregate state of the system is obtained through the Schauder-Tychonoff Theorem. The results are related to models available in the literature.

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1. INTRODUCTION

Stochastic dynamic programming, as the study of optimal solutions to stochastic problems that evolve with time, may be applied to several areas of research. For instance, to Portfolio Analysis, as in Hakansson (1970), to inventory control, as in Bellman (1957), or to macroeconomic models with consumers and producers, as in Stokey and Lucas with Prescott (1989). The dynamic programming problem studied here, although applicable to the above areas, will be presented in the macroeconomic setting. The basic competitive dynamic macroeconomic model is given by two agents: a representative individual, whose behavior is the result of a dynamic maximization process through time, and a representative firm, that maximizes a given function in each time period. The optimal control process of the representative individual has been usually described by either the Hamiltonian theory or the recursive programming theory, which is the approach followed here. The main feature that we want to know about such macroeconomic system is if the system has equilibria. In this case (since the firm's problem is static) the equilibrium is an optimal action function $\gamma(x_t)$ that determines the action taken at time t by the individual when the state of the system is x_t. The existence of equilibria for these economies is well known, and Stokey and Lucas with Prescott (1989) (SLP henceforth) provides the standard approach for the problem. Although the above approach provides excellent results to model economic systems, several shortcomings are apparent. The most evident is the fact that two agents is an extreme simplification of the dynamic system.

There exists a lot of recent work in macroeconomics that poses incomplete marktet economies with a continuum of agents. The state variable in these economies includes a probability measure that describes the individual states of all agents in the economy. Economies of this type includes Hugget (1993), Aiyagary (1994), Krusell and Smith (1994), Hugget (1997), Castañeda et al. (1998), and Storesletten, Telmer and Yaron (2001), just to name a few. Although sometimes existence of a steady state has been established under particular conditions, these economies are often looked outside the steady state using a collection of numerical procedures to compute something that looks like an equilibrium. The existence of such an equilibrium has not been established.

This paper presents a proof of existence of equilibria, and shows that steady states always exists. Furthermore, the proof uses a recursive representation of the equilibria which relates it closely to the numerical methods used by researches. The result uses the Schauder fixed point theorem providing a fixed point of a map that provides new decision rules from old decision rules. The difficulty that I overcome lies essentially in the finding of a topology under which the appropriate family of functions is equicontinuous. The closest tool available in the literature is the proof of existence of equilibria with a finite number of heterogeneous agents studied in Duffie, Geanakoplos, Mas-Colell, and McLennan (1994).

The rest of the paper is organized as follows. Section 2 describes the heterogeneous economy, defines the equilibrium concepts, and states the assumptions. Section 3.1, and 3.2 proves the existence of the defined equilibria. Section 3.3 analyzes the steady state. Section 4 provides the link with computable heterogenous economies of the literature and comments on previous results of existence of equilibria. Section 5 concludes. Appendix A defines key concepts and summarizes some basic theorems used in this paper. Appendix B contains technical results. Definitions and Assumptions are numbered consecutively. The same holds for lemmas and theorems.

2. THE ECONOMY

The state of a stochastic heterogeneous economy is a collection of elements

$$(\{k_t^i\}, \{z_t^i\}, \{\mu_t\})_{i\in I, t\in T}$$

where t is an index for time (which is a sequence $t \in \mathbf{N}$) and i is an index for individuals (which lives in the interval $[0,1]$). k_t^i is the state of individual i at time t, z_t^i is the individual stochastic shock for agent i, and μ_t is the aggregate state of the system. The future individual state k_{t+1}^i is the action taken by individual i, which is the result of a stochastic maximization problem under constraints. The maximization problem can be stated as a stochastic dynamic program whose solution is given by solving the Bellman's functional equation (which will be called the BE problem hereafter)

$$\begin{aligned}(B\ f)(k_t^i, z_t^i, \mu_t) \quad &= \quad sup\ \ F(k_t^i, z_t^i, \mu_t, k_{t+1}^i) + \beta E(f(k_{t+1}^i, z_{t+1}^i, \mu_{t+1})|z_t^i) \\ \text{subject to} \quad & \qquad k_{t+1}^i \in \Omega(k_t^i, z_t^i, \mu_t) \\ & \qquad \mu_{t+1} = G(\mu_t),\end{aligned}$$

where the variable $(k_t^i, z_t^i, \mu_t) \equiv s_t^i$ lives in the space $K \times Z \times \mathcal{P} \equiv S$. The real function f is in the space $\mathcal{V}$ and B is an operator $B : \mathcal{V} \to \mathcal{V}$. We always take the discount factor $\beta \in (0,1)$. The dynamics of individual shocks z_t^i follows an exogenous process described by a transition function $\pi(z; A)$ which provides the probability for $z_{t+1} \in A$ given that $z_t = z$. Therefore, $E(f(k_{t+1}^i, z_{t+1}^i, \mu_{t+1})|z_t^i) = \int_Z f(k_{t+1}^i, z_{t+1}^i, \mu_{t+1})\pi(z_t^i; dz')$. The aggregate state μ_t is a probability measure $\mu_t \in \mathcal{P}$ over the individuals' variables $(k_t^i, z_t^i) \in K \times Z$. The dynamics of the aggregate measure is given by the integral operator $G : \mathcal{P} \to \mathcal{P}$, with kernel $\Gamma(k, z; A_1, A_2; \mu)$ given by

$$(G\mu)(A_1, A_2) = \int_{K\times Z} \Gamma(k, z; A_1, A_2; \mu)\ \mu(dk, dz). \tag{1}$$

The solution of the equation BE is a fixed point v, such that $Bv = v$. Given the fixed point v, we obtain from it the (possibly multivalued) optimal policy function γ by

$$\gamma(s) = \{k' \in \Omega(s) \mid v(s) = F(s,k') + \beta E(v(k',z',\mu')|z),\ \mu' = G(\mu)\} \tag{2}$$

where the function γ lives in some space $\mathcal{F}$ and provides the optimal action to be taken by individual i at time t when the state is s_t^i. The optimal policy γ determines the dynamics of the individual variables, and will be called an individual equilibrium of the system. An aggregate equilibrium have to satisfy also a restriction that links the policy γ with the dynamics of the aggregate state μ.

The following defintion provides a precise description of concepts of equilibria and steady state that will be analyzed. Assumption 2 describes the basic spaces K, Z, and X, and families of subsets on them, as well as the function π. The lemmas and theorems below are proved under such assumptions. See Appendix A for a definition of the Feller property. Definition 3 describes the spaces of functions defined on the basic spaces. Definition 4 adds a wider space S.

Definition 1:
(i) An individual equilibrium to the BE problem is an optimal policy γ provided by the fixed point v to the problem, as it is given in (2).
(ii) An aggregate equilibrium to the BE problem is an individual equilibrium γ^* that generates sequences $\{k_t^i\}$ that are consistent with the rest of the sequences $\{z_t^i\}$, and $\{\mu_t^i\}$.
(iii) An aggregate steady state is a pair (γ^*, μ^*) such that γ^* is an aggregate solution and μ^* is a constant value for the aggregate equilibrium.

Assumption 2:
(a) The individual state k is in K, a convex compact Borel subset of a euclidean space $\mathbf{R}^N$. The Borel sigma-algebra of K is $\mathcal{K}$. The individual shock z is in Z, a compact Borel subset of a euclidean space $\mathbf{R}^M$. The Borel sigma-algebra of Z is $\mathcal{Z}$. Denote $K \times Z \equiv X$ and the product sigma-algebra $\mathcal{K} \times \mathcal{Z} \equiv \mathcal{X}$.
(b) The function $\pi : Z \times \mathcal{Z} \to [0,1]$ is a transition function for the process $\{z_t^i\}$, i.e. $\pi(\cdot; A)$ is a probability measure, and $\pi(z; \cdot)$ is a measurable function. The transition function π has the Feller property.

Definition 3:
(a) The space $\mathcal{C}$ is $\mathcal{C} = \{g : X \to \mathbf{R} \mid \text{is continuous}\}$.
(b) The space $\mathcal{M}$ is $\mathcal{M} = \{\mu : \mathcal{X} \to \mathbf{R} \mid \text{is finite regular Borel signed measure}\}$.
(c) The space $\mathcal{P}$ is $\mathcal{P} = \{\mu : \mathcal{X} \to [0,1] \mid \text{is probability measure}\}$.

Definition 4: Define $K \times Z \times \mathcal{P} \equiv S$.
(a) The space $\mathcal{V}$ is $\mathcal{V} = \{f : S \to \mathbf{R} \mid \text{is continuous}\}$.

(b) The space $\mathcal{G}$ is $\mathcal{G} = \{G : \mathcal{P} \to \mathcal{P} \mid \text{is continuous}\}$.
(c) The space $\mathcal{F}$ is $\mathcal{F} = \{\tau : S \to K \mid \text{is continuous}\}$.

The topology for the space $\mathcal{P}$ is defined in Appendix B. An important point is that this topology makes $\mathcal{P}$ a compact metric space and then X and S are compact metric spaces in the product topology. The topology on $\mathcal{F}$ defined by

$$\|f\|_\infty = \sum_{i=1}^{N} \|f_i\|_\infty, \quad \|f_i\|_\infty = sup_{s \in S} |f_i(s)|$$

where $f = (f_1, ..., f_N)$, makes $(\mathcal{F}, \|\cdot\|_\infty)$ a Banach space. In all cases continuity is understood with respect to the product topology. One more topology will be defined for the space $\mathcal{G}$, when we need it. Note that the domains X and S are compact and then continuous functions on X or S are bounded. Note also that if f is a continuous function on X or S then it is Borel measurable, and since f is bounded then it is (finitely) integrable with respect to any finite measure. Note that the relevant sigma-algebra for $[0, 1]$ is the Borel one.

The following assumption provides conditions on F, Ω and Γ. The conditions on F and Ω are close to the ones given in SLP, as one would expect.

Assumption 5:
(a) The correspondence $\Omega : S \to K$ is non-empty, compact-valued and continuous.
(b) The function $F : S \times K \to \mathbf{R}$ is bounded and continuous, and for all $\theta \in (0, 1)$, $\overline{z} \in Z$, and $\overline{\mu} \in \mathcal{P}$,

$$F(\theta x + (1-\theta)x', \overline{z}, \overline{\mu}, \theta y + (1-\theta)y') \geq \theta F(x, \overline{z}, \overline{\mu}, y) + (1-\theta)F(x', \overline{z}, \overline{\mu}, y')$$

and the inequality is strict if $x \neq x'$.
(c) For all $\overline{z} \in Z, \overline{\mu} \in \mathcal{P}$, if $k_1' \in \Omega(k_1, \overline{z}, \overline{\mu})$, $k_2' \in \Omega(k_2, \overline{z}, \overline{\mu})$, then $\theta k_1' + (1-\theta)k_2' \in \Omega(\theta k_1 + (1-\theta)k_2, \overline{z}, \overline{\mu})$.
(d) The kernel Γ depends on the transition function π and on a given function $\tau \in \mathcal{F}$, in the form

$$\Gamma(k, z; A_1, A_2; \mu) = \pi(z; A_2)\chi(\tau(k, z, \mu); A_1) \tag{3}$$

where $\chi(x; A)$ is the indicator function with value 1 if $x \in A$ and 0 otherwise.

Now we have a precise description of the elements, and in the following lines we comment the definitions of equilibria. The individual equilibrium γ becomes an aggregate equilibrium, say γ^*, when it provides a law of motion for the individual variable k_t that is consistent with the law of motion of the aggregate state μ. This definition is the natural one if we interpret the aggregate state μ as a measure on the states k, so it naturally links the dynamics for k obtained from the optimization process with the dynamics assumed for μ. Formally, with the kernel defined in (3) the function G defined in (1) becomes

$$(G(\mu;\tau))(A_1,A_2) = \int_{K\times Z} \pi(z;A_2)\chi(\tau(k,z,\mu);A_1)\ \mu(dk,dz) \tag{4}$$

so it is parametrized by a function $\tau \in \mathcal{F}$. The individual equilibrium to problem BE provides an optimal policy γ as a function of the initial policy τ, say $\Phi(\tau) = \gamma_\tau$. The aggregate equilibrium γ^* can be defined as the fixed point of the function Φ, i.e. $\Phi(\tau) = \gamma_\tau = \tau$. For this fixed point the kernel $\pi(z;A_2)\chi(\tau(k,z,\mu);A_1)$ provides a dynamics for μ that is consistent with the dynamics of the individual k.

3. EXISTENCE OF EQUILIBRIA

Once we have defined the problem posed by a heterogeneous economy, the question is if there exist solutions to the problem, i.e. if there exists equilibria. We will see in Section 3.1 that the assumptions defined in Section 2 guarantee the existence of individual equilibria. The question of aggregate equilibria has an answer in Section 3.2 We also guarantee the existence of a steady state in Section 3.3.

3.1. *The Optimal Policy Function*

Theorem 3 proves the existence and uniqueness of the individual equilibrium associated with the BE problem. The scheme of the proof follows SLP (1989) i.e. a Theorem of the Maximum ensures a smooth behavior of the operator B. Lemma 1 states that $G(\mu;\tau) \equiv G_\tau$ is a well defined operator $G_\tau : \mathcal{P} \rightarrow \mathcal{P}$. Lemma 2 states that G belongs to $\mathcal{G}$. Note that since the kernel Γ depends on the measure μ, then G_τ is a non-linear operator.

Lemma 1: For any fixed $\tau \in \mathcal{F}$ and any $\mu \in \mathcal{P}$ the value $G(\mu;\tau)$ is in $\mathcal{P}$.
Proof:
By assumption $\Gamma(k,z;A_1,A_2;\mu) = \pi(z;A_2)\chi(\tau(k,z,\mu);A_1)$. Since $\tau \in \mathcal{F}$ then for any $(k,z,\mu) \in S$ and $A \in \mathcal{K}$, the function $\chi(\tau(k,z,\mu);A)$ is measurable. Since $\pi(z;\cdot)$ is a transition function then the proof of SLP Theorem 9.13 proves that $\Gamma(k,z;\cdot,\cdot;\mu) \in \mathcal{P}$ and that $\Gamma(\cdot,\cdot;A_1,A_2;\mu)$ is measurable.

Therefore, as a set function $\Gamma : \mathcal{X} \rightarrow [0,1]$, we have $\Gamma \geq 0$, $\Gamma(\emptyset) = 0$, $\Gamma(k, z; K, Z; \mu) = 1$. Write $\mu' = G(\mu; \tau)$, then along the lines of SLP Theorem 8.2 we have $\mu' \geq 0$ $\mu'(\emptyset) = 0$, $\mu'(K, Z) = 1$, and μ' is countably additive, and so $\mu' \in \mathcal{P}$. *Q.E.D.*

To prove the following lemma we use that $(\mathcal{P}, \mathcal{T}^*)$ is metrizable as explained in Appendix B. A specific metric will be introduced later.

Lemma 2: For any fixed $\tau \in \mathcal{F}$, the function $G(\cdot; \tau) \in \mathcal{G}$.
Proof:
We prove that $\mu_n \overset{w}{\rightarrow} \mu$ implies $\mu'_n \overset{w}{\rightarrow} \mu'$, where $\mu'_n = G(\mu_n; \tau)$. We write the integral $\int f \; d\mu'_n$ as

$$\int_{K,Z} f(k,z) \; \mu'_n(dk, dz) = \int_{K,Z} f(k,z) \int_{\widetilde{K},\widetilde{Z}} \pi(\widetilde{z}; dz) \chi(\tau(\widetilde{k}, \widetilde{z}, \mu_n); dk) \; \mu_n(d\widetilde{k}, d\widetilde{z}).$$

Now we can write $| \int f \; d\mu'_n - \int f \; d\mu'|$ as

$$\begin{aligned} &| \int_{\widetilde{K},\widetilde{Z}} \int_Z \int_K f(k,z) \chi(\tau(\widetilde{k}, \widetilde{z}, \mu_n); dk) \pi(\widetilde{z}; dz) \mu_n(d\widetilde{k}, d\widetilde{z}) \\ &- \int_{\widetilde{K},\widetilde{Z}} \int_Z \int_K f(k,z) \chi(\tau(\widetilde{k}, \widetilde{z}, \mu); dk) \pi(\widetilde{z}; dz) \mu(d\widetilde{k}, d\widetilde{z})| \; . \end{aligned}$$

We need the following result. The indicator function $\chi(y; B)$ is a probability measure if $B \in \mathcal{K}$, then for any positive integrable function $f^+(k)$ which is the pointwise limit of an increasing sequence of simple functions [2] we have

$$\begin{aligned} \int_K f^+(k) \chi(y; dk) &= \int lim_m \sum_{i=1}^{N_m} \alpha_{m,i} \chi(k; E_{m,i}) \chi(y; dk) \\ &= lim_m \sum_{i=1}^{N_m} \alpha_{m,i} \int \chi(k; E_{m,i}) \chi(y; dk) = lim_m \sum_{i=1}^{N_m} \alpha_{m,i} \chi(y; E_{m,i}) = f^+(y), \end{aligned}$$

since $\int \chi(k; E_{m,i}) \chi(y; dk) = \int_{E_{m,i}} \chi(y; dk) = \chi(y; E_{m,i})$, and where the Monotone Convergence Theorem allows us to commute the limit and the integral.

Now we apply the result to the decomposition $f = f^+ - f^-$ where $f^+ = max\{f, 0\}$ and $f^- = max\{-f, 0\}$, to get $| \int f \; d\mu'_n - \int f \; d\mu'|$ as

$$\begin{aligned} &| \int_{\widetilde{K},\widetilde{Z}} \int_Z f(\tau(\widetilde{k}, \widetilde{z}, \mu_n), z) \pi(\widetilde{z}; dz) \mu_n(d\widetilde{k}, d\widetilde{z}) - \int_{\widetilde{K},\widetilde{Z}} \int_Z f(\tau(\widetilde{k}, \widetilde{z}, \mu), z) \pi(\widetilde{z}; dz) \mu(d\widetilde{k}, d\widetilde{z})| \\ = \; &| \int_{\widetilde{K},\widetilde{Z}} h(\widetilde{k}, \widetilde{z}, \mu_n) \mu_n(d\widetilde{k}, d\widetilde{z}) - \int_{\widetilde{K},\widetilde{Z}} h(\widetilde{k}, \widetilde{z}, \mu) \mu(d\widetilde{k}, d\widetilde{z})| \end{aligned}$$

[2] i.e. $f^+(k) = \lim_m \phi_m(k)$, where $\phi_m(k) = \sum_{i=1}^{N_m} \alpha_{m,i} \chi(k; E_{m,i})$, $E_{m,i} \in \mathcal{K}$, and $\phi_m \geq \phi_{m-1} \geq 0$.

where $h(\widetilde{k},\widetilde{z},\mu_n) = \int_Z f(\tau(\widetilde{k},\widetilde{z},\mu_n),z)\pi(\widetilde{z};dz)$ is a continuous function of $(\widetilde{k},\widetilde{z},\mu_n)$ since f and τ are continuous functions on compact domains, and so they are uniformly continuous. Then

$$
\begin{aligned}
&|\int f\ d\mu'_n - \int f\ d\mu'| = |\int h(\widetilde{k},\widetilde{z},\mu_n)\mu_n(d\widetilde{k},d\widetilde{z}) - \int h(\widetilde{k},\widetilde{z},\mu)\mu(d\widetilde{k},d\widetilde{z})| \leq \\
&|\int h(\widetilde{k},\widetilde{z},\mu_n)\mu_n(d\widetilde{k},d\widetilde{z}) - \int h(\widetilde{k},\widetilde{z},\mu_n)\mu(d\widetilde{k},d\widetilde{z})| \\
&\quad +|\int h(\widetilde{k},\widetilde{z},\mu_n)\mu(d\widetilde{k},d\widetilde{z}) - \int h(\widetilde{k},\widetilde{z},\mu)\mu(d\widetilde{k},d\widetilde{z})| \leq \\
&|\int h(\widetilde{k},\widetilde{z},\mu_n)\mu_n(d\widetilde{k},d\widetilde{z}) - \int h(\widetilde{k},\widetilde{z},\mu_n)\mu(d\widetilde{k},d\widetilde{z})| \\
&\quad + \int |h(\widetilde{k},\widetilde{z},\mu_n) - h(\widetilde{k},\widetilde{z},\mu)|\mu(d\widetilde{k},d\widetilde{z})\ \equiv\ \varepsilon_1 + \varepsilon_2
\end{aligned}
$$

and ε_1 goes to zero as $\mu_n \xrightarrow{w} \mu$, by definition of weak convergence, and ε_2 goes to zero as $\mu_n \xrightarrow{w} \mu$, because h is continuous. This completes the proof. *Q.E.D.*

The following theorem proves the existence of an individual equilibrium for problem BE.

Theorem 3: (Existence and Uniqueness of the Optimal Action Function) The operator B, given in the BE problem, maps $\mathcal{V}$ into $\mathcal{V}$, and has a unique fixed point $v \in \mathcal{V}$. The optimal action correspondence $\gamma \in \mathcal{F}$, defined by v as in (2), is a continuous function.

Proof:

Using the terminology of the Theorem of the Maximum in Appendix A, define $S = K \times Z \times \mathcal{P}$, $R = K \times Z \times \mathcal{P} \times K$. For $s \in S$, we write $\Sigma(s) = (k,z,\mu,\Omega(k,z,\mu))$, and for $r \in R$, we write $\phi(r) = F(k,z,\mu,k') + \beta E(f(k',z',G(\mu;\tau))|z)$. By Lemma 2 G is continuous so f is continuous and the same argument that the one in SLP Lemma 9.5, can be used for any compact metric space. Therefore ϕ is a continuous function on a compact domain so its maximum is attained, hence by the Maximum Theorem $B : \mathcal{V} \rightarrow \mathcal{V}$. It is clear that B is a contraction of modulus β, then by the Banach Contraction Theorem, B has a unique fixed point $v \in \mathcal{V}$. The Theorem of the Maximum implies that the optimal action function γ is an u.h.c. correspondence. Denote $\mathcal{V}'$ the subspace of $\mathcal{V}$ of concave functions, and $\mathcal{V}''$ the subspace of $\mathcal{V}'$ of strictly concave functions. Now if $f(\cdot,z,\mu) \in \mathcal{V}'$ then $Bf \in \mathcal{V}''$ from Assumption 5 (b) along the lines in SLP Theorem 9.8. Then, the unique fixed point v is in $\mathcal{V}''$ and since F is concave and Ω is convex the optimum is attained at a unique k', hence γ is single valued, and therefore a continuous function. *Q.E.D.*

3.2 *The Aggregate Equilibria*

Once we have the individual solution to problem BE we can attack the problem of an aggregate equilibrium. The tool here is the Schauder Fixed Point Theorem. Two different versions of Schauder's theorem are presented in Appendix A, the Schauder Theorem and the Schauder-Tychonoff Theorem, and both will be used. The first can be found in Dugundji and Granas (1982) while the second is in Dunford and Schwartz (1958).

In order to prove the existence of a fixed point for the function Φ, we study the continuity of Φ. Let us write $\Phi : \mathcal{F} \to \mathcal{F}$ as $\Phi = \Phi_2 \circ \Phi_1$, such that $\Phi_1 : \mathcal{F} \to \mathcal{G}$, maps $\tau \mapsto G(\cdot;\tau)$, and $\Phi_2 : \mathcal{G} \to \mathcal{F}$, maps $G(\cdot;\tau) \mapsto \gamma_\tau$. The continuity of Φ is implied by the continuity of Φ_1 and Φ_2.

The topology relevant for $\mathcal{F}$ is the one defined by $\|\cdot\|_\infty$. The space $(\mathcal{P}, \mathcal{T}^*)$ can be metrized by $\rho(\mu, \nu)$, as explained in Appendix B. Define a metric for $\mathcal{G}$ by

$$\sigma(G, H) = sup_{\mu \in \mathcal{P}}\ \rho(G(\mu), H(\mu)).$$

Since $\mathcal{P}$ is compact with $\mathcal{T}^*$, ρ is continuous in the product topology defined by itself, and G, H are in $\mathcal{G}$, then *sup* is attained and σ is well defined. The topology relevant for $\mathcal{G}$ is the one defined by σ.

Lemma 4: The function $\Phi_1 : \mathcal{F} \to \mathcal{G}$, which maps $\tau \mapsto G(\cdot;\tau)$, is continuous.
Proof:
We want to see that $\tau_n \to \tau$ in $\|\cdot\|_\infty$ implies $G(\mu;\tau_n) \to G(\mu;\tau)$ in the topology defined by σ. Denote $\mu_n' \equiv G(\mu;\tau_n)$ and $q_i(\mu_n' - \mu') \equiv |\int f_i\ d(\mu_n' - \mu')|$ where $\{f_i\}_{i \in I}$ is a countable separating subset of the space $(\mathcal{C}, \|\cdot\|_\infty)$ as explained in Appendix B. Now $q_i(\mu_n' - \mu')$ becomes

$$\begin{aligned}
&|\int_{K,Z} f_i(k,z)\mu_n'(dk,dz) - \int_{K,Z} f_i(k,z)\mu'(dk,dz)| = \\
&|\int_{K,Z}\int_{\widetilde{K},\widetilde{Z}} f_i(k,z)\pi(\widetilde{z};dz)\Big(\chi(\tau_n(\widetilde{k},\widetilde{z},\mu);dk) - \chi(\tau(\widetilde{k},\widetilde{z},\mu);dk)\Big)\mu(d\widetilde{k},d\widetilde{z})| = \\
&|\int_{\widetilde{K},\widetilde{Z}}\int_Z \Big(f_i(\tau_n(\widetilde{k},\widetilde{z},\mu),z) - f_i(\tau(\widetilde{k},\widetilde{z},\mu),z)\Big)\pi(\widetilde{z};dz)\mu(d\widetilde{k},d\widetilde{z})| = \\
&|\int_{\widetilde{K},\widetilde{Z}}\int_Z \Big((\frac{\partial}{\partial k}f_i(k^0,z))(\tau_n(\widetilde{k},\widetilde{z},\mu) - \tau(\widetilde{k},\widetilde{z},\mu))\Big)\pi(\widetilde{z};dz)\mu(d\widetilde{k},d\widetilde{z})|
\end{aligned}$$

where the last two lines follow the steps of Lemma 2 and apply The Mean Value Theorem to obtain the result for some $k^0 \in K$.

Although the separating family $\{f_i\}$ is not equicontinuous, it is enough for our purpose to relabel the sequence of functions such that the slope $f_i'(\cdot, z)$ grows at a slow enough rate. The polynomials with rational coefficients is a separating subset, which can be denumerated such that

$sup_{k,z}|(\partial/\partial k)f_i(k,z)| \leq \alpha(i)B^{i\theta}$, where $\alpha(i)$ grows linearly with i and B is a bound with $0<\theta<1$. See Appendix B for the proof of this statement. In this case $q_i(\mu_n' - \mu')$ is

$$\begin{aligned} q_i(\mu_n' - \mu') &\leq \int_{\widetilde{K},\widetilde{Z}}\int_Z f_i'(k^0,z)|\tau_n(\widetilde{k},\widetilde{z},\mu) - \tau(\widetilde{k},\widetilde{z},\mu)|\pi(\widetilde{z};dz)\mu(d\widetilde{k},d\widetilde{z}) \\ &\leq \alpha(i)B^{i\theta}\int_{\widetilde{K},\widetilde{Z}}\int_Z |\tau_n(\widetilde{k},\widetilde{z},\mu) - \tau(\widetilde{k},\widetilde{z},\mu)|\pi(\widetilde{z};dz)\mu(d\widetilde{k},d\widetilde{z}) \leq \alpha(i)B^{i\theta}\varepsilon \end{aligned}$$

where ε is as small as we want since $\tau_n \rightarrow \tau$ uniformly. Therefore, if we take $M = B$ in the definition of the metric ρ and apply the ratio test to the series $\sum_{i=0}^{\infty}\alpha(i)B^{-i+i\theta}$ (as is suggested by Lemma B.3) we obtain

$$\begin{aligned} sup_{\mu\in\mathcal{P}}\sum_{i=0}^{\infty}B^{-i}\frac{q_i(G(\mu;\tau_n) - G(\mu;\tau_n))}{1+q_i(G(\mu;\tau_n) - G(\mu;\tau_n))} &\leq \sum_{i=0}^{\infty}B^{-i}\frac{\alpha(i)B^{i\theta}\varepsilon}{1+\alpha(i)B^{i\theta}\varepsilon} \\ &\leq \varepsilon\sum_{i=0}^{\infty}B^{-i+i\theta}\alpha(i) \leq \varepsilon\,\overline{B} \end{aligned}$$

for some constant $\overline{B}$ that does not depend on the measures μ_n, μ nor the functions τ_n, τ. Therefore $G(\mu_n;\tau)\rightarrow G(\mu;\tau)$ with respect to ρ. *Q.E.D.*

Lemma 5: The function $\Phi_2 : \mathcal{G}\rightarrow\mathcal{F}$, which maps $G(\cdot;\tau) \mapsto \gamma_\tau$, is continuous.
Proof:
We should understand the problem BE as a particular case of a problem BE' given by

$$\begin{aligned} (\widetilde{B}\ \widetilde{f})(k_t^i, z_t^i, \mu_t, G_\tau) &= sup\ \ F(k_t^i, z_t^i, \mu_t, k_{t+1}^i) + \beta\ E\Big(\widetilde{f}(k_{t+1}^i, z_{t+1}^i, \mu_{t+1}, G_\tau)|z_t^i\Big) \\ \text{subject to} \quad & k_{t+1}^i \in \Omega(k_t^i, z_t^i, \mu_t) \\ & \mu_{t+1} = \Delta(G_\tau, \mu_t) \end{aligned}$$

where $(k_t^i, z_t^i, \mu) \in S$ and $G_\tau \equiv G(\cdot;\tau) \in \mathcal{G}$. If we denote $\mathcal{W} = \{\overline{f} : S\times\mathcal{G}\rightarrow\mathbf{R} \mid \text{is continuous}\}$, then $\mathcal{V}\subset\mathcal{W}$. Now we have $\widetilde{B} : \mathcal{W}\rightarrow\mathcal{W}$, and $B : \mathcal{V}\rightarrow\mathcal{V}$.

Assume for the moment that the function $\Delta(G_\tau,\mu)$ is a continuous function of (G_τ,μ), then along the lines of Theorem 3, the solution to BE' provides an optimal action function $\widetilde{\gamma}$ which is a continuous function of (k,z,μ,G_τ). If we consider $\Delta(G_\tau,\mu) = G_\tau(\mu)$ then the problem BE' coincides with the problem BE, and the optimal action $\gamma_\tau(k,z,\mu)$ of problem BE equals the optimal action $\widetilde{\gamma}$ of problem BE', and $\widetilde{\gamma}(\cdot,\cdot,\cdot,G_\tau) \in \mathcal{V}$. Since $\widetilde{\gamma}$ is a continuous function of G_τ, the result is proved.

Let us prove that the function $\Delta : \mathcal{G}\times\mathcal{P}\rightarrow\mathcal{P}$, which maps $(G_\tau,\mu) \mapsto G_\tau(\mu)$, is a continuous function of (G_τ,μ). Let us fix $G\in\mathcal{G}, \mu\in\mathcal{P}$. Assume sequences $\{G_n\}$ and $\{\mu_m\}$ such that $G_n\rightarrow G$ and $\mu_m\rightarrow\mu$ as $n,m\rightarrow\infty$, in the respective metrics σ and ρ. Then

$$\rho(G_n(\mu_m), G(\mu)) \leq \rho(G_n(\mu_m), G(\mu_m)) + \rho(G(\mu_m), G(\mu)).$$

The first term is smaller or equal than $\sigma(G_n, G)$, so goes to zero because $G_n \to G$ with σ. The second goes to zero by continuity of G (as we saw in Lemma 2) and because $\mu_n \to \mu$ with ρ. So Δ is continuous and this completes the proof. *Q.E.D.*

Now we prove that there exists a function $\gamma^* \in \mathcal{F}$ such that $\Phi(\gamma^*) = \gamma^*$. In other words, the function $\Phi : \mathcal{F} \to \mathcal{F}$ has a fixed point γ^*. The following Lemma provides an auxiliary result.

Lemma 6: The family of functions $\Phi(\mathcal{F})$ is an equicontinuous family.
Proof:
Note that the family $\{\gamma_\tau \mid \tau \in \mathcal{F}\}$ provided by $\Phi : \mathcal{F} \to \mathcal{F}$, which maps $\tau \mapsto \gamma_\tau$, is an equicontinuous family if the function $\Psi : S \times \mathcal{F} \to K$, which maps $(s, \tau) \mapsto \gamma_\tau(s)$, is a continuous function.
We have seen in Lemma 5 that $\tilde{\gamma}$ is a continuous function of (k, z, μ, G_τ) and G_τ is continuous in τ, so $(s, \tau) \mapsto (s, G_\tau) \mapsto \tilde{\gamma}(s, G_\tau)$ provides a continuous function. *Q.E.D.*

Theorem 7: The function $\Phi : \mathcal{F} \to \mathcal{F}$ has a fixed point γ^* such that $\Phi(\gamma^*) = \gamma^*$.
Proof:
To prove the existence of such a fixed point we apply the Schauder fixed point theorem, stated in Appendix A. Note that (i) $\Phi(\mathcal{F})$ is an equicontinuous family, and (ii) $\Phi(\mathcal{F})$ is bounded, since $\|\gamma_\tau\|_\infty \leq sup_{k \in K}|k|$, then by Arzela-Ascoli Theorem, $\overline{\Phi(\Gamma)}$ is compact.
Define $A = \Phi(\mathcal{F})$ and $C_1 = \overline{co}(A) = \overline{co(A)}$. Note that C_1 is the closed convex hull containing A, and in a Banach space, as our $(\mathcal{F}, \|\cdot\|_\infty)$, if A is a compact set, then the closed convex hull containing A, $\overline{co(A)}$, is also compact. Now consider $\Phi : C_1 \to C_1$ then $\Phi(C_1)$ is a compact map, hence the Schauder fixed point theorem may be applied, which provides the result. *Q.E.D.*

3.3. *The Aggregate Steady State*

The proof a the existence of a steady state rests on the results obtained in Section 3.1 on the function $G_\tau : \mathcal{P} \to \mathcal{P}$.

Consider the operator $T^* : \mathcal{P} \to \mathcal{P}$ defined by

$$(T^*\mu)(A_1, A_2) = \int_{K \times Z} \pi(z; A_2)\chi(\tau'(k, z); A_1)\ \mu(dk, dz)$$

where τ' is a function $\tau' : K \times Z \to K$, and π and χ are as given in Assumption 2 and 5. Since $T^* : \mathcal{P} \to \mathcal{P}$, and π has the Feller property, then T^* is a continuous linear operator, and in this case the existence of a fixed point μ^*, such that $T^*\mu^* = \mu^*$, is provided in SLP Theorem 12.10. The fixed point measure μ^* is obtained by taking the sequence $\{\frac{1}{N}\sum_{n=0}^{N-1} T^{*n}\lambda_0\}$ for any initial measure λ_0 on a compact set X and applying duality arguments for the linear operator T^*.

The operator G defined in (4) is a nonlinear operator, which prevent us from using the usual duality arguments. However, the existence of an invariant measure μ^* such that $G_\tau(\mu^*) = \mu^*$ can still be proved by applying the Schauder-Tychonoff Fixed Point Theorem stated in Appendix A. The Schauder-Tychonoff Theorem provides an extension of the Schauder Theorem from normed spaces to locally convex linear topological spaces (LTS). This is exactly what we need, since the $\mathcal{T}^*$ topology makes $(\mathcal{M}, \mathcal{T}^*)$ a locally convex LTS.

Theorem 8: Let G_τ be the operator $G_\tau : \mathcal{P} \to \mathcal{P}$ defined in (4). There exists a measure $\mu^* \in \mathcal{P}$ such that $G_\tau(\mu^*) = \mu^*$.

Proof:

The proof reduces to check the hypotheses of the Schauder-Tychonoff Theorem. The $\mathcal{T}^*$ topology is a locally convex topology (see for instance Rudin (1973) Theorem 3.10) and the space of measures $\mathcal{P}$ is a compact convex subset of $\mathcal{M}$ in the topology $\mathcal{T}^*$. Since we have proved in Section 3.1 that the operator G is a continuous operator, then the required hypotheses hold and then the operator G_τ has a fixed point μ^*, such that $G_\tau(\mu^*) = \mu^*$. *Q.E.D.*

4. HETEROGENEOUS ECONOMIES

The purpose of this section is to provide a link with the computable heterogenous economies studied elsewhere and with the results of existence of equilibria available in the literature. Three examples are presented, and the solutions proposed by some authors are described. These examples are also helpful to give content to the general functions defined above. In the examples the aggregate state variable μ is naturally identified with a probability measure over some domain, and then the relation between the dynamics of μ and the dynamics of the individual states k_t^i (and the aggregate equilibrium) becomes apparent.

4.1. *The Simplest Production Economy*

In this example we consider a very simple non-stochastic production economy. In this economy there are a continuum of consumers, all identical in everything, that own an amount k_t of capital, and at each time period choose either to consume or to invest a fraction of their income. The individuals solve their problem taking as given the capital per capita of the economy, say x_t, and the policy function of the others, say $\tau(\cdot)$. Specifically, the income consist in the payment of wages $f(x_t) - x_t f'(x_t)$ and the income from capital $k_t f'(x_t)$, and the BE becomes

$$\begin{aligned} (B\ f)(k_t^i, x_t) &= sup\ \ U(w_t + (1+r_t)k_t^i - k_{t+1}^i) + \beta f(k_{t+1}^i, x_{t+1}) \\ \text{subject to} &\quad k_{t+1}^i \in [0, w_t + (1+r_t)k_t^i] \\ &\quad x_{t+1} = \tau(x_t) \end{aligned}$$

where $(1 + r_t) = f'(x_t)$ and $w_t = f(x_t) - x_t f'(x_t)$. In this economy (with no distortions) the equilibrium is the one provided by the optimally planned economy with no prices by substituting $w_t + r_t = f(x_t)$, and the problem fits into the Ramsey classical problem solved by the methods of SLP.

If a tax θ on capital income is introduced in this economy, and the proceeds returned to the consumers as a lump-sum transfer T, the problem becomes

$$\begin{aligned} (B\ f)(k_t^i, x_t) &= sup\ \ U(w_t + (1-\theta)(1+r_t)k_t^i + T - k_{t+1}^i) + \beta f(k_{t+1}^i, x_{t+1}) \\ \text{subject to} &\quad k_{t+1}^i \in [0, w_t + (1+r_t)k_t^i + T] \\ &\quad x_{t+1} = \tau(x_t) \end{aligned}$$

where $T = \theta x_t f'(x_t)$. In this case the Euler equation shows that the problem is not equivalent to the Ramsey problem and the existence of a solution (in our terminology, an aggregate solution) is not guaranteed by the classical results. Chapter 14 in SLP is devoted to the analysis of this economy, and we provide here an alternative solution directly from the results obtained above. In this case

$$(G(\mu;\tau))(A) = \int_X \chi(\tau(x;\mu), A)\ \mu(dx)$$

and the measure μ_t becomes the Dirac measure $\delta_{x,t}$ degenerate at point x and the function τ reduces to the aggregate savings function of the economy and $G(\mu;\tau)$ merely updates the state x_t using τ.

4.2. *An Endowment Heterogeneous Economy*

The economy in this case is similar to the endowment economy studied in Huggett (1993). In this case there is a continuum of agents $I = [0, 1]$ and time is discrete $T = \{0, 1, 2, ...\}$. In each period t, each agent receives an exogenous endowment z_t^i that can be consumed to increase the utility $U(c_t^i)$, or invested to increase the amount of assets k_t^i. Since the endowment is perishable, there are no more uses for it. The aggregate state μ_t is a probability distribution over the space $K \times Z$ of capital and shocks through the individuals, with associated sigma-algebras $\mathcal{K}$ and $\mathcal{Z}$. The total amount of assets k_t^i of each individual produces $q(\mu)k_t^i$. If we substitute consumption c_t^i of the individual in terms of the future capital k_{t+1}^i, the individual's problem can be stated in the following terms

$$\begin{aligned} (B\ f)(k_t^i, z_t^i, \mu_t) &= sup\ \ U(q(\mu_t)k_t^i + z_t^i - k_{t+1}^i) + \beta E(f(k_{t+1}^i, z_{t+1}^i, \mu_{t+1})|z_t^i) \\ \text{subject to} \quad & \quad k_{t+1}^i \in [0, q(\mu_t)k_t^i + z_t^i] \\ & \quad \mu_{t+1} = G(\mu_t) \end{aligned}$$

where $k_0^i > 0$ for all i, and for all $t \in T, i \in I$ we have $k_t^i \in [0, \overline{k}]$ for some $\overline{k} > 0$. The random shock z_t^i follows a Markov process (the same for all i) with transition matrix $\pi : Z \times \mathcal{Z} \to [0, 1]$, given by $\pi(z; A) = Prob(z_{t+1}^i \in A | z_t^i = z)$.

In this problem the dynamics of the aggregate state $\mu_{t+1} = G(\mu_t)$ is given by

$$\mu_{t+1}(A_1, A_2) = \int_{K \times Z} P(k, z; A_1, A_2)\ \mu_t(dk, dz) \tag{5}$$

where $P(k, z; A_1, A_2)$ is a Markovian transition function $P : (K \times Z) \times (\mathcal{K} \times \mathcal{Z}) \to [0, 1]$.

An individual equilibrium for this problem is an optimal action function $\gamma(k, z, \mu)$, for each individual that provides the consumption c_t when the state is (k, z, μ). The individual equilibrium induces sequences of capital for each individual k_t^i, and with the function π we obtain a Markov process for k_t^i and z_t^i with transition function $\Gamma(k, z; A_1, A_2) = \pi(z; A_2)\chi(\gamma(k, z, \mu); A_1)$. The aggregate equilibrium is obtained when the transition function Γ, provided by the optimal action γ, coincides with the transition function P that defines $G(\mu)$, and the aggregate restrictions hold. The aggregate steady state is given by an aggregate state μ^* plus an individual function γ^*, such that μ^* is stable for the transition function G, i.e. $G(\gamma^*; \mu^*) = \mu^*$.

The solution proposed by Huggett is a steady state equilibrium (μ^*, γ^*). In this case the problem of finding the aggregate equilibrium γ^* is greatly simplified because the constraint Ω can be written $\Omega(k_t^i, z_t^i, \mu_t) = [0, q^{-1}k_t^i + z_t^i]$ for a fixed constant function $q(\mu) = q$, which is, of course, a continuous function of μ. This fact is crucial for the solution of the problem. Huggett proposes the following algorithm to find the steady state: (i) solve the individual problem with a fixed value q^* to obtain the optimal action γ, (ii) obtain a fixed point $\mu^* = G(\gamma; \mu^*)$ where the transition Γ is provided by the equilibrium γ, (iii) check a set of aggregate restrictions $R(\mu^*) = 0$. If they hold then γ defines the aggregate solution γ^*. If they do not hold go to (i) and update q^*. Note that step (i) is a standard solution of recursive dynamic optimization, since the problem becomes parametrized by q, and the solution is justified by the methods of SLP. Step (ii) is more complicated since to obtain a fixed point it is needed more stringent conditions. In particular, in SLP or Hoppenhayn and Prescott (1989) compactness of the domain of measures is needed in order to obtain a unique state μ^* associated with γ. To prove such compactness, Huggett imposes conditions on q^* and the transition π that guarantee a compact domain for k_t^i. In other words, the conditions guarantee an endogenously determined bound $\overline{k}$ for the accumulation of capital.

As we see, the individual solution and the steady state (the aggregate solution is implicit) can be solved by basic methods because Ω and G are very simple functions, Ω depends only on q which is a constant function of μ and G is a linear operator on μ given by $G(\mu) = \int \pi(z; A_2)\chi(\gamma(k,z); A_1)\mu(dk, dz)$.

Finally, the storage-technology economy ($q(\mu_t)$ constant) of this section becomes a monetary economy if we allow $q(\mu_t)$ to vary. This amounts to a constraint $\Omega(k_t^i, z_t^i, \mu_t)$ changing with μ_t, which is the case of the next section.

4.3. *A Production Heterogeneous Economy*

The following production economy is basically the one presented in Huggett (1997), Aiyagary (1996), or Krusell and Smith (1998). The individuals are described at time t by their capital assets k_t^i and its individual shocks z_t^i that affect its wages w_t. They choose their consumption c_t^i by solving the problem

$$\begin{aligned} (B\ f)(k_t^i, z_t^i, \mu_t) &= sup\ \ U(z_t^i w_t + (1+r_t)k_t^i - k_{t+1}^i) + \beta E(f(k_{t+1}^i, z_{t+1}^i, \mu_{t+1})|z_t^i) \\ \text{subject to} &\quad k_{t+1}^i \in [0, z_t^i w_t + (1+r_t)k_t^i] \\ &\quad \mu_{t+1} = G(\mu_t) \end{aligned}$$

and $k_t^i \geq 0$. In this case the constraint set $\Omega(k_t^i, z_t^i, \mu_t)$ is $[0, z_t^i w_t + (1+r_t)k_t^i]$, and the variables w_t and r_t depend on the aggregate state μ_t in the form $w_t = f(\overline{k}_t) - \overline{k}_t f'(\overline{k}_t)$, and $r_t = f'(\overline{k}_t) - 1$, and where $\overline{k}_t = \int_{K\times Z} k\ \mu_t(dk, dz)$. Note that in this particular case the restriction Ω depends only on $\overline{k}$, i.e. the mean of the distribution μ_t, as noted in Krusell and Smith (1998). The dynamics of the aggregate state is as in the past example. In this case w_t and r_t are the competitive prices of labor and capital, obtained by a first order condition of a maximizing firm. Note also that in this model there are no aggregate uncertainty, since the agents only have uncertainty with respect to their individual shocks. However, it is straightforward to extend the model presented here by including an aggregate shock $\xi_t \in Y$, for instance in the aggregate production function obtaining $f(\overline{k}_t, \xi_t)$, such that now we have $S = K \times Z \times Y \times \mathcal{P}$.

5. CONCLUSIONS

The workhorse of classical macroeconomic theory has been the representative agent economies, where the representative agent solves a dynamic optimization problem subject to constraints. The theory that justifies the existence of equilibria for those economies has been developed by several authors, and is presented in Stokey and Lucas with Prescott (1989). In recent years the models have been extended to richer environments where economies are composed of a continuum of agents. There are a number of simulations experiments for those economies (see the references in Section 4) but a lack of analytic results for them. In the same way that representative economies are examples of more general problems, the heterogeneous economies are particular cases of more general dynamic programming problems, defined as heterogeneous systems. The theory developed here provides a framework to study such heterogeneous systems.

An heterogeneous system is defined by a collection of individuals, individual states and aggregate states, through time, such that the individual states change according to a dynamic optimization process and the aggregate state updates according to the individual behavior. This paper defines two different concepts of equilibria for those systems, and proves the existence of them under some conditions. The results obtained in Section 3 are applied to the heterogeneous economies provided in Section 4. In those cases the proof of the existence of equilibria is important to guarantee that the computer simulations performed for these type of models, approach solutions that exists.

As it use to be the case, existence of equilibria is proved through a fixed point theorem. The main difficulty in this case is to provide a suitable space where the fixed point can be found. This is done by metrizing the space of measures and defining in it the appropriate functions that represent the problem of a heterogenous system. Since we obtain an infinite dimensional space Schauder Theorem is used, and in this case it is required that the function not only is continuous, but also compact. A related fixed point theorem, Schauder-Tychonoff Theorem, guarantees the existence of a steady state for the aggregate state of the system.

APPENDIX A

The terminology used in this paper is standard in mathematical analysis. For instance, the closure of a subset $A \subset X$ of a topological space X is denoted $\overline{A}$, its convex hull is $co(A)$, etc. In order to refresh some concepts, this appendix provides some basic definitions and important theorems that are used in this paper. For any other term (contractive function, locally convex linear topological space, equicontinuity, algebra, etc.) the reader may consult Reed and Simon (1980) or Dunford and Schwartz (1958).

Given a mapping between topological spaces X and Y, the term function will be used for a point valued mapping, say $\phi : X \to Y$, whereas correspondence will mean a set valued mapping, denoted $\Sigma : X \to \wp(Y)$, where $\wp(Y)$ is the power set of Y.

Definition A.1: Let $\Sigma : X \to \wp(Y)$ be a correspondence.
(1) Σ is upper hemi-continuous (u.h.c.) at $x_0 \in X$ if for each open subset $V \subseteq Y$, with $\Sigma(x_0) \subseteq V$, exists $U \subseteq X$, with $x_0 \in U$, such that $y \in U$ implies $\Sigma(y) \subseteq V$.
(2) Σ is lower hemi-continuous (l.h.c.) at $x_0 \in X$ if for each open set $V \subseteq Y$, with $V \cap \Sigma(x_0) \neq \emptyset$, exists $U \subseteq X$, with $x_0 \in U$, such that $y \in U$ implies $\Sigma(y) \cap V \neq \emptyset$.
(3) Σ is continuous at $x_0 \in X$ if it is both upper and lower hemi-continuous.

Definition A.2: The correspondence $\Sigma : X \to \wp(Y)$ is upper hemi-continuous (or lower hemi-continuous) written u.h.c. (or l.h.c.) if it is u.h.c. (or l.h.c.) at each point $x \in X$.

Theorem: (Theorem of the Maximum, Berge (1963) Ch.6)
Let X and Y be topological spaces. Assume $\phi : Y \to \mathbf{R}$ is a continuous function, and $\Sigma : X \to \wp(Y)$ is a continuous corresponcence with compact values and with $\Sigma(x) \neq \emptyset$ for all $x \in X$. Define

$$\begin{aligned} M(x) &= max\ \{\phi(y) \mid y \in \Sigma(x)\} \\ \gamma(x) &= \{y \in \Sigma(x) \mid \phi(y) = M(x)\}\ . \end{aligned}$$

Then $M(x)$ is a continuous function and $\gamma(x)$ is an u.h.c. corresponcence.

Theorem: (Banach Contraction Theorem, Dugundji and Granas (1982) I.1.1)
Let (X, d) be a complete metric space and $F : X \to X$ a contractive function. Then F has a unique fixed point x_0 and $F^n(x) \to x_0$ as $n \to \infty$ for each $x \in X$.

Definition A.3: A continuous function $F : X \to Y$ between Hausdorff topological spaces X and Y is called compact if $\overline{F(X)}$ is a compact subset of Y.

Theorem: (Schauder Fixed-Point Theorem, Dugundji and Granas (1982) II.4.3 Theorem 3.2)

Let C be a convex subset of a normed linear space E. Each compact map $F : C \rightarrow C$ has at least one fixed point.

Definition A.4: A topological space X is said to have the fixed point property if for every continuous function $f : X \rightarrow X$, there exists a $x \in X$ with $x = f(x)$.

Theorem: (Schauder-Tychonoff Fixed-Point Theorem, Dunford and Schwartz (1959) Ch.V)
A compact convex subset of a locally convex linear topological space has the fixed point property.

Theorem: (Arzela-Ascoli Theorem)
Let X be a compact metric space and $\mathcal{C}(X) = \{f : X \rightarrow \mathbf{R} \mid \text{is continuous}\}$. A subset $A \subseteq \mathcal{C}(X)$ is compact iff it is closed, bounded and equicontinuous.

Definition A.5: (Feller property) A transition function $\pi : Z \times \mathcal{Z} \rightarrow [0, 1]$ has the Feller property if the associated operator T_π, defined $(T_\pi f)(z) = \int f(z')\ \pi(z; dz')$, maps $\mathcal{C}_Z$ into $\mathcal{C}_Z$, where $\mathcal{C}_Z = \{f : Z \rightarrow \mathbf{R} \mid \text{is continuous}\}$.

Definition A.6: Let $F = \{f\}$ be a family of functions $f : X \rightarrow Y$. The family F separates points of X if for all $x, x' \in X$, $x \neq x'$, exists $f \in F$ such that $f(x) \neq f(x')$.

Definition A.7: Let $\mathcal{C}_X$ be the algebra of real functions $f : X \rightarrow \mathbf{R}$. Let $F = \{f\}$ be a subset of $\mathcal{C}_X$. The algebra generated by F, $alg(F)$, is the unique smallest subalgebra of $\mathcal{C}_X$ containing F.

Note that $alg(F)$ can be described as the set of all finite linear combinations of reals and functions $\sum_{i=1}^{N} a_i f_{i,1} \cdots f_{i,M_i}$, for $a_i \in \mathbf{R}$ and $f_{i,j} \in F$.

Theorem: (Stone-Weierstrass Theorem, Reed and Simon (1980) Theorem IV.9)
Let X be a compact topological space. Let $\mathcal{C}_X$ be the algebra of functions $f : X \rightarrow \mathbf{R}$. Let F be a family of functions $F \subset \mathcal{C}_X$. If F contains 1 (the identity element) and separates the points of X, then the algebra generated by F, $alg(F)$, is dense in $\mathcal{C}_X$.

APPENDIX B

As we saw, the aggregate state μ is a probability measure on some sigma-algebra. In order to deal with problems involving such measures, we will endow the space of measures with the structure of a metric space. The purpose of this appendix is to detail the process.

Define $\mathcal{C}$, $(X, \mathcal{X})$, $\mathcal{M}$, and $\mathcal{P}$ as in Section 2. Note that $\mathcal{C}$ is a real vector space. Denote **R** the real line, and **Q** the set of rational numbers. The space $\mathcal{C}$ is a Banach space with the norm $\|f\|_\infty = sup_{x\in X}|f(x)|$. We know from the Riesz Representation Theorem that $\mathcal{C}^*$ is isometrically isomorphic to $\mathcal{M}$. The weak* topology $\mathcal{T}^*$ defined on $\mathcal{M}$ by $\mathcal{C}$ has as a local base the open sets of the form

$$U^{f_1,...,f_N}_{\varepsilon_1,...,\varepsilon_N}(\mu) = \{\nu \mid |\int f_i d\nu - \int f_i d\mu| < \varepsilon_i;\ i = 1, ..., N;\ f_i \in \mathcal{C}\}.$$

The space $(\mathcal{M}, \mathcal{T}^*)$ is not metrizable, but the space $(\mathcal{P}, \mathcal{T}^*)$ is it. It should be recalled that $(\mathcal{P}, \mathcal{T}^*)$ is a compact space. See for instance Parthasarathy (1967) Th.6.7.

The topology of the space $(\mathcal{P}, \mathcal{T}^*)$ is the one that provides the usual weak convergence of probability measures. This convergence will be denoted $\mu_n \xrightarrow{w} \mu$. Recall that $\mu_n \xrightarrow{w} \mu$ iff $\forall \varepsilon > 0, \exists N_\varepsilon, \forall f \in \mathcal{C}, \forall \mu_n, n \geq N_\varepsilon,\ |\int f\ d\mu_n - \int f d\ \mu| < \varepsilon$. Different metrics have been proposed for the space of probabilities, see for instance Prohorov (1956) or Dudley (1966). In this paper we consider the metric that is usually applied to a metrization of a dual space, as in Dunford and Schwartz (1959). Define $\mathcal{M}_1 = \{\mu \in \mathcal{M} \mid \|\mu\| \leq 1\}$. We will state the following theorem in terms of $\mathcal{M}$ and $\mathcal{C}$.

Theorem B.1: (Dunford and Schwartz, Theorem V.5.1) If $\mathcal{C}$ is a Banach space, then the $\mathcal{T}^*$ topology of the closed unit sphere $\mathcal{M}_1$ of $\mathcal{M}$ is a metric topology if and only if $\mathcal{C}$ is separable.

If the hypothesis of Theorem B.1 holds, and $\{f_n\}$ is a separating subset of $\mathcal{C}$, an appropriate metric for $(\mathcal{M}_1, \mathcal{T}^*)$ is given by

$$\rho(\mu, \nu) = \sum_{n=1}^{\infty} \frac{1}{M^n} \frac{q_n(\mu, \nu)}{1 + q_n(\mu, \nu)}$$

where M is a real number greater than 1 and $q_n(\mu, \nu) = |\int f_n\ d(\mu - \nu)|$.

We provide now a countable dense subset of $\mathcal{C}$. Let $\mathcal{Q}$ be the subalgebra of $\mathcal{C}$ of the polynomials with rational coefficients in the unknowns $k_1, ..., k_N, z_1, ..., z_M$. Since $\mathcal{Q}$ contains 1 and separate the points of X, then the closure of $alg(\mathcal{Q})$, $\overline{alg(\mathcal{Q})}$, equals $\mathcal{C}$. See Reed and Simonn (1980) Ch. IV. Now the thesis we are going to prove is that the set $\mathcal{Q}$ also separates $\mathcal{C}$.

It is clear that if $f_1, ..., f_M \in \mathcal{Q}$ then the product $f_1 \cdots f_M \in \mathcal{Q}$, so the elements in $\overline{alg(\mathcal{Q})}$ are of the form $\sum_{i=1}^{n} r_i p_i$ for real numbers r_i, and with $p_i \in \mathcal{Q}$, and the elements in $\mathcal{Q}$ are of the form $\sum_{i=1}^{n} q_i p_i$ for rational numbers q_i. Now, let us see that for any $f \in \mathcal{C}$ and any $\varepsilon > 0$, exists $q \in \mathcal{Q}$ such that $\|f - q\|_\infty < \varepsilon$. Take $r \in \overline{alg(\mathcal{Q})}$ such that $\|f - r\|_\infty < \varepsilon/2$, and choose $q \in \mathcal{Q}$ such that $\|r - q\|_\infty < \varepsilon/2$, then

$$\|f - q\|_\infty \leq \|f - r\|_\infty + \|r - q\|_\infty < \varepsilon$$

where $\|r - q\|_\infty < \varepsilon/2$ is due to the fact that $\mathbf{Q}$ is dense in $\mathbf{R}$.

Now, since $\mathcal{P} \subset \mathcal{M}_1$, we have proved the following

Lemma B.2: The space $(\mathcal{P}, \mathcal{T}^*)$ is metrizable, and the metric ρ defined by

$$\rho(\mu, \nu) = \sum_{n=1}^{\infty} \frac{1}{M^n} \frac{q_i(\mu, \nu)}{1 + q_i(\mu, \nu)}$$

where M is a real number greater than 1, $q_n(\mu, \nu) = |\int f_n \; d(\mu - \nu)|$, and $\{f_n\}$ is the family of polynomials in $k_1, ..., k_N, z_1, ..., z_M$ with rational coefficients, induces the topology $\mathcal{T}^*$.

The objective of this appendix is to provide a metric for the space $\mathcal{P}$. The metric is based on a countable set of functions (polynomials in this case.) The following lemma proves that the set can be rearranged such that a certain condition (that is used in the paper) holds. Up to this point the theorems have been proved for the case $K \subset \mathbf{R}^N$ and $Z \subset \mathbf{R}^M$. In the proof of the following lemma we will specialize to the case $N = 1 = M$, although the result is obviously more general.

Lemma B.3: Let $\{p_n(k, z)\}$ be a countable sequence of polynomials. Then the sequence can be rearranged such that

$$sup_z |\frac{\partial}{\partial k} p_n(k, z)| \leq \alpha(n) B^{\theta n}$$

where $0 < \theta < 1$ and the sequence $\alpha(n)$ grows at rate $\frac{\alpha(n+1)}{\alpha(n)} < B^{1-\theta}$.

Proof: We write

$$p_n(k, z) = \sum_{i_n=1}^{R_n} \sum_{j_n=1}^{S_n} a_{i_n, j_n} k^{i_n} z^{j_n}$$

therefore

$$|\frac{\partial}{\partial k} p_n(k, z)| = |\sum_{i_n=1}^{R_n} \sum_{j_n=1}^{S_n} b_{i_n, j_n} k^{i_n - 1} z^{j_n}|$$

where b_{i_n,j_n} depends on a_{i_n,j_n} and i_n, and so

$$sup_z|\frac{\partial}{\partial k}p_n(k,z)| \quad \leq \quad |\sum_{i_n=1}^{R_n}\sum_{j_n=1}^{S_n} b_{i_n,j_n}| \; B^{i_n+j_n-1}$$

where $B = max\{sup_k|k|, sup_z|z|\}$, and then to ensure

$$sup_z|\frac{\partial}{\partial k}p_n(k,z)| \leq \alpha(n)B^{\theta n}$$

it is enough to ensure $i_n + j_n - 1 \leq \theta n$, and $|\sum_{i_n=1}^{R_n}\sum_{j_n=1}^{S_n} b_{i_n,j_n}| \leq \alpha(n)$. *Q.E.D.*

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