

## Jan 23rd, 2007

- What is a model? A model is 'Toy World': a simplified specification of the world, endowed with (i) an environment, (ii) agents and (iii) characteristics of the agents. Once a model is defined, we need to know what happens, i.e., an equilibrium. Generally, by equilibrium we mean that we want to know what happens with allocations: Are they (in some sense) 'optimal? Do they exist?
- What is an equilibrium?  
An equilibrium is a statement about what the outcome of an economy is. Tells us **what happens in an economy**, and by an economy we mean a well defined environment in terms of primitives such as preferences and technology.  
Then an equilibrium is a particular mapping from the environment (preference, technology, information, market structure) to allocations where,
  1. Agents maximize
  2. Agents' actions are compatible with each other.
- One of the important questions is, given the environment what type of equilibria we should look at. The economist doesn't have the right to choose what happens, but is free to define the environment.
- For the theory to be able to predict precisely what is going to happen in a well defined environment, the outcome we define as the equilibrium needs to **exist** and must be **unique**. For this reason uniqueness is property that we want the equilibrium to have. We also know with certain assumptions that will be covered we can ensure the existence and uniqueness of an equilibrium outcome.

## 1 Growth Model

The basic model we deal with in 702 is the neoclassical growth model. We will discuss the basic environment and then ask what happens in this 'toy world': does an equilibrium exist? is it optimal?

## 1.1 Technology

- Agents have 1 unit of labor and own capital which can be transformed in output.
- Production function:

$$f : R_+^2 \rightarrow R_+ \quad \text{such that } c_t + k_{t+1} = f(k_t, n_t) \quad (1)$$

- We assume (i) Constant Returns to Scale (CRS, or homogeneous of degree one, meaning  $f(\lambda k, \lambda n) = \lambda f(k, n)$ ), (ii) strictly increasing in both arguments, and (iii) INADA condition, if necessary)

## 1.2 Preferences

- We assume infinitely-lived representative agent (RA).<sup>1</sup>
- We assume that preference of RA is (i) time-separable (with constant discount factor  $\beta < 1$ ), (ii) strictly increasing in consumption (iii) strictly concave
- Our assumptions let us use the utility function of the following form:

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \quad (2)$$

- Initial capital stock  $k_0$  is given.

With these in hand the problem is,

$$\max_{\{c_t, n_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \quad (3)$$

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<sup>1</sup>For now, let's treat the economy as if there were only one agent in the economy. We might interpret it as the result of normalization (so the number of population is 1) of the economy with FINITE number of identical (sharing the same technology, preference, and allocation) agents. If we proceed to the economy with mass of zero measure agents, things will be not so trivial because changing allocation of one agent does not change the aggregate amount of resources in the economy (since, by assumption, measure of an agent is zero), but let's forget it for now.

subject to<sup>2</sup>

$$k_{t+1} + c_t = f(k_t, n_t) \quad (4)$$

$$c_t, k_{t+1} \geq 0 \quad (5)$$

$$k_0 \text{ is given} \quad (6)$$

A solution to this problem is a sequence of consumption and capital accumulation decisions  $\{c_t^*, k_{t+1}^*\}_{t=0}^{\infty}$ . We would like this solution to exist, be unique and be Pareto Optimal. How do we do that? We can show these properties by rewriting the model in Arrow-Debreu language and using the theorems we know from 701.

**Jan 25th, 2007**

## 2 Arrow-Debreu (AD) Equilibrium

- In macroeconomics, we are interested in infinite-dimensional commodity spaces. We want to look at the relationship between competitive equilibrium and Pareto optimality in models with infinite-dimensional spaces. You looked at competitive equilibrium and Pareto optimality in 701, but the proofs of the FBWT and SBWT were done in the context of finite-dimensional commodity spaces, like the **Neoclassical Growth Model**. Here we want to show that the welfare theorems hold for economies with infinite dimensional spaces. To do this, we introduce the equilibrium concept of '*valuation (or AD) equilibrium*'.
- Before defining valuation equilibrium, we first need to define the environment, unlike the social planner problem, which is a problem of allocation, in a AD world we will have exchange among agents. This requires definition of markets in which the relevant commodities to be defined are traded.

1.  $\mathcal{L}$ , Commodity space:  
 $\mathcal{L}$  is a topological vector space.

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<sup>2</sup>We can also define  $f$  (the production function) as including depreciation of capital. In the 1st class, Victor actually took this approach, but I modified the notation to make notation consistent across classes.

**Definition 1 (Vector Space)** *A vector space is a space where the operations addition and scalar multiplication are defined, and where the space is closed under these two operations. i.e. If we take two sequences  $a = \{a_i\} \in \mathcal{L}$  and  $b = \{b_i\} \in \mathcal{L}$ , it must be that  $a + b \in \mathcal{L}$ . And if we take  $k \in \mathcal{R}^+, k > 0$ , it must be that  $a \in \mathcal{L} \Rightarrow d = ka \in \mathcal{L} \forall k > 0$ .*

**Definition 2 (Topological Vector Space)** *A topological vector space is a vector space which is endowed with a topology such that the maps  $(x, y) \rightarrow x + y$  and  $(\lambda, x) \rightarrow \lambda x$  are continuous. So we have to show the continuity of the vector operations addition and scalar multiplication.*

2.  $X \subset \mathcal{L}$ , Consumption Possibility Set:  
Specification of the 'things' that people could do (that are feasible to them).  $X$  contains every (individually) technologically feasible consumption point.  
Characteristics of  $X$ : non-empty, closed and convex. Also, note that we will use the convention that output is positive while inputs are negative.
3.  $U : X \rightarrow \mathcal{R}$ , Specifies the preference ordering (utility function)
4.  $Y$ , Production possibility set, which must be non-empty, closed, convex and must have an interior point.

A simplifying assumption (which will be relaxed in a couple of weeks) we'll impose, is that there are many identical firms and agents. With this, we guarantee that they act competitively (take prices as given) and we only have to consider a representative agent who chooses what everyone else chooses (although everyone could do differently).

## 2.1 Prices

Prices ( $p$ ) are continuous linear functions that are defined on our commodity space. More specifically,  $p \in \mathcal{L}^*$ , where  $\mathcal{L}^*$  is the 'dual' of  $\mathcal{L}$  (the set of all linear functions over  $\mathcal{L}$ ); it may not always be possible to find a sequence of real numbers to represent this function as a dot product formulation (as we think of prices in finite dimensions)

**Remark 1** if  $\mathcal{L} = \mathcal{R}^n \Rightarrow \mathcal{L}^* = \mathcal{R}^n$ . In other words  $p(\ell) = \sum_{n=1}^N p_n \ell_n$  defines the value of a bundle  $\ell$ , composed of  $\{\ell_1, \ell_2, \dots, \ell_N\}$

if  $\mathcal{L} = \mathcal{R}^n \Rightarrow \mathcal{L}^* = \mathcal{R}^n$ . In particular

$$p(\ell) = \sum_{n=1}^N p_n \ell_n$$

### 2.1.1 Definition of the Arrow-Debreu equilibrium

**Definition 3 (Arrow-Debreu/Valuation Equilibrium)** ADE equilibrium is a feasible allocation  $(x^*, y^*)$  and a continuous linear function  $p^*$  such that,

1.  $x^*$  solves the consumer's problem:

$$x^* \in \text{Arg max}_{x \in X} u(x)$$

st

$$p^*(x) \leq 0$$

2.  $y^*$  solves the firm's problem:

$$y^* \in \text{Arg max}_{y \in Y} p^*(y)$$

3. markets clear (compatibility of actions)

$$x^* = y^*$$

## 2.2 Welfare Theorems

**Theorem 1 (First Basic Welfare Theorem)** Suppose that for all  $x \in X$  there exists a sequence  $\{x_n\}_{n=0}^{\infty}$  in  $X$  converging to  $x$  with  $u(x_n) \geq u(x)$  for all  $n$  (local nonsatiation). If an allocation  $(x^*, y^*)$  and a continuous linear functional  $p$  constitute a competitive equilibrium, then the allocation  $(x^*, y^*)$  is Pareto optimal.

The FBWT tells us that the market is a good mechanism to allocate resources, which imply that 'you cannot do better'.

**Definition 4 (Feasible allocation)** For all  $x = y$  st  $x \in X$  and  $y \in Y$

$$F(\varepsilon) = X \cap Y$$

where  $\varepsilon$  describes, through primitives, a particular economy

**Theorem 2** If  $u$  is strictly concave and  $X \cap Y$  is convex and compact, then the solution to

$$\max_{x \in X \cap Y} u(x) \tag{7}$$

exists and its unique.

This last result plus the FBWT imply that if an Arrow Debreu equilibrium exists, then it's a Pareto Optimal allocation.

From (7) we can calculate the allocation  $(x^*, y^*)$ . Nevertheless, to construct an AD equilibrium, we still need a price function. The Second Basic Welfare Theorem will provide us with one.

**Theorem 3 (Second Basic Welfare Theorem)** If (i)  $X$  is convex, (ii) preference is convex (for  $\forall x, x' \in X$ , if  $x' < x$ , then  $x' < (1-\theta)x' + \theta x$  for any  $\theta \in (0, 1)$ ), (iii)  $U(x)$  is continuous, (iv)  $Y$  is convex, (v)  $Y$  has an interior point, then with any PO allocation  $(x^*, y^*)$  such that  $x^*$  is not a saturation point, there exists a continuous linear functional  $p^*$  such that  $(x^*, y^*, p^*)$  is a Quasi-Equilibrium with transfers ((a) for  $x \in X$  which  $U(x) \geq U(x^*)$  implies  $p^*(x) \geq p^*(x^*)$  and (b)  $y \in Y$  implies  $p^*(y) \leq p^*(y^*)$ )

Note that an additional assumption we are making for SBWT to go through in infinitely dimensional spaces is that  $Y$  has an interior point i.e.

$$\exists \bar{y} \in Y, B \subset Y, B \text{ open and } \bar{y} \in B$$

Also that the SBWT states that under certain conditions listed above, we can find prices to support any Pareto optimal allocation as a quasi equilibrium with transfers. Transfers are not relevant in our case since we are

working in an representative agent environment with identical households. Taking care of the transfers still leaves us with Quasi-Equilibrium so SBWT by itself it does not say anything about the existence of Arrow-Debreu equilibrium. The following lemma takes care of this.

**Lemma 1** *If, for  $(x^*, y^*, \nu^*)$  in the theorem above, the budget set has cheaper point than  $x^*$  ( $\exists x \in X$  such that  $\nu(x) < \nu(x^*)$ ), then  $(x^*, y^*, \nu^*)$  is a ADE.*

With the SBWT, we established that there exists a  $p$  that will support our PO allocation as a competitive equilibrium. What's the problem with this approach? SBWT only tells us that such a  $p$  exists, it doesn't tell us what it is. Also, we are not sure that  $p$  has a dot product representation. The next theorem deals with this nuance

**Theorem 4** *(based on Prescott and Lucas 1972) If, in addition to the conditions of the SBWT, agents discount remote and/or unlikely states and  $u$  is bounded, then  $\exists \{q_t\}$  such that*

$$p(x) = \sum_{t=0}^{\infty} q_t x_t \tag{8}$$

*i.e. the price system has an inner product representation.*

Remember, our main purpose is to be able to apply the welfare theorems to the most commonly used models in macroeconomics where we have an infinite-dimensional commodity space. Until now, we set up an environment (Arrow-Debreu economy) which consisted of the commodity space, consumption possibility set, production possibility set, and preferences) with infinite-dimensional commodity space and we stated that under certain conditions the Welfare Theorems hold in this environment. Now we will map the growth model into the environment that we talked about until here, and show that in the context of the growth model the assumptions we need for the Welfare Theorems are satisfied. Then we can conclude that any competitive equilibrium allocation is Pareto optimal and moreover we can support a PO allocation with some prices as a competitive equilibrium. This result is very important in macroeconomics. It helps us in solving for the equilibria.

With the FBWT and SBWT, we can just solve for the PO allocations and then get the prices. This makes life much easier.

What is an allocation in this environment? An allocation is a pair  $(x, y)$ . On the other hand, a feasible allocation is  $(x, y)$  such that  $x=y$  (agents' actions need to be compatible). What are the commodities we need to make tradable in this environment? Output, labor services, capital services. So lets define the commodity space.

$$\mathcal{L} = \{ \{ \ell_t \}^{t=0, \infty} = \{ \ell_{it} \}_{i=1,2,3}^{t=0, \infty}, s_{it} \in R : \sup_t |s_t| < \infty \}$$

so our commodity space will be the set of bounded sequences in sup norm. Subindexes 1,2 and 3 represent output, capital services and labor services respectively. The interested reader can refer to Stokey and Lucas (1989) for the reasons behind the choice of this particular space. Next is the definition of consumption possibility set  $X$

$$X(k_0) = \{ x \in \mathcal{L} = l_\infty^3 : \exists \{ c_t, k_{t+1} \}_{t=0}^\infty \geq 0 \text{ such that}$$

$$\begin{aligned} k_{t+1} + c_t &= x_{1t} & \forall t & \quad (9) \\ x_{2t} &\in [-k_t, 0] & \forall t & \quad (10) \\ x_{3t} &\in [-1, 0] & \forall t & \end{aligned}$$

Now, let's define the production possibility set  $Y$ . The firm's problem is relatively simple as firms do not have intertemporal decisions. Firms just rent production factors and produce period by period.

$$Y = \prod_{t=0}^\infty \widehat{Y}_t : \widehat{Y}_t = \{ y_{1t} \geq 0, y_{2t}, y_{3t} \leq 0 : y_{1t} \leq f(-y_{2t}, -y_{3t}) \} \quad (11)$$

Finally, preferences over this space  $U : X \rightarrow R$

$$U(x) = \sum_{t=0}^\infty \beta^t u(c_t(x)) \quad (12)$$

$c_t$  is unique given  $x$ , because each  $x$  implies a sequence  $\{ c_t, k_{t+1} \}_{t=0}^\infty$ . If  $x_{2t} = k_t$ ,  $c_t = x_{1t} - x_{2t+1}$ .



Jan 30th, 2007

### 3 Existence and Pareto Optimality in the Growth Model

To support a Pareto Optimal allocation as a solution to the growth model presented before, we have to take care of certain issues that arise when we apply the SBWT to get our equilibrium. Those issues/solutions are listed below:

- What are the 'transfers' of the conclusion of the SBWT in terms of the growth model? / we don't need transfers; agents are homogeneous, so even if they can act differently, they choose to do the same as everyone else.
- Do we have to worry about the 'Quasi' part of the equilibrium? / If we can find a cheaper point in the feasible set, then the Quasi equilibrium is equivalent to the AD equilibrium
- representation of prices/ if we can check the conditions of the Prescott & Lucas Theorem, then we have a dot product representation of prices.

#### 3.1 Characterization of the solution to the growth model

The solution to the growth model is triplet of sequences  $\{c_t^*, k_{t+1}^*, q_t^*\}_{t=0}^{\infty}$ . As you proved in the homeworks, you can use the Arrow-Debreu apparatus in order to argue that such an equilibrium exists. To characterize more carefully the equilibrium, we have to impose additional restrictions:

- $u, f$  are  $C^2$  (twice continuously differentiable)
- Inada conditions (see the Stockey and Lucas textbook for specifics)

With these conditions, we can restrict our attention to interior solutions, which means that first order conditions are sufficient to characterize equilibria.

Rewriting the growth model (replacing consumption in the utility function using the budget constraint):

$$\max_{\{k_{t+1}\}_{t=0}^{\infty}} u[f(k_t) - k_{t+1}]$$

Taking the FOC with respect to  $k_{t+1}$  and replacing for  $c_t$  to ease notation, we get (note that we are using variables with \* to denote that the following are *equilibrium* conditions)

$$-\beta^t u'[c_t^*] + \beta^{t+1} u'[c_{t+1}^*] f'(k_{t+1}^*) = 0$$

rearranging terms

$$\frac{u'[c_t^*]}{\beta u'[c_{t+1}^*]} = f'(k_{t+1}^*) \quad (13)$$

Therefore, the solution to the growth model has to satisfy the condition in (13).

Now, for prices, we can rewrite the budget equation from the AD setting (if the conditions of Prescott and Lucas are satisfied so that prices have a dot product representation) as follows

$$p(x) \equiv \sum_{t=0}^{\infty} (q_{1t} x_{1t} + q_{2t} x_{2t} + q_{3t} x_{3t}) \leq 0 \quad (14)$$

Since  $c_t + k_{t+1} = x_{1t}$ ,  $k_t \geq -x_{2t} \geq 0$  and  $1 \geq -x_{3t} \geq 0$ , (14) becomes

$$\sum_{t=0}^{\infty} (q_{1t}^* (c_t + k_{t+1}) - q_{2t}^* k_t - q_{3t}^*) \leq 0 \quad (15)$$

Note that in (15), we have used the fact that there is no waste (agents rent their full capital and labor services) and that agents take the equilibrium prices as given. The maximization problem now can be set as a Lagrangian:

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \mathcal{L} = \sum_{t=0}^{\infty} \beta^t u[c_t] - \lambda \left\{ \sum_{t=0}^{\infty} q_{1t}^* (c_t + k_{t+1}) + q_{2t}^* k_t + q_{3t}^* \right\} \quad (16)$$

The first order conditions of this problem with respect to  $c_t$  and  $k_{t+1}$  are respectively

$$\frac{\beta^t u'[c_t^*]}{q_{1t}^*} = \lambda \quad (17)$$

$$\lambda q_{1t}^* - \lambda q_{2,t+1}^* = 0 \quad (18)$$

Note that (18) implies that  $q_{1t}^* = q_{2,t+1}^*$ <sup>3</sup>, which pins down one sequence of prices (specifically, the price of capital services). From (17), we get

$$\begin{aligned} \lambda &= \frac{\beta^t u'[c_t^*]}{q_{1t}^*} = \frac{\beta^{t+1} u'[c_{t+1}^*]}{q_{1,t+1}^*} \\ &\Rightarrow \frac{u'[c_t^*]}{\beta u'[c_{t+1}^*]} = \frac{q_{1t}^*}{q_{1,t+1}^*} \end{aligned}$$

From before, we know that the left hand side of the last equation equals  $f'(k_{t+1}^*)$ . Hence

$$\frac{q_{1t}^*}{q_{1,t+1}^*} = f'(k_{t+1}^*) \quad (19)$$

Since  $f'$  represents the (technical) rate of exchange between goods today and goods tomorrow, (19) tells us exactly what the sequence of output prices should be. Finally, to obtain  $q_{3t}^*$ , we turn to problem of the producer

$$\begin{aligned} \max_{y \in Y_t} q^*(y) &= q_{1t}^* y_{1t} + q_{2t}^* y_{2t} + q_{3t}^* y_{3t} \\ &st \\ y_{1t} &= f(-y_{2t}, -y_{3t}) \end{aligned}$$

Again, we know that  $\{k_{t+1}^*, 1\}_{t=0}^\infty$  solve this problem. Then, the problem of the firm is equivalent to

$$\max_{y \in Y_t} f(k_t, y_{3t}) - q_{2t}^* k_t - q_{3t}^* y_{3t}$$

Taking FOCs with respect to  $k_t^*$  and  $y_{3t}$  respectively

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<sup>3</sup>This condition is misleading, since we don't have depreciation. In the more general case when  $\delta \neq 1$ , the condition is

$$\frac{q_{1t}^*}{q_{1,t+1}^*} = 1 - \delta + \frac{q_{2,t+1}^*}{q_{1,t+1}^*}$$

$$q_{1t}^* f_k(k_t^*, 1) = q_{2t}^*$$

$$q_{1t}^* f_n(k_t^*, 1) = q_{3t}^*$$

Hence, the price of labor services must satisfy  $f_n(k_t^*, 1) = q_{3t}^*/q_{1t}^* \forall t$ .

We know now how to characterize the sequence of prices at equilibrium in the growth model. The problem with the AD framework however, is that we have a triple infinite (!) number of prices. Together with the assumption that all trade takes place at  $t = 0$ , this implies that agents must know a triple infinite number of prices in order to solve their problem.

We want to depart from this assumption of all trading happening at the beginning of time, so we will define *sequential markets* and a corresponding *sequential markets equilibrium* (SME). Note that we would like to maintain existence, uniqueness and optimality of the equilibrium, so we would like  $ADE \Leftrightarrow SME$ .

### 3.2 Sequential Markets Equilibrium

- We need a spot market at every period of time where agents would be able to trade output, capital and labor services and a new good (which we will specify below) which are 'loans'.
- Agents must be able to move resources across time.

Clearly, the budget constraint will change from the previous setup. In ADE

$$\sum_{i=1}^3 \sum_{t=0}^{\infty} q_{it} x_{it} \leq 0$$

In SME, we introduce the concept of 'loans' ( $l$ ), to enable agents to move resources across time. Loans are rights to a  $R$  units of output/consumption tomorrow, in exchange of 1 unit of output/consumption today. So, the budget constraint becomes

$$-l_t R_t + l_{t+1} + \sum_{i=1}^3 q_{it} x_{it} \leq 0 \quad \forall t$$

**Definition 5** A Sequence of Markets Equilibrium is  $\{x_{it}^*, q_t^*, y_{it}^*, l_{t+1}^*, R_t^*\}_{t=0}^{\infty}$  such that

- Agents maximize, i.e.

$$\{x_{it}^*, l_{t+1}^*\} \in \arg \max_{x \in X} \sum_{t=0}^{\infty} \beta^t u[c_t(x)]$$

st

$$c_t + k_{t+1} + l_{t+1}^* = R_t^* l_t + q_{2t}^* k_t + q_{3t}^* \\ k_0, l_0 \text{ given}$$

- Firms maximize
- $x^* = y^*$  (market clears)
- $l_{t+1}^* = 0 \quad \forall t$  (loan market clears)

To show that  $ADE \Leftrightarrow SME$ , we need to check that allocations and choices of the agents in both worlds are the same. In the  $SME \Rightarrow ADE$  direction, it's easy to see that if we have a  $SME$ , we can construct an  $ADE$  just by ignoring  $\{l_{t+1}\}$  (it's zero at equilibrium anyway).

Conversely ( $ADE \Rightarrow SME$ ), if we have an  $ADE$ , we need  $l_{t+1}^*$  and  $R_t^*$  to construct a  $SME$ . Again, given the condition for the clearing of the loans market,  $l_{t+1}^*$  comes trivially. For  $R_t^*$ , we use an arbitrage condition: since loans and capital perform the same function (move resources from one period of time to another), then their price should be the same. Specifically

$$R_t = \frac{q_{1t}^*}{q_{1,t+1}^*}$$

Finally, we have a close relationship between prices between both equilibriums. If  $\{x^*, y^*, q^*\}$  is an  $ADE$  and  $\{x^*, y^*, \hat{q}^*, R^*, l^*\}$  is a  $SME$ , the following is true since at a  $SME$ , the budget constraint is priced with respect to output/consumption at each point of time

$$\hat{q}_{it}^* = \frac{q_{it}^*}{q_{1t}^*} \quad \forall t$$

### 3.3 SME 'easy'

Now we will define a simpler version of the *SME*. Basically, we will simplify the definition of equilibrium by ignoring loans and using the properties of the production function

**Definition 6** A *SMEE* is  $\{c_t^*, k_{t+1}^*, w_t^*, R_t^*\}$  such that

- *Agents maximize:*

$$\{c_t^*, k_{t+1}^*\} \in \arg \max_{\{c_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u[c_t]$$

*st*

$$c_t + k_{t+1} = R_t^* k_t + w_t^*$$

$k_0$  given

- *Firms maximize:*

$$\{k_{t+1}^*, 1\} \in \arg \max_{k_t, n_t} f(k_t, n_t) - R_t^* k_t - w_t^* n_t$$

- *Market clearing:*

$$c_t^* + k_{t+1}^* = f(k_t^*, 1)$$

Note that the last condition is redundant, because the production function is homogenous of degree one, i.e., production is exhausted in the payment to production factors.

After all this work, we still have the problem of how to calculate the equilibrium. From the FOCs we know that to get a solution, we have to solve a second order difference equation, with an initial conditions plus a transversality condition. Nevertheless, the Growth model has infinite dimensions, which complicate things a bit.

The next step is to reformulate the problem in a recursive form. This is much better, since we will be able to solve the problem recursively, that is, every new period, the agent faces the same problem.

## 4 Stochastic Processes

### 4.1 Markov Process

In this course, we will concentrate on Markov productivity shocks. Considering shocks is really a pain, so we want to use less painful ones. A Markov shock is a stochastic process with the following properties:

1. there are FINITE number of possible states for each time. More intuitively, no matter what happened before, tomorrow will be represented by one finite set.
2. the only thing that matters for the realization of tomorrow's shock is today's state. More intuitively, no matter what kind of history we have, the only thing you need to predict the realization of the shock tomorrow is TODAY's realization.

More formally, for each period, suppose either  $z^1$  or  $z^2$  happens<sup>4</sup>. Denote  $z_t$  is the state today and  $Z_t$  is the set of possible states today, i.e.  $z_t \in Z_t = \{z^1, z^2\}$  for all  $t$ . Since the shock follows a Markov process, the state tomorrow will only depend on today's state. So let's write the probability that  $z^j$  will happen tomorrow, conditional on today's state being  $z^i$  as  $\Gamma_{ij} = \text{prob}[z_{t+1} = z^j | z_t = z^i]$ . Since  $\Gamma_{ij}$  is a probability, we know that

$$\sum_j \Gamma_{ij} = 1 \quad \text{for } \forall i \quad (20)$$

Notice that a 2-state Markov process is summarized by 6 numbers:  $z^1, z^2, \Gamma_{11}, \Gamma_{12}, \Gamma_{21}, \Gamma_{22}$ .

The great beauty of using a Markov process is that we can use the explicit expression of probability for future events, instead of using the ambiguous operator called expectation, which very often people don't know what it means when they use it.

### 4.2 Representation of History

- Let's concentrate on a 2-state Markov process. In each period the state of the economy is  $z_t \in Z_t = \{z^1, z^2\}$ .

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<sup>4</sup>Here we restrict our attention to the 2-state Markov process, but increasing the number of states to any finite number does not change anything fundamentally.

- Denote the history of events up to  $t$  (which one of  $\{z^1, z^2\}$  happened from period 0 to  $t$ , respectively) by  $h_t = \{z_1, z_2, \dots, z_t\} \in H_t = Z_0 \times Z_1 \times \dots \times Z_t$ .
- In particular,  $H_0 = \emptyset$ ,  $H_1 = \{z^1, z^2\}$ ,  $H_2 = \{(z^1, z^1), (z^1, z^2), (z^2, z^1), (z^2, z^2)\}$ .
- Note that even if the state today is the same, past history might be different. By recording history of events, we can distinguish the two histories with the same realization today but different realizations in the past (think that the current situation might be "you do not have a girl friend", but we will distinguish the history where "you had a girl friend 10 years ago" and the one where you didn't
- Let  $\Pi(h_t)$  be the unconditional probability that the particular history  $h_t$  does occur. By using the Markov transition probability defined in the previous subsection, it's easy to show that (i)  $\Pi(h_0) = 1$ , (ii) for  $h_t = (z^1, z^1)$ ,  $\Pi(h_t) = \Gamma_{11}$  (iii) for  $h_t = (z^1, z^2, z^1, z^2)$ ,  $\Pi(h_t) = \Gamma_{12}\Gamma_{21}\Gamma_{12}$ .

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## 5 Stochastic Growth Model

With this, we can rewrite the growth model when these shocks affect the production function (usual convention in Macro). Preferences are given by the usual von Neumann-Morgenstern utility

$$u(x) = \sum_t \beta^t \sum_{h_t \in H_t} \pi(h_t) u[c_t(h_t)].$$

In an Arrow -Debreu world the constraint is<sup>5</sup>

$$\sum_t \sum_{h_t \in H_t} \sum_j p_t^j(h_t) x_t^j(h_t) \leq 0, \quad \text{where } j = 1, 2, 3.$$

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<sup>5</sup>The definition of the relevant commodity spaces and consumption/production sets is in the second problem set



In a SM setting we need to give to the agent enough tools, so that she can consume different quantities in different states of the world. In other words, we have to make sure that whatever she was able to do in an AD setting, she will also be able to do it in the SM setting. To that end, we will introduce loans and state contingent claims (Also known as *Arrow* securities or bonds). For example,  $b_t(h_{t-1}, z^i)$  is a claim that the agent bought in period  $t-1$ , and will pay 1 unit of consumption for sure if state  $i$  occurs. In the SM world, the budget constraint will be

$$c_t(h_t) + k_{t+1}(h_t) + \ell_t(h_t) + \sum_{z_{t+1}} q_t(h_t, z_{t+1}) b_{t+1}(h_t, z_{t+1}) = k_t(h_{t-1}) R_t^k(h_t) + \ell_{t-1}(h_{t-1}) R_t^\ell(h_t) + w_t(h_t) + b_t(h_{t-1}, z_t),$$

where  $q_t(h_t, z_{t+1})$  is the price of the state contingent claim that pays 1 in period  $t+1$  if state  $z_{t+1}$  occurs.

From the definition of these assets, we can derive immediately a no-arbitrage condition for loans and the state contingent securities. On the one hand, if we save one unit of consumption today and get a loan, the gross return is given by

$$\frac{R_{t+1}^\ell(h_t)}{1}$$

, since tomorrow the loan will pay some interests. On the other hand, by using that same unit of consumption and investing in Arrow securities, we get a gross return of

$$\frac{\sum_{z_{t+1}} q_t(h_t, z_{t+1}) \bar{b}}{\bar{b}} = \sum_{z_{t+1}} q_t(h_t, z_{t+1})$$

where  $\bar{b}$  is the level of *certain* future consumption one can get by spending 1 unit of current consumption. Since these two ways of savings are the same, the no-arbitrage condition is

$$1 = R_{t+1}^\ell(h_t) \sum_{z_{t+1}} q_t(h_t, z_{t+1})$$

The importance of the no arbitrage conditions is that we can eliminate (shut down) some markets, since they can be perfectly replicated by other

markets. In the example above, we can close the state contingent market, since we already have a loan market.

Furthermore, given our representative agent assumption, in equilibrium we don't need any additional markets, since

$$\ell^* = b^* = 0 \quad \forall t, (h_t, z_{t+1})$$

We can also derive this no-arbitrage condition through first order conditions. Using the lagrangian of the problem (with multipliers  $\lambda$  for each  $t$  and  $h_t$ ) and taking a FOC with respect to  $b_t(h_t, z_{t+1})$

$$\begin{aligned} q_t(h_t, z_{t+1})\lambda_t(h_t) &= \lambda_{t+1}(h_t, z_{t+1}) \\ \Rightarrow q_t(h_t, z_{t+1}) &= \frac{\lambda_{t+1}(h_t, z_{t+1})}{\lambda_t(h_t)} \end{aligned} \quad (21)$$

Next, taking a first order condition with respect to  $\ell_t(h_t)$  we get

$$\lambda_t(h_t) = \sum_{z_{t+1}} \lambda_{t+1}(h_t, z_{t+1}) R_{t+1}^\ell(h_t)$$

Since the return on loans is 'set' before the shock is known (hence, that return is NOT state-dependent), we can take  $R^\ell$  outside the sum:

$$1 = R_{t+1}^\ell(h_t) \sum_{z_{t+1}} \frac{\lambda_{t+1}(h_t, z_{t+1})}{\lambda_t(h_t)}$$

Finally, using (21) we arrive to the same no-arbitrage condition as before:<sup>6</sup>

$$1 = R_{t+1}^\ell(h_t) \sum_{z_{t+1}} q_t(h_t, z_{t+1})$$

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<sup>6</sup>In problem set # 2, you derived an analogous condition for the return on capital  $R_t^k(h_t)$ , but since this IS state dependent, the condition is slightly different:

$$1 = \sum_{z_{t+1}} q_t(h_t, z_{t+1}) R_{t+1}^k(h_t, z_{t+1})$$

**Feb 5th, 2007**

Let's write the SPP problem in sequence representation:

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(f(k_t) - k_{t+1})$$

$k_0$  given

Because of the INADA conditions we know that the solution is interior. So if  $\{k_{t+1}^*\}$  is a solution then it satisfies the first order conditions,

$$-\beta^t u_c(f(k_t) - k_{t+1}) + \beta^{t+1} f_k(k_t + 1) u_c(f(k_{t+1}) - k_{t+2}) = 0$$

But notice that these conditions are not sufficient. The above is a second order difference equation. It has two degrees of freedom. Therefore this equation is not enough to find the solution. There can be many sequences that satisfy it. We need to more conditions to pin down the right solution: The initial condition  $k_0$  and the transversality condition.

### How to solve the SPP?

In Greenwood's class you learned how to solve this problem through dynamic programming.

Denote  $R(k, k') = u(f(k) - k')$ .

The Bellman equation,

$$V(k) = \max_{k'} R(k, k') + \beta V(k')$$

FOC:

$$R'_2(k, g(k)) + \beta R'_1(g(k), g(g(k))) = 0$$

You can verify that FOC's of these problems imply each other (HW: Verify that this is true).

## 6 Recursive Competitive Equilibrium

As you have seen in 704, the beauty of the recursive representation lies in the fact that, in a stationary environment, the nature of the problem do not

change with passage of time. Unlike the sequential formulation, in which the solution to the problem depends on at what point in time you solve it, the solution to the recursive problem do not depend on time and we do not have to keep track of time.

So what do we keep track of? Everything that matters to the structure of our problem. These are the variables that our agents respond to either directly or indirectly and we call them the STATE VARIABLES. State variables need to satisfy the following criteria:

1. PREDETERMINED: when decisions are made, the state variables are taken as given and cannot be effected by the agent.
2. It must MATTER for decisions of agents: there is no sense of adding irrelevant variables as state variable.
3. It must VARY across time and state: otherwise, we can just take it as a parameter.

One important thing is to be able to distinguish the aggregate and individual state variables. Aggregate state is not affected by individual choice. But aggregate state should be consistent with the individual choice (we will consider the meaning of "consistency" more formally later), because aggregate state represents the aggregated state of individuals. In particular, in our RA-NGM aggregate state turns out to be the same as individual state in equilibrium, but this does not mean that the agent decide the aggregate state or the agent is forced to follow the average behavior, but rather the behavior of the agent turns out to be the aggregate behavior, in equilibrium.

Also note that prices (wages, and rental rates of capital) is determined by aggregate capital, rather than individual capital, and since individual takes aggregate state as given, she also takes prices as given (because they are determined by aggregate state). Again, the aggregate capital turns out to coincide with the individual choice, but it is not because of the agent's choice, rather it is the result of consistency.

One notational note. Victor is going to use  $a$  for individual capital and  $K$  for aggregate capital, in order to avoid the confusion between  $K$  and  $k$ . But the problem with aggregate and individual capital is often called as "big-K, small-k" problem, because the difference of aggregate capital and individual capital is crucial. So for our case, the counterpart is "big-K, small-a" problem.

What does the agent has to know in order to solve her problem? Our agent has to know how wealthy she is. She also needs to know the prices but we do not need  $\{R, w\}$  directly. Why? Because they are redundant:  $K$  is a sufficient statistic to calculate  $\{R, w\}$  as they must be equal to marginal products in the firm problem at equilibrium. If we put  $K$  as a state variable instead of these prices, we do not need  $\{R, w\}$ . So are we done? Not yet. As the problem of the HH is formulated below, our agent not only needs to know  $\{R, w\}$  but  $\{R', w'\}$  as well thus  $K'$ . But this is a variable that our agent has no control over and the best she can do is to have a 'belief' about it. These beliefs in our model are parameterized by the  $G^F$  function which maps today's state to a unique belief about next period's value. As it is formulated, it is an exogenous parameter, i.e. we can solve this problem for any sort of beliefs under which the problem is well defined. But from the beginning of this course we want to be able to 'predict' the outcome, once we setup our environment as precise as possible. To continue to be able to do so, as we will see later, we will impose an additional constraint on the beliefs and make them an equilibrium object as well, i.e. endogenize them.

The recursive form of the problem of the agent (that does not value leisure) is:

$$\begin{aligned}
 V(K, a; G^F) &= \max_{c, a'} [u(c) + \beta V(K', a'; G^F)] \\
 \text{s.t.} \quad &c + a' = Ra + w \\
 &R = R(K) = F_K(K, 1) \\
 &w = w(K) = F_N(K, 1) \\
 &K' = G^F(K)
 \end{aligned}$$

Note that  $F$  is indexing the beliefs. This means that for different beliefs  $F$ , different equilibria will arise. Of course, in this course we will only focus on rational expectations equilibria (which we will define below).

How do we deal with the problem written above? The first step is to assume that the solution (for the decision rule) is of the form,  $a' = g(a, K; G^F)$ . So use this fact to write:

$$V(K, a; G^F) = u(R(K)a - g(a, K; G^F)) + \beta V(G^F(K), g(a, K; G^F); G^F) \quad (1)$$

Note that the FOC implies that:

$$-u_c(c) + \beta \frac{\partial V}{\partial a'} = 0 \quad (2)$$

and in order to obtain a more useful expression for the derivative we use the Envelope condition. This is where equation (1) is useful. From (1), take the derivative with respect to  $a$  :

$$\frac{\partial V}{\partial a} = u_c(c) \left[ R(K) - \frac{\partial g}{\partial a} \right] + \beta \frac{\partial V}{\partial a'} \frac{\partial g}{\partial a} = u_c(c) R(K) + \frac{\partial g}{\partial a} \left[ -u_c(c) + \beta \frac{\partial V}{\partial a'} \right],$$

$$\text{so by (2): } \frac{\partial V}{\partial a} = u_c(c) R(K).$$

We will assume that there exist a  $\bar{K}$ , such that  $\bar{K} \geq F(\bar{K}, 1) + (1 - \delta)\bar{K}$ . In words  $\bar{K}$  is the amount of capital stock which is impossible to reproduce. The assumption is that such a value of  $K$  exists.

*Definition 1:*

A Recursive Competitive Equilibrium with arbitrary beliefs  $G^F$  is a list of functions  $\{V^*(.), g^*(.), G^*(.), R(.), w(.)\}$  such that:

- 1) Given  $\{G^F(.), R(.), w(.)\}$ ,  $\{V^*(.), g^*(.)\}$  solves the household problem above,
- 2)  $\{R(.), w(.)\}$  are characterized by the optimal decisions of firms, and
- 3)  $G^*(K; G^F) = g^*(K, K; G^F)$  (agent is representative).

Some comments on the third condition. This condition is called 'Representative agent condition' and is a specific case of the compatibility condition that any equilibrium must satisfy. It basically means that if a consumer turns out to be average this period (her individual capital stock is  $K$ , which is aggregate capital stock), the consumer will choose to be average in the next period (she chooses  $G^*(K)$ , which is a belief on the aggregate capital stock in the next period if today's aggregate capital stock is  $K$ ). This condition guarantees that in an equilibrium, individual choice turns out to be consistent with the aggregate law of motion. This is true not because our agent is constrained to do so but because the prices are such that she chooses to do so.

**Feb 8th, 2007**

*Definition*

A RCE with rational expectations is a list of functions  $\{V^*(.), g^*(.), G^*(.), R(.), w(.)\}$  such that conditions 1) to 3) from the above definition are satisfied, plus:

4)  $G^*(K) = G^F(K)$ , in other words:

A RCE with rational expectations is a list of functions  $\{V^*(.), g^*(.), G^*(.), R(.), w(.)\}$  such that:

- 1) Given  $\{G^*(.), R(.), w(.)\}$ ,  $\{V^*(.), g^*(.)\}$  solves the household problem above,
- 2)  $\{R(.), w(.)\}$  are characterized by the optimal decisions of firms, and
- 3)  $G^*(K; G^*) = g^*(K, K; G^*)$ .

*Comment:* Why do we need the RCE? Isn't it more easy to work with the Social Planner's Problem? The answer is yes. However, there are lots of cases where the solution to SPP does not coincide with that of the Competitive Economy. The most characteristic example is the presence of an externality or a distortionary tax. Moreover, we will also see models where we don't have a representative agent (heterogeneous agents models), and so we don't really know what the SP Problem looks like. In all the above cases the SPP is not helpful anymore, and we have to "attach" the Competitive Equilibrium directly.

The most common way to find and characterize A RCE (given its complexity) is to use computational methods. But we will not cover any of these techniques in 702.

Suppose now that leisure appears in the utility function of the agent. How does the analysis of the previous lecture change? The firm's problem (and thus their optimal choices) surely don't change. However,  $l$  (or equivalently  $n$ , hours worked) will now be a choice variable for the agent. The problem of the agent now is the following:

$$V(K, a; G, H) = \max_{c, l, a'} [u(c, l) + \beta V(K', a'; G, H)]$$

$$s.t: \quad c + a' = Ra + (1 - l)w$$

$$R = R(K) = F_K(K, N)$$

$$w = w(K) = F_N(K, N)$$

$$K' = G(K)$$

$$N = N(K) = H(K)$$

Note that introducing leisure in the utility function created a potential problem: Now  $N$ , the aggregate labor supply is present in the production function. This means that our problem is not well defined (remember: a recursive problem is ill defined if every variable is either a state, or a control, or an explicit function of the above). We overcome this problem by adding the last condition which gives  $N$  as a function of the aggregate state.

*Definition*

A RCE in this context is a list  $\{V^*(.), g^*(.), h^*(.), G^*(.), H^*(.), R(.), w(.)\}$  such that:

- 1) Given  $\{G^*(.), H^*(.), R(.), w(.)\}$ ,  $\{V^*(.), g^*(.), h^*(.)\}$  solves the household problem above,
- 2)  $\{R(.), w(.)\}$  are characterized by the optimal decisions of firms, and
- 3) Agent is representative, which here means:  
 $G^*(K) = g^*(K, K; G^*, H^*)$  and  $H^*(K) = h^*(K, K; G^*, H^*)$ .

*Introducing Government*

The simplest model with government is the one where a constant tax is imposed. Also suppose government spends this money for households.

The recursive formulation of this simple problem is as follows (from now on we will not repeat the  $G$  function as an argument (more precisely an index) of the value function):

$$V(K, a) = \max_{c, l, a'} [u(c, l, p) + \beta V(K', a')]$$

$$\begin{aligned} s.t : \quad & T + c + a' = Ra + wl \\ & R = R(K) = F_K(K, N) \\ & w = w(K) = F_N(K, N) \\ & p = \bar{P} \\ & K' = G(K) \\ & N = N(K) = H(K) \end{aligned}$$

A RCE in this context is a list  $\{V^*(.), g^*(.), h^*(.), G^*(.), H^*(.), R(.), w(.)\}$  such that:

- 1) Given  $\{G^*(.), H^*(.), R(.), w(.)\}$ ,  $\{V^*(.), g^*(.), h^*(.)\}$  solves the household problem above,
- 2)  $\{R(.), w(.)\}$  are characterized by the optimal decisions of firms, and
- 3) Agent is representative, which here means:



$$G^*(K) = g^*(K, K; G^*, H^*) \text{ and } H^*(K) = h^*(K, K; G^*, H^*).$$

4)  $\bar{P} = T$  (Government budget constraint)

Now let's introduce a more interesting model with government. Suppose a constant income tax on wages is imposed. Typically the government returns the earnings in the form of a lump sum tax. But it wouldn't make any difference if the tax earnings were thrown into the ocean:

$$V(K, a) = \max_{c, l, a'} [u(c, n) + \beta V(K', a')]$$

$$s.t : \quad c + a' = (1 - \tau)w(K)n + aR$$

$$R = R(K), w = w(K)$$

$$K' = G(K)$$

$$N = H(K)$$

$$\tau = \tau(K)$$

Here a RCE is a list  $\{V^*(.), g^*(.), h^*(.), G^*(.), H^*(.), \tau^*(.), R(.), w(.)\}$  such that:

- 1) Given  $\{G^*(.), H^*(.), \tau^*(.), R(.), w(.)\}$ ,  $\{V^*(.), g^*(.), h^*(.)\}$  solves the household problem above,
- 2)  $\{R(.), w(.)\}$  are characterized by the optimal decisions of firms,
- 3) Agent is representative:  $G^*(K) = g^*(K, K)$  and  $H^*(K) = h^*(K, K)$ , and
- 4) Government budget constraint is satisfied:

$$\tau^*(K)w^*(K)H^*(K) = P$$

## Feb 13th, 2007

Consider an economy in which there are two types of agents, rich and poor, who differ in their initial wealth. The fraction of rich people in this economy is  $\mu^R$ . Then aggregate state variables are aggregate capital (K) and share of this aggregate capital hold by rich guys (s).

Households solve the following problem:

$$V(K, s, a) = \max_{a', c} u(c) + \beta V(K', s', a)$$

$$c + a' = aR + w$$

$$K' = G^K(K, s)$$

$$s' = G^s(K, s)$$

$$R = R(K)$$

$$w = w(K)$$

$a' = g(K, s, a)$  solves this problem.

A RCE of this economy is a list  $\{V(), g(), G^K(), G^s(), R(), w()\}$  s.t.

1. Given  $\{G^K(), G^s(), R(), w()\}$ ,  $\{V(), g()\}$  solves HH's problem.
2. Prices are characterized by firm's problem:  $R(K) = F_K(K, 1)$ ,  $w(K) = F_L(K, 1)$ .
3. Consistency:

$$G^K(K, s) = \mu^R g(K, s, \frac{sK}{\mu^r}) + (1 - \mu^R) g(K, s, \frac{(1-s)K}{(1-\mu^r)})$$

$$G^s(K, s) = \frac{\mu^R g(K, s, \frac{sK}{\mu^r})}{G^K(K, s)}$$

What else in agents can differ? Agents may have different utility functions (preferences),  $u^i(c)$ . They may differ in patience,  $\beta^i$  and in efficiency endowment of labor,  $\varepsilon^i$ .

*An economy with two countries that have different production functions*

We will assume that there is perfect capital integration, but no labor mobility. Also assume equal size of the two countries. The main difference between this model and the previous one is that now it matters in which country you are. The reason is that there is no labor mobility and hence the wage will be different in these countries. For  $i = A, B$  the recursive specification of the problem is:

$$V_i(a, K^A, K^B, x) = \max_{c, a'} [u(c) + \beta V_i(a', K'^A, K'^B, x')]$$

$$\begin{aligned} & \text{s.t. : } c + a' = Ra + w^i, \\ R &= F_K^A(K^A, 1) = F_K^A(K^A, 1) \text{ (from perfect capital integration)} \\ & w^i = F_N^i(K^i, 1) \\ & K'^A = G^A(K^A, K^B, x) \\ & K'^B = G^B(K^A, K^B, x) \\ & x' = \chi(K^A, K^B, x) \end{aligned}$$

In the problem above we know how much capital there is in each country, but not who owns it (whether it is owned by a country A or B citizen). This is why we introduce the new variable  $x$  which denotes the percentage of capital that belongs to the citizens of country A<sup>7</sup>.

Here a RCE will be as usual a set of functions such that 1) given prices and the laws of motion of the aggregate states agents maximize, 2) firms maximize, plus the following equilibrium conditions:

$$\chi^*(K^A, K^B, x) [g^{A*}(x(K^A + p^*K^B), K^A, K^B, x) = G^{*A}(K^A, K^B, x) + p'^*G^{*B}(K^A, K^B, x)] \quad (1)$$

$$(1 - \chi^*(K^A, K^B, x)) [g^{B*}((1-x)(K^A + p^*K^B), K^A, K^B, x) = G^{*A}(K^A, K^B, x) + p'^*G^{*B}(K^A, K^B, x)] \quad (2)$$

$$R^A(G^{*A}(K^A, K^B, x)) = R^B(G^{*B}(K^A, K^B, x)) \quad (3)$$

$$p^*((K^A, K^B, x)) = 1 \quad (4)$$

Note that  $p$  is the relative price of capital between the countries. We use it to make sure that we don't add "apples" with "oranges". However, by the assumption of perfect capital integration, in equilibrium  $p^* = 1$ .

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<sup>7</sup>In class to overcome this problem we introduced the aggregate variable  $A^1$  instead of  $x$ , where  $A^1$  is the total wealth of people live in country 1.

## Feb 15th, 2007

We will continue in the same spirit as in the previous lecture to define RCE for different environments.

Assume that in the economy of this example the firms own the capital. They also own land and there is no market for this commodity. Moreover, set  $L = 1$ . Of course, the firms are owned by the households. In this setting the firms will have a dynamic problem. This is given by:

$$\begin{aligned}\Omega(K, k) &= \max_{k'} \{F(k, 1) - k' + q(K')\Omega(K', k')\} \\ \text{s.t: } &K' = G(K)\end{aligned}$$

The households problem is:

$$\begin{aligned}V(K, a) &= \max_{a', c} \{u(c) + \beta V[K', a']\} \\ \text{s.t.} & \\ &c + a'q(K') = a \\ &K' = G(K)\end{aligned}$$

Assume that the solutions are of the form  $a' = y(a, K)$ . The equilibrium conditions here are:

$$\Omega() \text{ and } g() \text{ solve firm's problem } (K, 1) \quad (1)$$

$$V() \text{ and } y() \text{ solve HH's problem } (K, 1) \quad (2)$$

$$g(K, K) = G(K) \quad (3) \text{ (firm is representative)}$$

$$y(\Omega(K, k), K) = \Omega(G(K), G(K)) \quad (4) \text{ (household owns the firm and is representative)}$$

$$q^{-1}(K') = F_1(K, 1) \quad (5)^8$$

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<sup>8</sup>We indeed don't need this condition for an equilibrium, equilibrium implies it.

### Government Debt

In this last example we consider a model with government that issues debt and imposes a consumption tax. What are the state variables for this economy?

Aggregate states:  $B, K$ , individual states:  $a$   
The problem of the representative agent is:

$$V(k, B, a) = \max_{c, a'} [u(c) + \beta V(k', B', a')]$$

$$\begin{aligned} s.t : \quad & c(1 - \tau) + a' = w(k) + aR(k) \\ & k' = G(k, B) \\ & B' = H(k, B) \end{aligned}$$

$$\begin{aligned} w(k) &= F_2(k, 1) \\ R(k) &= F_1(k, 1) \\ \tau &= \tau(k, B) \end{aligned}$$

Let the solution have the form  $a' = g(a, k, B)$ .

*DEFINITION:* A Recursive Competitive Equilibrium for policy  $(\tilde{G}, \tau(K, B))$  is a list of functions  $\{V^*(k, B, a), g^*(k, B, a), G^*(k, B), H^*(k, B)$  such that:

1) Given  $G^*(k, B), H^*(k, B), \tau(k, B)$  the functions  $V^*(k, B, a), g^*(k, B, a)$

solve the agent's maximization problem.

2) The Government Budget Constraint is balanced

$$R(k)B + \tilde{G} = H^*(k, B) + \tau(k, B)[F(k, 1) - G^*(k)]$$

3) (Consistency Equilibrium Condition)

$$g^*(k + B, k, B) = G^*(k, B) + H^*(k, B)$$

Note that the unknown functions here are three  $(G^*, H^*, \tau)$ , but the equilibrium conditions are only two (namely (2) and (3) above). What's missing? The problem here is that we don't have a theory for government, i.e, we don't have a theory of how  $\tau$  is specified. In other words, the above equilibrium is indexed by  $\tau$ . Only after specifying the government's objective can we more precisely about how the tax equation looks like. However, we can redefine the above set of RCE by restricting attention only to feasible policies. So we add the following NO Ponzi condition:

4) There exist  $\underline{B}, \bar{B}$  and  $\underline{k}, \bar{k}$  such that for every  $(k, B) \in [0, \bar{k}] \times [\underline{B}, \bar{B}]$ ,  $G(k, B) \in [0, \bar{k}]$  and  $H(k, B) \in [\underline{B}, \bar{B}]$ .

**Feb. 20th, 2007**

## 7 Recursive, Stochastic Growth Model

In previous sections, we discussed random shocks, stochastic processes and histories. Now we will introduce those concepts into the growth model and analyze the recursive formulation.

In the growth model, the aggregate state variables are  $\{z, K\}$ , the technology shock and the total amount of capital. The individual state variables are  $\{a, b_z\}$ , the amount of assets of the household and the amount of state contingent claims (also known as Arrow securities<sup>9</sup>)

The recursive stochastic growth model is

$$V(z, K, a, b_z) = \max_{c, a', b_{z'}} [u(c) + \beta \sum_{z'} \Gamma_{zz'} V(z', K', a', b_{z'})]$$

st

---

<sup>9</sup>These securities pay 1 unit of the good when a particular state happens, and zero otherwise

$$c + a' + \sum_{z'} q^{z'}(z, K) b_{z'} = w(z, K) + R(z, K) a + b_z$$

$$K' = G(z, K)$$

where  $q^{z'}(z, K)$  is the price of the state contingent claim related to state  $z'$  tomorrow. The solution to this problem for the household are functions that relate state variables to optimal asset accumulation

$$a' = g(z, K, a, b_z)$$

and to optimal security consumption for all future states

$$b_{z'}(z') = b(z, K, a, b_z)(z')$$

Looking at equilibrium conditions, we find that this way of writing the stochastic growth model has an 'overkill'. In equilibrium we get that

$$G(z, K) = g(z, K, K, 0)$$

$$b(z, K, a, 0)(z') = 0 \quad \forall z'$$

the first condition is just the representative agent condition we have seen before. The second condition says that state contingent securities are not traded, so their demand is zero for all states of the world. This is due the fact that we are in a world of identical agents without uninsurable risks. Another way of looking at this is to use the no-arbitrage condition

$$1 = \sum_{z'} q(z, K) R(z', G(z, K))$$

which says that the household can save equally by buying securities or saving through capital.

Finally, we can reduce the individual state space for the household and rewrite the model as follows:

$$V(z, K, a) = \max_{c, a'} [u(c) + \beta \sum_{z'} \Gamma_{zz'} V(z', K', a')]$$

st

$$\begin{aligned}c + a' &= w(z, K) + R(z, K) a \\ K' &= G(z, K)\end{aligned}$$

This is because the household can secure herself a unit of consumption for sure next period either by saving or having a portfolio that pays 1 unit for sure next period at each possible state. Since the last option is an overkill, we drop it in order to work with our usual formulation.

Again, a solution to this problem is an optimal policy for asset accumulation  $a' = g(z, K, a)$

**Definition 7** *A RCE with stochastic shocks is a list  $\{V, G, g, w, R\}$  such that*

1. *Given  $\{G, R, w\}$ ,  $V$  and  $g$  solve the consumer problem*
2.  *$R$  and  $w$  solve the firm's problem*
3. *Representative agent condition is satisfied, i.e.*

$$g(z, K, K) = G(z, K)$$

## 8 Lucas Tree Model (Lucas 1978)

### 8.1 The Model

Suppose there is a tree which produces random amounts of fruit every period. We can think of these fruits as dividends and use  $d_t$  to denote the stochastic process of fruit production. Further, assume  $d_t$  follows a Markov process. Formally:

$$d_t \sim \Gamma(d_{t+1} = d_i \mid d_t = d_j) = \Gamma_{ji} \quad (22)$$

Let  $h_t$  be the history of realization of shocks, i.e.,  $h_t = (d_0, d_1, \dots, d_t)$ . Probability that certain history  $h_t$  occurs is  $\pi(h_t)$ .



The representative household in the economy consumes the only good, which is the fruit. Consumers maximize:

$$\sum_t \beta^t \sum_{h_t \in H_t} \pi(h_t) u(c_t) \quad (23)$$

Since we are assuming a representative agent in the economy who possesses no storage technology, in the unique equilibrium the representative household eats all the dividends every period. Hence, the lifetime utility of the household will be:

$$\sum_t \beta^t \sum_{h_t \in H_t} \pi(h_t) u(d_t) \quad (24)$$

with

$$p_0 = 1 \quad (25)$$

Note that we are considering the Arrow-Debreu market arrangement, with consumption goods in period 0 as a numeraire.

## 8.2 First Order Condition

Take first order condition of the above maximization problem:

$$\frac{p(h_t)}{p_0} = p_t(h_t) = \frac{\beta^t \pi(h_t) u'(c(h_t))}{u'(c(h_0))} \quad (26)$$

By combining this FOC with the following equilibrium condition:

$$c(h_t) = d_t \quad \forall t, h_t \quad (27)$$

We get the expression for the price of the state contingent claim in the Arrow-Debreu market arrangement.

$$p_t(h_t) = \frac{\beta^t \pi(h_t) u'(d(h_t))}{u'(d(h_0))} \quad (28)$$

### 8.3 LT in Sequential Markets

In sequential markets, the household can buy and sell fruits in every period, and the tree (the asset). To consider the trade of the asset, let  $s_t$  be share of asset and  $q_t$  be the asset price at period  $t$ . The budget constraint at every time-event is then:

$$q_t s_{t+1} + c_t = s_t(q_t + d_t) \quad (29)$$

Thus, the consumer's optimization problem turns out to be:

$$\max_{\{c_t(h_t), s_{t+1}(h_t)\}_{t=0}^{\infty}} \sum_t \beta^t \sum_{h_t \in H_t} \pi(h_t) u(c_t(h_t)) \quad (30)$$

subject to

$$q_t(h_t) s_{t+1}(h_t) + c_t(h_t) = s_t(h_{t-1}) [q_t(h_t) + d_t] \quad (31)$$

Feb. 22nd, 2007

## 8.4 Lucas Tree in Recursive Language

Looking at the recursive version of the same problem with denoting discrete state variable as subscripts (a note on notation: this is the same as having  $V(d, s)$ , but since the amount of fruit is linked one to one to the shock, we can drop  $d$  and use the state as a subscript)

$$V_i(s) = \max_{s', c} u(c) + \beta \sum_{d'} \Gamma_{ij} V_j(s')$$

$$s.t. \quad c + s'q_i = s[q_i + d_i]$$

In equilibrium, the solution has to be such that  $c = d$  and  $s' = 1$ . Impose these on the FOC and get the prices that induce the agent to choose that particular allocation. Then the FOC for a particular state  $i$  would imply,

$$q_i = \beta \sum_j \Gamma_{ij} \frac{u'(d_j)}{u'(d_i)} [q_j + d_j] \quad (32)$$

where

$$q_i = \frac{p(h_{t-1}, d_i)}{p(h_t)}$$

and  $p(\cdot)$  are the prices we derived from the AD setting.

A closer look tells us that we can calculate all prices in just one system of equations. Taking FOCs, at an equilibrium we have

$$p_i u_c(d_i) + \beta \sum_j \Gamma_{ij} \frac{\partial V^j(s')}{\partial s'} = 0$$

using the envelope condition

$$\frac{\partial V^j(s')}{\partial s'} = [p_j + d_j] u_c(d_j)$$

hence,

$$p_i u_c(d_i) = \beta \sum_j \Gamma_{ij} [p_j + d_j] u_c(d_j) \quad \forall i$$

Stacking each equation and forming matrices

$$\begin{bmatrix} u_c(d_1) & 0 & \cdots & 0 \\ 0 & u_c(d_2) & \ddots & \vdots \\ 0 & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & u_c(d_{n_d}) \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_{n_d} \end{bmatrix} = \beta \begin{bmatrix} u_c(d_1) & 0 & \cdots & 0 \\ 0 & u_c(d_2) & \ddots & \vdots \\ 0 & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & u_c(d_{n_d}) \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_{n_d} \end{bmatrix} + \beta \Gamma \begin{bmatrix} u_c(d_1) & 0 & \cdots & 0 \\ 0 & u_c(d_2) & \ddots & \vdots \\ 0 & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & u_c(d_{n_d}) \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{n_d} \end{bmatrix}$$

In matrix notation

$$\begin{aligned} p &= \beta u_c^{-1} \Gamma u_c p + \beta u_c^{-1} \Gamma u_c d \\ [I - \beta u_c^{-1} \Gamma u_c] p &= \beta u_c^{-1} \Gamma u_c d \\ p &= [I - \beta u_c^{-1} \Gamma u_c]^{-1} [\beta u_c^{-1} \Gamma u_c d] \end{aligned}$$

## 8.5 Asset Pricing

Because in a complete market any asset can be reproduced by buying and selling contingent claims at every node, we can use this model as a powerful asset pricing formula. For example, take the option of selling shares at price  $\bar{p}$  tomorrow. Since tomorrow we'll have the option to sell, we exercise only if

$$\bar{p} - p_i > 0 \quad \forall i$$

Then, the value of this option (if we are in state  $i$ ), is

$$\varphi_i(\bar{p}) = \sum_j q_{ij} \max\{\bar{p} - p_j, 0\}$$

where  $q_{ij} = \beta \Gamma_{ij} u_c(d_j) [u_c(d_i)]^{-1}$ .

Our next example is the option that can only be executed 2 periods from now. In that case, we have

$$\varphi_i^2(\bar{p}) = \sum_j q_{ij} \sum_l q_{jl} \max\{\bar{p} - p_l, 0\}$$

Finally, take the option that can be exercised tomorrow or the day after tomorrow. The day after tomorrow, we exercise the option iff

$$\bar{p} - p_l > 0$$

where  $l$  is the state the day after tomorrow. At the previous node (if we haven't exercised the option yet and the state is  $j$ ), the value of the option is

$$\sum_l q_{jl} \max \{ \bar{p} - p_l, 0 \}$$

Hence, if the state today is  $i$ , the value of the option is

$$\hat{\varphi}_i^2(\bar{p}) = \sum_j q_{ij} \max \left\{ \bar{p} - p_j, \sum_l q_{jl} \max \{ \bar{p} - p_l, 0 \} \right\}$$

## 8.6 Aggregate Rates of Return

Given the notation and the previous analysis, we can ask some questions on rates of return in the model economy. On the one side, consider an asset that gives a certain return next period, risk free (e.g., a treasury bond, that pays 1 no matter the state tomorrow) which we can denote as  $r^f$ . Using the no-arbitrage condition we know that:

$$\begin{aligned} 1 + r^f &= \left[ \sum_j q_{i,j} 1 \right]^{-1} \\ &= \left[ \sum_j \beta \Gamma_{ij} \frac{u_c[d_j]}{u_c[d_i]} \right]^{-1} \end{aligned}$$

On the other hand, consider a risky asset, which pays proportionally to the aggregate state of the economy tomorrow (fruit is  $d_j / \sum \Gamma_{ij} d_j$ ). Denote its return as  $r^R$ . By the same argument as before, we have that

$$\begin{aligned}
1 + r^R &= \left[ \sum_j q_{i,j} \frac{d_j}{\sum \Gamma_{ij} d_j} \right]^{-1} \\
&= \left[ \sum_j \beta \Gamma_{ij} \frac{u_c[d_j]}{u_c[d_i]} \frac{d_j}{\sum \Gamma_{ij} d_j} \right]^{-1}
\end{aligned}$$

By Jensen's inequality

$$\sum_j \beta \Gamma_{ij} \frac{u_c[d_j]}{u_c[d_i]} > 1$$

which means that  $1 + r^f < \frac{1}{\beta}$

Also,

$$\sum_j \beta \Gamma_{ij} \frac{u_c[d_j]}{u_c[d_i]} > \sum_j \beta \Gamma_{ij} \frac{u_c[d_j]}{u_c[d_i]} \left( \frac{d_j}{\sum \Gamma_{ij} d_j} \right)$$

Hence,  $r^f < r^R$ , or the equity premium. The intuition is that in a world with risk averse individuals, the demand for risk free assets is higher, which drives its price (today) upwards and then its return downwards.

If we go to the data, we find a BIG equity premium, which cannot be accounted by the model in any way. This is called the equity premium puzzle. For further discussion, see Mehra and Prescott (1985)

**Feb. 27th, 2007**

## 9 Economy with Heterogeneous Agents

### 9.1 Introduction

So far, in environments we have analyzed, the type of agents is the same always. If the number of type of agents is small (as the example we did in class with only two different types) it's easy to keep track of all the types, and so is to define an equilibrium. From now on, we will consider economies with (i) many agents who are very different among themselves at a given time (cross-section), and (ii) change their types over time.

An immediate question is: what is a considered MANY agents?

- $N$  large, countable
- a continuum, uncountable and infinite

We will use models with continuum of agents, since we can use math tools (such as measure theory) and other useful concepts (like continuity) to describe agents.

## 9.2 Introduction to Measure Theory

### 9.2.1 Intuition

Measure theory can be understood nicely by comparing to the notion of weight. Measure is about "measuring" a mass in a mathematically consistent way, which is similar to weighting a mass. Therefore, intuitively the following properties are expected to be satisfied by measures:

1.  $measure(nothing) = 0$
2. if  $A \cap B = \emptyset \Rightarrow measure(A + B) = measure(A) + measure(B)$

These properties are intuitive with weight. The weight of nothing is zero. If a body is 200 pounds, and you chop off a hand from the body and put the hand and the rest of the body together on the scales, they must weight 200 pounds. Now consider an economy with many agents. The measure of nobody in the economy is zero. If a measure of the total population is normalized to one, and you take away the rich people from the population and measure the sum of rich people and the rest of the population, they must have measure one.

In macro models with heterogenous agents, we are interested in how to measure agents with different characteristics (wealth, earnings, etc.).

### 9.2.2 Definitions

**Definition 8** For a set  $S$ ,  $\mathcal{S}$  is a set of subsets of  $S$ .

**Definition 9**  $\sigma$ -algebra  $\mathcal{S}$  is a set of subsets of  $S$ , with the following properties:

1.  $S, \emptyset \in \mathcal{S}$

2.  $A \in \mathcal{S} \Rightarrow A^c \in \mathcal{S}$  (closed in complementarity)
3. for  $\{B_i\}_{i=1,2,\dots}$ ,  $B_i \in \mathcal{S} \Rightarrow [\cap_i B_i] \in \mathcal{S}$  (closed in countable intersections)

The intuition of the property 2 of  $\sigma$ -algebra is as follows. If we chop off a hand from a body, and if the hand is an element of  $\mathcal{S}$ , the rest of the body is also an element of  $\mathcal{S}$ . Soon we will define measure as a function from  $\sigma$ -algebra to a real number. Then the property of  $\sigma$ -algebra implies that if we can measure the chopped hand, we can measure also the rest of the body.

Examples of  $\sigma$ -algebra are the follows:

1. Everything, aka the power set (all the possible subsets of a set  $S$ )
2.  $\{\emptyset, S\}$
3.  $\{\emptyset, S, S_{1/2}, S_{2/2}\}$  where  $S_{1/2}$  means the lower half of  $S$  (imagine  $S$  as an closed interval on  $\mathcal{R}$ ).

If  $S = [0, 1]$  then the following is NOT a  $\sigma$ - algebra

$$\mathcal{S} = \left\{ \emptyset, [0, \frac{1}{2}), \left\{ \frac{1}{2} \right\}, \left[ \frac{1}{2}, 1 \right], S \right\}$$

**Remark 2** A convention is (i) use small letters for elements, (ii) use capital letters for sets, (iii) use "fancy" letters for set of subsets.

**Definition 10** A measure is a function  $x : \mathcal{S} \rightarrow \mathcal{R}_+$  such that

1.  $x(\emptyset) = 0$
2. if  $B_1, B_2 \in \mathcal{S}$  and  $B_1 \cap B_2 = \emptyset \Rightarrow x(B_1 \cup B_2) = x(B_1) + x(B_2)$
3. if  $\{B_i\}_{i=1}^{\infty} \in \mathcal{S}$  and  $B_i \cap B_j = \emptyset$  for all  $i \neq j \Rightarrow x(\cup_i B_i) = \sum_i x(B_i)$  (countable additivity)

In English, countable additivity means that measure of the union of countable disjoint sets is the sum of the measure of these sets.



**Definition 11** *Borel- $\sigma$ -algebra is (roughly) a  $\sigma$ -algebra which is generated by a family of open sets (generated by a topology).*

Since a Borel- $\sigma$ -algebra contains all the subsets generated by intervals, you can recognize any subset of a set using Borel- $\sigma$ -algebra. In other words, Borel- $\sigma$ -algebra corresponds to complete information.

You might find that a  $\sigma$ -algebra is similar to a topology. Topology is also a set of subsets, but its elements are open intervals and it does not satisfy closedness in complementarity (complement of an element is not an element of a topology). Very roughly, the difference implies that topologies are useful in dealing with continuity and  $\sigma$ -algebra is useful in dealing with measure.

**Definition 12** *Probability (measure) is a measure such that  $x(A) = 1$*

Lets apply these basic notions of measure theory to a simple set of problems: industry equilibria with many firms

## 10 Partial Equilibrium Industry Theory

We will consider models where prices are given exogenously, hence our analysis is a partial equilibrium one. For that reason, think that the environment is a small industry producing "flip flops" (things nobody cares about).

There is an inverse demand function  $y^d(p)$ , where  $p$  is the price of the good. A firm in this industry is indexed by its productivity  $s \in S = [\underline{s}, \bar{s}]$  and produces according to  $sf(n)$  (the production function depends on labor only). Firms are competitive in the output market as well as in the labor market.

Problem of the firm is

$$\max_n psf(n) - wn$$

from the FOC, we get  $psf'(n) = w$ , which implicitly defines the solution of the firm  $n^* = n(s, p)$ . The profits are defined

$$\Pi(s, p) = psf(n^*[s, p]) - wn[s, p]$$

Given a price  $p$ , to calculate the output of the industry we need a measure  $X$  of firms (the distribution of firms according to their  $s$ ).

Let

- $\mathcal{S}$  be the Borel  $\sigma$ -algebra of  $[\underline{s}, \bar{s}]$
- $X : \mathcal{S} \rightarrow \mathbf{R}$  be a measure

Then, the supply of the industry is defined as

$$y^s(p) = \int_{\underline{s}}^{\bar{s}} sf(n[s, p]) X(ds)$$

In this economy, the measure of firms with respect to its type  $X(s)$  is an uninteresting object since its given exogenously. We need to add some tweaks in order to have some economics.

## 11 Mar. 1st, 2007

Now suppose that the firm will only operate next period with probability  $(1 - \delta)$ . With probability  $\delta$  it will die. In that case, the two period profit of the firm is,

$$\pi_2 = [p^* s f(n^*) - w n^*] \left[ 1 + \frac{1 - \delta}{1 + r} \right]$$

Now consider the infinite periods profit of the firm,

$$\begin{aligned} \pi_\infty &= [p^* s f(n^*) - w n^*] \sum_{t=0}^{\infty} \left( \frac{1 - \delta}{1 + r} \right)^t \\ &= [s f(n^*) - w n^*] \left( \frac{1 + r}{r + \delta} \right) \end{aligned}$$

The zero profit condition is that the profit from entry is equal to the cost of entry, denoted by  $c_e$ . This condition says that there are no further incentives to enter the industry:

$$c_e = \pi_\infty$$

Define  $x : \mathcal{S} \rightarrow R$  as the measure of firms, where  $\mathcal{S}$  is the  $\sigma$ -algebra defined on the set  $S$

An industry equilibrium is a set  $\{p^*, y^*, n^*, x^*(s)\}$ , such that:

- 1)  $p^* = p(y^*)$  (demand is satisfied)
- 2)  $y^* = x^*(s, p^*)$   $s f(n^*)$  (feasibility)
- 3) Firms optimize:  $n^* \in \arg \max_n p^* s f(n) - w^* n$
- 4) Zero profit condition:  $c_e = \pi_\infty$

So far we still have no interesting dynamics. The measure of firms is exogenous and no economic decisions are involved. In order to introduce more interesting elements, we need some more mathematical tools:

## 11.1 Digression on Transitions and Updating Operators

**Definition 13** A function  $f : S \rightarrow R$  is measurable with respect to the  $\sigma$ -Algebra  $\mathcal{S}$ , if for every  $a \in R$ ,  $B \equiv \{s \in S : f(s) \leq a\} \in \mathcal{S}$ .

**Definition 14** A transition function is a mapping  $Q : S \times \mathcal{S} \rightarrow [0, 1]$ , such that:

- 1)  $Q(s, \cdot)$  is a probability measure for every  $s \in S$ .
- 2)  $Q(\cdot, B)$  is a measurable function with respect to  $\mathcal{S}$ , for every  $B \in \mathcal{S}$ .

Based on the transition function  $Q$  we can define an updating operator  $T$ . This new object satisfies  $x'(B) = T(x(B), Q)$ . In words,  $T$  gives us the measure of (here) firms in the subset  $B$  in the next period ( $x(B)$ ), based on the transition function and the measure of firms in this subset today.

## 11.2 Industry Equilibria, Tweak #1

Suppose that each firm has to pay a cost of entry  $c_\infty$ , and the productivity shock is drawn from the distribution  $\gamma(s)$ . Once the firm draws  $s$  it keeps it forever.

An industry equilibrium is a set  $\{p^*, n^*(s, p^*), N_e^*, x^*\}$ , such that:

- 1)  $n^*(s, p^*) \in \arg \max [p^* s f(n) - wn]$
- 2)  $y^D(p^*) = \int_S s f(n^*(s, p^*)) dx^*$
- 3)  $c_\infty = \int_S \hat{\Pi}_\infty(s, p^*) d\gamma(s)$ , where  $\hat{\Pi}_\infty = \sum_{t=0}^{\infty} \left(\frac{1-\delta}{1+r}\right)^t \Pi(s, p^*)$ .
- 4)  $x^*(B) = (1 - \delta) x^*(B) + N_e^* \gamma(B)$ , for all  $B \in \mathcal{S}$ , and
- 5)  $N_e^* = \delta x^*(S)$ .

Note that here, the distribution of firms completely reflects the distribution from which they draw their productivity shocks,  $\gamma(s)$ . This is because what types of firms remain or what types of firms exit is not an issue since there is exogenous entry and exit. For example, if exit was endogenous we would expect the 'bad' firms to exit and the better ones to stay, and therefore

the type distribution of incumbent firms would be different than the initial distribution  $\gamma(s)$ . But in our case, the distribution of incumbents and the initial type distribution are identical.

So this model is not interesting because it STILL has no economics.

### 11.3 Ind. Eq., Tweak #2

Here  $s$  is drawn from  $\gamma(s)$  as before, but after the initial shock is obtained,  $s' \sim \Gamma_{ss'}$ . We will assume that  $\Gamma$  satisfies First Order Stochastic Dominance. This means that

$$\text{For } s_1, s_2 \in S, \quad s_1 < s_2 \Rightarrow \int_{\bar{s}}^{\bar{s}} \Gamma(s_1, s) ds \leq \int_{\bar{s}}^{\bar{s}} \Gamma(s_2, s) ds$$

Moreover, we introduce a fixed cost that the firm has to pay in every period,  $c_f$ .

The recursive formulation of the firm's problem is:

$$\Omega(s) = \max \left[ 0, -c_f + \max_n \{ p s f(n) - w n \} + \frac{1}{1+r} \int \Omega(s') \Gamma(ds' | s) \right]$$

**Definition 15** *A stationary equilibrium for the economy described by Tweak 2 is a list*

$\{p^*, N_e^*, s^*, n^*(s, p^*), x^*, Q^*\}$  *such that:*

- 1)  $n^*(s, p^*)$  maximizes profits and  $s^* = s^*(p^*)$  (threshold is optimal).
- 2) Free entry is satisfied:  $c_\infty = \Pi_\infty$ .
- 3) Market clearing:  $y^D(p^*) = \int_S s f(n^*(s, p^*)) dx^*$ .
- 4) The measure of firms is stationary:  $x^*(B) = \int_S Q(s, B) dx^* + N_e^* \gamma(B \cap [s^*, \bar{s}])$
- 5)  $Q(s, B) = \Gamma \left[ s, B \mid [s, s^*] \right]$ , (where  $\Gamma$  is implied by the Markovian transition matrix).

**Mar. 15th, 2007**

### 11.4 Ind. Eq. Tweak #3

In this section we discuss Industry equilibria with employment and capital as a state variables. The firm owns capital and rents labor in a competitive market. The key ingredient here is that there are costs of adjusting the level of capital and employment. Hence, the state variables for a particular firm are  $\{s, k, n\}$ . The recursive problem of a firm is given by

$$\begin{aligned} \Omega(s, k, n) &= \max_{x_k, x_n} p s f(k, n) - w n - x_k - x_n + \frac{1}{1+r} \sum_{s'} \Gamma_{ss'} \Omega(s', k', n') \\ &s.t. \\ k' &= (1-\delta)k + \phi(x_k) \\ n' &= (1-\rho)n + \psi(x_n) \\ s' &\sim \Gamma \quad (Markov) \end{aligned}$$

where  $\delta$  is the depreciation rate and  $\rho$  is the exogenous separation rate. Functions  $\phi, \psi$  represent non-linear adjustment costs of modifying employment/capital levels. Some assumptions:

- $\phi(0) = \psi(0) = 0$
- $\phi', \psi' \in [0, 1]$
- $\phi'', \psi'' < 0$  (convex costs)

The solution to this problem are two policy functions for  $x_k^* = x_k^*(s, k, n)$  and  $x_n^* = x_n^*(s, k, n)$

In this economy, the transition function is<sup>10</sup>

$$Q(\{s, k, n\}, \mathcal{B}) = \sum_{s' \in B_s} 1_{\{\phi(x_k(s, k, n)) + (1-\delta)k \in B_k\}} 1_{\{\psi(x_n(s, k, n)) + (1-\rho)n \in B_n\}}$$

where  $1_{\{\cdot\}}$  is the indicator function. Then, the updating operator is given by

---

<sup>10</sup>In order to ease notation, here we take the case where  $B_s, B_k, B_n$  are rectangles

$$X' = \int_s \int_k \int_n Q(\{s, k, n\}, \mathcal{B}) X(ds, dk, dn)$$

Note that in this model, all firms have value, since there is no endogenous exit by assumption. Some key predictions of the model:

- when a firm receives a good shock, employment and capital react sluggishly, since its costly to reverse levels
- Aggregate production is lower than in the previous models

**March 20th, 2007**

### Goat Farmers Economy

Imagine a economy that has a continuum of goat farmers. The goat farmers get an endowment  $s$  each period which follows a Markov process with transition  $\Gamma_{ss'}$  and,

$$s \in \{s^1, \dots, s^{n_s}\}$$

There is a storage technology such that, if the goatfarmer store  $q$  units of goats today, they get 1 unit of goat tomorrow.  $(s, a)$  is the type of a goat farmer and the set consisting of all possible such pairs is,

$$S \times A = \{s^1, s^2, \dots, s^n\} \times [\underline{a}, \bar{a}]$$

Let  $\mathcal{A}$  be the set of Borel sets on  $S \times A$ . And define a probability measure  $x$  on  $\mathcal{A}$ ,

$$x : \mathcal{A} \rightarrow [0, 1]$$

The goat farmer's problem is:

$$V(s, a) = \max_{c, a' \geq 0} u(c) + \beta \sum_{s'} \Gamma_{ss'} V(s', a')$$

subject to

$$\begin{aligned} c + qa' &= e + a \\ c \geq 0 \quad \text{and } a' &\in [0, \bar{a}] \end{aligned}$$

With the decision rule  $a' = g(e, a)$  and the transition matrix for the endowment process  $\Gamma_{ss'}$

The First Order Conditions are,

$$u_c(s + a - qa') = \frac{\beta}{q} \sum_{s'} \Gamma_{ss'} u_c(s' + a' - q'a'')$$

You'll notice that  $a' \in [0, \bar{a}]$  is already one of the constraints of the above maximization problem. But now rather than just imposing such a constraint, we will find a natural reason that savings should have a lower bound and we will consider a condition that ensures an upper bound for savings.

For the lower bound, we assume that there is no technology which allows negative amount of saving and this sounds natural since storing a negative amount of goat does not make much sense. So savings has a lower bound because Mother Nature says so.

Here, the goat farmer has the risk of getting a very bad shock tomorrow. So the goat farmer would save just in case he has this bad shock; he would want to store some fish today in order to insure himself against getting very small number of goat tomorrow so he is not hungry in case that happens. In this case we need to think more about how to put an upper bound on savings, because with uncertainty even if  $\beta < q$ , the fisherman is willing to save due to gains from insurance. The kind of savings to protect oneself from risk in the future in the absence of state contingent commodity markets which can be used to insure against any contingency to make sure consumption is constant across states, which is usually called precautionary savings. In order to ensure an upper bound for savings, we need to bound the gains from insurance somehow. The way to do this is to impose the condition on the utility function that its negative curvature (keeping in mind that the utility function is concave) is diminishing as wealth increases. This means that wealthier agents are less risk-averse. Formally, that  $u'$  is convex. The wealthier the agent is, the smaller the variance of his endowment next period proportional to his wealth so he doesn't want to save if he is very wealthy. This is simply because of the fact that the wealth is not subject to any uncertainty but income is thus as the income wealth ratio rise, the overall uncertainty the agent faces diminishes.

So in the economy with uncertainty, in order to have an upper bound on savings, we need the first derivative of the utility function to be convex so



that the following Jensen's Inequality holds:

$$\frac{\beta}{q} \int \Gamma_{ss'} u_c(c') > \frac{\beta}{q} u_c \left( \int \Gamma_{ss'} c' \right)$$

**Theorem:** If  $\beta < q$  and  $u'$  is convex then  $\exists \bar{a}$  such that  $a_0 < \bar{a}$ ,  $g(s,a) < \bar{a}$   $\forall s$ .

Let  $a' = g(s, a)$  solves the goat farmer's problem.  $g(s, a)$  is monotone in  $a$  and if stochastic dominance (i.e. exogenous Markov chain doesn't feature negatively correlated income) holds then  $g(s, a)$  is also monotone in  $a$ .

The transition function for this economy is:

$$Q((s, a), B) = \sum_{s' \in B_s} \Gamma_{ss'} 1_{\{g(s,a) \in B_a\}}$$

and the updating operator is:

$$x'(B) = \sum_{s' \in B_s} \int_{S \times A} \Gamma_{ss'} 1_{\{g(s,a) \in B_a\}} dx(s, a)$$

**Definition:** An equilibrium for goat farmers' economy is a list  $\{V(\cdot), g(\cdot)\}$  s.t. given  $q$ ,  $\{V(\cdot), g(\cdot)\}$  solves an individual goat farmer's problem.

**Definition:** An stationary equilibrium for goat farmers' economy is a list  $\{V(\cdot), g(\cdot), x^*(\cdot)\}$  s.t.

1. Given  $q$ ,  $\{V(\cdot), g(\cdot)\}$  solves an individual goat farmer's problem.
2.  $x^*$  is a stationary distribution:

$$x^*(B) = \sum_{s' \in B_s} \int_{S \times A} \Gamma_{ss'} 1_{\{g(s,a) \in B_a\}} dx^*(s, a)$$

In this case idiosyncratic shocks affect income (endowment). How else could we model economies with idiosyncratic shocks of households:

- Agents with different discount factors, i.e.  $\beta(s)$ .
- Agents with different returns of storage, i.e.  $q(s)$ .
- Agents with different utility functions, i.e.  $\frac{c^{1-\sigma(s)}-1}{1-\sigma(s)}$

# March 22th, 2007

## Huggett Economy

Now consider the case of lower bound for saving less than zero. Suppose we let the goat farmers borrow and lend to each other but not store any goats, how can we make sure that our agents always has the capability to pay back what they owe. What would be the endogenous lower bound to ensure this? Such a condition would make sure that in the worst case scenario our agent should be able pay the interest rate on its debt and roll over the same amount (the lowest possible amount). Thus, letting the lower bound be  $\underline{a}$  and the lowest possible shock be  $\underline{s}$  then,

$$\begin{aligned} 0 + qa &= \underline{a} + \underline{s} \\ \underline{a} &= \frac{\underline{s}}{q-1} \end{aligned}$$

This is called the solvency constraint rather than a borrowing constraint. Note that when we let the fishermen to get into lending contracts with each other, we need a consistency condition to make sure agents actions are compatible with each other. Here the price will be endogenously determined will ensure the agents hold just the right amount of assets.

The stationary equilibrium of such an economy is defined as,

**Definition:** A stationary equilibrium for an Huggett(1993) economy is a set  $\{q^*, x^*(q^*), Q(s, a, B; q^*), g(s, a; q^*)\}$  such that

1. (Agent Optimization) Given  $q^*$ ,  $g(s, a; q^*)$  solves the agent's problem.
2. (Consistency)  $Q(s, a, B; q^*)$  is a transition matrix associated with  $\Gamma_{ss'}$  and  $g(s, a; q^*)$ .
3. (Stationarity)  $x^*$  is the unique stationary distribution associated with  $Q(s, a, B; q^*)$ , that is  $x^* = T(x^*, Q)$ .

4. (Market clear)

$$\int a dx^*(q^*) = 0$$

(Please note that  $g(e, \bar{a}) = \bar{a}$ , and that  $g(e, a)$  never crosses the 45 degree line again (here  $s = e$  stands for the state of employment, i.e. the good state).)

## Aiyagari Economy

Finally we incorporate growth in the Huggett economy setting. Here we still have heterogeneity and each agent obtains an idiosyncratic shock,  $s$ . We can think of this as a shock on the efficiency of labor provided by that particular agent. The difference is that now the agent can work and also rent his capital ( $a$ ) to firms at the market price  $R$ :

$$\begin{aligned}
 V(s, a, \frac{K}{N}) &= \max_{a', c, l} u(c, l) + \beta \sum_{s'} \Gamma_{ss'} V(s', a', \frac{K'}{N'}) \\
 c + a' &= (1 - l)ws + (1 + r)a \\
 w &= W(\frac{K}{N}), \quad r = R(\frac{K}{N}) \\
 \frac{K'}{N'} &= \Phi(\frac{K}{N}) \\
 a' &\in (\underline{a}, \bar{a}), \quad c \geq 0 \\
 a' &= g(s, a, \frac{K}{N}), \quad l = h(s, a, \frac{K}{N}) \text{ solve the HH problem}
 \end{aligned}$$

**Definition:** A stationary equilibrium for the Aiyagari economy is a list  $\{V(\cdot), g(\cdot), h(\cdot), R(\cdot), W(\cdot), \Phi(\cdot), \{x^*\}$  such that

1. Given  $\{R(\cdot), W(\cdot), \Phi(\cdot)\}$ ,  $\{V(\cdot), g(\cdot), h(\cdot)\}$  solves the HH problem.
2.  $R(\frac{K}{N}) = \theta(\frac{K}{N})^{\theta-1}$ ,  $W(\frac{K}{N}) = (1 - \theta)(\frac{K}{N})^\theta$
3.  $K = \int g(a, s, \frac{K}{N}) dx^*(a, s)$ ,  $N = \int (1 - h(a, s, \frac{K}{N})) dx^*(a, s)$ ,  $\frac{K}{N} = \Phi(\frac{K}{N})$
- 4.

$$Q((s, a), B) = \sum_{s' \in B_s} \Gamma_{ss'} 1_{\{g(s, a) \in B_a\}};$$

and  $x^*$  is a stationary distribution:

$$x^*(B) = \sum_{s' \in B_s} \int_{S \times A} \Gamma_{ss'} 1_{\{g(s, a) \in B_a\}} dx^*(s, a)$$

**March 27th, 2007**

## **Technology Improvements in the Goat Farmers Economy (Transitions-simple)**

In the model we were studying, we get a stationary distribution  $X^*$  for the assets of all goat farmers. This equilibrium object clearly depends on the 'fundamentals' of the economy, namely  $\{\beta, \Gamma, q, u(c)\}$ . In general, the problem is defined as

$$V(s, a; q) = \max_{a'} \left\{ u[c] + \beta \sum_{s'} \Gamma_{ss'} V(s', a'; q) \right\}$$

subject to

$$c + qa' = s + a$$

with solution  $y(s, a; q)$

Note that we are showing explicitly the importance of  $q$  in the value functions. Now imagine that the economy is in a steady state with some  $q^0$ . We would like to know what happens if unexpectedly the cost of storage decreases permanently from  $q^0$  to  $q^1$ . In a first step, we can solve the above problem with this new  $q^1 < q^0$  and get a new policy function  $y(s, a; q^1)$ . Applying the updating operator for this new economy

$$T[X^*(q^0), y(s, a; q^1)] \neq X^*(q^0)$$

Clearly, since a fundamental of the economy has changed, the distribution of the fishermen will move away from  $X^*(q^0)$  into a new stationary distribution. This will be true if  $\Gamma$  satisfies the 'AD-AN (American Dream-American Nightmare (Monotone Mixing Condition))' conditions. Then, we know for sure that

$$\lim_{n \rightarrow \infty} T^n[X(q^0), y(\cdot)] = X^*(q^1)$$

How do we understand transitions for this economy? Some economist forget about transitions and just compare steady states, which is not correct if we are interested in welfare. The correct way to proceed is to guess

some  $N$  (number of periods necessary to achieve the new stationary equilibrium) and then compare the sequence of distributions  $\{T^n [X(q^0), y(\cdot)]\}_{n=1}^N$  to  $X^*(q^0)$ . Note that we can do this because we can calculate the decision rules independently from  $X^*(\cdot)$ .

## Welfare Questions

How much would this society be willing to pay for a reduction of  $q^0$  to  $q^1$ ? We can answer this for a particular household and for the whole society. For a particular household, we have three different values (call them  $\mathcal{V}$ )

- Utility gain from the switch

$$\mathcal{V}_1 = V(s, a; q^1) - V(s, a; q^0)$$

Note that this is a calculation that we can perform for particular households, defined by particular  $(s, a)$ .

- Consumption/assets willing to be sacrificed (for each household) in order to have  $q^1$

$$\mathcal{V}_2 = V(s, a - z; q^1) - V(s, a; q^0)$$

where  $z$  represents the amount of consumption/assets that the household would sacrifice in order to get the better technology. Note that we don't make the distinction between consumption and assets, since in this economy we have only one good. Also,  $\mathcal{V}_1$  is not defined for households with  $a = 0$  i.e., no assets. One way of avoiding this problem is to use the next calculation

- Consumption/assets needed to compensate the household if it stays with the old technology

$$\mathcal{V}_3 = V(s, a; q^1) - V(s, a + z; q^0)$$

Now, for the whole society, the gains from technical improvements are given by the sum of the individual gains. The formula is given by

$$\mathcal{V}_{soc} = \int z(s, a) dX^*(q^0)$$

where  $z(s, a)$  is the one calculated from  $\mathcal{V}_3$ , for each household (pairs of  $s, a$ ) and the integration is with respect to  $X^*(q^0)$ , the distribution of fishermen at the beginning.

## Technology Improvements in the Fishermen Economy (Transitions-difficult)

Suppose now that instead of storage technology, we have an aggregate technology of the form

$$Y = A_0 K^\theta N^{1-\theta}$$

where  $K = \int a dX^*$  and  $L = \int s dX^*$ . In this case, if the economy receives an improvement in total factor productivity (i.e.,  $A_1 > A_0$ ), answering the welfare questions becomes much harder. Why? simply, because now the measure matters. Recall that in the previous example, the measure of fishermen was exogenous to the calculation of the policy functions. Now, given the new value  $A_1$ , we will be moving away from a steady state, which means that the interest rate ( $F_K = \partial Y / \partial K$ ) and the wage ( $F_N = \partial Y / \partial N$ ) are not constant. Hence, we need the entire sequences of prices to get the policy function. The problem is that to get prices, we need the whole distribution (measure) of agents in the economy, which makes this a non-trivial problem. The problem is still solvable with computer-intensive methods and comparing steady states is still wrong.

## Growth

In our analysis so far, we have used Neo-classical Growth Model as our benchmark model and built on it for the analysis of more interesting economic questions. One peculiar characteristic of our benchmark model, unlike its name suggested, was lack of growth. Many interesting questions in economics are

related to the cross-country differences of growth rates. We will cover some models that will allow for growth so that we will be able to attempt to answer such questions.

## Exogenous growth

What does it take for an economy to grow? Before answering that question, we know in our standard NGM there is basically two ways of growth, one in which everything grows, which is not necessarily a per-capita growth, and the other is per-capita growth. We will be focusing on per-capita growth, hence, the next definition is useful

**Balanced Growth Path** is a situation where all variables of a model grow at a constant rate (not necessarily at the same rate)

The title exogenous growth refers to the structure of models in which growth rate is determined exogenously, and is not an outcome of the model. First and the simplest one of these is one in which the determinant of the growth rate is population growth.

## Growth with population

Suppose the population of our economy grows at rate  $\gamma$  and we have the classical CRTS technology in capital and labor inputs.

$$\begin{aligned} Y_t &= AF(K_t, N_t) \\ N_t &= N_0 * \gamma^t \end{aligned} \tag{33}$$

Note that our economy is no longer stationary but as we will see, within the exogenous growth framework we can make these economies look like stationary ones by re-normalizing the variables. Thus, at the end of the day it will only be a mathematical twist on our standard growth model. Once we do that, we will be looking for the counterpart of a steady state that we have in our stationary economies, the Balanced Growth Path, in which all the variables will be growing at constant rates but not necessarily equal. Back to our population growth model, we know

$$AF(K, N) = A[KF_k(K, N) + NF_N(K, N)] \tag{34}$$

Question is, if  $N$  is growing at rate  $\gamma$ , can this economy have a balanced growth path. Can we construct one? We know by CRTS property  $F_K$  and  $F_N$  are homogenous of degree zero. If we assume capital stock grows at rate  $\gamma$  as well, then prices stay constant and per-capita variables are constant and output grow at the same rate. So we get growth on a balanced growth path without per-capita growth. One question is how do we model population growth in our representative agent model. One way is to assume there is a constant proportion of immigration to our economy from outside but this has to assume the immigrants are identical to our existing agents in our economy, which is a bit problematic. The other way could be to assume growing dynasties which preserves the representative agent nature of our economy. If we do so, the problem of the social planner becomes,

$$\begin{aligned} \max \quad & \sum_{t=0}^{\infty} \beta^t N_t U\left(\frac{C_t}{N_t}\right) \\ \text{st} \quad & C_t + K_{t+1} = AF(K_t, N_t) + (1 - \delta)K_t \end{aligned} \quad (35)$$

To transform the budget set to per capita terms, divide all terms by  $N_t$  and to make the environment stationary by dividing all the variables by  $\gamma^t$  and assume  $N_0 = 1$ , we get,

$$\begin{aligned} \max \quad & \sum_{t=0}^{\infty} (\beta\gamma)^t U(\hat{c}_t) \\ \text{st} \quad & \hat{c}_t + \gamma\hat{k}_{t+1} = AF(\hat{k}_t, 1) + (1 - \delta)\hat{k}_t \end{aligned} \quad (36)$$

So how is this transformed model any different than our NGM? By the discount factor, the agents in this economy with growth discounts the future less but everything else is identical to NGM of course with the exception of this economy growing at a constant rate.

**March 29th, 2007**

## Labor-Productivity Growth

Now suppose we have a 'labor augmenting' productivity growth with constant population normalized to one, i.e. have the following CRTS technology,

$$Y_t = AF(K_t, \gamma^t N_t) \quad (37)$$

$$AF(K_t, \gamma^t N_t) = A[K_t F_k(K_t, \gamma^t N_0) + \gamma^t N_0 F_N(K_t, \gamma^t N_0)] \quad (38)$$



Can we have an BGP? The problem is,

$$\begin{aligned} & \max \sum_{t=0}^{\infty} \beta^t U(C_t) \\ \text{st} \quad & C_t + K_{t+1} = AF(K_t, \gamma^t N_0) + (1 - \delta)K_t \end{aligned} \quad (39)$$

and since we have a population of one, these variables are already per-capita terms. For stationarity, we have to normalize the variables to 'per productivity' units, by dividing all by  $\gamma^t$ . Then the problem becomes,

$$\begin{aligned} & \max \sum_{t=0}^{\infty} \beta^t U(\gamma^t \hat{c}_t) \\ \text{st} \quad & \hat{c}_t + \gamma \hat{k}_{t+1} = AF(\hat{k}_t, 1) + (1 - \delta)\hat{k}_t \end{aligned} \quad (40)$$

Suppose we have a CRRA preferences, then the question is how can we represent the preferences as a function of  $\hat{c}_t$  only. Writing the CRRA,

$$\sum_{t=0}^{\infty} \beta^t \frac{(\gamma^t \hat{c}_t)^{1-\sigma} - 1}{1 - \sigma} = \sum_{t=0}^{\infty} (\beta(\gamma^{1-\sigma}))^t \frac{\hat{c}_t^{1-\sigma} - 1}{1 - \sigma} \quad (41)$$

and the problem becomes,

$$\begin{aligned} & \max \sum_{t=0}^{\infty} (\beta(\gamma^{1-\sigma}))^t \frac{\hat{c}_t^{1-\sigma} - 1}{1 - \sigma} \\ \text{st} \quad & \hat{c}_t + \gamma \hat{k}_{t+1} = AF(\hat{k}_t, 1) + (1 - \delta)\hat{k}_t \end{aligned} \quad (42)$$

and once again it is exact same problem as the NGM with a different discount factor. Note that the existence of a solution to this problem depends on  $\beta(\gamma^{1-\sigma})$ . In this set-up we get per-capita growth at rate  $\gamma$ . Also note that CRRA is the only functional form for preferences that is compatible with BGP. This is because as per-capita output grows, for consumption to grow at a constant rate, our agent has to face the same trade-off at each period.

Now suppose we have the TFP growing at rate  $\gamma$  with a CRTS Cobb-Douglas technology

$$\begin{aligned} Y_t &= A_t F(k_t, 1) \\ \frac{A_{t+1}}{A_t} &= \gamma \end{aligned}$$

What would be the growth rate of this economy? We can show that like the previous cases the growth rate of the economy is the growth rate for the productivity of labor, which is  $\gamma^{\frac{1}{1-\alpha}}$  in this case.

## 11.5 Endogenous Growth

So far in the models we covered growth rate has been determined exogenously. Next we will look to models in which the growth rate is chosen by the model itself. We do know for a fixed amount of labor, the curvature of our technology limits the growth due to diminishing marginal return on capital and with depreciation there is an upper limit on (physical) capital accumulation. So if our economy is to experience sustainable growth for a long period of time, we either give up the curvature assumption on our technology or we have to be able to shift our production function up. Given a fixed amount of labor, this shift is possible either by an increasing TFP parameter or increasing labor productivity, . The simplest of such models where we can see that is the AK model, where the technology is linear in capital stock so that diminishing marginal return on capital does not set in.

## 11.6 AK Model

We have the usual social planner's problem with linear technology and full depreciation,

$$\max \sum_{t=0}^{\infty} \beta^t U(C_t)$$

such that

$$C_t + K_{t+1} = AK_t$$

and the FOCs

$$(c_t) : \beta^t U_c(\cdot) = \lambda_t \tag{43}$$

$$(k_{t+1}) : \lambda_t = \lambda_{t+1} \tag{44}$$

together implies the Euler equation,

$$U_c(c_t) = A\beta U_c(c_{t+1}) \quad (45)$$

and on the BGP with consumption growing at rate  $\gamma$  with CRRA utility we get,

$$\gamma = (A\beta)^{1/\sigma} \quad (46)$$

and the growth rate is determined by the model parameters endogenously. Note that capital also grows at rate  $\gamma$  and the fate of the economy is determined by the fundamentals of the model. The capital stock will diverge to infinity if  $(A\beta)^{1/\sigma} > 1$  or the economy is destined to vanish if  $(A\beta)^{1/\sigma} < 1$ . Also note that there is no transitional dynamics in this model (we lose the state variables in the Euler equation after substituting for the balanced growth rate relation) and conditional on  $\gamma$ , asymptotically all economies are the same regardless of the initial capital level. If we de-centralize this economy we know wages will be zero since labor has no use and gross rental rate of capital will be fixed at  $A$ . This is at odds with what we observe in the real world. We would rather like to have a model that allows for both transitional dynamics, labor and growth at the same time. Allowing for labor implies that we need a variable that proxies the increasing productivity of labor endogenously and be reproducible in terms of output, such that we are able to shift our production function continually in the output-capital space without hitting a natural bound.

## 11.7 Human Capital and Growth

Another way of getting our models to grow "endogenously", is by introducing the variable 'Human Capital' as an input of production. This will proxy continuous and endogenous increasing labor efficiency. We have two ways of modeling human capital:

- one way is to see it very much like physical capital, in the sense output has to be invested to increase the existing stock of human capital. That is the Lucas' approach, in which you can think of investing in education by building more schools as a way to increase the existing human capital stock.

- The alternative way would be to reserve a part of the leisure time for increasing the human capital stock, which can be thought of studying harder to get better in a fraction of the leisure time. Unfortunately, the second approach puts limit on the rate human capital can grow and might fail to generate sustainable endogenous growth. Next, we look at the Lucas' human capital model.

**Lucas' Human Capital Model** We have an Cobb-Douglas technology with CRTS and human capital (H) as an input of production instead of labor and the laws of motion for the inputs,

$$F(H, K) = AK^\alpha H^{1-\alpha} \quad (47)$$

$$K' = i_k + (1 - \delta_k)K \quad (48)$$

$$H' = i_h + (1 - \delta_h)H \quad (49)$$

Now that there is no limit to the accumulation of human capital and sustainable growth on a BGP is feasible. Furthermore, an analysis of the characterization of the balanced growth path will indicate that this model indeed has transitional dynamics, so unlike the AK model if economy starts out of this optimal growth path economy can adjust and converge to it by responding to prices in a de-centralized setting. If we model the law of motion for human capital as,

$$H' = (1 - N) + (1 - \delta_h)H \quad (50)$$

where  $(1 - N)$  is the time devoted to accumulating human capital, say by studying harder, we see there is a natural limit to the growth of human capital and such an economy might not have a BGP. The key ingredient of endogenous growth with labor is then the reproducibility of the human capital without such a limit.

**Apr. 3rd, 2007**

## 12 Growth Through Externalities in Capital Accumulation

We have seen in the AK model that the growth rate is endogenous and determined solely by model primitives. Still, it is not directly or indirectly

determined by the agents' choices in our model. In Lucas' human capital model, the growth rate is determined by the choice of agents, specifically by the optimal ratio of human and physical capital. The source of growth in Lucas' model is reproducibility of human capital. In this next model, Romer introduces the notion of externality generated by the aggregate capital stock to go through the problem of diminishing marginal returns to aggregate capital. In this model, the source of growth will be the aggregate capital accumulation, which is possible with a linear aggregate technology in capital as we saw in the AK model. The firms in our model will not be aware of this externality and will have the usual CRTS technology and observe the source of growth coming from the TFP parameter. As usual with externalities, the equilibrium outcome will not be optimal. Each firm has the following technology,

$$y_t = AK_t^{1-\alpha} k_t^\alpha n_t^{1-\alpha} \quad (51)$$

but since the firms are not aware of the positive externality they are facing they are solving the problem with the following technology.

$$y_t = \bar{A}_t k_t^\alpha n_t^{1-\alpha} \quad (52)$$

$$(53)$$

where

$$\bar{A}_t = A_t K_t^{1-\alpha}$$

We can see that the social planner in fact is solving an AK model in per-capita terms. So does the de-centralized version of this economy have a BGP and if it does, how would it look like? Assuming CRRA preferences without leisure we can derive the BGP condition and pin down the growth rate from the euler equation of a typical household,

$$1 = \beta \gamma^{-\sigma} (1 + r) \quad (54)$$

where  $\gamma = \frac{c_{t+1}}{c_t}$  is the growth rate at the balanced path as usual and  $r = MP_k$ . So to find out the marginal product of capital for the firm we differentiate the technology w.r.t.  $k_t$ ,

$$1 + r_t = \alpha AK_t^{1-\alpha} k_t^{\alpha-1} n_t^{1-\alpha} + (1 - \delta) \quad (55)$$

and since the prices are determined by aggregate state variables  $K_t = k_t$  gives,

$$A\alpha - \delta = r \tag{56}$$

and substituting this into the euler equation we get the growth rate of consumption.

$$[(A\alpha - \delta + 1)\beta]^{\frac{1}{\sigma}} = \gamma \tag{57}$$

Solving the AK problem the SP faces we can verify the optimal growth rate for consumption is,

$$[(A - \delta + 1)\beta]^{\frac{1}{\sigma}} = \gamma^{sp}. \tag{58}$$

The important properties of the decentralized model are,

1. It is sub-optimal due to firms' unawareness of the externality they are facing and thus have lower growth rate.
2. Once again, the rental rate does not depend on the capital stock (due to the aggregate linear technology, the states variables drop out from the euler equation) and there is no transitional dynamics generated by the model.

To sum up what we have done so far, we have started with models that had exogenous growth and saw that we can make these models look and behave like our NGM after appropriate transformation. Then we went on to look at models that generate growth endogenously and saw that a prerequisite for growth in these models is linearity of the technology in reproducible factors. We looked at the simple AK model, where the technology is linear in capital stock and analyzed the BGP of such an economy. Then we looked at Lucas' human capital model, in which we had two forms of capital, human and physical, both of which are reproducible in terms of output. Then we analyzed the model by Romer, which again has linearity in the reproducible factor at the aggregate level (capital stock), but firms were facing the CRTS technology with diminishing marginal return on capital and not aware of the positive externality they face. Next we will see another model by Romer with monopolistic competition and a R&D sector which can generate endogenous growth.

## 13 Monopolistic Competition, Endogenous Growth and R&D

Romer's monopolistic competition model has three production sectors, the final goods production, intermediate goods production and R&D i.e. variety production. Our usual TFP parameter in production function will represent the 'variety' in production inputs and as we will see, the growth of varieties through research and development firms will make sure a balanced growth path is sustainable. The production function in this economy is,

$$Y_t = L_{1t}^\alpha \int_0^{A_t} x_t(i)^{1-\alpha} di \quad (59)$$

where  $x_t(i)$  is the type  $i$  intermediate good and there is a measure  $A_t$  of different intermediate goods and  $L_{1t}$  is the amount of labor allocated to the final good production. The production function exhibits CRTS. The intermediate goods are produced with the following linear technology,

$$\int_0^{A_t} \eta x_t(i) di = K_t \quad (60)$$

Now suppose the variety of goods grows at rate  $\gamma$ , that is  $A_{t+1} = \gamma A_t$ . Is long run sustainable growth possible? The answer to this question will depend whether our final goods production technology is linear in growing terms. We do know that by the curvature of the technology, optimality implies equal amount of each variety will be used in production,  $x_t(i) = x_t$ , then we have,

$$A_t \eta x_t = K_t \quad (61)$$

and our output at this equal variety becomes,

$$Y_t = L_{1t}^\alpha A_t x_t^{1-\alpha} \quad (62)$$

then substituting for  $x_t$  we have,

$$Y_t = \frac{L_{1t}^\alpha}{\eta^{1-\alpha}} A_t^\alpha K_t^{1-\alpha} \quad (63)$$

thus if both  $A_t$  and  $K_t$  are growing at rate  $\gamma$ , then production function is linear in growing terms and long run balanced growth is feasible. Note that

this model becomes very similar to our previous exogenous labor productivity growth under these assumptions. The purpose of this model is to determine  $\gamma$  endogenously. *What will be the source of growth, where does  $\gamma$  come from?* As we will see, there will be incentives for R&D firms to produce new 'varieties' because there will be a demand for them. These new varieties will be patented to intermediate good production firms, where a patent will mean exclusive rights to produce that intermediate good. So we will have monopolistic competition in the intermediate goods production. Now suppose the law of motion for 'varieties', which is the technology in R&D sector, is given by

$$A_{t+1} = (1 + L_{2t}\zeta)A_t \quad (64)$$

where  $L_{2t}$  is the labor employed in R&D sector. Note that this is not a regular law of motion in the sense that every new variety produced helps the production of further new varieties. Hence, there is a positive externality to variety production. Also, assume leisure is not valued and we have an aggregate feasibility condition for labor

$$L_{2t} + L_{1t} = 1 \quad (65)$$

## Apr. 5th, 2007

The period  $t$  problem of a firm in the competitive final good production sector is

$$\max_{x_t(i), L_{1t}} \left\{ L_{1t}^\alpha \int_0^{A_t} x_t(i)^{1-\alpha} di - w_t L_{1t} - \int_0^{A_t} q_t(i) x_t(i) di \right\} \quad (66)$$

and since we have CRTS with perfect competition we have zero profit with following FOCs,

$$w_t = \alpha L_{1t}^{\alpha-1} \int_0^{A_t} x_t(i)^{1-\alpha} di \quad (67)$$

$$q_t(i) = (1 - \alpha) L_{1t}^\alpha x_t(i)^{-\alpha} \quad (68)$$

notice that the inverse demand function for good of variety  $i$  is,

$$\left( \frac{q_t(i)}{(1 - \alpha) L_{1t}^\alpha} \right)^{\frac{-1}{\alpha}} = x_t(i) \quad (69)$$



The intermediate goods industry will show monopolistic competition, in which there is only one firm, that is one patent holder, producing each variety. Each firm takes the demand of its variety and prices as given, and solves the following problem each period

$$\begin{aligned} \Pi_t(i) &= \max_{x_t(i), K_t(i)} \{q_t(i)x_t(i) - R_t K_t(i)\} \\ \text{s.t. } x_t(i) &= \frac{K_t(i)}{\eta} \end{aligned} \quad (70)$$

plugging in the inverse demand function and the technology constraint, the FOC is,

$$(1 - \alpha)^2 x_t(i)^{-\alpha} L_{1t} = R_t \eta \quad (71)$$

and because of the symmetry we mentioned ( $x_t(i) = x_t = \frac{K_t}{\eta A_t}$ ) we can write this FOC as,

$$(1 - \alpha)^2 \left(\frac{K_t}{\eta A_t}\right)^{-\alpha} L_{1t} = R_t \eta \quad (72)$$

i.e. the rental price of capital is not equal to it's marginal product and there is opportunities for positive profit. But also remember there is a fixed cost of entering this industry, namely the price paid for the patent. Then as we will see, the relation between the two will be one of our equilibrium conditions. Now lets look at the problem of R&D firms,

$$\max_{A_{t+1}, L_{2t}} \{p_t^P (A_{t+1} - A_t) - w_t L_{2t}\} \quad (73)$$

$$\text{s.t. } A_{t+1} = (1 + L_{2t}\zeta)A_t$$

where  $p_t^P$  is the patent of the price. Free entry is assumed, thus there will be zero profits in equilibrium. Notice also the R&D firm is solving a static problem without realizing the positive externality this period's decision creates on next periods production. As we will see, this and the monopoly power of the patent owners will be the sources of sub-optimality in the decentralized solution. The FOC is,

$$p_t^P = \frac{w_t}{\zeta A_t} \quad (74)$$

where the wage ( $w_t$ ) is determined in the final goods market and given this price, equilibrium quantity will come from the demand function. As we mentioned before, one equilibrium condition will be that at any point in

time, total profit a patent generates will be equal to price of it such that there will also be zero profit in the intermediate goods market.

$$p_t^P = \sum_{\tau=t}^{\infty} \frac{\Pi_{\tau}(i)}{(1+r)^{\tau-t}} \quad (75)$$

These conditions with constant growth equations for the growing variables is sufficient to characterize the equilibrium growth rate of this economy.

## 14 Economies with Frictions

In this part of the course, we will discuss some type of models with frictions. We are interested in mainly two types of frictions:

1. Hidden Actions
2. Lack of Commitment

The first model we study, labeled the 'optimal unemployment insurance' problem, deals with the first type of friction. In the next section, we deal directly with problems of commitment (only ONE-sided)

### April 4th to April 12th, 2006 (Intensive Week)

## 15 Optimal unemployment insurance

### 15.1 (with observable effort)

Consider an economy where the probability of finding a job  $p(a)$  is a function of effort  $a \in [0, 1]$ . And we assume that once the agent gets a job, she will have wage  $w$  for ever. Thus, the individual problem is

$$\max_{a_t} E \sum_t \beta^t [u(c_t) - a_t]$$

There are two cases: when the agent has got a job, she will pay no effort and enjoy  $w$  for ever. The life long utility is

$$V^E = \sum_t \beta^t u(w) = \frac{u(w)}{1-\beta} \quad (76)$$

When the agent is still unemployed, she will have nothing to consumer. Her problem is

$$V^u = \max_a \{u(0) - a + \beta [p(a) V^E + (1 - p(a) V^u)]\} \quad (77)$$

If the optimal solution of  $a$  is interior,  $a \in (0, 1)$ , then the first order condition gives

$$-1 + \beta p'(a) (V^E - V^u) = 0 \quad (78)$$

And since the  $V^u$  is stationary,

$$V^u = \max_a \{u(0) - a + \beta [p(a) V^E + (1 - p(a) V^u)]\} \quad (79)$$

Solving (78)(79) gives the optimal  $a$  and  $V^u$ . Another way is to successively substitute  $a$  and obtain a solution because the problem defines a contraction mapping operator. We can fix  $V_0^u$ , then solve (79) to get  $a(V_0^u)$  and obtain  $V_1^u$ . Keeping going until  $V_n^u = V_{n+1}^u$ . In a word, optimal effort level  $a^*$  solves (79) with  $V^u = V^u$ .

The probability of finding a job  $p(a)$  is called hazard rate. If agents did not find a job with effort level  $a^*$ , next period, she will still execute the same effort level  $a^*$ . Why? Because the duration of unemployment is not state variable in agent's problem. (If agents do not have enough realization about the difficulty of getting a job. With learning, their effort  $a$  will increase as they revise their assessment of the difficulty. But such revision of belief is not in this model.)

Now suppose resource is given to people who is unemployed to relive her suffering by a benevolent planner. This planner has to decide the minimal cost of warranting agent a utility level  $V$ ,  $\Psi(V)$ . To warrant utility level  $V$ , the planner tells the agent how much to consume, how much effort to exert and how much utility she will get if she stay unemployed next period. Obviously, the cost function  $\Psi(V)$  is increasing in  $V$ .

The cost minimization problem of the planner can be written in the following recursive problem:

$$\Psi(V) = \min_{c, a, V^u} c + [1 - p(a)] \frac{1}{1 + r} \Psi(V^u) \quad (80)$$

subject to

$$V = u(c) - a + \beta [p(a) V^E + (1 - p(a)) V^u] \quad (81)$$

To solve the problem, construct Lagrangian function

$$\mathcal{L} = c + [1 - p(a)] \beta \Psi(V^u) + \theta [V - u(c) + a - \beta [p(a) V^E + (1 - p(a)) V^u]]$$

FOC: (c)

$$\theta = \frac{1}{u_c} \quad (82)$$

(a)

$$\Psi(V^u) = \theta \left[ \frac{1}{\beta p'(a)} - (V^E - V^u) \right] \quad (83)$$

(V<sup>u</sup>)

$$\Psi'(V^u) = \theta \quad (84)$$

Envelope condition

$$\Psi'(V) = \theta \quad (85)$$

We will work on some implication of these conditions:

1. Compare (83) and (78), we can see that the substitution between consumption and effort is different from the one in agent's problem without unemployment insurance. This is because the cost of effort is higher for work than it is from the viewpoint of planner.

2. (84) tells us that the marginal cost of warranting an extra unit of utility tomorrow is  $\theta$ , provided that tomorrow  $V^u$  is optimally chosen when today's promise is  $V$ . And (85) tells us that the marginal cost of warranting an extra unit of  $V$  today is  $\theta$ .

3. Given that  $\Psi$  is strictly concave,  $V = V^u$ .

4. Regardless of unemployment duration,  $V = V^u$ . So, effort required the planner is the same over time. Hazard rate is still constant.

Next, we will study the case when effort is not observable. Planner can only choose consumption and  $V^u$ . Effort level is chosen optimally by worker and it is unobservable.

## 15.2 (Problem with UN-observable effort)

When  $a$  is not observable, planner can only choose  $c$  and  $V^u$ . And households choose  $a$  optimally. Now it becomes a principle-agent problem. We will solve the problem backward.

If given  $c$  and  $V^u$ , the agent will solve

$$\max_a u(c) - a + \beta [p(a) V^E + (1 - p(a)) V^u] \quad (86)$$

FOC is

$$[p'(a) \beta]^{-1} = V^E - V^u \quad (87)$$

This FOC gives an implicit function of  $a$  as a function of  $V^u$ :  $a = g(V^u)$ . (Because  $c$  and  $a$  are separate in the utility function,  $a$  is not a function of  $c$ ).

Then, the planner solve her cost minimization problem, in which the optimality condition is also one constraint.

$$\Psi(V) = \min_{c, a, V^u} c + [1 - p(a)] \beta \Psi(V^u)$$

subject to

$$V = u(c) - a + \beta [p(a) V^E + (1 - p(a)) V^u] \quad (88)$$

$$1 = [p'(a) \beta] [V^E - V^u] \quad (89)$$

Lagrangian is

$$c + [1 - p(a)] \beta \Psi(V^u) + \theta [V - u(c) + a - \beta [p(a) V^E + (1 - p(a)) V^u]] \\ + \eta [1 - [p'(a) \beta] [V^E - V^u]]$$

FOC: (c)

$$\theta^{-1} = u_c$$

(a)

$$\Psi(V^u) = \theta \left[ \frac{1}{\beta p'(a)} - (V^E - V^u) \right] - \eta \frac{p''(a)}{p'(a)} (V^E - V^u) \quad (90)$$

( $V^u$ )

$$\Psi'(V^u) = \theta - \eta \frac{p'(a)}{1 - p(a)} \quad (91)$$

Envelope condition

$$\Psi'(V) = \theta \quad (92)$$

Again, (91) tells the marginal cost to warrant additional amount of delayed promise. (92) gives the marginal cost to increase today's utility. The

Lagrangian multiplier associated with constraint (89) is positive,  $\eta > 0$ , which means that the constraint is binding. So,

$$\eta \frac{p'(a)}{1-p(a)} > 0$$

Therefore, we have

$$\Psi'(V^u) < \Psi'(V) \Rightarrow V^u < V$$

from the strict concavity of  $\Psi(\cdot)$ . The delayed promised utility decreases over time.

Let  $\theta^u = \theta - \eta \frac{p'(a)}{1-p(a)}$ , then  $\theta^u < \theta$ , which tells us about the consumption path. Consumption decreases over time because  $\theta^{-1} = u_c$ .

Overall, we get the following model implications: optimal unemployment insurance says that longer unemployment period the agent stays, the less insurance she will be insured for. In this way, the planner induces the higher effort level. Although you cannot let people do what is optimal, such behavior can be achieved by giving out less consumption and promised utility over time. This model implies that time-varying unemployment insurance plan is optimal, under which the replacement rate  $\theta$  goes down over time.

## 16 Models with one sided lack of commitment

We now study an economy where the agent cannot commit to the contract that the Social Planner offers her. This means that as long as what the Planner offers is better than what the agent can do alone she stays around. If in some period the shock that the agent receives is so good that her value under autarky is higher than what the planner is offering she will be willing to walk away and be on her own (at no cost). On the other hand, if the planner wants to keep her around, he now has to offer even more. In this model there is no storage or financial markets, and so the agent consumes the fruit (or fish) which is just the shock that she receives in each period.

Consider a village of agents, each of them receive  $s \in \{s_1, s_2, \dots, \bar{s}\}$  in every period. We assume that  $s$  is iid. The probability that a certain  $s$  is realized is  $\Pi_s$ .  $h_t$  is a history of shocks up to period  $t$ .

First, if the agent stays in autarky, she will enjoy total utility,

$$V_{AUT} = \sum_{t=0}^{\infty} \beta^t \sum_s \Pi_s u(s) = \frac{\sum_s \Pi_s u(s)}{1 - \beta}$$

Note that here  $V^A$  is the utility of the agent before endowment shock is realized.

Now we assume that the Insurer offers a contract to the agent, which transfer resources and provide insurance to her. The Insurer can commit. But the agent may leave the insurer (Commitment problem). Thus, this model has one-sided commitment model: an agent can walk away from a contract but the other cannot. Therefore, the contract should be always in the interest of the agent for her to stay.

We define a contract  $f_t : H_t \rightarrow c \in [0, \tau]$ . We will see next class that incentives compatibility constraint requires that at each node of history  $H_t$ , the contract should guarantee a utility which is higher than that in autarky.

Notice that the problem is different from Lucas tree model because of the shock realization timing. In Lucas tree model, shock is state variable because action takes place after shock is realized. Thus, action is indexed by shock. Here action is chosen before shock realization. Therefore, shock is not a state variable and action is state contingent.

In Lucas tree model,  $V(s) = \max_c u(c) + \beta \sum_{s'} \Pi_{ss'} V(s')$ . Here, if we write the problem recursively, it is  $V = \max_{c_s} \sum_s \Pi_s u(c_s) + \beta V$ .

Remember, the Insurer will make a deal with the agent. They sign a contract to specify what to do in each state.  $h_t \in H_t$ . Contract is thus a mapping  $f_t(h_t) \rightarrow c(h_t)$ . With this contract, the agent gives  $y_t$  to the Insurer and receives  $c_t = f_t(h_{t-1}, y_t)$ . But if the agent decided not to observe the contract, she consumes  $y_t$  this period and cannot enter a contract in the future, i.e. she has to live in autarky in the future.

For the Insurer to keep the agent around, the contract has to be of interest to the agent (commitment issue). There are two possible outcome if this contract is broken. One is that the agent goes away with current and future endowment. The other is that they renegotiate. We ignore the second possibility as no renegotiation is allowed, since it would introduce a lot of messy algebra.

The first best outcome is to warrant a constant consumption  $c_t$  to the agent who is risk averse. But because of the one-side lack of commitment, the first best is not achievable. The contract should always be attractive to

the agent, otherwise, when she gets lucky with a high endowment  $y_s$ , she will feel tempted to leave. So, this is a dynamic contract problem which the Insurer will solve in order to induce good behavior from the agent. The contract is dynamic because nature keeps moving.

We say the contract  $f_t(h_t)$  is incentive compatible or satisfies participation constraint if for all  $h_t$ ,

$$u(f_t(h_t)) + \sum_{\tau=1}^{\infty} \beta^{\tau} \sum_s \Pi_s u(f_{t+\tau}(h_{t+\tau})) \geq u(y_s(h_t)) + \beta V^A \quad (93)$$

The left hand side is utility guaranteed in the contract. And the right hand side is the utility that the agent can get by herself. The participation constraint is not binding if  $y_s$  is low. And when  $y_s$  is high, PC is binding.

## 16.1 Problem of the Insurer

In this model, problem of the Insurer is to find an optimal contract that maximizes the value of such a contract of warranting  $V$  to her. We define the problem using recursive formula. Firstly, let's define the value of contract to the Insurer if she promised  $V$  to the agent.  $\Phi(V)$  can be defined recursively as the following:

$$\Phi(V) = \max_{\{c_s, \omega_s\}_{s=1}^S} \sum_s \Pi_s [(s - c_s) + \beta \Phi(\omega_s)] \quad (94)$$

subject to

$$u(c_s) + \beta \omega_s \geq u(s) + \beta V^A \quad \forall s \quad (95)$$

$$\sum_s \Pi_s [u(c_s) + \beta \omega_s] \geq V \quad (96)$$

Notice that there are  $1 + S$  constraints. The choice variables  $c_s, \omega_s$  are state-contingent where  $\omega_s$  is the promised utility committed to the agent in each state. In the objective function,  $\sum_s \Pi_s (s - c_s)$  is the expected value of net transfer.

There are two sets of constraints. (95) is IC and (96) is promise keeping constraint.

The First Order Conditions to the problem are:



$$(c_s) \quad \Pi_s = (\theta_s + \lambda \Pi_s) u'(c_s) \quad (97)$$

$$(\omega_s) \quad -\Pi_s \Phi'(\omega_s) = \lambda \Pi_s + \theta_s \quad (98)$$

$$(\mu) \quad \sum_s \Pi_s [u(c_s) + \beta \omega_s] = V \quad (99)$$

$$(\lambda) \quad u(c_s) + \beta \omega_s \geq u(s) + \beta V^A \quad (100)$$

In addition, Envelope Theorem tells that:

$$\Phi'(v) = -\lambda \quad (101)$$

Interpreting the first order conditions:

1. (97) tells that in an optimal choice of  $c_s$ , the benefit of increasing one unit of  $c$  equals the cost of doing so. The benefit comes from two parts: first is  $\lambda \Pi_s u'(c_s)$  as increasing consumption helps the Insurer to fulfill her promise and the second part is  $\theta_s u'(c_s)$  since increase in consumption helps alleviate the participation constraint. And the cost is the probability of state  $s$  occurs.

2. (98) equates the cost of increasing one unit of promised utility and the benefit. The cost to the Insurer is  $-\Pi_s \Phi'(\omega_s)$  and the benefit is  $\mu \Pi_s + \lambda_s$  which helps the Insurer deliver promise utility and alleviate the participation constraint.

How about the contract value  $\Phi(V)$ . First,  $\Phi(V)$  can be positive or negative.

Claim: There exists  $V$  such that  $\Phi(V) > 0$ .

What's the largest  $V$  we will be concerned with? When PC will be binding for sure. If PC binds for the best endowment shock  $s$ , then PC holds for all the shock  $s$ . When the agent gets the best shock, the best autarky value is then

$$V_{AM} = u(\bar{s}) + \beta V_A$$

And the cheapest way to guarantee  $V_{AM}$  is to give constant consumption  $\bar{c}_s$ , such that

$$V_{AM} = \frac{u(\bar{c}_s)}{1 - \beta}$$

From this case, we can see that because of lack of commitment, the Insurer will have to give more consumption in some states. While when there is no lack of commitment, strict concavity of  $u(\cdot)$  implies that constant stream of consumption beats any  $\{c_t\}$  that have the same present value, as there is no PC.

## 16.2 Characterizing the Optimal Contract

We will characterize the optimal contract by considering the two cases: (i)  $\theta_s > 0$  and (ii)  $\theta_s = 0$ .

Firstly, if  $\theta_s = 0$ , we have the following equations from FOC and EC:

$$\Phi'(\omega_s) = -\mu \quad (102)$$

$$\Phi'(V) = -\mu \quad (103)$$

Therefore, for  $s$  where PC is not binding,

$$V = \omega_s$$

$c_s$  is the same for all  $s$ . For all  $s$  such that the Participation Constraint is not binding, the Insurer offers the same consumption and promised future value.

Let's consider the second case, where  $\theta_s > 0$ . In this case, the equations that characterize the optimal contract are:

$$u'(c_s) = \frac{-1}{\Phi'(\omega_s)} \quad (104)$$

$$u(c_s) + \beta\omega_s = u(s) + \beta V^A \quad (105)$$

Note that this is a system of two equations with two unknowns ( $c_s$  and  $\omega_s$ ). So these two equations characterize the optimal contract in case  $\theta_s > 0$ . In addition, we can find the following properties by carefully observing the equations:

1. The equations don't depend on  $V$ . Therefore, if a Participation Constraint is binding, promised value does not matter for the optimal contract.

2. From the first order condition with respect to  $\omega_s$ ,  $\Phi'(\omega_s) = \Phi'(V) - \frac{\theta_s}{\Pi_s}$ , where  $\frac{\theta_s}{\Pi_s}$  is positive. Besides, we know that  $\Phi$  is concave. This means that  $V < \omega_s$ . In words, if a Participation Constraint is binding, the moneylender promises more than before for future.

Combining all the results we have got, we can characterize the optimal contract as follows:

1. Let's fix  $V_0$ . We can find a  $s^*(V_0)$ , such that  $\forall s < s^*(V_0)$ , the participation constraint is not binding and  $\forall s \geq s^*(V_0)$ , the constraint is binding, i.e.  $\theta_s > 0$ .
2. The optimal contract that the Insurer offers to an agent is the following:  
 If  $s_t \leq s^*(V_0)$ , the Insurer gives  $(V_0, c(V_0))$ . Both of them are the same as in the previous period. In other words, the Insurer offers the agent the same insurance scheme as before.  
 If at some point in time  $s_t > s^*(V_0)$ , the moneylender gives  $(V_1, c(V_1))$ , where  $V_1 > V_0$  and  $c$  doesn't depend on  $V_0$ . In other words, the moneylender promises larger value to the agent to keep her around.  
 So the path of consumption and promised value for an agent is increasing with steps.

**April 12th, 2007**

## **Overlapping Generations (OLG or OG ) Models**

So far we been utilizing the NGM with infinite horizon with no demographic details. This wasn't because we didn't have the tools to consider a finite horizon. Many important questions in macroeconomics should be approached within a framework where these details matter. These models contain agents who are born at different dates and have finite lifetimes, even though the economy goes on forever. This induces a natural heterogeneity across individuals at a point in time, as well as nontrivial life-cycle considerations for a given individual across time. These features of the model can also generate differences from models where there is a finite set of time periods and agents, or from models where there is an infinite number of time periods but agents live forever.

Let's now first consider a very basic OLG model: Suppose that the economy goes on forever, and that at every date  $t$  there is born a new generation of individuals ( $t$ -born agents) who live for two periods. This is the simplest case where the generations overlap. There is also a generation of initial old guys at  $t = 1$  who only live for one period. For now, every generation consists of a  $[0, 1]$  continuum of homogeneous agents. In the 1st and 2nd periods of life, and let  $w_1 = 3$  and  $w_2 = 1$  be (time-invariant) endowments in the

1st and 2nd periods of life of a  $t$ -born agent. Let  $c_t^y$  and  $c_{t+1}^o$  denote the consumption of a  $t$ -born agent when he is young in period  $t$  and when he is old in period  $t + 1$ , respectively. The utility function of a  $t$ -born agent is  $\log(c_t^y) + \log(c_{t+1}^o)$ . Initial old guys consume only  $c_0^o$  and are endowed  $e_2 = 1$ .

**Definition:** A SME for this economy is a list of sequences,  $\{p_t^*, c_t^{y*}, c_{t+1}^{o*}\}_{t=0}^\infty$  s.t.

1. Given prices,  $\{c_t^{y*}, c_{t+1}^{o*}\}$  solves the  $t$ -born agents problem: for  $t = 0, 1, 2..$

$$\begin{aligned} & \max_{c_t^y, c_{t+1}^o} \log(c_t^y) + \log(c_{t+1}^o) \\ \text{s.t.} \quad & p_t^* + 3p_{t+1}^* = c_t^y p_t^* + c_{t+1}^o p_{t+1}^* \end{aligned}$$

2.  $c_0^o = 1$

3. Markets are clear:  $c_t^{y*} + c_t^{o*} = 4, \quad t = 0, 1, ..$

Autarky is the unique equilibrium of this economy: Since only option for the initial olds is to eat their endowment, by market clearance condition every future generation consume their endowment. Then by the FOC of the  $t$ -born agents problem, we characterize the prices assuming  $p_0^* = 1$ :

$$\frac{c_{t+1}^{o*}}{c_t^{y*}} = \frac{p_t^*}{p_{t+1}^*} = \frac{1}{3} \quad t = 0, 1, 2..$$

Is this equilibrium (autarky allocation) Pareto optimal? Obviously not (if at every period youngs give olds  $\epsilon \leq 1$  of their endowment, everyone would be better off). Then what is wrong with the First Basic Welfare Theorem? Indeed there is nothing wrong with FBWT, but it just doesn't apply to this environment. Why it doesn't apply to this environment? Because autarky is a SME not an Arrow-Debrou equilibrium.

Now we introduce money (an intrinsically valueless piece of paper) into our model:

**Definition:** A monetary equilibria is a list of sequences,  $\{q_t^*, R_t^*, c_t^{y*}, c_{t+1}^{o*}, m_t^*, b_t^*\}_{t=0}^\infty$  s.t.

1. Given prices,  $\{q_t^*, R_t^*, \}, \{c_t^{y*}, c_{t+1}^{o*}, b_t^*\}$  solves the  $t$ -born agents problem: for  $t = 0, 1, 2..$

$$\max_{c_t^y, c_{t+1}^o} \log(c_t^y) + \log(c_{t+1}^o)$$

$$s.t. \quad c_t^y + q_t^* m_t + b_t = 3$$

$$c_{t+1}^o = q_{t+1}^* m_t + R_t^* b_t + 1$$

$$2. \quad c_0^o = 1 + q_0^* M$$

$$3. \quad \text{Markets are clear: } c_t^{y*} + c_t^{o*} = 4, \quad m_t = M, \quad b_t^* = 0 \quad t = 0, 1, \dots$$

**Definition:** A non-monetary equilibria is a monetary equilibria where  $q_t^* = 0, \forall t$ .

**Definition:** A stationary monetary equilibria is a monetary equilibria where  $q_t^* = q^*, \forall t$ .

Note: Please check the Problem set 9 Q3 for the difference equations that characterize the monetary equilibria.

## April 16th, 2007

### 16.3 Labor Earnings

What is a good theory on  $\epsilon$ ? If we look at the average wage per hour at the different age ( $w\epsilon$ ), the wage per hour increases with age, peaks at around 40, and slowly decreases until the retirement. Since  $w$  is assumed to be same for all agents, we need a theory that explains the difference in  $\epsilon$  to replicate the hump shape of the average wage profile. What kind of theory do we have? There are two ways, in general:

1. Hormones: Take  $\{\epsilon\}$  as exogenous; i.e., assuming that the young agents are useless because they are young.
2. Human capital theory. Assume that the difference in capital stock between the young agents and the old agents yields the difference in  $\epsilon$ . There are three branches:

- (a) **Learning-by-doing:** Assume that agents accumulate human capital,  $\epsilon$  by working. Agents learn something which enhances their human capital stock while they are working. Imagine interns of doctor. The young doctors learn how to do operations by actually working at hospitals. This idea is represented by:

$$\epsilon_{i+1} = \phi_i(\epsilon_i; n_i)$$

where  $n_i$  is hours worked of agents of age  $i$ .  $\phi_i$  is indexed by  $i$  because learning ability can be different depending on age.

The problem of an agent is :

$$V_i(a, \epsilon) = \max_{c, n} u(c) + \nu(n) + \beta V_{i+1}(a', \epsilon')$$

$$c + a' = w\epsilon'n + (1 + r)a$$

$$\epsilon' = \phi_i(\epsilon, n)$$

- (b) **Learning-by-not-doing:** Assume that agents accumulate human capital by actually learning (which is different from working or enjoying leisure). This idea is represented by:

$$\epsilon_{i+1} = \phi(\epsilon_i; l_i)$$

where  $l_i$  is the time spent on learning, which is different from working or enjoying leisure. Agents allocate their time in learning to accumulate human capital.

The problem of an agent is:

$$V_i(a, \epsilon) = \max_{c, n} u(c) + \nu(n + l) + \beta V_{i+1}(a', \epsilon')$$

$$c + a' = w\epsilon'n + (1 + r)a$$

$$\epsilon' = \phi_i(\epsilon, l)$$

- (c) **Schooling:** the difference from learning models above is that most of education is acquired in the early stage of life. Keane and Ken Wolpin showed that 90% of people's fate is determined before age 16, by using structurally estimated model of the career choice. Non college guys start earning 4-5 years earlier than college graduates, however college graduates make more than non college guys. Then why some people decide to go to college or some of them not?

Borrowing constraint

Discount factor: College graduates are more patient than non college guys

For non college guys it is more costly to go to college in terms of intelligence

Mostly parents make the college decision.

## April 17th, 2007

$\sigma_y$ , standard deviation of income among people, increases by age. What could be the possible reasons?

- The skill levels are amplified: Someone with better skill level at the beginning augment his skills better than others.
- Luck: It is all luck, luck leads to more variance. Also luck is persistent:  $\epsilon$  with  $\sigma_\epsilon$  then the variance of  $\rho\epsilon_1 + \epsilon_2$  has variance of  $(1 + \rho^2)\sigma_\epsilon^2$ .

## OLG in Business Cycles

Agents live  $I$  periods, their endowment of labor in period  $i$  is  $\epsilon_i$ . Newborns born with zero asset.

State variables:  $i$  (age of the agent),  $z$  (Productivity shock),  $A$  (Vector of aggregate asset levels of  $I$  cohorts),  $a$  (individual asset level).

$$V_i(z, A, a) = \max_{c, a'(z')} u(c) + \beta \sum_{z'} \Gamma_{zz'} V_{i+1}(z', A'(z'), a'(z'))$$

$$s.t. \quad c + \sum_{z'} q_{z'}(z, A) a'(z') = R(z, K) a + w(z, K) \epsilon_i$$

$$A'(z') = G_{z'}(z, A)$$

$$R(z, K) = (1 - \delta) + z F_1\left(\sum_{i=2}^I, \sum_{i=1}^I \epsilon_i\right)$$

$$w(z, K) = z F_2\left(\sum_{i=2}^I, \sum_{i=1}^I \epsilon_i\right)$$

$$\text{with solution } a'(z') = g_{z'}^i(z, A, a)$$

**Definition:** A RCE of this equilibrium is a list of  $\{V^{i*}(\cdot), g^{i*}(\cdot), G^*(\cdot), q^*(\cdot), R^*(\cdot), w^*(\cdot)\}$  s.t.

1. Given  $\{G^*(\cdot), q^*(\cdot), R^*(\cdot), w^*(\cdot)\}$ ,  $\{V^{i*}(\cdot), g^{i*}(\cdot)\}$  solves the agent's problem.
2.  $G_{z'}^i(z, A) = g_{z'}^{i-1}(z, A, A_{i-1})$

3. There are  $n_z - 1$  market clearing conditions:

$$\sum_i G_{z'}^i(z, A) = \sum_i G_{\tilde{z}'}^i(z, A) \quad \forall \hat{z}', \tilde{z}' \in Z$$

4.  $\sum_{z'} q_{z'}(z, A) = 1$  (No Arbitrage condition)

Now let's introduce the probability of surviving from one period to next. Let  $s_i$  is the probability of surviving from  $i$  to  $i + 1$ . (So we know that  $s_I = 0$ .) Then the probability of living in the  $i^{th}$  period is  $\mu_i = \prod_{j=0}^{i-1} s_j$ . But what happens to the assets of the death people. There are few options to incorporate this issue into our model:

1. Pharaoh Model: Bury the capital with the death. Then the problem of the agent is:

$$V_i(z, A, a) = \max_{c, a'(z')} u(c) + \beta s_i \sum_{z'} \Gamma_{zz'} V_{i+1}(z', A'(z'), a'(z'))$$

$$s.t. \quad c + \sum_{z'} q_{z'}(z, A) a'(z') = R(z, K)a + w(z, K)\epsilon_i$$

2. The capital of death is reallocated to cohort  $j$  (cohort  $j$  is arbitrarily chosen):

$$V_i(z, A, a) = \max_{c, a'(z')} u(c) + \beta s_i \sum_{z'} \Gamma_{zz'} V_{i+1}(z', A'(z'), a'(z'))$$

$$s.t. \quad c + \sum_{z'} q_{z'}(z, A) a'(z') = R(z, K)a + w(z, K)\epsilon_i + 1_{i=j} B(z, A)$$

$$B(z, A) = \sum_i \mu_i (1 - s_i) A_i$$

3. Perfect Annuity Market: The assets of the death is equally reallocated to all survivors:

$$V_i(z, A, a) = \max_{c, a'(z')} u(c) + \beta s_i \sum_{z'} \Gamma_{zz'} V_{i+1}(z', A'(z'), a'(z'))$$

$$s.t. \quad c + \sum_{z'} q_{z'}(z, A) a'(z') s_i = R(z, K)a + w(z, K)\epsilon_i$$



## April 18th, 2007

What are the important things in life?

- Health
- Money
- Love
- Children

### Health

Suppose  $s$  is the health condition of an agent which is assumed to be Markov process. Then the agent's problem is:

$$V_i(s, a) = \max_{c, a'} u(c, s) + \gamma^i(s) \beta \sum_{s'} \Gamma_{ss'} V_{i+1}(s', a')$$

$$\Psi(s) + c + a' = (1 + r)a + w\phi^i(s)$$

where  $\gamma^i(s)$  is the probability of survival of a cohort  $i$  agent from  $i$  to  $i + 1$ ,  $\Psi(s)$  is the cost of being in a health condition of  $s$ , and  $\phi^i(s)$  is the labor endowment function of cohort  $i$  agent which depends on health condition.

### Love (Marriage)

Value function of a female in cohort  $i$  married with a guy in cohort  $j$ :

$$V_{i,j}^f(a) = \max_{c, n, f} u^f(c, n^f) + \beta V_{i+1, j+1}^f(a')$$

$$c + a' = \epsilon_j w^m + \epsilon_i n^f w^f + a(1 + r)$$

Let's introduce divorce into this function:

$$V_{i,j}^f(a, z) = \max_{c, n, f} u^f(c, n^f) + \beta \sum_{z'} \Gamma_{zz'} V_{i+1, j+1}^f(a', z')$$

$$c + a' = \epsilon_j w^m 1_{\{z=j\}} + \epsilon_i n^f w^f + a(1 + r)$$

## April 19th, 2007

A model of single people with health under partial control:

$$V(a, h) = \max_{c, c', y} u(c, y) + \beta \{ \phi(h') u(c', h') + (1 - \phi(h')) u_D \}$$

$$h' = \psi(h, y), \quad \phi' > 0$$

$$a = c + c'$$

Why people marry?

- Match specific quality,  $q$ , of the partner.
- Individual education level.
- $q, e$  are female's attributes, and  $q^*, e^*$  are the male's.
- $V^{fm}$  is the value function of married female.
- $V^{fs}$  is the value function of single female.
- $V^{mm}$  is the value function of married male.
- $V^{ms}$  is the value function of single male.

$$V^{fm}(e, e^*, q, q^*) - V^{fs}(e) > 0$$

$$V^{mm}(e^*, e, q^*, q) - V^{ms}(e^*) > 0$$