Jan 23rd, 2007

- What is a model? A model is 'Toy World': a simplified specification of the world, endowed with (i) an environment, (ii) agents and (iii) characteristics of the agents. Once a model is defined, we need to know what happens, i.e., an equilibrium. Generally, by equilibrium we mean that we want to know what happens with allocations: Are they (in some sense) 'optimal? Do they exist?

- What is an equilibrium?
  An equilibrium is a statement about what the outcome of an economy is. Tells us **what happens in an economy,** and by an ecomomy we mean a well defined environment in terms of primitives such as preferences and technology.
  Then an equilibrium is a particular mapping from the environment (preference, technology, information, market structure) to allocations where,

  1. Agents maximize
  2. Agents' actions are compatible with each other.

- One of the important questions is, given the environment what type of equilibria we should look at. The economist doesn't have the right to choose what happens, but is free to define the environment.

- For the theory to be able to predict precisely what is going to happen in a well defined environment, the outcome we define as the equilibrium needs to **exist** and must be **unique**. For this reason uniqueness is property that we want the equilibrium to have. We also know with certain assumptions that will be covered we can ensure the existence and uniqueness of an equilibrium outcome.

1 Growth Model

The basic model we deal with in 702 is the neoclassical growth model. We will discuss the basic environment and then ask what happens in this 'toy world': does an equilibrium exist? is it optimal?

1.1 Technology

- Agents have 1 unit of labor and own capital which can be transformed in output.
- Production function:

$$f : R_+^2 \to R_+ \quad \text{such that } c_t + k_{t+1} = f(k_t, n_t) \tag{1}$$

- We assume (i) Constant Returns to Scale (CRS, or homogeneous of degree one, meaning $f(\lambda k, \lambda n) = \lambda f(k, n)$), (ii) strictly increasing in both arguments, and ((iii) INADA condition, if necessary)

1.2 Preferences

- We assume infinitely-lived representative agent (RA).[1]
- We assume that preference of RA is (i) time-separable (with constant discount factor $\beta < 1$), (ii) strictly increasing in consumption (iii) strictly concave
- Our assumptions let us use the utility function of the following form:

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \tag{2}$$

- Initial capital stock k_0 is given.

  With these in hand the problem is,

$$\max_{\{c_t, n_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{3}$$

subject to[2]

$$k_{t+1} + c_t = f(k_t, n_t) \tag{4}$$

[1]For now, let's treat the economy as if there were only one agent in the economy. We might interpret it as the result of normalization (so the number of population is 1) of the economy with FINITE number of identical (sharing the same technology, preference, and allocation) agents. If we proceed to the economy with mass of zero measure agents, things will be not so trivial because changing allocation of one agent does not change the aggregate amount of resources in the economy (since, by assumption, measure of an agent is zero), but let's forget it for now.

[2]We can also define f (the production function) as including depreciation of capital. In the 1st class, Victor actually took this approach, but I modified the notation to make notation consistent across classes.

$$c_t, k_{t+1} \geq 0 \tag{5}$$

$$k_0 \text{ is given} \tag{6}$$

A solution to this problem is a sequence of consumption and capital accumulation decisions $\left\{c_t^*, k_{t+1}^*\right\}_{t=0}^{\infty}$. We would like this solution to exists, be unique and be Pareto Optimal. How do we do that? We can show these properties by rewriting the model in Arrow-Debreu language and using the theorems we know from 701.

Jan 25th, 2007

2 Arrow-Debreu (AD) Equilibrium

- In macroeconomics, we are interested in infinite- dimensional commodity spaces. We want to look at the relationship between competitive equilibrium and Pareto optimality in models with infinite-dimensional spaces. You looked at competitive equilibrium and Pareto optimality in 701, but the proofs of the FBWT and SBWT were done in the context of finite-dimensional commodity spaces, like the **Neoclassical Growth Model**. Here we want to show that the welfare theorems hold for economies with infinite dimensional spaces. To do this, we introduce the equilibrium concept of '*valuation (or AD) equilibrium*'.

- Before defining valuation equilibrium, we first need to define the environment, unlike the social planner problem, which is a problem of allocation, in a AD world we will have exchange among agents. This requires definition of markets in which the relevant commodities to be defined are traded.

1. $\mathcal{L}$, Commodity space:
   $\mathcal{L}$ is a topological vector space.

   Definition 1 (Vector Space)**.** *A vector space is a space where the operations addition and scalar multiplication are defined, and where the space is closed under these two operations. i.e. If we take two sequences $a = \{a_i\} \in \mathcal{L}$ and $b = \{b_i\} \in \mathcal{L}$, it must be that $a + b \in \mathcal{L}$. And if we take $k \in \mathcal{R}^+, k > 0$, it must be that $a \in \mathcal{L} \Rightarrow d = ka \in \mathcal{L} \ \forall\, k > 0$.*

Definition 2 (Topological Vector Space)**.** *A topological vector space is a vector space which is endowed with a topology such that the maps* $(x, y) \rightarrow x+y$ *and* $(\lambda, x) \rightarrow \lambda x$ *are continuous. So we have to show the continuity of the vector operations addition and scalar multiplication.*

2. $X \subset \mathcal{L}$, Consumption Possibility Set:
   Specification of the 'things' that people could do (that are feasible to them). X contains every (individually) technologically feasible consumption point.
   Characteristics of X: non-empty, closed and convex. Also, note that we will use the convention that output is positive while inputs are negative.

3. $U : X \rightarrow \mathcal{R}$, Specifies the preference ordering (utility function)

4. Y, Production possibility set, which must be non-empty, closed, convex and must have an interior point.

A simplifying assumption (which will be relaxed in a couple of weeks) we'll impose, is that there are many identical firms and agents. With this, we guarantee that they act competitively (take prices as given) and we only have to consider a representative agent who chooses what everyone else chooses (although everyone could do differently).

2.1 Prices

Prices (p) are continuous linear functions that are defined on our commodity space. More specifically, $p \in \mathcal{L}^*$, where $\mathcal{L}^*$ is the 'dual' of $\mathcal{L}$ (the set of all linear functions over $\mathcal{L}$); it may not always be possible to find a sequence of real numbers to represent this function as a dot product formulation (as we think of prices in finite dimensions)

Remark 1. *if* $\mathcal{L} = \mathcal{R}^n \Rightarrow \mathcal{L}^* = \mathcal{R}^n$*. In other words* $p(\ell) = \sum_{n=1}^{N} p_n \ell_n$ *defines the value of a bundle* ℓ*, composed of* $\{\ell_1, \ell_2, ..., \ell_N\}$

if $\mathcal{L} = \mathcal{R}^n \Rightarrow \mathcal{L}^* = \mathcal{R}^n$. In particular

$$p(\ell) = \sum_{n=1}^{N} p_n \ell_n$$

2.1.1 Definition of the Arrow-Debreu equilibrium

Definition 3 (Arrow-Debreu/Valuation Equilibrium). *ADE equilibrium is a feasible allocation (x^*, y^*) and a continuous linear function p^* such that,*

1. *x^* solves the consumer's problem:*

$$x^* \in Arg \max_{x \in X} u(x)$$

st

$$p^*(x) \leq 0$$

2. *y^* solves the firm's problem:*

$$y^* \in Arg \max_{y \in Y} p^*(y)$$

3. *markets clear (compatibility of actions)*

$$x^* = y^*$$

2.2 Welfare Theorems

Theorem 1 (First Basic Welfare Theorem). *Suppose that for all $x \in X$ there exists a sequence $\{x_n\}_{n=0}^{\infty}$ in$X$ converging to x with $u(x_n) \geq u(x)$ for all n (local nonsatiation). If an allocation (x^*, y^*) and a continuous linear functional p constitute a competitive equilibrium, then the allocation (x^*, y^*) is Pareto optimal.*

The FBWT tells us that the market is a good mechanism to allocate resources, which imply that 'you cannot do better'.

Definition 4 (Feasible allocation). *For all $x = y$ st $x \in X$ and $y \in Y$*

$$F(\varepsilon) = X \cap Y$$

where ε describes, through primitives, a particular economy

Theorem 2. *If u is strictly concave and $X \cap Y$ is convex and compact, then the solution to*

$$\max_{x \in X \cap Y} u(x) \tag{7}$$

exists and its unique.

This last result plus the FBWT imply that if an Arrow Debreu equilibrium exists, then it's a Pareto Optimal allocation.

From (7) we can calculate the allocation (x^*, y^*). Nevertheless, to construct an AD equilibrium, we still need a price function. The Second Basic Welfare Theorem will provide us with one.

Theorem 3 (Second Basic Welfare Theorem)**.** *If (i) X is convex, (ii) preference is convex (for $\forall x, x' \in X$, if $x' < x$, then $x' < (1-\theta)x' + \theta x$ for any $\theta \in (0,1)$), (iii) $U(x)$ is continuous, (iv) Y is convex, (v)Y has an interior point, then with any PO allocation (x^*, y^*) such that x^* is not a saturation point, there exists a continuous linear functional p^* such that (x^*, y^*, p^*) is a Quasi-Equilibrium with transfers((a) for $x \in X$ which $U(x) \geq U(x^*)$ implies $p^*(x) \geq p^*(\nu^*)$ and (b) $y \in Y$ implies $p^*(y) \leq p^*(y^*)$)*

Note that an additional assumption we are making for SBWT to go through in infinitely dimensional spaces is that Y has an interior point i.e.

$$\exists \bar{y} \in Y,\ B \subset Y,\ B \text{ open and } \bar{y} \in B$$

Also that the SBWT states that under certain conditions listed above, we can find prices to support any Pareto optimal allocation as a quasi equilibrium with transfers. Transfers are not relevant in our case since we are working in an representative agent environment with identical households. Taking care of the transfers still leaves us with Quasi-Equilibrium so SBWT by itself it does not say anything about the existence of Arrow-Debreu equilibrium. The following lemma takes care of this.

Lemma 1. *If, for (x^*, y^*, ν^*) in the theorem above, the budget set has cheaper point than x^* ($\exists x \in X$ such that $\nu(x) < \nu(x^*)$), then (x^*, y^*, ν^*) is a ADE.*

With the SBWT, we established that there exists a p that will support our PO allocation as a competitive equilibrium. What's the problem with this approach? SBWT only tells us that such a p exists, it doesn't tell us what it is. Also, we are not sure that p has a dot product representation. The next theorem deals with this nuance

Theorem 4. *(based on Prescott and Lucas 1972) If, in addition to the conditions of the SBWT, agents discount remote and/or unlikely states and u is bounded, then* $\exists \{q_t\}$ *such that*

$$p(x) = \sum_{t=0}^{\infty} q_t x_t \tag{8}$$

i.e. the price system has an inner product representation.

Remember, our main purpose is to be able to apply the welfare theorems to the most commonly used models in macroeconomics where we have an infinite-dimensional commodity space. Until now, we set up an environment (Arrow-Debreu economy) which consisted of the commodity space, consumption possibility set, production possibility set, and preferences) with infinite-dimensional commodity space and we stated that under certain conditions the Welfare Theorems hold in this environment. Now we will map the growth model into the environment that we talked about until here, and show that in the context of the growth model the assumptions we need for the Welfare Theorems are satisfied. Then we can conclude that any competitive equilibrium allocation is Pareto optimal and moreover we can support a PO allocation with some prices as a competitive equilibrium. This result is very important in macroeconomics. It helps us in solving for the equilibria. With the FBWT and SBWT, we can just solve for the PO allocations and then get the prices. This makes life much easier.

What is an allocation in this environment? An allocation is a pair (x, y). On the other hand, a feasible allocation is (x, y) such that x=y (agents' actions need to be compatible). What are the commodities we need to make tradable in this environment? Output, labor services, capital services. So lets define the commodity space.

$$\mathcal{L} = \{\{\ell_t\}^{t=0,\infty} = \{\ell_{it}\}_{i=1,2,3}^{t=0,\infty},\ s_{it} \in R : \sup_t |s_t| < \infty\ \}$$

so our commodity space will be the set of bounded sequences in sup norm. Subindexes 1,2 and 3 represent output, capital services and labor services respectively. The interested reader can refer to Stokey and Lucas (1989) for the reasons behind the choice of this particular space. Next is the definition of consumption possibility set X

$$X(k_0) = \{x \in \mathcal{L} = l_\infty^3 : \exists \{c_t, k_{t+1}\}_{t=0}^{\infty} \geq 0 \text{ such that}$$

$$
\begin{aligned}
k_{t+1} + c_t &= x_{1t} \qquad \forall t && (9) \\
x_{2t} &\in [-k_t, 0] \qquad \forall t && (10) \\
x_{3t} &\in [-1, 0] \qquad \forall t\}
\end{aligned}
$$

Now, let's define the production possibility set Y. The firm's problem is relatively simple as firms do not have intertemporal decisions. Firms just rent production factors and produce period by period.

$$
Y = \Pi_{t=0}^{\infty} \widehat{Y}_t : \widehat{Y}_t = \{y_{1t} \geq 0, y_{2t}, y_{3t} \leq 0 : y_{1t} \leq f(-y_{2t}, -y_{3t})\} \tag{11}
$$

Finally, preferences over this space $U : X \to R$

$$
U(x) = \sum_{t=0}^{\infty} \beta^t u(c_t(x)) \tag{12}
$$

c_t is unique given x, because each x implies a sequence $\{c_t, k_{t+1}\}_{t=0}^{\infty}$. If $x_{2t} = k_t$, $c_t = x_{1t} - x_{2t+1}$.