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1 Recursive, Stochastic Growth Model

In previous sections, we discussed random shocks, stochastic processes and histories. Now we will introduce those concepts into the growth model and analyze the recursive formulation.

In the growth model, the aggregate state variables are $\{z, K\}$, the technology shock and the total amount of capital. The individual state variables are $\{a, b_z\}$, the amount of assets of the household and the amount of state contingent claims (also known as Arrow securities[1])

The recursive stochastic growth model is

$$V(z,K,a,b_z) = \max_{c,a',b_{z'}} [u(c) + \beta \sum_{z'} \Gamma_{zz'} V(z',K',a',b_{z'})]$$

st

$$\begin{aligned} c + a' + \sum_{z'} q^{z'}(z,K)\, b_{z'} &= w(z,K) + R(z,K)\, a + b_z \\ K' &= G(z,K) \end{aligned}$$

where $q^{z'}(z,K)$ is the price of the state contingent claim related to state z' tomorrow. The solution to this problem for the household are functions that relate state variables to optimal asset accumulation

$$a' = g(z,K,a,b_z)$$

and to optimal security consumption for all future states

$$b_{z'}(z') = b(z,K,a,b_z)(z')$$

Looking at equilibrium conditions, we find that this way of writing the stochastic growth model has an 'overkill'. In equilibrium we get that

$$\begin{aligned} G(z,K) &= g(z,K,K,0) \\ b(z,K,a,0)(z') &= 0 \quad \forall z' \end{aligned}$$

the first condition is just the representative agent condition we have seen before. The second condition says that state contingent securities are not traded, so their demand is zero for all states of the world. This is due the fact that we are in a world of identical agents without uninsurable risks. Another way of looking at this is to use the no-arbitrage condition

[1] These securities pay 1 unit of the good when a particular state happens, and zero otherwise

$$1 = \sum_{z'} q(z,K)R(z',G(z,K))$$

which says that the household can save equally by buying securities or saving through capital.

Finally, we can reduce the individual state space for the household and rewrite the model as follows:

$$V(z,K,a) = \max_{c,a'}[u(c) + \beta \sum_{z'} \Gamma_{zz'} V(z',K',a')]$$

st

$$\begin{aligned} c + a' &= w(z,K) + R(z,K)\,a \\ K' &= G(z,K) \end{aligned}$$

This is because the household can secure herself a unit of consumption for sure next period either by saving or having a portfolio that pays 1 unit for sure next period at each possible state. Since the last option is an overkill, we drop it in order to work with our usual formulation.

Again, a solution to this problem is an optimal policy for asset accumulation $a' = g(z,K,a)$

Definition 1 *A RCE with stochastic shocks is a list* $\{V, G, g, w, R\}$ *such that*

1. *Given* $\{G, R, w\}$*,* V *and* g *solve the consumer problem*
2. R *and* w *solve the firm's problem*
3. *Representative agent condition is satisfied, i.e.*

$$g(z,K,K) = G(z,K)$$

2 Lucas Tree Model (Lucas 1978)

2.1 The Model

Suppose there is a tree which produces random amounts of fruit every period. We can think of these fruits as dividends and use d_t to denote the stochastic process of fruit production. Further, assume d_t follows a Markov process. Formally:

$$d_t \sim \Gamma(d_{t+1} = d_i \mid d_t = d_j) = \Gamma_{ji} \tag{1}$$

Let h_t be the history of realization of shocks, i.e., $h_t = (d_0,\ d_1,\ ...,\ d_t)$. Probability that certain history h_t occurs is $\pi(h_t)$.

The representative household in the economy consumes the only good, which is the fruit. Consumers maximize:

$$\sum_t \beta^t \sum_{h_t \in H_t} \pi(h_t) u(c_t) \tag{2}$$

Since we are assuming a representative agent in the economy who posses no storage technology, in the unique equilibrium the representative household eats all the dividends every period. Hence, the lifetime utility of the household will be:

$$\sum_t \beta^t \sum_{h_t \in H_t} \pi(h_t) u(d_t) \tag{3}$$

with

$$p_0 = 1 \tag{4}$$

Note that we are considering the Arrow-Debreu market arrangement, with consumption goods in period 0 as a numeraire.

2.2 First Order Condition

Take first order condition of the above maximization problem:

$$\frac{p(h_t)}{p_0} = p_t(h_t) = \frac{\beta^t \pi(h_t) u'(c(h_t))}{u'(c(h_0))} \tag{5}$$

By combining this FOC with the following equilibrium condition:

$$c(h_t) = d_t \ \forall t, h_t \tag{6}$$

We get the expression for the price of the state contingent claim in the Arrow-Debreu market arrangement.

$$p_t(h_t) = \frac{\beta^t \pi(h_t) u'(d(h_t))}{u'(d(h_0))} \tag{7}$$

2.3 LT in Sequential Markets

In sequential markets, the household can buy and sell fruits in every period, and the tree (the asset). To consider the trade of the asset, let s_t be share of asset and q_t be the asset price at period t. The budget constraint at every time-event is then:

$$q_t s_{t+1} + c_t = s_t(q_t + d_t) \tag{8}$$

Thus, the consumer's optimization problem turns out to be:

$$\max_{\{c_t(h_t), s_{t+1}(h_t)\}_{t=0}^{\infty}} \sum_t \beta^t \sum_{h_t \in H_t} \pi(h_t) u(c_t(h_t)) \tag{9}$$

subject to

$$q_t(h_t)s_{t+1}(h_t) + c_t(h_t) = s_t(h_{t-1})[q_t(h_t) + d_t] \tag{10}$$

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2.4 Lucas Tree in Recursive Language

Looking at the recursive version of the same problem with denoting discrete state variable as subscripts (a note on notation: this is the same as having $V(d,s)$, but since the amount of fruit is linked one to one to the shock, we can drop d and use the state as a subscript)

$$V_i(s) = \max_{s',c} u(c) + \beta \sum_{d'} \Gamma_{ij} V_j(s')$$

$$s.t. \quad c + s\prime q_i = s[q_i + d_i]$$

In equilibrium, the solution has to be such that $c = d$ and $s^{'} = 1$. Impose these on the FOC and get the prices that induce the agent to choose that particular allocation. Then the FOC for a particular state i would imply,

$$q_i = \beta \sum_j \Gamma_{ij} \frac{u'(d_j)}{u'(d_i)} [q_j + d_j] \tag{11}$$

where

$$q_i = \frac{p(h_{t-1}, d_i)}{p(h_t)}$$

and $p(.)$ are the prices we derived from the AD setting.

A closer look tells us that we can calculate all prices in just one system of equations. Taking FOCs, at an equilibrium we have

$$p_i u_c(d_i) + \beta \sum_j \Gamma_{ij} \frac{\partial V^j(s')}{\partial s'} = 0$$

using the envelope condition

$$\frac{\partial V^j(s')}{\partial s'} = [p_i + d_i] u_c(d_i)$$

hence,

$$p_i u_c(d_i) = \beta \sum_j \Gamma_{ij} [p_j + d_j] u_c(d_j) \qquad \forall i$$

Stacking each equation and forming matrices

$$\begin{bmatrix} u_c(d_1) & 0 & \cdots & 0 \\ 0 & u_c(d_2) & \ddots & \vdots \\ 0 & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & u_c(d_{n_d}) \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_{n_d} \end{bmatrix} = \beta \begin{bmatrix} u_c(d_1) & 0 & \cdots & 0 \\ 0 & u_c(d_2) & \ddots & \vdots \\ 0 & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & u_c(d_{n_d}) \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_{n_d} \end{bmatrix} + \beta\Gamma \begin{bmatrix} u_c(d_1) & 0 & \cdots & 0 \\ 0 & u_c(d_2) & \ddots & \vdots \\ 0 & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & u_c(d_{n_d}) \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{n_d} \end{bmatrix}$$

In matrix notation

$$\begin{aligned} p &= \beta u_c^{-1} \Gamma u_c p + \beta u_c^{-1} \Gamma u_c d \\ \left[I - \beta u_c^{-1} \Gamma u_c\right] p &= \beta u_c^{-1} \Gamma u_c d \\ p &= \left[I - \beta u_c^{-1} \Gamma u_c\right]^{-1} \left[\beta u_c^{-1} \Gamma u_c d\right] \end{aligned}$$

2.5 Asset Pricing

Because in a complete market any asset can be reproduced by buying and selling contingent claims at every node, we can use this model as a powerful asset pricing formula. For example, take the option of selling shares at price $\bar{p}$ tomorrow. Since tomorrow we'll have the option to sell, we exercise only if

$$\bar{p} - p_i > 0 \qquad \forall i$$

Then, the value of this option (if we are in state i), is

$$\varphi_i(\bar{p}) = \sum_j q_{ij} \max\{\bar{p} - p_j, 0\}$$

where $q_{ij} = \beta \Gamma_{ij} u_c(d_j) [u_c(d_i)]^{-1}$.

Our next example is the option that can only be executed 2 periods from now. In that case, we have

$$\varphi_i^2(\bar{p}) = \sum_j q_{ij} \sum_l q_{jl} \max\{\bar{p} - p_l, 0\}$$

Finally, take the option that can be exercised tomorrow or the day after tomorrow. The day after tomorrow, we exercise the option iff

$$\bar{p} - p_l > 0$$

where l is the state the day after tomorrow. At the previous node (if we haven't exercised the option yet and the state is j), the value of the option is

$$\sum_l q_{jl} \max\{\bar{p} - p_l, 0\}$$

Hence, if the state today is i, the value of the option is

$$\hat{\varphi}_i^2(\bar{p}) = \sum_j q_{ij} \max\left\{\bar{p} - p_j, \sum_l q_{jl} \max\{\bar{p} - p_l, 0\}\right\}$$

2.6 Aggregate Rates of Return

Given the notation and the previous analysis, we can ask some questions on rates of return in the model economy. On the one side, consider an asset that gives a certain return next period, risk free (e.g., a treasury bond, that pays 1 no matter the state tomorrow) which we can denote as r^f. Using the no-arbitrage condition we know that:

$$\begin{aligned} 1 + r^f &= \left[\sum_j q_{i,j} 1\right]^{-1} \\ &= \left[\sum_j \beta \Gamma_{ij} \frac{u_c[d_j]}{u_c[d_i]}\right]^{-1} \end{aligned}$$

On the other hand, consider a risky asset, which pays proportionally to the aggregate state of the economy tomorrow (fruit is $d_j / \sum \Gamma_{ij} d_j$). Denote its return as r^R. By the same argument as before, we have that

$$\begin{aligned} 1 + r^R &= \left[\sum_j q_{i,j} \frac{d_j}{\sum \Gamma_{ij} d_j}\right]^{-1} \\ &= \left[\sum_j \beta \Gamma_{ij} \frac{u_c[d_j]}{u_c[d_i]} \frac{d_j}{\sum \Gamma_{ij} d_j}\right]^{-1} \end{aligned}$$

By Jensen's inequality

$$\sum_j \beta \Gamma_{ij} \frac{u_c[d_j]}{u_c[d_i]} > 1$$

which means that $1 + r^f < \frac{1}{\beta}$
Also,

$$\sum_j \beta\Gamma_{ij}\frac{u_c[d_j]}{u_c[d_i]} > \sum_j \beta\Gamma_{ij}\frac{u_c[d_j]}{u_c[d_i]}\left(\frac{d_j}{\sum \Gamma_{ij} d_j}\right)$$

Hence, $r^f < r^R$, or the equity premium. The intuition is that in a world with risk averse individuals, the demand for risk free assets is higher, which drives its price (today) upwards and then its return downwards.

If we go to the data, we find a BIG equity premium, which cannot be accounted by the model in any way. This is called the equity premium puzzle. For further discussion, see Mehra and Prescott (1985)