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1 Economy with Heterogeneous Agents

1.1 Introduction

So far, in environments we have analyzed, the type of agents is the same always. If the number of type of agents is small (as the example we did in class with only two different types) it's easy to keep track of all the types, and so is to define an equilibrium. From now on, we will consider economies with (i) many agents who are very different among themselves at a given time (cross-section), and (ii) change their types over time.

An immediate question is: what is a considered MANY agents?

- N large, countable
- a continuum, uncountable and infinite

We will use models with continuum of agents, since we can use math tools (such as measure theory) and other useful concepts (like continuity) to describe agents.

1.2 Introduction to Measure Theory

1.2.1 Intuition

Measure theory can be understood nicely by comparing to the notion of weight. Measure is about "measuring" a mass in a mathematically consistent way, which is similar to weighting a mass. Therefore, intuitively the following properties are expected to be satisfied by measures:

1. $measure(nothing) = 0$
2. if $A \cap B = \emptyset \Rightarrow measure(A + B) = measure(A) + measure(B)$

These properties are intuitive with weight. The weight of nothing is zero. If a body is 200 pounds, and you chop off a hand from the body and put the hand and the rest of the body together on the scales, they must weight 200 pounds. Now consider an economy with many agents. The measure of nobody in the economy is zero. If a measure of the total population is normalized to one, and you take away the rich people from the population and measure the sum of rich people and the rest of the population, they must have measure one.

In macro models with heterogenous agents, we are interested in how to measure agents with different characteristics (wealth, earnings, etc.).

1.2.2 Definitions

Definition 1 *For a set S, $\mathcal{S}$ is a set of subsets of S.*

Definition 2 *σ-algebra $\mathcal{S}$ is a set of subsets of S, with the following properties:*

1. $S, \emptyset \in \mathcal{S}$
2. $A \in \mathcal{S} \Rightarrow A^c \in \mathcal{S}$ *(closed in complementarity)*
3. *for* $\{B_i\}_{i=1,2\ldots}$, $B_i \in \mathcal{S} \Rightarrow [\cap_i B_i] \in \mathcal{S}$ *(closed in countable intersections)*

The intuition of the property 2 of σ-algebra is as follows. If we chop off a hand from a body, and if the hand is an element of $\mathcal{S}$, the rest of the body is also an element of $\mathcal{S}$. Soon we will define measure as a function from σ-algebra to a real number Then the property of σ-algebra implies that if we can measure the chopped hand, we can measure also the rest of the body.

Examples of σ-algebra are the follows:

1. Everything, aka the power set (all the possible subsets of a set S)
2. $\{\emptyset, S\}$
3. $\{\emptyset, S, S_{1/2}, S_{2/2}\}$ where $S_{1/2}$ means the lower half of S (imagine S as an closed interval on $\mathcal{R}$).

If $S = [0, 1]$ then the following is NOT a $\sigma-$ algebra

$$\mathcal{S} = \left\{\emptyset, [0, \frac{1}{2}), \left\{\frac{1}{2}\right\}, \left[\frac{1}{2}, 1\right], S\right\}$$

Remark 3 *A convention is (i) use small letters for elements, (ii) use capital letters for sets, (iii) use "fancy" letters for set of subsets.*

Definition 4 *A measure is a function $x : \mathcal{S} \to \mathcal{R}_+$ such that*

1. $x(\emptyset) = 0$
2. *if* $B_1, B_2 \in \mathcal{S}$ *and* $B_1 \cap B_2 = \emptyset \Rightarrow x(B_1 \cup B_2) = x(B_1) + x(B_2)$
3. *if* $\{B_i\}_{i=1}^{\infty} \in \mathcal{S}$ *and* $B_i \cap B_j = \emptyset$ *for all* $i \neq j \Rightarrow x(\cup_i B_i) = \sum_i x(B_i)$ *(countable additivity)*

In English, countable additivity means that measure of the union of countable disjoint sets is the sum of the measure of these sets.

Definition 5 *Borel-σ-algebra is (roughly) a σ-algebra which is generated by a family of open sets (generated by a topology).*

Since a Borel-σ-algebra contains all the subsets generated by intervals, you can recognize any subset of a set using Borel-σ-algebra. In other words, Borel-σ-algebra corresponds to complete information.

You might find that a σ-algebra is similar to a topology. Topology is also a set of subsets, but its elements are open intervals and it does not satisfy closedness in complementarity (complement of an element is not an element of a topology). Very roughly, the difference implies that topologies are useful in dealing with continuity and σ-algebra is useful in dealing with measure.

Definition 6 *Probability (measure) is a measure such that $x(A) = 1$*

Lets apply these basic notions of measure theory to a simple set of problems: industry equilibria with many firms

2 Partial Equilibrium Industry Theory

We will consider models were prices are given exogenously, hence our analysis is a partial equilibrium one. For that reason, think that the environment is a small industry producing "flip flops" (things nobody cares about).

There is an inverse demand function $y^d(p)$, where p is the price of the good. A firm in this industry is indexed by it's productivity $s \in S = [\underline{s}, \bar{s}]$ and produces according to $sf(n)$ (the production function depends on labor only). Firms are competitive in the output market as well as in the labor market.

Problem of the firm is

$$\max_n psf(n) - wn$$

from the FOC, we get $psf'(n) = w$, which implicitly defines the solution of the firm $n^* = n(s,p)$. The profits are defined

$$\Pi(s,p) = psf(n^*[s,p]) - wn[s,p]$$

Given a price p, to calculate the output of the industry we need a measure X of firms (the distribution of firms according to their s).

Let

- $\mathcal{S}$ be the borel σ-algebra of $[\underline{s}, \bar{s}]$
- $X : \mathcal{S} \to \mathbb{R}$ be a measure

Then, the supply of the industry is defined as

$$y^s(p) = \int_{\underline{s}}^{\bar{s}} sf(n[s,p])\, X(ds)$$

In this economy, the measure of firms with respect to its type $X(s)$ is an uninteresting object since its given exogenously. We need to add some tweaks in order to have some economics.

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Now suppose that the firm will only operate next period with probability $(1-\delta)$. With probability δ it will die. In that case, the two period profit of the firm is,

$$\pi_2 = [p^* sf(n^*) - wn^*]\left[1 + \frac{1-\delta}{1+r}\right]$$

Now consider the infinite periods profit of the firm,

$$\begin{aligned}\pi_\infty &= [p^* sf(n^*) - wn^*] \sum_{t=0}^{\infty}\left(\frac{1-\delta}{1+r}\right)^t \\ &= [sf(n^*) - wn^*]\left(\frac{1+r}{r+\delta}\right)\end{aligned}$$

The zero profit condition is that the profit from entry is equal to the cost of entry, denoted by c_e. This condition says that there are no further incentives to enter the industry:

$$c_e = \pi_\infty$$

Define $x : \mathcal{S} \to R$ as the measure of firms, where $\mathcal{S}$ is the $\sigma - algebra$ defined on the set S

An industry equilibrium is a set $\{p^*, y^*, n^*, x^*(s)\}$, such that:

1) $p^* = p(y^*)$ (demand is satisfied)

2) $y^* = x^*(s, p^*)\ \ s\ f(n^*)$ (feasibility)
3) Firms optimize: $n^* \in \arg\max_n \ \ p^* sf(n) - w^* n$

4) Zero profit condition: $c_e = \pi_\infty$
So far we still have no interesting dynamics. The measure of firms is exogenous and no economic decisions are involved. In order to introduce more interesting elements, we need some more mathematical tools:

3.1 Digression on Transitions and Updating Operators

Definition 7 *A function $f : S \to R$ is measurable with respect to the $\sigma - A\lg ebra$ $\mathcal{S}$, if for every $a \in R$, $B \equiv \{s \in S : \ f(s) \leq a\} \in \mathcal{S}$.*

Definition 8 *A transition function is a mapping* $Q : S \times \mathcal{S} \to [0,1]$*, such that:*

1) $Q(s,.)$ is a probability measure for every $s \in S$.
2) $Q(.,B)$ is a measurable function with respect to $\mathcal{S}$, for every $B \in \mathcal{S}$.

Based on the transition function Q we can define an updating operator T. This new object satisfies $x'(B) = T(x(B), Q)$. In words, T gives us the measure of (here) firms in the subset B in the next period $(x(B))$, based on the transition function and the measure of firms in this subset today.

3.2 Industry Equilibria, Tweak #1

Suppose that each firm has to pay a cost of entry c_∞, and the productivity shock is drawn from the distribution $\gamma(s)$. Once the firm draws s it keeps it forever.

An industry equilibrium is a set $\{p^*, n^*(s,p^*), N_e^*, x^*\}$, such that:

1) $n^*(s,p^*) \in \arg\max\ [p^* s f(n) - wn]$
2) $y^D(p^*) = \int\limits_S s\ f(n^*(s,p^*))\ dx^*$
3) $c_\infty = \int\limits_S \hat{\Pi}_\infty(s,p^*) d\gamma(s)$, where $\hat{\Pi}_\infty = \sum_{t=0}^{\infty} \left(\frac{1-\delta}{1+r}\right)^t \Pi(s,p^*)$.
4) $x^*(B) = (1-\delta)\, x^*(B) + N_e^* \gamma(B)$, for all $B \in \mathcal{S}$, and
5) $N_e^* = \delta x^*(S)$.

Note that here, the distribution of firms completely reflects the distribution from which they draw their productivity shocks, $\gamma(s)$. This is because what types of firms remain or what types of firms exit is not an issue since there is exogenous entry and exit. For example, if exit was endogenous we would expect the 'bad' firms to exit and the better ones to stay, and therefore the type distribution of incumbent firms would be different than the initial distribution $\gamma(s)$. But in our case, the distribution of incumbents and the initial type distribution are identical.

So this model is not interesting because it STILL has no economics.

3.3 Ind. Eq., Tweak #2

Here s is drawn from $\gamma(s)$ as before, but after the initial shock is obtained, $s' \sim \Gamma_{ss'}$. We will assume that Γ satisfies First Order Stochastic Dominance. This means that

$$For\ s_1, s_2 \in S,\ \ s_1 < s_2 \Rightarrow \int_{\tilde{s}}^{\bar{s}} \Gamma(s_1, s)\, ds \leq \int_{\tilde{s}}^{\bar{s}} \Gamma(s_2, s)\, ds$$

Moreover, we introduce a fixed cost that the firm has to pay in every period, c_f . The recursive formulation of the firm's problem is:

$$\Omega(s) = \max\left[0, -c_f + \max_n\{psf(n) - wn\} + \tfrac{1}{1+r}\int \Omega(s')\,\Gamma(ds' \mid s)\right]$$

Definition 9 *A stationary equilibrium for the economy described by Tweak 2 is a list* $\{p^*, N_e^*, s^*, n^*(s,p^*), x^*, Q^*\}$ *such that:*

1) $n^*(s,p^*)$ maximizes profits and $s^* = s^*(p^*)$ (threshold is optimal).
2) Free entry is satisfied: $c_\infty = \Pi_\infty$.
3) Market clearing: $y^D(p^*) = \int\limits_S s\ f(n^*(s,p^*))\,dx^*$.
4) The measure of firms is stationary: $x^*(B) = \int\limits_S Q(s,B)\ dx^* + N_e^*\gamma(B \cap [s^*, \bar{s}])$
5) $Q(s,B) = \Gamma\left[s, B \mid [\underline{s}, s^*)\right]$, (where Γ is implied by the Markovian transition matrix).

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3.4 Ind. Eq. Tweak #3

In this section we discuss Industry equilibria with employment and capital as a state variables. The firm owns capital and rents labor in a competitive market. The key ingredient here is that there are costs of adjusting the level of capital and employment. Hence, the state variables for a particular firm are $\{s,k,n\}$. The recursive problem of a firm is given by

$$\begin{aligned}
\Omega(s,k,n) &= \max_{x_k,x_n} psf(k,n) - wn - x_k - x_n + \frac{1}{1+r}\sum_{s'}\Gamma_{ss'}\Omega(s',k',n') \\
s.t. & \\
k' &= (1-\delta)k + \phi(x_k) \\
n' &= (1-\rho)n + \psi(x_n) \\
s' &\sim \Gamma \quad (Markov)
\end{aligned}$$

where δ is the depreciation rate and ρ is the exogenous separation rate. Functions ϕ, ψ represent non-linear adjustment costs of modifying employment/capital levels. Some assumptions:

- $\phi(0) = \psi(0) = 0$
- $\phi', \psi' \in [0,1]$

- $\phi'', \psi'' < 0$ (convex costs)

The solution to this problem are two policy functions for $x_k^* = x_k^*(s,k,n)$ and $x_n^* = x_n^*(s,k,n)$

In this economy, the transition function is[1]

$$Q(\{s,k,n\},\mathcal{B}) = \sum_{s'\in B_s} 1_{\{\phi(x_k(s,k,n))+(1-\delta)k\in B_k\}} 1_{\{\psi(x_n(s,k,n))+(1-\rho)n\in B_n\}}$$

where $1_{\{.\}}$ is the indicator function. Then, the updating operator is given by

$$X' = \int_s \int_k \int_n Q(\{s,k,n\},\mathcal{B})X(ds,dk,dn)$$

Note that in this model, all firms have value, since there is no endogenous exit by assumption. Some key predictions of the model:

- when a firm receives a good shock, employment and capital react sluggishly, since its costly to reverse levels
- Aggregate production is lower than in the previous models

[1]In order to ease notation, here we take the case where B_s, B_k, B_n are rectangles