Suggested Solutions to Macro 702, Sp 2007, Midterm.

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Growth Models (Catching up with the Jones)

There is an economy with many identical consumers and infinite time. Consumers have preferences

$$E\left\{\sum_{t=0}^{\infty}\beta^t\, u(c_t, n_t, C_{t-1})\right\}$$

where c_t is own consumption at time t, n_t is the fraction of time worked by the agent at time t and C_{t-1} is the economy wide average consumption in period $t-1$. The first partial derivative of u is positive while the others are negative. These agents hate the idea that other people have consumed a lot in the past. Output can be produced with labor and capital according to a standard neoclassical production function

$$z_t\, F(K_t, N_t)$$

where K_t, is capital. Shocks to productivity z have finite support and follow a Markov chain with transition matrix Γ. Capital depreciates at rate δ. Output can be used either for consumption or for investment purposes.

1. (10 points) **Define an Arrow-Debreu competitive equilibrium. Carefully define the commodity space, and the consumption and production possibility sets.**

   The shocks to the production function have a bounded support $z_t \in Z_t = \{z^0, z^1, ..., z^{n_z}\}$. Then, we can define histories of length t as $h_t = \{z_0, z_1, ..., z_t\}$ with $h_t \in H_t = Z_0 \times Z_1 \times \cdots \times Z_t$. Finally, define a probability measure over histories, $\pi(h_t)$. Then, the commodity space is

$$L = \left\{(l_{1t}(h_t), l_{2t}(h_t), l_{3t}(h_t)) \in \mathbb{R}^3 \ \ \forall t, h_t: \ \ \sup|l_{it}(h_t)| < \infty \ \ \forall i, t, h_t\right\}$$

   Note that this commodity space is not modified by the existence of non-standard preferences. It's a general object, identical to the usual case.

   The consumption possibility IS modified:

$$\begin{aligned} X = \{x \in L : \exists \{c_t(h_t), k_{t+1}(h_t)\}_{t,h_t} \ \ s.t. & \\ x_{1t}(h_t) + (1-\delta)k_t(h_{t-1}) &= c_t(h_t) + k_{t+1}(h_t) \\ x_{2t}(h_t) &\in [-k_t(h_{t-1}), 0] \\ x_{3t}(h_t) &\in [-1, 0] \\ k_0, C_{-1} \ \ given\} & \end{aligned}$$

   The last line takes into account that an initial condition for aggregate consumption is needed. Alternatively, one can think of the more 'natural' case where $C_{-1} = 0$, in which case some assumption must be made in order to guarantee that $u[c, n, 0] \neq \infty$ or we would be dealing with a problem which is not well defined.

Next, the production possibility set is standard: $Y = \Pi_{t,h_t} Y_t(h_t)$ where

$$Y_t(h_t) = \{(y_{1t}(h_t), y_{2t}(h_t), y_{3t}(h_t)) : 0 \le y_{1t}(h_t) \le z_t F(-y_{2t}(h_t), -y_{3t}(h_t))\}$$

Preferences are defined as

$$U(x) \equiv \sum_{t=0}^{\infty} \beta^t \sum_{h_t \in H_t} \pi(h_t) u\left[c_t(x), n_t(x), C_{t-1}(x)\right]$$

where it is understood that $c_t(x), n_t(x), C_{t-1}(x)$ are the streams of consumption, labor services and past consumption which are derived from a particular $x \in X$. Since we are in a representative agent economy, note that $c_t(x) = C_t(x)$.

Finally, we can define equilibrium:

Definition (AD Equilibrium):

An Arrow-Debreu (Valuation) equilibrium is an allocation x^*, y^* and a continuous linear functional v^* such that

1. given v^*, x^* solves the problem of the household:

$$\begin{aligned} x^* &\in \arg\max_{x \in X} U(x) \\ &s.t. \\ v^*(x) &\le 0 \end{aligned}$$

2. given v^*, y^* solves the problem of the firm:

$$y^* \in \arg\max_{y \in Y} v^*(y)$$

3. markets clear:

$$x^* = y^*$$

2. (5 points) **State the two welfare theorems.**

Definition (FBWT):

IF

- the preferences of consumers arenonsatiated ($\exists \{x_n\} \in X$ that converges to $x \in X$ such that $U(x_n) > U(x)$)

THEN An allocation (x^*, y^*) of an ADE (v^*, x^*, y^*) is Pareto Optimal.

Definition (SBWT):

IF

- X is convex

- preference is convex (for $\forall x, x' \in X$, if $x' < x$, then $x' < (1-\theta)x' + \theta x$ for any $\theta \in (0,1)$)
- $U(x)$ is continuous
- Y is convex
- Y has an interior point

THEN With any PO allocation (x^*, y^*) such that x^* is not a satiation point, there exists a continuous linear functional p^* such that (x^*, y^*, p^*) is a Quasi-Equilibrium ((a) for $x \in X$ which $U(x) \geq U(x^*)$ implies $p^*(x) \geq p^*(x^*)$ and (b) $y \in Y$ implies $p^*(y) \leq p^*(y^*)$)

3. (5 points) **Briefly describe what may go wrong for the first welfare theorem to hold.**

   Notice that we know for sure that the FBWT doesn't hold. The competitive equilibrium will have FOC's where the agent doesn't consider C_{t-1} as a control variable. The social planner's problem solution (which is PO), makes $c = C$, hence the FOC's are different.

   The reason for the failure of the FBWT, is the presence of the externality C_{t-1}: individual agents don't take into account that they will hate it tomorrow if they consume a lot today.

Suppose now that the household owns capital and rents it to firms.

4. (10 points) **Define a recursive competitive equilibrium. Make sure that you list the state variables. Briefly describe what is NOT standard in this problem.**

   In order to define a Recursive Competitive equilibrium, we have to identify first the state variables[1]:

   -Aggregate: z, K, C_{-1}

   -Individual: a (assets)

   Also, since we are in a representative agent economy, we can shut down all financial markets (in order to ease notation).

   Now, the problem of the household is

$$
\begin{aligned}
V(z, K, C_{-1}, a) &= \max_{c,n,a'} \; u[c, n, C_{-1}] + \beta \sum_{z'} \Gamma_{zz'} V(z', K', C, a') \\
s.t. & \\
c + a' &= R(z, K, C_{-1})a + w(z, K, C_{-1})n \\
K' &= G(z, K, C_{-1}) \\
C &= \mathcal{C}(z, K, C_{-1}) \\
z' &\sim \Gamma \quad (Markov)
\end{aligned}
$$

[1] In recursive notation, let C_{t-1} be denoted as C_{-1}

with solutions

$$\begin{aligned} a' &= g(z, K, C_{-1}, a) \\ n &= h(z, K, C_{-1}, a) \\ c &= \phi(z, K, C_{-1}, a) \end{aligned}$$

Strictly speaking, function ϕ is redundant, since

$$\phi = R(\cdot)a + w(\cdot)h(\cdot) - g(\cdot)$$

Definition (RCE):

A Recursive Competitive Equilibrium for this economy is a list of functions $\{V^*, g^*, h^*, \phi^*, R^*, w^*, G^*, \mathcal{C}^*\}$ such that

1. Given $\{R^*, w^*, G^*, \mathcal{C}^*\}$, $\{V^*, g^*, h^*, \phi^*\}$ solve the problem of the household, i.e.

$$V^* = u\left[R^*a + w^*h^* - g^*\right] + \beta \sum_{z'} \Gamma_{zz'} V(z', G^*, \mathcal{C}^*, g^*)$$

2. Firms optimize, i.e.

$$\begin{aligned} R^* &= zF_1(K, h^*(z, K, C_1, K)) + 1 - \delta \\ w^* &= zF_2(K, h^*(z, K, C_1, K)) \end{aligned}$$

3. Consistency:

$$\begin{aligned} g^*(z, K, C_{-1}, K) &= G^*(z, K, C_{-1}) \\ \phi^*(z, K, C_{-1}, K) &= \mathcal{C}^*(z, K, C_{-1}) \\ &= zF(K, h^*(z, K, C_{-1}, K)) + (1-\delta)K - G^*(z, K, C_{-1}) \end{aligned}$$

Now assume that the government taxes/subsidizes labor at rate τ and returns the proceeds in a lump sum manner.

5. (10 points) **Write a formula that links the equilibrium transfer as a function of the τ and the state variables.**

Let's write the individual's budget constraint with this new policy, where Tr is the lump sum transfer from the government:

$$c + a' = R(z, K, C_{-1})a + (1-\tau)w(z, K, C_{-1})n + Tr(z, K, C_{-1})$$

Since τ is fixed, the equilibrium transfer depends on the assumption with respect to the governments budget. The easiest way to go is to impose period by period tight budgets (no debt). Then, the aggregate transfer is:

$$Tr(z, K, C_{-1}) = \tau w^*(z, K, C_{-1})h^*(z, K, C_{-1}, K)$$

6. (10 points) **Imagine that the utility function is separable in all its arguments. Does this imply any simplification to your answer to the definition of recursive competitive equilibrium? Explain.**

Note that the only reason we need the lagged aggregate consumption as a state variable is because it is assumed to effect the optimal decision of the household AT THE MARGIN. If the utility function is separable in all its arguments then the FOC with respect to n and a' and the envelope condition (thus the euler equation) will not depend on C_{t-1} and the household no longer needs this information to determine its optimal behavior. Thus we can drop C_{t-1} as a state variable and simplify our definition of RCE.

Note that preferences still show 'habit' persistence, but the household optimally ignores it.

Lucas tree

Assume there is a representative agent economy. Each agent owns a tree that produces fruit d_t which follows a Markov chain with transition matrix Γ. In addition, each agent has a backyard that yields one unit of a special type of fruit that gives the same utility to the agents as the one from the tree but that cannot be traded due to regulations by the Health Department. The agent has preferences given by

$$E\left\{\sum_{t=0}^{\infty}\beta^t \frac{c_t^{1-\sigma}}{1-\sigma}\right\}$$

7. (5 points) Define equilibria recursively.

d_t has a finite support: $d_t \in \{d_1, d_2, ..., d_I\}$, $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$. Then the RCE of this economy is a list of functions, $\{V(s,i), g^s(s,i), g^c(s,i), p(i)\}$ s.t.

1. Given the prices, $p(i)$, $\{V(s,i), g^s(s,i), g^c(s,i)\}$ solves the agent's problem:

$$V(s,i) = \max_{c,s'} u(c+1) + \beta \sum_{j=1}^{I} \Gamma_{ij} V(s', j)$$

$$\texttt{s.t.} \qquad c + p(i)s' = (p(i) + d_i)s$$

$$s' = g^s(s,i), c = g^c(s,i) \texttt{ solve the problem}$$

2. Consistency: Prices are s.t. $g^s(1,i) = 1$, $g^c(1,i) = d_i$.

8. (15 points) How much would an individual agent pay to have the restriction on sales of backyard fruit lifted?

Firstly let's define the RCE for this new economy (i.e. agents can buy/sell backyard fruits). The RCE of the new economy is a list of functions, $\{\Phi(s,i), \tilde{g}^s(s,i), \tilde{g}^c(s,b,i), p(i)\}$ s.t.

1. Given the prices,$\{p(i)\}$, $\{\Phi(s,i), \tilde{g}^s(s,b,i), \tilde{g}^c(s,b,i)\}$ solves the agent's problem:

$$\Phi(s,i) = \max_{c,s'} u(c) + \beta \sum_{j=1}^{I} \Gamma_{ij} \Phi(s',j)$$

$$\texttt{s.t.} \qquad c + p(i)s' = (p(i) + d_i)s + 1$$

$$s' = \tilde{g}^s(s,i), c = \tilde{g}^c(s,i) \texttt{ solve the problem}$$

2. Consistency: Prices, $\{p(i)\}$, are s.t. $\tilde{g}^s(1,i) = 1$, $\tilde{g}^c(1,i) = d_i + 1$.

So in both of the equilibriums (the one with the restriction and the other one without restriction) representative agent (who holds 1 share of tree and 1 backyard fruit) choose to hold her/his tree and consume $d_i + 1$ fruits every period (i.e. $\Phi(1,i) = V(1,i)$). Then she/he wouldn't pay anything to have the restriction lifted. However the question asks "how much an individual agent would pay to have the restriction lifted", doesn't mention the representative agent who holds 1 share of the tree. Then an individual agent in state i who holds s share of tree would pay x (in terms of tree shares) to have the restriction lifted s.t.:

$$\Phi(s - x, i) = V(s,i)$$

Also the cost of having the restriction lifted is $p(i)x$ in terms of today's consumption.

9. (5 points) Write a formula for an option to buy land tomorrow at price p_1 and then reselling it at price p_2 the period after.

Firstly let's introduce the model with agents trading backyards:

The RCE of this new economy is a list of functions, $\{\Phi(s,b,i), g^s(s,b,i), g^b(s,b,i), g^c(s,b,i), p(i), \tilde{p}(i)\}$,(where $g^b(s,b,i)$ is the number of backyards agent keeps for the next period, and $\tilde{p}(i)$ is the price of 1 backyard in terms of today's consumption) s.t.

1. Given the prices,$\{p(i), \tilde{p}(i)\}$, $\{\Phi(s,b,i), g^s(s,b,i), g^b(s,b,i), g^c(s,b,i)\}$ solves the agent's problem:

$$\Phi(s,b,i) = \max_{c,s',b'} u(c) + \beta \sum_{j=1}^{I} \Gamma_{ij} \Phi(s',b',j)$$

$$\texttt{s.t.} \qquad c + p(i)s' + \tilde{p}(i)b' = (p(i) + d_i)s + (\tilde{p}(i) + 1)b$$

$$s' = g^s(s,b,i), b' = g^b(s,b,i), c = g^c(s,b,i) \texttt{ solve the problem}$$

2. Consistency: Prices, $\{p(i), \tilde{p}(i)\}$, are s.t. $g^s(1,1,i) = 1$, $g^b(1,1,i) = 1$, $g^c(1,1,i) = d_i + 1$.

Now let's introduce Arrow securities, and the prices of these state contingent claims:

$$q_{ij} = \beta \Gamma_{ij} \frac{u'(d_j + 1)}{u'(d_i + 1)}$$

Now we can price the option mentioned in the question by using these Arrow securities. From now on today is t, tomorrow is $t+1$ and the day after tomorrow is $t+2$.

What I assume about the option is that if an agent buys land in $t+1$ at price p_1 using the option, he can sell this land in $t+2$ either at price p_2 or at price $\tilde{p}(t+2)$. However if the agent didn't buy land in $t+1$ using this option (i.e. at price p_1), he can't sell land in $t+2$ at the price p_2.

We'll use backward induction kind of strategy: Suppose that an agent buys land using this option in $t+1$ at price p_1. Then the return of this asset (not only the option but the combo of land and option) will be $\max\{p_2 + 1; \tilde{p}(k) + 1\}$ in period $t+2$ in state k (i.e. this agent sells the land, and gets 1 fruit from the land).

Then the price of this option in period t, state i is:

$$\hat{p}_i = \sum_j q_{ij} (\max\{0, -p_1 + \sum_k q_{jk} \max\{p_2 + 1; \tilde{p}(k) + 1\}\}$$

10. (5 points) Make any assumptions that you want to ensure that in equilibrium the amount of consumption is constant.

In the equilibrium (in both of the equilibriums, w/ restriction and w/o restriction) amount of consumption is:

$$c(i) = d_i + 1$$

Then the only assumption that makes consumption constant is that $d_i = d_j$, $\forall i, j$ (which means that d_t is a degenerate stochastic process).

11. (5 points) Under the assumptions of the previous question, characterize tree prices.

By the FOC from the agent's problem in question 7 w.r.t s':

$$p(i) = \beta \sum_{j=1}^{I} \Gamma_{ij} \frac{u'(c)}{u'(c')} [p(j) + d_j] \qquad \forall i$$

Since $c = c'$ and $d_i = d\ \forall i$, then:

$$p(i) = \beta \sum_{j=1}^{I} \Gamma_{ij} p(j) + \beta d \qquad \forall i$$

In matrix notation

$$
\begin{aligned}
p &= \beta\Gamma p + \beta 1_d \quad \texttt{where } 1_d \texttt{ is a column vector with all d's} \\
[I - \beta\Gamma]\, p &= \beta 1_d \\
p &= [I - \beta\Gamma]^{-1} [\beta 1_d]
\end{aligned}
$$

Industry Equilibria

Imagine that the shock that affects a firm's productivity can take 10 values $\{s^1, \cdots, s^{10}\}$ with transition matrix Γ_{ij}. Imagine that the optimal policy is to quit if the shock is ever s^1 or s^2. Imagine that a measure .3 of firms enter each period and that they all enter with shock s^4. Give formulas for

12. (5 points) The transition function that characterizes the evolution of firms. Please verify that what you construct is indeed a transition.

A transition function is a mapping $Q : S \times \mathcal{S} \rightarrow [0, 1]$, such that:

1) $Q(s, .)$ is a probability measure for every $s \in S$.

2) $Q(., B)$ is a measurable function with respect to $\mathcal{S}$, for every $B \in \mathcal{S}$.

$$Q(s^i, B) = \Gamma[s^i, B] = \sum_{s^j \in B} \Gamma_{ij}$$

$Q(s^i, .)$ is a probability measure for every $s^i \in S$ since $0 \leq Q(s^i, .) \leq 1$.
Also $Q(., B)$ is a measurable function with respect to $\mathcal{S}$, for every $B \in \mathcal{S}$ since for every $a \in [0, 1]$, $\{s \in S : \; Q(s, B) \leq a\} \in \mathcal{S}$.

(Note: There is some inconsistency (typo) in last year's notes about this question: In 2006 notes the probability of a firm being in a state s.t. firm quits is not added to the probability of being in set B even if these states are in the set of B. But obviously this is not true w.r.t the above definition of transition function since the total probability doesn't add up to 1. However this is not really a very big deal as long as you define the updating operator correct and consistent with transition function.)

13. (5 points) The updating operator for the distribution of firms in the industry

Based on the transition function Q we can define an updating operator T. This new object satisfies $x'(B) = T(x, Q)$. In words, T gives us the measure of firms in the subset B in the next period, based on the transition function and the measure of firms today:

$$x'(B) = T(x,Q) = \sum_{s^i \in S} Q(s^i, B \backslash \{s^1, s^2\}) x(s^i) + 0.3 * 1_{\{s^4 \in B\}} x(S)$$
$$= \sum_{s^i \in S} \sum_{s^j \in B \backslash \{s^1, s^2\}} \Gamma_{ij} x(s^i) + 0.3 * 1_{\{s^4 \in B\}} x(S)$$

14. (5 points) What conditions and what kind of object would the stationary distribution of firms be?

The stationary distribution of firms, $x^\star$ is a distribution of firms s.t.

$$x^\star(B) = 0.3 * 1_{\{s^4 \in B\}} x^\star(S) + \sum_{s^i \in S} \sum_{s^j \in B \backslash \{1,2\}} \Gamma_{ij} x^\star(s^i) \quad \forall B \in \mathcal{S}$$

15. (5 points) What price would be the one that gave zero profit given a production function, a cost of entry and a wage?

In addition to a cost of entry, c_∞, we introduce a fixed cost that the firm has to pay in every period, c_f, in order to capture the idea that the optimal policy is to quit if the shock is ever s^1 or s^2 (another way is to assume that s^1 and s^2 are negative). Then the recursive formulation of the firm's problem is:

$$\Omega(s^i) = \max[0, -c_f + \max_n \{p s^i f(n) - wn\} + \tfrac{1}{1+r} \textstyle\sum_{j=1}^{10} \Omega(s^j) \Gamma_{ij}]$$

Then the zero profit condition is $\Omega(s^4) = c_\infty$, and the price would give zero profit condition is the one solves the following equation:

$$\Omega(s^4) = \max[0, -c_f + \max_n \{p s^4 f(n) - wn\} + \frac{1}{1+r} \sum_{j=1}^{10} \Omega(s^j) \Gamma_{4j}] = c_\infty$$