Econ 702, Spring 2007
Problem set 2
Suggested Solutions[1]

Problem 1. *Characterize the relationship between commodity prices (output/consumption, capital services and labor services) in the valuation equilibrium framework, when the depreciation rate is not equal to 1, i.e., when the problem looks like the following:*

$$\max_{x\in X} \sum_t \beta^t u\left[c_t(x)\right]$$

such that

$$\sum_t \left[p_{1t}\{c_t + k_{t+1} - (1-\delta)k_t\} - p_{2t}k_t - p_{3t}\right] \leq 0$$

Suggested Solution

First, the budget constraint in this case is of the 'valuation equilibrium' form, hence, we can set up the lagrangian as follows:

$$\max_{x\in X} \sum_t \beta^t u\left[c_t(x)\right] + \lambda\left[\sum_t \left[p_{1t}\{c_t + k_{t+1} - (1-\delta)k_t\} - p_{2t}k_t - p_{3t}\right]\right]$$

As in class, we can take a first order condition with respect to k_{t+1}:

$$\lambda\left[p_{1t} - (1-\delta)p_{1,t+1} - p_{2,t+1}\right] = 0$$

Note that if $\delta = 1$, the condition reduces to $p_{1t} = p_{2,t+1}$. More generally, if $\delta \neq 1$

$$\frac{p_{1t}}{p_{1,t+1}} = 1 - \delta + \frac{p_{2,t+1}}{p_{1,t+1}}$$

The condition changes because $\delta = 1$ is the extreme case of full depreciation, hence the above condition doesn't relate to prices of consumption services in the future: there is no capital left from what the firm used, hence, no resale value (no need to worry about $p_{1,t+1}$.

Problem 2. *Consider the sequential market formulation of the deterministic growth model:*

$$\max_{\{c_t,k_{t+1}\}_{t=0}^{\infty}} \sum_t \beta^t u\left[c_t\right]$$

s.t.

$$c_t + k_{t+1} + \ell_t = \ell_{t-1}R^{\ell}_{t-1} + k_t R^k_t + w_t \quad \forall t$$

Write the definition of a sequential market equilibrium (SME) as completely as you can.

[1] Prepared by Se Kyu Choi. Questions, comments and "Typo police" should email me at sechoi@econ.upenn.edu

Suggested Solution

A Sequential Market Equilibrium (SME) is a list of sequences $\{c_t^*, k_{t+1}^*, \ell_t^*, R_{t-1}^{\ell *}, R_t^{k*}, w_t^*\}_{t=0}^{\infty}$ (allocations and prices) such that,

1. given prices, households maximize, i.e.:

$$\{c_t^*, k_{t+1}^*, \ell_t^*\}_{t=0}^{\infty} \in argmax_{\{c_t, k_{t+1}, \ell_t\}_{t=0}^{\infty}} \sum_t \beta^t u\left[c_t\right]$$

$$s.t.$$

$$c_t + k_{t+1} + \ell_t = \ell_{t-1} R_{t-1}^{\ell *} + k_t R_t^{k*} + w_t^* \quad \forall t$$

2. given prices, firms maximize, i.e.:

$$\{k_t^*, 1\} \in argmax_{k_t, n_t} \;\; F(k_t, n_t) - R_t^{k*} k_t - w_t^* n_t \quad \forall t$$

3. markets clear:

$$c_t^* + k_{t+1}^* = F(k_t^*, 1)$$

This condition implies that returns on loans $R^{\ell *}$ are such that

$$\ell_t^* = 0 \quad \forall t$$

Problem 3. *For the stochastic growth model, verify the conditions on L, X and Y (commodity space, consumption possibility set and production possibility set) such that the uniqueness and welfare theorems apply.*

Suggested Solution

This is too long of a proof for the scope of this problem set. However, the spaces in the stochastic growth model are isomorphic to those of the deterministic model, hence the proofs are analogous.[2]. Below are the definitions of the spaces and possibility sets.

The shocks to the production function have a bounded support $z_t \in Z_t = \{z^0, z^1, ..., z^{n_z}\}$. Then, we can define histories of length t as $h_t = \{z_0, z_1, ..., z_t\}$ with $h_t \in H_t = Z_0 \times Z_1 \times \cdots \times Z_t$. Finally, define a probability measure over histories, $\pi(h_t)$. Then, the commodity space is

$$L = \left\{ \left(l_{1t}\left(h_t\right), l_{2t}\left(h_t\right), l_{3t}\left(h_t\right)\right) \in \mathbb{R}^3 \;\; \forall t, h_t : \;\; \sup |l_{it}(h_t)| < \infty \;\; \forall i, t, h_t \right\}$$

The consumption possibility set:

[2]You can check the conditions on the spaces for the deterministic growth model here: http://www.econ.upenn.edu/ vr0j/70206/tas/solpr1ec70206.pdf

$$
\begin{aligned}
X = \{x \in L : \exists \{c_t(h_t), k_{t+1}(h_t)\}_{t,h_t} \;\; s.t. & \\
x_{1t}(h_t) + (1-\delta) k_t(h_{t-1}) &= c_t(h_t) + k_{t+1}(h_t) \\
x_{2t}(h_t) &\in [-k_t(h_{t-1}), 0] \\
x_{3t}(h_t) &\in [-1, 0] \\
k_0 \;\; given\}
\end{aligned}
$$

And the production possibility set $Y = \Pi_{t,h_t} Y_t(h_t)$ where

$$
Y_t = \{(y_{1t}(h_t), y_{2t}(h_t), y_{3t}(h_t)) : 0 \leq y_{1t}(h_t) \leq F(-y_{2t}(h_t), -y_{3t}(h_t))\}
$$

Problem 4. *Write the definition of SME for the stochastic growth problem.*

Suggested Solution

A Sequence of Markets Equilibrium for the Stochastic Growth Model (with full depreciation) is a list of sequences

$\{c_t^*(h_t), k_{t+1}^*(h_t), b_t^*(h_t, z_{t+1}), \ell_t^*(h_t), q_t^*(h_t, z_{t+1}), R_t^{k*}(h_t), R_t^{\ell*}(h_{t-1}), w_t^*(h_t)\}$ for every period t and history h_t such that

1. Households maximize:

$$
\{c_t^*(h_t), k_{t+1}^*(h_t), b_t^*(h_t, z_{t+1})\ell_t^*(h_t)\}_{t,h_t} \in \arg\max \sum_{t=0}^{\infty} \beta^t \sum_{h_t \in H_t} \pi(h_t) u[c_t(h_t)]
$$

$$
st
$$

$$
\begin{aligned}
c_t(h_t) + k_{t+1}(h_t) + \ell_t(h_t) + \sum_{z_{t+1}} q_t(h_t, z_{t+1}) b_t(h_t, z_{t+1}) = \\
R_t^k(h_t) k_t(h_{t-1}) + R_t^\ell(h_{t-1}) \ell_{t-1}(h_{t-1}) + w_t(h_t) + b_{t-1}(h_{t-1}, z_t) \\
k_0, b_0(h_0, z_1), \ell_0 \quad given
\end{aligned}
$$

2. Firms maximize:

$$
\{k_t^*(h_t), 1\} \in \arg\max \;\; z_t F(k_t(h_t), n_t(h_t)) - R_t^{k*}(h_t) - w_t^*(h_t) n_t(h_t)
$$

for all t and h_t

3. Market clears, i.e., for all t and h_t

$$
c_t^*(h_t) + k_{t+1}^*(h_t) = z_t F(k_t^*(h_t), 1)
$$

which imply (given the representative agent assumption) that

$$
b_t^*(h_t, z_{t+1}) = \ell_t^*(h_t) = 0
$$

Problem 5. *We saw in class that the sequential budget constraint for the stochastic growth model is:*

$$c_t(h_t) + k_{t+1}(h_t) + \ell_t(h_t) + \sum_{z_{t+1}} q_t(h_t, z_{t+1}) b_t(h_t, z_{t+1}) =$$
$$R^k_t(h_t) k_t(h_{t-1}) + R^\ell_t(h_{t-1}) \ell_{t-1}(h_{t-1}) + w_t(h_t) + b_{t-1}(h_{t-1}, z_t)$$

Characterize equilibrium return rates ($R^k_t(h_t)$ and $R^\ell_t(h_{t-1})$) in terms of $q_t(h_t, z_{t+1})$

Suggested Solution

Like all sequential market problems, we have a budget constraint for each period t and history h_t, i.e., a lagrange multiplier $\lambda_t(h_t)$.

The first order condition wrt $b_t(h_t, z_{t+1})$

$$\begin{aligned} q_t(h_t, z_{t+1}) \lambda_t(h_t) &= \lambda_{t+1}(h_t, z_{t+1}) \\ \Rightarrow q_t(h_t, z_{t+1}) &= \frac{\lambda_{t+1}(h_t, z_{t+1})}{\lambda_t(h_t)} \end{aligned} \tag{1}$$

The first order condition wrt $\ell_t(h_t)$

$$\lambda_t(h_t) = \sum_{z_{t+1}} \lambda_{t+1}(h_t, z_{t+1}) R^\ell_{t+1}(h_t)$$

Since the return on loans is 'agreed' upon from period to period (and is not shock dependent), we get the following expression

$$R^\ell_{t+1}(h_t) = \frac{1}{\sum_{z_{t+1}} \frac{\lambda_{t+1}(h_t, z_{t+1})}{\lambda_t(h_t)}} = \frac{1}{\sum_{z_{t+1}} q_t(h_t, z_{t+1})}$$

On the other hand, the FOC wrt $k_{t+1}(h_{t+1})$ yields

$$\lambda_t(h_t) = \sum_{z_{t+1}} \lambda_{t+1}(h_t, z_{t+1}) R^k_{t+1}(h_t, z_{t+1})$$

Note that the return on capital is state dependent, hence the condition for the prices boils down to

$$1 = \sum_{z_{t+1}} q_t(h_t, z_{t+1}) R^k_{t+1}(h_t, z_{t+1})$$

Problem 6. *Write a modification to the stochastic growth model with two locations (All remaining/neccessary assumptions are yours to make). Write the commodity space, the consumption possibility set and the production possibility set and pose the maximization problem with both a 'Valuation-equilibrium' budget constraint and a 'sequential market equilibrium' budget constraint.*

(note: this question is left open intentionally. There is no correct answer, but CONSISTENT answers, i.e., your response must be consistent with your assumptions)