Econ 702, Spring 2007
Problem set 4
Suggested Solutions[1]

Problem 1. 1. *Consider the growth model with two different social classes as seen in class. Show that if $g^i(K, a^i), i = \{R, P\}$ (the optimal saving decisions for R = rich and P = poor) are linear in their second arguments, then aggregate capital is the only necessary aggregate state variable.*

2. *Is this the case when utility is quadratic? ($u(c) = -\frac{1}{2}(c-b)^2$ where b is a constant) When the utility is CRRA? ($u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$)*

Suggested Solution

1. First, notice that given our assumptions, the only difference between individuals of different classes is their initial wealth; the two social classes are identical in every other aspect. This means that the optimal policies are the same, i.e.

$$g^R(K, a) = g^P(K, a) = g(K, a) \quad \forall a$$

Now, take the aggregate law of motion for capital:

$$K' = \mu^R g(K, a^R) + (1 - \mu^P) g(K, a^P)$$

Assuming linear policy functions ($g(K, a) = \alpha_0 + \alpha_1 K + \alpha_2 a$)

$$\begin{aligned}
K' &= \mu^R \left[\alpha_0 + \alpha_1 K + \alpha_2 a^R\right] + (1 - \mu^R) \left[\alpha_0 + \alpha_1 K + \alpha_2 a^P\right] \\
&= \alpha_0 + \alpha_1 K + \mu^R \alpha_2 a^R + (1 - \mu^R)\alpha_2 a^P \\
&= \alpha_0 + \alpha_1 K + \alpha_2 \left[\mu^R a^R + (1 - \mu^R) a^P\right] \\
&= \alpha_0 + \alpha_1 K + \alpha_2 [K] \\
&= g(K, K)
\end{aligned}$$

which is simply the optimal policy for the median agent in the economy. Note that this result is due to the fact that, if $x \in X$ is a random variable, and $f : X \to \mathbb{R}$ then

$$\mathbb{E}[f(x)] = f(\mathbb{E}[x])$$

iff f is linear. [2]

2. Basically, we have to check if the optimal policies under these particular utility functions are linear or not. First, we know that in equilibrium the euler equation has to hold with equality at interior solutions

$$u_c[c] = \beta u_c[c'] R(K')$$

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[2] This is the same idea behind Jensen's Inequality

where $c = aR(K) + w(K)$. Using the Implicit Function Theorem, we know that $\frac{\partial a'}{\partial a}$ exists. Define $F = u_c[c] - \beta u_c[c']R(K') = 0$, then

$$\begin{aligned}
\frac{\partial a'}{\partial a} &= -\frac{\partial F/\partial a}{\partial F/\partial a'} \\
&= \frac{u_{cc}[c]R(K)}{u_{cc}[c] + \beta u_{cc}[c']R(K)}
\end{aligned}$$

Clearly, the optimal policy function $a' = g$ is linear if $\frac{\partial a'}{\partial a}$ is constant in a (given K and K'). Since a appears as an argument of the utility function, it all depends on the second derivative of $u[c]$: if utility is quadratic

$$u_{cc}[c] = -1$$

and then $\frac{\partial a'}{\partial a}$ being constant in $a \Rightarrow$ policy function is linear.

If utility is CRRA:

$$u_{cc}[c] = -\sigma c^{-\sigma-1}$$

so, by the same argument as above, the policy function is NOT linear in this case, since $\frac{\partial a'}{\partial a}$ is a function of a.

Problem 2. *Redo the analysis of the above model (setup and definition of Recursive Competitive Equilibrium) using $\{k^R, k^P\}$ (capital owned by Rich and capital owned by Poor) as aggregate state variables.*

Suggested Solution

We only need two aggregate states. Let's define them as $\{k^R, k^P\}$, the TOTAL capital owned by Riche and Poor people respectively, so total capital is $K = k^R, k^P$.

The individual state variable is still just a^i for $i = R, P$.

Then, the following bellman equations follow for both social classes:

$$\begin{aligned}
V^i(k^R, k^P, a^i) &= \max_{c^i, a'^i} \quad u[c] + \beta V^i(k'^R, k'^P, a'^i) \\
& \quad s.t. \\
c^i + a'^i &= a^i R(k^R, k^P) + w(k^R, k^P) \\
k'^R &= G^R(k^R, k^P) \\
k'^P &= G^P(k^R, k^P)
\end{aligned}$$

with solution $g = (k^R, k^P, a^i)$.[3]. Now we can define equilibrium:

Definition: A Recursive Competitive Equilibrium for the economy with two social classes is a list of functions $\{V^{*R}, V^{*P}, g^*, G^{*R}, G^{*P}, R^*, w^*\}$ such that

[3]The optimal policy is the same for both classes, since everything in the problem -but initial wealth- is the same for Rich and Poor classes

1. Given price functions $\{R^*, w^*\}$, $\{V^{*i}, g^*\}$ solve the problem of the households, for $i = R, P$ respectively, i.e.

$$\begin{aligned} V^{*i}(k^R, k^P, a^i) = u[a^i R^*(k^R, k^P) + w^*(k^R, k^P) - g^*(k^R, k^P, a^i)] \\ +\beta V^{*i}(G^{*R}(k^R, k^P), G^{*P}(k^R, k^P), g^*(k^R, k^P, a^i)) \end{aligned}$$

   for $i = R, P$

2. Markets clear:

$$\begin{aligned} R^*(k^R, k^P) &= F_1(k^R + k^P, 1) \\ w^*(k^R, k^P) &= F_2(k^R + k^P, 1) \end{aligned}$$

3. Representative agent conditions:

$$\begin{aligned} G^{*R}(k^R, k^P) &= \mu^R g(k^R, k^P, k^R) \\ G^{*P}(k^R, k^P) &= (1 - \mu^R) g(k^R, k^P, k^P) \end{aligned}$$

Problem 3. *(optional) In the growth model with 2 social classes, define steady state. What are the conditions for a Steady State to exist?*

Problem 4. *In the growth model with land, show that condition (5) is redundant from the given definition of equilibrium*

Suggested Solution

The bellman equation for firms is:

$$\begin{aligned} \Omega(K, k) &= \max_{k'} \quad F(k, 1) - k' + q(K')\Omega(K', k') \\ & s.t. \\ K' &= G(K) \end{aligned}$$

The first order condition:

$$-1 + q(K')\Omega_2(K', k') = 0$$

The envelope condition:

$$\Omega_2(K, k) = F_1(k, 1)$$

Hence, the euler condition is

$$1 = q(K')F_1(k', 1)$$

which is exactly condition (5) from the lecture notes, hence that condition is redundant, since it is stating simply that firms optimize (already part of the equilibrium definition in the notes).

Problem 5. *Take the growth model with government debt (last model seen in class). Show, by way of finding 'wedges', that the competitive equilibrium is inefficient*

Suggested Solution

Let's write the problem of the household (given some government policy) as seen in class:

$$
\begin{aligned}
V(K,B,a) &= \max_{c,a'} \quad u[c] + \beta V(K',B',a') \\
& \quad s.t. \\
(1+\tau)c + a' &= aR(K) + w(K) \\
K' &= G^K(K,B) \\
B' &= G^B(K,B) \\
\tau &= \tau(K,B)
\end{aligned}
$$

Using the FOC and the envelope condition, we get an euler equation of the form:

$$\frac{u_c[c]}{\beta u_c[c']} = R(K')\frac{1+\tau(K,B)}{1+\tau(K',B')} \tag{1}$$

On the other hand, the usual condition of optimality (from the social planner's problem) is

$$\frac{u_c[c]}{\beta u_c[c']} = R(K')$$

Or in other words, in equilibrium the marginal rate of substitution (left hand side) has to equal the marginal rate of transformation (right hand side). Hence, the 'wedge' is $\frac{1+\tau(K,B)}{1+\tau(K',B')}$, an indicator of how taxes are evolving.

Note that if $\tau(K,B) = \tau(K',B') = \tau$, the wedge disappears, since the marginal decisions are not affected: consumption is taxed every period in the same amount, hence there is no point in altering optimal saving decisions.

On the other hand, take the case when $\tau(K,B) > \tau(K',B')$. By (1) we get that

$$\frac{u_c[c]}{\beta u_c[c']} > R(K')$$

By concavity of u, this means that savings in this case are greater than socially optimal. Intuitively, since the tax profile is decreasing in time and the households want to avoid tax payments, there is increased savings in the economy, since consumption is taxed while asset accumulation (a') is not.

The analysis with $\tau(K,B) < \tau(K',B')$ is analogous, so is left to you.

Problem 6. *Write recursively a simple growth model with utility from leisure and government, where the government wants to throw a party of size $\overline{G}$ every period (not valued by the households), financed by either consumption taxes (τ_c) OR labor income taxes (τ_w) (the gov. budget is balanced from period to period). Find the relation between taxes in equilibrium. In terms of utility, when will τ_c be preferred over τ_w?*

Suggested Solution

Lets start with a generic recursive growth model, with labor/leisure choice:

$$\begin{aligned}
V(K,a) &= \max_{c,a'} \quad u[c,n] + \beta V(K',a') \\
& \quad\; s.t. \\
c + a' &= aR(K,N) + nw(K,N) \\
K' &= G(K) \\
N &= H(K)
\end{aligned}$$

The solution to this problem will be characterized (almost entirely) by two equations: an euler equation for asset accumulation and a static condition for hours worked.

Labor Income Taxes

The budget equation is

$$c + a' = aR(K,N) + nw(K,N)(1 - \tau_w(K))$$

The euler equation:

$$\frac{u_c[c,n]}{\beta u_c[c',n']} = R(K',N') \tag{2}$$

The condition for hours worked:

$$\frac{1}{1-\tau_w(K)} = -\frac{w(K,N)u_c[c,n]}{u_n[c,n]} \tag{3}$$

Consumption Taxes

The budget equation is

$$(1 + \tau_c(K))c + a' = aR(K,N) + nw(K,N)$$

The euler equation:

$$\frac{u_c[c,n]}{\beta u_c[c',n']} = R(K',N')\frac{1+\tau_c(K)}{1+\tau_c(K')} \tag{4}$$

The condition for hours worked:

$$1 + \tau_c(K) = -\frac{w(K,N)u_c[c,n]}{u_n[c,n]} \tag{5}$$

Labor income taxes should be preferred to consumption taxes since the latter affects two margins -equations (4) and (5)- instead of just one -equation (3)-, as labor income taxes do.

But depending on the aggregate wealth level, consumption taxes could be preferred to labor income ones (to be completed).