Econ 702, Spring 2007
Problem set 6
Suggested Solutions[1]

Problem 1. ***Consider the model of industry equilibrium as described in Class. Shocks to productivity have a finite support and follow a Markov process. Also, there is a fixed cost of operating each period c_f, so the firm has the option of quitting the market forever.***

1. ***Show the existence of a reservation property for the firm, i.e., there is a unique s^* such that if $s < s^*$, the firm quits the market and stays otherwise. State clearly all your assumptions on parameters.***

2. ***Define transitions and the condition for the stationary distribution.***

Suggested Solution
1. The problem of the firm is as follows:

$$\Omega(s) = \max\{0, \pi(s)\}$$

where

$$\pi(s) = \max_n psf(n) - wn - c_f + \left(\frac{1}{1+r}\right) \sum_{s'} \Gamma_{ss'} \Omega(s')$$

the existence of the fixed cost c_f makes the problem of the firm non-trivial, since the firm can chose to leave the market forever. Basically, to show the reservation property, we need to show that $\pi(s)$ is strictly increasing in s.

We need the following assumptions:

- if $s_i < s_j \Rightarrow \sum_{s'} \Gamma_{s_i s'} s' < \sum_{s'} \Gamma_{s_i s'} s'$ or first order stochastic dominance of shocks (i.e., receiving a good shock is also good because it increases the expected value of future good shocks. In other words, there is persistence of shocks)

- $\Omega(s)$ non decreasing in s

Given the first assumption, it follows directly that the continuation value $\left(\left(\frac{1}{1+r}\right) \sum_{s'} \Gamma_{ss'} \Omega(s')\right)$ is non-decreasing in s.

Next, define the current benefit of the firm as

$$y(s) = psf(n) - wn$$

so

$$\frac{\partial y(s)}{\partial s} = pf(n) + (psf'(n) - w) \frac{\partial n}{\partial s}$$

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the second part of the RHS is zero, since it's the FOC from the problem of the firm (which in equilibrium must be equal to zero) times the derivative of n wrt s, which is positive (just using the implicit function theorem on the FOC).

Then, we just keep $pf(n)$ which is strictly positive. Hence, we have shown that $\pi(s)$ is strictly increasing in s. Now, the existence of a unique threshold value for s depends on whether $\pi(\underline{s}) < 0$ and $\pi(\overline{s}) < 0$, but that's a matter of parameterization (c_f vs. p and w).

2. Transitions:

$$Q(s, \mathcal{B}) = \sum_{s' \in \mathcal{B}} Gamma_{ss'}$$

Stationary distribution:

$$X^*(\mathcal{B}) = \int_{\mathcal{S}} Q(s, \mathcal{B}) dX^* + \left[\int_{\underline{s}}^{\overline{s}} dX^* \right] \int_{\mathcal{S}} \sum_{s' \in \mathcal{B}} \Gamma_{ss'} dG(s)$$

where G is the cdf from where new firms draw their initial s

Problem 2. ***In the 'goat-farmer' economy, with only two income shocks*** $\{s_l, s_h\}$ ***with*** $s_l < s_h$ ***and*** $\beta/q < 1$

1. ***Is the decision rule (***$g(s, a)$***) monotonic in a? What are minimal sufficient conditions to prove this?***
2. ***Is*** $g(s_l, \cdot)$ ***concave? what about*** $g(s_h, \cdot)$***?***

Suggested Solution

1. In the goat farmer economy, we have the restriction that $g(\cdot) \in [0, \overline{a}]$, or in other words, agents cannot accumulate debt; this kink in the decision space transforms the optimality condition on accumulation

$$u_c[a + s - qg(s, a)] = \max\{u_c[a + s], \frac{\beta}{q} \sum_{s'} \Gamma_{ss'} u_c[s' + g - qg(g)]\}$$

the above equation states that, either we are at an interior point so that we get the usual FOC for asset accumulation

$$u_c[a + s - qg(s, a)] = \frac{\beta}{q} \sum_{s'} \Gamma_{ss'} u_c[s' + g - qg(g)]$$

or that we are at a point where the restriction binds (we would like to get into debt in order to consume more today, but we can't), so

$$\begin{aligned} u_c[a + s - qg(s, a)] &= u_c[a + s] \\ &> \frac{\beta}{q} \sum_{s'} \Gamma_{ss'} u_c[s' + g - qg(g)] \end{aligned}$$

the last inequality comes from strict concavity of u.

If we are in the first case (interior condition) we already proved that $g(s,\cdot)$ is increasing in a.

On the other hand, if we are in the binding portion of the state space, $g(\cdot) = 0$. Hence, the policy function is non-decreasing for $a = [0, \overline{a}]$

2. By the same argument as above, $g(s_l, a)$ has a kink; there is some $\widetilde{a}$ such that $\forall a < \widetilde{a}$, $g(s_l, a) = 0$ (optimal policy figures drawn by Victor), so it cannot be a concave function of a. In general, concavity of these functions (included $g(s_h, a)$) depends on the parameterization of the model, so it is usually checked ex-post (after simulation on a computer for example)

Problem 3. ***In the general setting of the goat farmer economy, we know that total assets in the economy is given by $\int_{SxA} a dX(s,a)$. Compute a formula for the share of wealth owned by the richest x% of farmers.***

Suggested solution

First, we don't need $X(s,a)$. That distribution is 'finer' than what we need. It suffices to work with a coarser measure $\mu(a)$, since we are only interested in the measure of goat-farmers with asset level a.

Define $q(x)$ such that

$$\int_{\underline{s}}^{q(x)} \mu(a) da = 1 - x$$

Hence, $q(x)$ is the wealth level of the $1 - x$ richest farmer in the economy. Then, the share of wealth owned by the richest $x\%$ farmers is:

$$\frac{\int_{\underline{s}}^{q(x)} a\mu(a)da}{\int_{\underline{s}}^{\overline{a}} a\mu(a)da}$$

Problem 4. ***In the Aiyagari economy***

1. ***Relate total labor input with fundamentals***
2. ***Define a Recursive stationary equilibrium***

Suggested solution

1. Since in the Aiyagari economy there is no utility from leisure, all individuals work every period, independently from their wealth status. Hence, in the stationary equilibrium, total labor input (N) is given by

$$N = \sum_{s \in S} s\gamma(s)$$

where S is the state space and $\gamma(s)$ is the stationary distribution associated with Γ, the transition function of the income process

2. A stationary recursive competitive equilibrium in the Aiyagari economy is a list $\{V^*, g^*, r^*, w^*, X^*\}$ such that

1. given prices $\{r^*, w^*\}$, $\{V^*, g^*\}$ solve the problem of the household

2. markets clear

$$\begin{aligned} r^* &= F_1(K, L) - \delta \\ w^* &= F_2(K, L) \\ where &= \\ K &= \int_{SxA} g(s, a) dX^* \\ L &= \int_{SxA} s dX^* \end{aligned}$$

3. The distribution of agents is stationary:

$$X^* = \int_{SxA} Q(\{s, a\}, \mathcal{B}) dX^*$$

Problem 5. ***Let $S = \{e, u\}$ where e is employed and u is unemployed. The transition probabilities are given by Γ_{ee}, Γ_{uu} and its complements. Calibrate the transition probabilities, such that in steady state, unemployment is 6% and average duration of unemployment is two periods***

Suggested solution

The transition matrix looks like this:

$$\Gamma = \begin{bmatrix} \Gamma_{ee} & 1 - \Gamma_{ee} \\ 1 - \Gamma_{uu} & \Gamma_{uu} \end{bmatrix}$$

Calibration means that we want to assign values for the free parameters (in this case, Γ_{ee} and Γ_{uu}) using statistics from the real data.

Using the fact that average unemployment duration is two periods, we can set Γ_{uu} as follows[2]:

$$\frac{1}{1 - \Gamma_{uu}} = 2$$

Now, we need to determine Γ_{ee}. To calibrate its value, we will use the second piece of information, that is the long-run average unemployment rate of 6%.

[2] This is due the fact that the hazard rate for unemployment (probability of exiting the state) is the inverse of the duration

To calculate the stationary distribution related to Γ, we can use the following system of equations, where $*$ denotes stationary distributions

$$\begin{aligned} e^* &= e^*\Gamma_{ee} + u^*(1-\Gamma_{uu}) \\ u^* &= e^*(1-\Gamma_{ee}) + u^*\Gamma_{uu} \end{aligned}$$

Using the normalization $e^* + u^* = 1$ we get that

$$\begin{pmatrix} e^* \\ u^* \end{pmatrix} = \begin{pmatrix} \frac{1-\Gamma_{uu}}{1-\Gamma_{ee}+1-\Gamma_{uu}} \\ \frac{\Gamma_{uu}}{1-\Gamma_{ee}+1-\Gamma_{uu}} \end{pmatrix}$$