Econ 702, Spring 2007
Problem set 8
Suggested Solutions[1]

Problem 1. ***In Romer's model of growth through Research and Development***

1. ***Show that the equilibrium interest rate in this model is different from the Neoclassical growth model case. Why is it this the case?***

2. ***Show that in equilibrium,*** $x_t(i) = x_t$ ***for all*** i***.***

3. ***How would you describe a balanced growth path for this economy?***

Suggested Solution

1. Let's guess (and verify later, in question 2) that the level of intermediate good used in the production of the final good is the same for all varieties, i.e., $x(i) = x, \forall i$. Given that the resource constraint for intermediate firms (as a sector) is given by

$$\int_0^{A_t} \eta x_t(i) di = K_t$$

we get (using the guess) that

$$\eta A_t x_t(i) = K_t$$

Back in the the production function of the final good firm:

$$\begin{aligned} Y_t &= N_{1t}^{1-\theta} A_t \left(\frac{K_t}{\eta A_t} \right)^{\theta} \\ &= \widetilde{A}_t N_{1t}^{1-\theta} K_t^{\theta} \end{aligned}$$

where $\widetilde{A}_t = A_t^{1-\theta} \eta^{-\theta}$. From here we can calculate the marginal productivity of capital (the interest rate in a Neoclassical Growth model with $A = \widetilde{A}$

$$R_t^{nc} = \theta \widetilde{A} N_{1t}^{1-\theta} K_t^{\theta - 1} \tag{1}$$

Now, from the problem of the final good producer, we get the inverse demand function for variety i (when price is $q_t(i)$):

$$x_t(i) = \left(\frac{q_t(i)}{\theta n_{1t}^{1-\theta}} \right)^{\frac{1}{\theta - 1}}$$

On the other hand, the first order condition of the maximization problem for producer of variety i is:

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$$\theta^2 n_{1t}^{1-\theta} x_t(i)^{\theta-1} = \eta R_t$$

Using again the guess that $x_t(i) = x_t \forall i$

$$\theta^2 n_{1t}^{1-\theta} \left(\frac{K_t}{\eta A_t}\right)^{\theta-1} = \eta R_t$$

Hence, we get that

$$R_t = \theta R_t^{nc}$$

and since $\theta \in (0,1)$ we have the result that $R_t < R_t^{nc}$. This comes from the fact that producers of intermediate varieties have monopolistic power and produce less than socially optimal, in order to get a better price. Comparing this economy with the neoclassical model case, less final output is being produced with the same level of capital, hence, the interest rate is lower.

2. Comes directly from the curvature of the production function of the final good firm. The marginal benefit of increasing any particular variety $x_t(i)$ is decreasing (because of θ) while its cost is linear. Since all intermediate good firms are identical (face a similar problem), prices $q_t(i)$ have the same structure and the optimal decision for the final good firm is to contract the same level from all varieties $x_t(i)$.

3. A (symmetric[2]) competitive equilibrium for this economy is described by the following system of equations:

$$\begin{aligned}
Y_t &= A_t^{1-\theta} \eta^{-\theta} N_{1t}^{1-\theta} K_t^{\theta} \\
R_t &= \theta^2 A_t^{1-\theta} \eta^{-\theta} N_{1t}^{1-\theta} K_t^{\theta-1} \\
w_t &= (1-\theta) N_{1t}^{-\theta} A_t^{1-\theta} \eta^{-\theta} K_t^{\theta} \\
p_t^{nv} &= \frac{w_t}{\xi A_t} \\
1 &= N_{1t} + N_{2t} \\
A_{t+1} &= (1 + N_{2t}\xi) A_t \\
Y_t &= C_t + K_{t+1} \\
u_c[c_t] &= \beta R_t u_c[c_{t+1}]
\end{aligned}$$

Where the last equation is the usual euler equation which determines capital accumulation[3]

This system of equations characterize paths for the endogenous variables in the economy. The balanced growth path would entail capital (K), total number of varieties (A) and consumption growing (C) at the same, constant rate (γ), while hours worked (in both sectors, N_1 and N_2), the price of a new variety (p^{nv})and the services of those varieties ($x(i)$) are stationary (don't grow).

[2]Symmetric, in the sense that all intermediate good production is identical, i.e., $x_t(i) = x_t \forall i$

[3]Comes from the recursive problem of the household, who consumes, saves and doesn't value leisure

Problem 2. ***Take the model of unemployment that we saw in Class***

1. ***State conditions on $p(a)$ for the first order condition on a***

$$-1 + \beta p'(a)\left[V^E - V^u\right] = 0$$

to be necessary and sufficient

2. ***Sketch a proof for the operator T being a contraction, where***

$$T(V^u) = \max_a u(0) - a + \beta\left[p(a)V^E + (1 - p(a))V^u\right]$$

3. ***Show that imposing linearity (i.e., $u[c,a] = u(c) - a$) does not imply loss of generality***

4. ***Pick a utility function and solve the problem of the agent.***

Suggested Solutions

1. Conditions are

- $p(0) = 0$
- $p' > 0$ and $p'' < 0$
- $\lim_{a\to 0} p'(a) = \infty$ and $\lim_{a\to\infty} p'(a) = 0$

The third condition guarantees an interior solution for the optimal amount of effort. The second condition states that the probability of finding a job is a concave function of the search efforts of the agent. Both conditions guarantee that the search effort problem is well defined, with an interior solution.

2. The purpose of this proof is to show that a solution to the recursive search effort problem exists, is unique and can be found by iterating the bellman equation. There are two ways of showing this: using Blackwell's sufficient conditions for a contraction (monotonicity and discounting) or providing a direct proof as follows: let a^* be the level of effort that solves the right hand side of the expression above, and let $\widetilde{V}$ an arbitrary value function, then

$$\begin{aligned}
TV^u &= u(0) - a^* + \beta\left[p(a^*)V^E + (1 - p(a^*))V^u\right] \\
&= u(0) - a^* + \beta\left[p(a^*)V^E + (1 - p(a^*))\widetilde{V}\right] - \beta(1 - p(a^*))\widetilde{V} + \beta(1 - p(a^*))V^u \\
&\leq \max_a u(0) - a^* + \beta\left[p(a^*)V^E + (1 - p(a^*))\widetilde{V}\right] + \beta(1 - p(a^*))\|V^u - \widetilde{V}\| \\
&= T\widetilde{V} + \beta(1 - p(a^*))\|V^u - \widetilde{V}\|
\end{aligned}$$

where $\|\cdot\|$ is the sup-norm. Since a^* is just a real number and $p(\cdot) \in (0,1)$, we get that

$$TV^u - T\widetilde{V} \leq \|V^u - \widetilde{V}\|$$

The same argument goes through when we switch the order of the variables, hence

$$\|TV^u - T\widetilde{V}\| \leq \|V^u - \widetilde{V}\|$$

which means that T is a contraction.