5 The Solow Growth Model

5.1 Models and Assumptions

• What is a model? A mathematical description of the economy.

• Why do we need a model? The world is too complex to describe it in every detail.

• What makes a model successful? When it is simple but effective in describing and predicting how the world works.

• A model relies on simplifying assumptions. These assumptions drive the conclusions of the model. When analyzing a model it is crucial to spell out the assumptions underlying the model.

• Realism may not be the property of a good assumption.
5.2 Basic Assumptions of the Solow Model

1. Continuous time.

2. Single good produced with a constant technology.

3. No government or international trade.

4. All factors of production are fully employed.

5. Labor force grows at constant rate $n = \frac{\dot{L}}{L}$.

6. Initial values for capital, $K_0$ and labor, $L_0$ given.
Production Function

• Neoclassical (Cobb-Douglas) aggregate production function:

\[ Y(t) = F[K(t), L(t)] = K(t)^\alpha \ L(t)^{1-\alpha} \]

• To save on notation write:

\[ Y = A \ K^\alpha \ L^{1-\alpha} \]

• Constant returns to scale:

\[ F(\lambda K, \lambda L) = \lambda \ F(K, L) = \lambda A \ K^\alpha \ L^{1-\alpha} \]

• Inputs are essential:

\[ F(0, 0) = F(K, 0) = F(0, L) = 0 \]
• Marginal productivities are positive:
\[
\frac{\partial F}{\partial K} = \alpha A K^{\alpha-1} L^{1-\alpha} > 0 \\
\frac{\partial F}{\partial L} = (1 - \alpha) A K^\alpha L^{-\alpha} > 0
\]

• Marginal productivities are decreasing,
\[
\frac{\partial^2 F}{\partial K^2} = (\alpha - 1) \alpha A K^{\alpha-2} L^{1-\alpha} < 0 \\
\frac{\partial^2 F}{\partial L^2} = -\alpha (1 - \alpha) A K^\alpha L^{-\alpha-1} < 0
\]
Per Worker Terms

- Define $x = \frac{X}{L}$ as a per worker variable. Then

$$y = \frac{Y}{L} = \frac{A K^\alpha L^{1-\alpha}}{L} = A \left(\frac{K}{L}\right)^\alpha \left(\frac{L}{L}\right)^{1-\alpha} = A k^\alpha$$

- Per worker production function has decreasing returns to scale.
Capital Accumulation

- Capital accumulation equation: \[ \dot{K} = sY - \delta K \]

- Important additional assumptions:
  1. Constant saving rate (very specific preferences: no \( r \))
  2. Constant depreciation rate
• Dividing by \( K \) in the capital accumu equation: \( \frac{\dot{K}}{K} = s \frac{Y}{K} - \delta \).

• Some Algebra:

\[
\frac{\dot{K}}{K} = s \frac{Y}{K} - \delta = s \frac{Y}{L} - \delta = s \frac{Y}{k} - \delta
\]

• Now remember that:

\[
\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L} = \frac{\dot{K}}{K} - n \Rightarrow \frac{\dot{K}}{K} = \frac{\dot{k}}{k} + n
\]

• We get

\[
\frac{\dot{k}}{k} + n = s \frac{y}{k} - \delta \Rightarrow \dot{k} = s y - (\delta + n) k
\]

• Fundamental Differential Equation of Solow Model:

\[
\dot{k} = s A k^\alpha - (\delta + n) k
\]
Graphical Analysis

- Change in $k$, $\dot{k}$ is given by difference of $sA k^\alpha$ and $(\delta + n)k$

- If $sA k^\alpha > (\delta + n)k$, then $k$ increases.

- If $sA k^\alpha < (\delta + n)k$, then $k$ decreases.

- Steady state: a capital stock $k^*$ where, when reached, $\dot{k} = 0$

- Unique positive steady state in Solow model.

- Positive steady state (locally) stable.
Steady State Analysis

• Steady State: \( \dot{k} = 0 \)

• Solve for steady state

\[
0 = s A (k^*)^\alpha - (n + \delta)k^* \Rightarrow k^* = \left( \frac{s A}{n + \delta} \right)^{\frac{1}{1-\alpha}}
\]

• Steady state output per worker \( y^* = \left( \frac{s A}{n+\delta} \right)^{\frac{\alpha}{1-\alpha}} \)

• Steady state output per worker depends positively on the saving (investment) rate and negatively on the population growth rate and depreciation rate.
Comparative Statics

• Suppose that of all a sudden saving rate $s$ increases to $s' > s$. Suppose that at period 0 the economy was at its old steady state with saving rate $s$.

• $(n + \delta)k$ curve does not change.

• $s \ A \ k^\alpha = sy$ shifts up to $s'y$.

• New steady state has higher capital per worker and output per worker.

• Monotonic transition path from old to new steady state.
Evaluating the Basic Solow Model

- Why are some countries rich (have high per worker GDP) and others are poor (have low per worker GDP)?

- Solow model: if all countries are in their steady states, then:
  1. Rich countries have higher saving (investment) rates than poor countries
  2. Rich countries have lower population growth rates than poor countries

- Data seem to support this prediction of the Solow model
The Solow Model and Growth

• No growth in the steady state

• Positive or negative growth along the transition path:

\[ \dot{k} = sA k^\alpha - (n + \delta)k \]

\[ g_k \equiv \frac{\dot{k}}{k} = sA k^{\alpha-1} - (n + \delta) \]
Introducing Technological Progress

- Aggregate production function becomes

\[ Y = K^\alpha (AL)^{1-\alpha} \]

- \( A \): Level of technology in period \( t \).

- Key assumption: constant positive rate of technological progress:

\[ \frac{\dot{A}}{A} = g > 0 \]

- Growth is exogenous.
Balanced Growth Path

• Situation in which output per worker, capital per worker and consumption per worker grow at constant (but potentially different) rates

• Steady state is just a balanced growth path with zero growth rate

• For Solow model, in balanced growth path $g_y = g_k = g_c$
Proof

• Capital Accumulation Equation \( \dot{K} = s Y - \delta K \)

• Dividing both sides by \( K \) yields \( g_K \equiv \frac{\dot{K}}{K} = s \frac{Y}{K} - \delta \)

• Remember that \( g_k \equiv \frac{\dot{k}}{k} = \frac{\dot{K}}{K} - n \)

• Hence

\[
g_k \equiv \frac{\dot{k}}{k} = s \frac{Y}{K} - (n + \delta)
\]

• In BGP \( g_k \) constant. Hence \( \frac{Y}{K} \) constant. It follows that \( g_Y = g_K \). Therefore \( g_Y = g_k \)
What is the Growth Rate?

- Output per worker

\[ y = \frac{Y}{L} = \frac{K^{\alpha} (AL)^{1-\alpha}}{L} = \frac{K^{\alpha} (AL)^{1-\alpha}}{L^\alpha \cdot L^{1-\alpha}} = k^\alpha A^{1-\alpha} \]

- Take logs and differentiate \( g_y = \alpha g_k + (1 - \alpha) g_A \)

- We proved \( g_k = g_y \) and we use \( g_A = g \) to get

\[ g_k = \alpha g_k + (1 - \alpha) g = g = g_y \]

- BGP growth rate equals rate of technological progress. No TP, no growth in the economy.
Analysis of Extended Model

• in BGP variables grow at rate $g$. Want to work with variables that are constant in long run. Define:

\[
\begin{align*}
\tilde{y} &= \frac{y}{A} = \frac{Y}{AL} \\
\tilde{k} &= \frac{k}{A} = \frac{K}{AL}
\end{align*}
\]

• Repeat the Solow model analysis with new variables:

\[
\begin{align*}
\tilde{y} &= \tilde{k}^\alpha \\
\dot{\tilde{k}} &= s\tilde{y} - (n + g + \delta)\tilde{k} \\
\dot{\tilde{k}} &= s\tilde{k}^\alpha - (n + g + \delta)\tilde{k}
\end{align*}
\]
Closed-Form Solution

- Repeating all the steps than in the basic model we get:

\[
\tilde{k}(t) = \left( \frac{s}{\delta + n + g} + \left( \tilde{k}_0^{1-\alpha} - \frac{s}{\delta + n + g} \right) e^{-\lambda t} \right) \frac{1}{1-\alpha}
\]

\[
\tilde{y}(t) = \left( \frac{s}{\delta + n + g} + \left( \tilde{k}_0^{1-\alpha} - \frac{s}{\delta + n + g} \right) e^{-\lambda t} \right) \frac{\alpha}{1-\alpha}
\]

- Interpretation.
Balanced Growth Path Analysis

• Solve for $\tilde{k}^*$ analytically

$$0 = s\tilde{k}^* - (n + g + \delta)\tilde{k}^*$$

$$\tilde{k}^* = \left( \frac{s}{n + g + \delta} \right)^{\frac{1}{1-\alpha}}$$

• Therefore

$$\tilde{y}^* = \left( \frac{s}{n + g + \delta} \right)^{\frac{\alpha}{1-\alpha}}$$
\[ k(t) = A(t) \left( \frac{s}{n + g + \delta} \right)^\frac{1}{1-\alpha} \]

\[ y(t) = A(t) \left( \frac{s}{n + g + \delta} \right)^\frac{\alpha}{1-\alpha} \]

\[ K(t) = L(t)A(t) \left( \frac{s}{n + g + \delta} \right)^\frac{1}{1-\alpha} \]

\[ Y(t) = L(t)A(t) \left( \frac{s}{n + g + \delta} \right)^\frac{\alpha}{1-\alpha} \]
Evaluation of the Model: Growth Facts

1. Output and capital per worker grow at the same constant, positive rate in BGP of model. In long run model reaches BGP.

2. Capital-output ratio $\frac{K}{Y}$ constant along BGP

3. Interest rate constant in balanced growth path

4. Capital share equals $\alpha$, labor share equals $1 - \alpha$ in the model (always, not only along BGP)

5. Success of Solow model along these dimensions, but source of growth, technological progress, is left unexplained.
Evaluation of the Model: Development Facts

1. Differences in income levels across countries explained in the model by differences in $s, n$ and $\delta$.

2. Variation in growth rates: in the model permanent differences can only be due to differences in rate of technological progress $g$. Temporary differences are due to transition dynamics.

3. That growth rates are not constant over time for a given country can be explained by transition dynamics and/or shocks to $n, s$ and $\delta$.

4. Changes in relative position: in the model countries whose $s$ moves up, relative to other countries, move up in income distribution. Reverse with $n$. 

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Interest Rates and the Capital Share

- Output produced by price-taking firms

- Hire workers $L$ for wage $w$ and rent capital $K$ from households for $r$

- Normalization of price of output to 1.

- Real interest rate equals $r - \delta$
Profit Maximization of Firms

$$\max_{K,L} K^\alpha (AL)^{1-\alpha} - wL - rK$$

- First order condition with respect to capital $K$

$$\alpha K^{\alpha-1} (AL)^{1-\alpha} - r = 0$$

$$\alpha \left( \frac{K}{AL} \right)^{\alpha-1} = r$$

$$\alpha \tilde{k}^{\alpha-1} = r$$

- In balanced growth path $\tilde{k} = \tilde{k}^*$, constant over time. Hence in BGP $r$ constant over time, hence $r - \delta$ (real interest rate) constant over time.
Capital Share

- Total income = $Y$, total capital income = $rK$

- Capital share
  \[
  \text{capital share} = \frac{rK}{Y} = \frac{\alpha K^{\alpha-1} (AL)^{1-\alpha} K}{K^\alpha (AL)^{1-\alpha}} = \alpha
  \]

- Labor share = $1 - \alpha$. 
• First order condition with respect to labor $L$

$$(1 - \alpha)K^\alpha(LA)^{-\alpha}A = w$$

$$(1 - \alpha)\tilde{k}^\alpha A = w$$

• Along BGP $\tilde{k} = \tilde{k}^*$, constant over time. Since $A$ is growing at rate $g$, the wage is growing at rate $g$ along a BGP.