Economics 244: Macro Modeling
Dynamic Fiscal Policy

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Most material developed by Dirk Krueger

University of Pennsylvania

Organizational Details (Material also in canvas)

- **Time/Room of Class:** Mon., Wed., 2:00 - 3:30pm in Chem 514.
- **Class Web Page:**
  `http://www.sas.upenn.edu/~vr0j/244-20/`
- **Class Syllabus:**
  `http://www.sas.upenn.edu/~vr0j/244-20/syllabus244.pdf`
- **Lecture notes:** Available at:
  `http://www.sas.upenn.edu/~vr0j/244-20/PennFiscalNew.pdf`
- **Class slides:** Available at:
  `http://www.sas.upenn.edu/~vr0j/244-20/h244-20.pdf`
- **Diary of what we did in class:** Available at:
  `http://www.sas.upenn.edu/~vr0j/244-20/diary.html`

People

- **Instructor:** Professor José Víctor Ríos Rull; 516 PCPSE Building
- **Email:** vr0j@upenn.edu
- **Office Hours:** Mon. 3:30-4:30pm and by appointment
- **TA:** Yaacov Wittman ywittman@sas.upenn.edu,
- **Office Hours:** Wednesday, 12:30-1:30pm, and Thursday, 9-10am. Both in PCPE 500.

- Advanced undergraduate class
- Prerequisites: Econ 101 and 102 and math background required to pass these classes (i.e. Math 114, 115 or equivalent, we use calculus)
- Study the impact of fiscal policy (taxation, government spending, government deficit and debt, social security) on individual household decisions and the macro economy as a whole
- Economics and Climate Change. We will look at the classic problem of an externality and study it in the context of climate change.
- Class consists of model-based analysis, motivated by real world data and policy reforms

Course Requirements and Grades

- 3 Homeworks and 3 midterms.

Homework	25%	75 points
Midterm 1	25%	75 points
Midterm 2	25%	75 points
Midterm 3	25%	75 points
Total	100%	300 points

Homeworks

- Due date stated on homework. Due in class or in my mailbox by the end of class of the specified date. Late homework is **not** accepted.

- Grading complaints: **within one week of return** of homework written statement specifying complaint in detail. I will regrade entire assignment. No guarantee that revised score higher than original score (and may be lower).

- Work in groups on homeworks permitted, but everybody needs to hand in **own** assignment. Please state whom you worked with.

Exams

- Three midterms each make up 25% of total grade.
- Not cumulative.
- Dates: Dates: February 18, April 1, May 1.

Grades

Points Achieved	Letter Grade
285 - 300	A +
270 - 284.5	A
255 - 269.5	A -
240 - 245.5	B +
225 - 239.5	B
210 - 224.5	B -
195 - 209.5	C +
180 - 194.5	C
165 - 179.5	C -
150 - 164.5	D +
135 - 149.5	D
less than 135	F

Content of Course

- Some Basic Empirical Facts about the Size of the Government
- A Simple Model of Intertemporal Choice (Part I)
- The Full Life Cycle Model (Part II)
- Positive Analysis of Fiscal Policy (Part III)
- Pigou Taxation (Part IV)
- Climate Change and the Economy (Part V)
- Optimal Policy (Part VI)

Part I

Introduction:

Facts and the Benchmark Model

$$
\begin{aligned}
C &= \text{Consumption} \\
I &= \text{(Gross) Investment} \\
G &= \text{Government Purchases} \\
X &= \text{Exports} \\
M &= \text{Imports} \\
Y &= \text{Nominal GDP} \\
Y &= C + I + G + (X - M)
\end{aligned}
$$

US 2018 Main Macro Aggregates Bureau of Economic Analysis

	Billions of dollars	Perc of GDP
Gross domestic product	20,500.6	100.00
Personal consumption expenditures	13,951.6	68.05
Goods	4,342.1	21.18
Services	9,609.4	46.87
Gross private domestic investment	3,652.2	17.82
Fixed investment	3,595.6	17.54
Nonresidential	2,800.4	13.66
Structures	637.1	3.11
Equipment	1,236.3	6.03
Intellectual property products	927.0	4.52
Residential	795.3	3.88
Change in private inventories	56.5	0.28
Net exports of goods and services	–625.6	-3.05
Exports	2,530.9	12.35
Imports	3,156.5	15.40
Government expenditures	3,522.5	17.18
Federal	1,319.9	6.44
National defense	779.0	3.80
Nondefense	540.9	2.64
State and local	2,202.6	10.74

Two Deficits

- Federal Government Budget Deficit (more below)

- Trade Deficit (or Current Account Deficit): Trade Balance (TB)

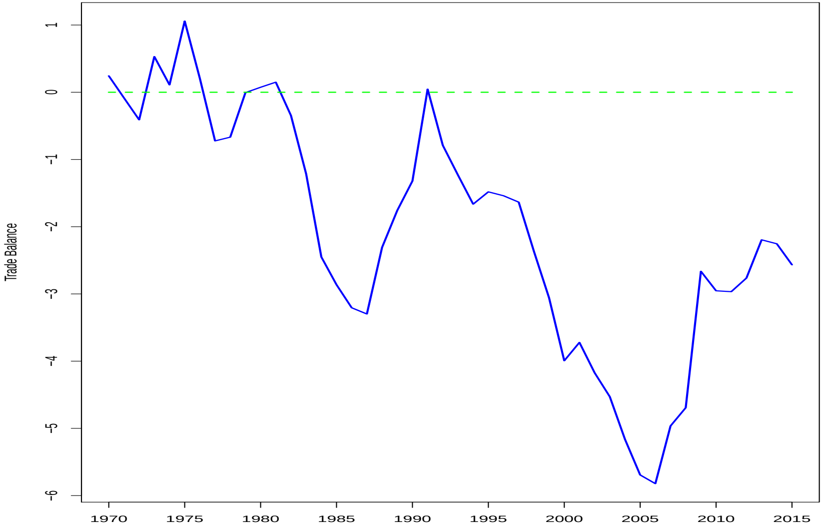
$$TB = X - M$$

$$\text{Current Account Balance} = \text{Trade Balance} + \text{Net Unilateral Transfers}$$

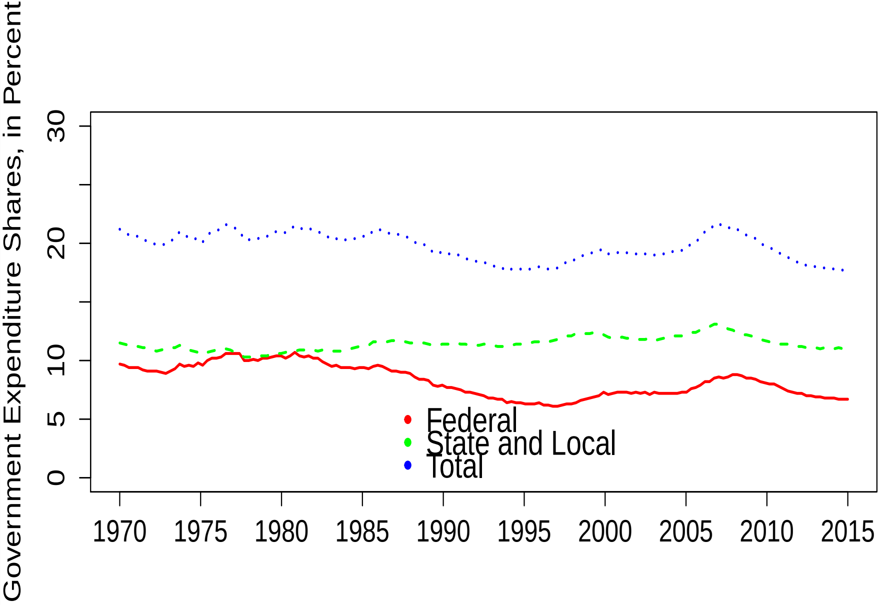
$$\begin{aligned}\text{Capital Account Balance this year} &= \text{Net wealth position at end of this year} \\ &\quad - \text{Net wealth position at end of last year}\end{aligned}$$

$$\text{Current Account Balance this year} = \text{Capital Account Balance this year}$$

Trade Balance as Share of GDP, 1970-2015



Government Spending as Fraction of GDP, 1970-2015



- Budget Deficit/Surplus

$$\begin{aligned} \text{Budget Surplus} &= \text{Total Federal Tax Receipts} \\ &\quad -\text{Total Federal Outlays} \end{aligned}$$

- Federal outlays

$$\begin{aligned} \text{Total Federal Outlays} &= \text{Federal Purchases of Goods and Services} \\ &\quad +\text{Transfers} \\ &\quad +\text{Interest Payments on Fed. Debt} \\ &\quad +\text{Other (small) Items} \end{aligned}$$

- Federal government deficits ever since 1969 (short interruption in late 90's)
- Federal debt and deficit are related by

$$\begin{aligned} \text{Fed. debt at end of this year} &= \text{Fed. debt at end of last year} \\ &\quad +\text{Fed. budget deficit this year} \end{aligned}$$

2015 Federal Budget (in billion $)

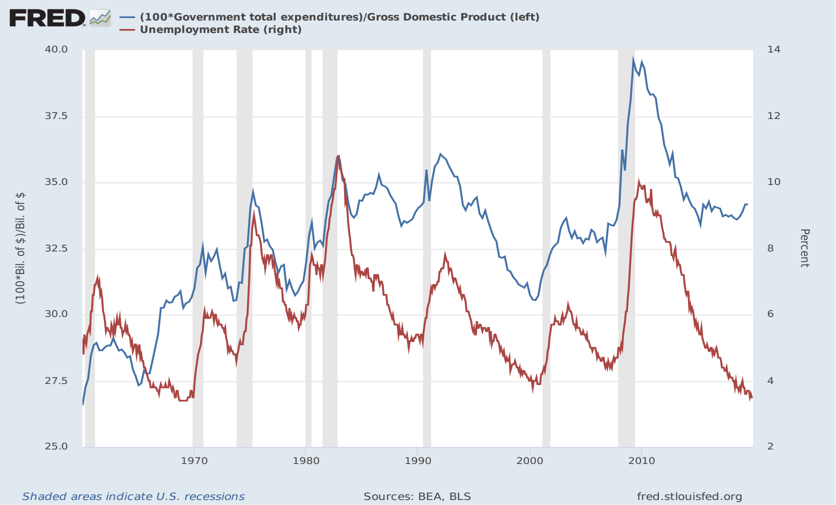
Receipts	3,453.3
Individual Income Taxes	1,532.7
Social Insurance Receipts	1,189.5
Corporate Income Taxes	344.7
Seignorage	110.4
Excise taxes	101.3
Customs duties	38.1
Other	136.6
Outlays	4,022.9
National Defense	705.6
International Affairs	45.7
Health	372.5
Medicare	485.7
Income Security	597.4
Social Security	730.8
Net Interest	230.0
Other	435.5
Surplus	−1,299.6

State and Local Budgets (in billion $)		
	2011	2013
Total Revenue	2,618	2,690
Property Taxes	445.8	445.4
Taxes on Production and Sales	464.0	496.4
Individual Income Taxes	285.3	338.5
Corporation Net Income Tax	48.4	53.0
Transfers from Federal Gov.	647.6	584.7
All Other	722.9	762.4
Total Expenditures	2,583.8	2,643.1
Education	862.27	876.6
Highways	153.9	158.7
Public Welfare	494.7	516.4
All Other	1,072.9	1,091.4
Surplus	34.2	47.3

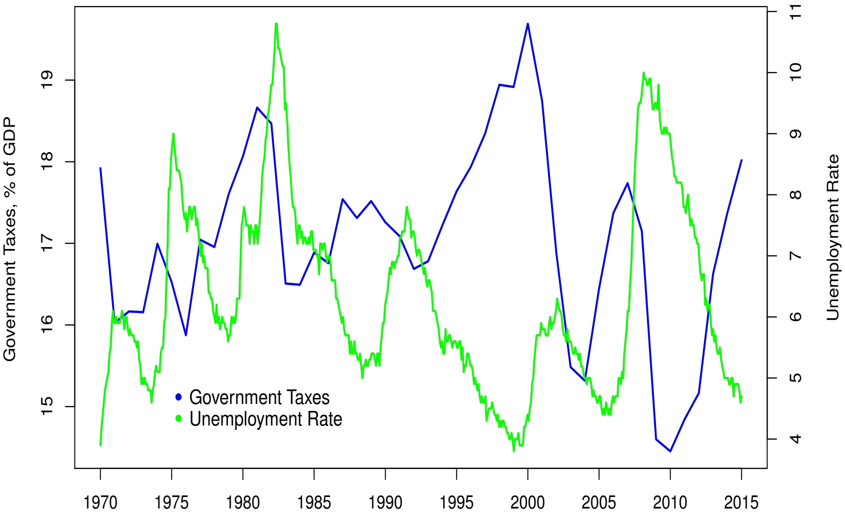
Fiscal Variables and the Business Cycle

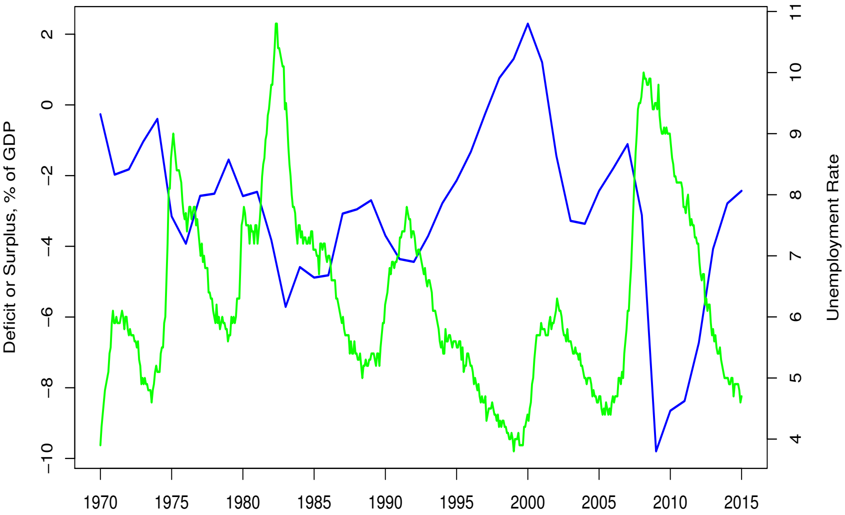
- Use the unemployment rate as indicator for the business cycle: high unemployment rates indicate recessions, low unemployment rates indicate expansions
- Does fiscal policy (government spending, taxes collected, government deficit) vary systematically over the business cycle?

Government Outlays and Unemployment Rate, 1960–2016



Gov Taxes and Unemployment Rate, 1970-2016



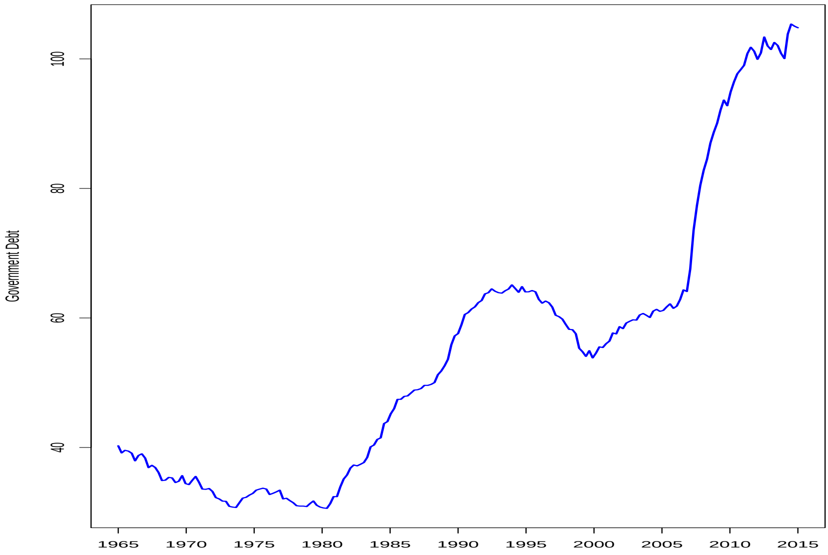

Deficit or Surplus, % of GDP
Unemployment Rate
2
0
−2
−4
−6
−8
−10
11
10
9
8
7
6
5
4
1970
1975
1980
1985
1990
1995
2000
2005
2010
2015

- $$\begin{aligned}\text{Government Outlays to GDP ratio} &= \frac{Outlays}{GDP}\\ \text{Deficit-GDP ratio} &= \frac{Deficit}{GDP}\\ \text{Debt-GDP ratio} &= \frac{Debt}{GDP}\end{aligned}$$
- $$\begin{aligned}\text{Debt at end of this year} &= \text{Debt at end of last year}\\ &\quad +\text{Budget deficit this year}\end{aligned}$$

Government Outlays to GDP ratio, 2006

- US: 36.4%
- Canada: 39.3%
- Japan: 36.0%
- Sweden: 54.3%, France: 52.7%, Germany: 45.3%

Debt to GDP Ratio, 1965-2015



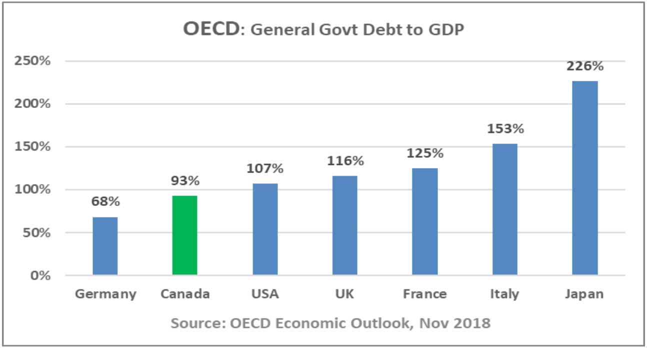
International Debt to GDP Ratios (OECD)

Includes Currency and Deposits (Overzealous Measure)

Country	2010	2011	2012	2013	2014	2015	2016	2017
Estonia	11.93	9.54	13.15	13.62	13.85	12.75	12.73	12.55
Chile	15.27	17.85	18.37	18.99	22.39	24.41	28.08	29.65
Denmark	53.44	60.11	60.62	56.73	59.14	53.79	52.60	49.96
Sweden	52.59	53.28	54.40	57.15	63.40	61.56	60.33	57.95
Australia	41.92	46.31	59.25	55.77	61.63	64.28	68.64	65.72
Germany	84.45	84.18	88.11	83.27	83.35	78.96	76.01	71.52
Ireland	83.50	111.46	129.36	131.73	121.20	88.52	84.14	77.24
Canada	105.22	107.88	111.54	107.51	108.54	114.75	114.13	109.10
Spain	66.56	77.69	92.53	105.73	118.41	116.31	116.52	114.66
United Kingdom	86.56	100.31	104.11	99.92	109.92	109.45	119.38	116.91
Belgium	107.98	110.60	120.47	118.48	131.11	127.67	128.44	121.90
France	101.00	103.81	111.94	112.47	120.16	120.83	125.46	124.25
United States	125.85	130.98	132.69	136.28	135.60	136.60	138.51	135.66
Portugal	104.07	107.85	137.10	141.43	151.40	149.15	145.32	145.38
Italy	124.88	117.94	136.24	143.69	156.06	157.03	154.90	152.61
Greece	128.97	110.91	164.11	179.69	180.82	182.94	185.79	188.73
Japan	207.52	222.31	230.39	233.22	238.46	237.39	234.55	
Mexico	31.15	37.14	41.13	47.11	50.06	53.33	51.79	
Switzerland	42.62	43.03	43.81	43.08	43.14	43.18	42.46	

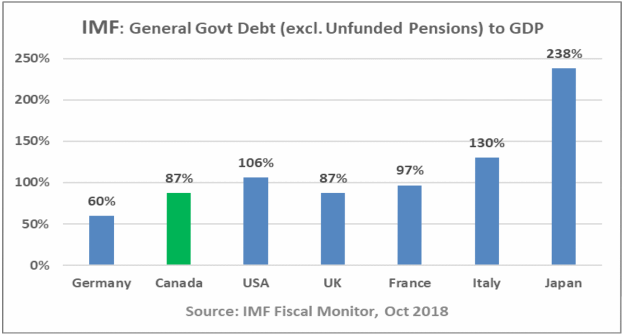
Public Debt Including Some Unfunded Public Sector Liabilities

OECD Nov 2018



Public Debt Including Some Public Sector Liabilities

IMF Nov 2018



- Why a model? Because now we want to *understand* the effects of government activity (not just simply describe them).

- Why a two period (dynamic) model? Because the government choice of policies today affect what it can do tomorrow (a tax cut today, together with a budget deficit, requires higher taxes or lower spending tomorrow). Therefore need a model where choices today affect choices tomorrow. Simplest such model is a two-period model.

- Model is due to Irving Fisher (1867-1947), extension due to Albert Ando (1929-2003) and Franco Modigliani (1919-2003) and Milton Friedman (1912-2006).

- Single household, lives for two periods (working life, retired life)
- Cares about consumption in first period, c_1, and second period, c_2.
- Utility function

$$U(c_1, c_2) = u(c_1) + \beta u(c_2)$$

  where $\beta \in (0, 1)$ measures household's impatience.
- Function u satisfies $u'(c) > 0$ (more is better) and $u''(c) < 0$ (but at a decreasing rate).
- Income $y_1 > 0$ in the first period and $y_2 \geq 0$ in the second period. Income is measured in units of the consumption good, not in terms of money.
- Starts life with initial wealth $A \geq 0$, due to bequests; measured in terms of the consumption good.
- Can save or borrow at real interest rate r

- Nominal and real interest rates

$$1 + r = \frac{1 + i}{1 + \pi}$$

- Approximately (as long as $r\pi$ is small)

$$\begin{aligned} i &= r + \pi \\ r &= i - \pi \end{aligned}$$

- Budget constraint in period 1

$$c_1 + s = y_1 + A$$

where s is household's saving (borrowing if $s < 0$).

- Second period budget constraint

$$c_2 = y_2 + (1 + r)s$$

- Decision problem of household:
Choose (c_1, c_2, s) to maximize lifetime utility, subject to the budget constraints.

- Simplify: consolidate two budget constraints into intertemporal budget constraint by substituting out saving: solve second budget constraint for s to obtain

$$s = \frac{c_2 - y_2}{1 + r}$$

- Substitute into first budget constraint:

$$c_1 + \frac{c_2 - y_2}{1 + r} = y_1 + A$$

  or

$$c_1 + \frac{c_2}{1 + r} = y_1 + \frac{y_2}{1 + r} + A$$

- Interpretation: price of consumption in first period is 1. Price of consumption in period 2 is $\frac{1}{1+r}$, equal to relative price of consumption in period 2, relative to consumption in period 1.

- Intertemporal budget constraint says that total expenditures on consumption goods $c_1 + \frac{c_2}{1+r}$, measured in prices of the period 1 consumption good, equal total income $y_1 + \frac{y_2}{1+r}$, measured in units of the period 1 consumption good, plus initial wealth. Sum of labor income $y_1 + \frac{y_2}{1+r}$ also referred to as human capital.

- Let $I = y_1 + \frac{y_2}{1+r} + A$ denote total lifetime income, consisting of human capital and initial wealth.

- Maximization problem

$$\max_{c_1, c_2} \quad \{u(c_1) + \beta u(c_2)\}$$

$$s.t. \quad c_1 + \frac{c_2}{1+r} = I$$

- Lagrangian method or substitution method

- Lagrangian

$$\mathcal{L} = u(c_1) + \beta u(c_2) + \lambda \left[I - c_1 - \frac{c_2}{1+r} \right]$$

- Taking first order conditions with respect to c_1 and c_2 yields

$$\begin{aligned} u'(c_1) - \lambda &= 0 \\ \beta u'(c_2) - \frac{\lambda}{1+r} &= 0 \end{aligned}$$

- We can rewrite both equations as

$$\begin{aligned} u'(c_1) &= \lambda \\ \beta(1+r)u'(c_2) &= \lambda \end{aligned}$$

- Combining yields

$$u'(c_1) = \beta(1+r)u'(c_2)$$

  or

$$u'\left(I - \frac{c_2}{1+r} \right) = (1+r)\beta u'(c_2)$$

- Existence of unique solution? Assume Inada condition

$$\lim_{c \to 0} u'(c) = \infty$$

define

$$f(c_2) = u'\left(I - \frac{c_2}{1+r}\right) - (1+r)\beta u'(c_2)$$

and use the Intermediate Value Theorem to show that there is a value for c_2 that makes $f(c_2) = 0$.

- Optimality condition

$$u'(c_1) = \beta(1+r)u'(c_2)$$

- Equalize marginal rate of substitution between consumption tomorrow and consumption today, $\frac{\beta(u'(c_2)}{u'(c_1)}$, with relative price of consumption tomorrow to consumption today, $\frac{\frac{1}{1+r}}{1} = \frac{1}{1+r}$.

- This condition, together with the intertemporal budget constraint, uniquely determines the optimal consumption choices (c_1, c_2), as a function of incomes (y_1, y_2), initial wealth A and the interest rate r.

What is next:

- Explicit solution for a simply example
- Graphic representation of general case
- Changes in income (y_1, y_2, A) and the interest rate r

An Example: Period utility is $u(c) = \log(c)$; $u'(c) = \frac{1}{c}$

- Optimality condition becomes

$$\begin{aligned} \frac{\beta * \frac{1}{c_2}}{\frac{1}{c_1}} &= \frac{1}{1+r} \\ \frac{\beta c_1}{c_2} &= \frac{1}{1+r} \\ c_2 &= \beta(1+r)c_1 \end{aligned}$$

- Inserting this into the lifetime budget constraint yields

$$\begin{aligned} c_1 + \frac{\beta(1+r)c_1}{1+r} &= I \\ c_1(1+\beta) &= I \\ c_1 &= \frac{I}{1+\beta} \\ c_1(y_1, y_2, A, r) &= \frac{1}{1+\beta}\left(y_1 + \frac{y_2}{1+r} + A\right) \end{aligned}$$

- Since $c_2 = \beta(1+r)c_1$ we find

$$c_2 = \frac{\beta(1+r)}{1+\beta} I = \frac{\beta(1+r)}{1+\beta}\left(y_1 + \frac{y_2}{1+r} + A\right)$$

- Finally, since savings $s = y_1 + A - c_1$

$$\begin{aligned} s &= y_1 + A - \frac{1}{1+\beta}\left(y_1 + \frac{y_2}{1+r} + A\right) \\ &= \frac{\beta}{1+\beta}(y_1 + A) - \frac{y_2}{(1+r)(1+\beta)} \end{aligned}$$

  which may be positive or negative, depending on how high first period income and initial wealth is compared to second period income.

- Optimal consumption choice today is simple: eat a fraction $\frac{1}{1+\beta}$ of total lifetime income I today and save the rest.
- Note: the higher is income y_1 relative to y_2, the higher is saving s.

Graphic Solution of the Model

- For general utility functions $u(.)$, we cannot solve for the optimal consumption and savings choices analytically.
- But we can do graphical analysis. Idea: make a plot with c_1 on x-axis and c_2 on y-axis.
- Plot budget line and indifference curve and derive tangency point, which is the optimal choice.
- The computer can always be used.

- Combination of all (c_1, c_2) that can be exactly afforded.

$$c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r} + A$$

- Suppose $c_2 = 0$. Then can afford $c_1 = y_1 + A + \frac{y_2}{1+r}$ in the first period.
- Suppose $c_1 = 0$. Then can afford $c_2 = (1+r)(y_1 + A) + y_2$ in the second period.
- Slope of the budget line is

$$\begin{aligned} \text{slope} &= \frac{c_2^b - c_2^a}{c_1^b - c_1^a} \\ &= \frac{(1+r)(y_1 + A) + y_2}{-\left(y_1 + A + \frac{y_2}{1+r}\right)} \\ &= -(1+r) \end{aligned}$$

- Utility function tells us how the household values consumption today and consumption tomorrow.
- Indifference curve is a collection of bundles (c_1, c_2) that yield the same utility:

$$v = u(c_1) + \beta u(c_2)$$

- Slope: totally differentiate with respect to (c_1, c_2) :

$$dc_1 * u'(c_1) + dc_2 * \beta u'(c_2) = 0$$

- Rewriting

$$\frac{dc_2}{dc_1} = -\frac{u'(c_1)}{\beta u'(c_2)} = \text{MRS}$$

- For example $u(c) = \log(c)$ we find

$$\frac{dc_2}{dc_1} = -\frac{c_2}{\beta c_1}$$

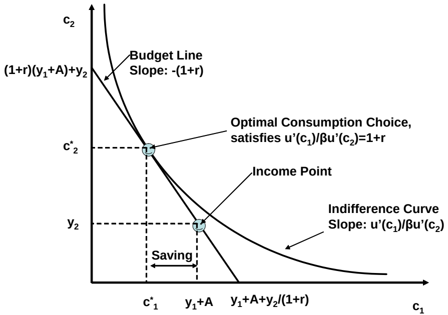
- Optimality condition

$$-\frac{u'(c_1)}{\beta u'(c_2)} = -(1+r) = \text{slope}$$

or

$$\text{MRS} = \frac{\beta u'(c_2)}{u'(c_1)} = \frac{1}{1+r}$$

- Interpretation: at the optimal consumption choice the cost, in terms of utility, of saving one more unit equals benefit of saving one more unit Cost of saving one more unit, and consume one unit less in first period, in terms of utility equals $u'(c_1)$. Saving one more unit yields $(1+r)$ more units of consumption tomorrow. In terms of utility, this is worth $(1+r)\beta u'(c_2)$. Equality of cost and benefit implies the optimality condition.



Optimal Consumption Choice

- Analyze how changes in income and the interest rate affect household consumption and savings decisions

- Why? Fiscal policy changes level and timing of after-tax income. Government deficits and monetary policy may change real interest rates.

Income Changes again for $u(c) = \log(c)$

$$
\begin{aligned}
I &= y_1 + \frac{y_2}{1+r} + A \\
c_1 &= \frac{I}{1+\beta} \\
c_2 &= \frac{\beta(1+r)}{1+\beta} I \\
s &= \frac{\beta}{1+\beta}(y_1 + A) - \frac{y_2}{(1+r)(1+\beta)}
\end{aligned}
$$

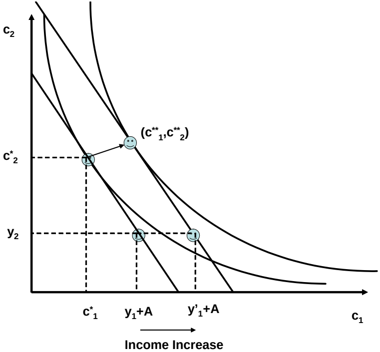
We have

$$
\begin{aligned}
\frac{dc_1}{dI} &= \frac{1}{1+\beta} > 0 \\
\frac{dc_1}{dI} &= \frac{\beta(1+r)}{1+\beta} > 0
\end{aligned}
$$

and thus

$$
\begin{aligned}
\frac{dc_1}{dA} &= \frac{dc_1}{dy_1} = \frac{1}{1+\beta} > 0 \text{ and } \frac{dc_1}{dy_2} = \frac{1}{(1+\beta)(1+r)} > 0 \\
\frac{dc_2}{dA} &= \frac{dc_2}{dy_1} = \frac{\beta(1+r)}{1+\beta} > 0 \text{ and } \frac{dc_2}{dy_2} = \frac{\beta}{1+\beta} > 0 \\
\frac{ds}{dA} &= \frac{ds}{dy_1} = \frac{\beta}{1+\beta} > 0 \text{ and } \frac{ds}{dy_2} = -\frac{1}{(1+\beta)(1+r)} < 0
\end{aligned}
$$

- Suppose income in the first period y_1 increases to $y_1' > y_1$.
- Budget line shifts out in a parallel fashion (since interest rate does not change).
- Consumption in both periods increases: positive income effect.
- Similar analysis for change in A or y_2.



A Change in Income

- Three effects, stemming from the budget constraint

$$c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r} + A \equiv I(r)$$

- Higher interest rate reduces the present discounted value of second period income, $\frac{y_2}{1+r}$. This is often called a (human capital) *wealth effect*.

- An increase in r reduces the price of second period consumption, $\frac{1}{1+r}$, which has two effects.
  - First, since the price of one of the two goods has declined, households can now afford more; this is an *income effect*.
  - Second, a decline in $\frac{1}{1+r}$ makes second period consumption cheaper, *relative* to first period consumption. Thus one would expect that the consumper substitutes second period consumption for first period consumption. This is called *substitution effect*.

	Incr. in r		Decr. in r	
Effect on	c_1	c_2	c_1	c_2
Wealth Effect	$-$	$-$	$+$	$+$
Income Effect	$+$	$+$	$-$	$-$
Substitution Effect	$-$	$+$	$+$	$-$

- Example $u(c) = \log(c)$. Optimal choices

$$
\begin{aligned}
c_1 &= \frac{1}{1+\beta} * I(r) \\
c_2 &= \frac{\beta(1+r)}{1+\beta} * I(r)
\end{aligned}
$$

- An increase in r reduces lifetime income $I(r)$, unless $y_2 = 0$. This is the negative wealth effect, reducing consumption in both periods.

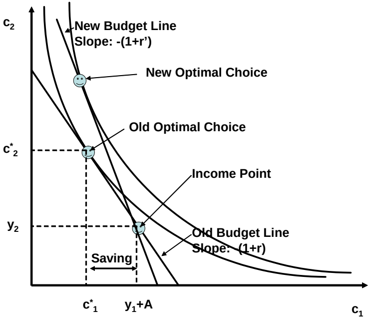
- For c_1 this is the only effect: absent a change in $I(r)$, c_1 does not change. For this special example income and substitution effect exactly cancel out, leaving only the negative wealth effect.

- For c_2 both income and substitution effect are positive. However, the wealth effect is negative, leaving the overall response of consumption c_2 in the second period to an interest rate increase ambiguous. Remembering that $I(r) = A + y_1 + \frac{y_2}{1+r}$, we see that

$$c_2 = \frac{\beta(1+r)}{1+\beta}(A + y_1) + \frac{\beta}{1+\beta} y_2$$

which is increasing in r.

- Increase in the interest rate from r to $r' > r$. Indifference curves do not change. Budget line gets steeper.
- Income point $c_1 = y_1 + A, c_2 = y_2$ remains affordable.
- Budget line tilts around the autarky point and gets steeper.



An Increase in the Interest Rate

Proposition

Let (c_1^*, c_2^*, s^*) denote the optimal consumption and saving choices associated with interest rate r. Furthermore denote by $(\widehat{c}_1^*, \widehat{c}_2^*, \widehat{s}^*)$ the optimal consumption-savings choice associated with interest $\widehat{r} > r$

1. *If $s^* > 0$ (that is $c_1^* < A + y_1$ and the agent is a saver at interest rate r), then $U(c_1^*, c_2^*) < U(\widehat{c}_1^*, \widehat{c}_2^*)$ and either $c_1^* < \widehat{c}_1^*$ or $c_2^* < \widehat{c}_2^*$ (or both).*
2. *Conversely, if $\widehat{s}^* < 0$ (that is $\widehat{c}_1^* > A + y_1$ and the agent is a borrower at interest rate $\widehat{r}$), then $U(c_1^*, c_2^*) > U(\widehat{c}_1^*, \widehat{c}_2^*)$ and either $c_1^* > \widehat{c}_1^*$ or $c_2^* > \widehat{c}_2^*$ (or both).*

Proof ($s^* > 0$) I

- Budget constraints read as

$$\begin{aligned} c_1 + s &= y_1 + A \\ c_2 &= y_2 + (1+r)s \end{aligned}$$

- (c_1^*, c_2^*, s^*) is optimal for r. If $\widehat{r} > r$, the agent can choose

$$\begin{aligned} \tilde{c}_1 &= c_1^* > 0 \\ \tilde{s} &= s^* > 0 \end{aligned}$$

  and

$$\begin{aligned} \tilde{c}_2 &= y_2 + (1+\widehat{r})\tilde{s} \\ &= y_2 + (1+\widehat{r})s^* \\ &> y_2 + (1+r)s^* = c_2^* \end{aligned}$$

Proof ($s^* > 0$) II

- Since $\tilde{c}_1 \geq c_1^*$ and $\tilde{c}_2 > c_2^*$ we have

$$U(c_1^*, c_2^*) < U(\tilde{c}_1, \tilde{c}_2)$$

- The optimal choice at $\widehat{r}$ is obviously no worse, and thus

$$U(c_1^*, c_2^*) < U(\tilde{c}_1, \tilde{c}_2) \leq U(\widehat{c}_1^*, \widehat{c}_2^*)$$

- But

$$U(c_1^*, c_2^*) < U(\widehat{c}_1^*, \widehat{c}_2^*)$$

requires either $c_1^* < \widehat{c}_1^*$ or $c_2^* < \widehat{c}_2^*$ (or both).

QED.

- So far assumed that household can borrow freely at interest rate r. Now suppose that household cannot borrow at all, that is, let us impose the additional constraint on the consumer maximization problem that

$$s \geq 0.$$

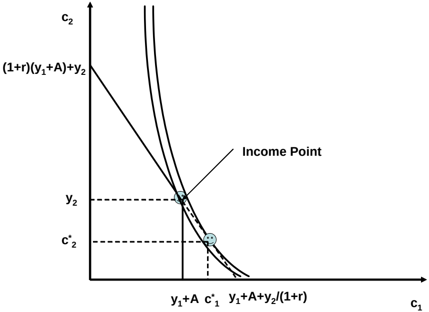
Let (c_1^*, c_2^*, s^*) denote the optimal consumption choice the household would choose *in the absence* of the borrowing constraint.

- If optimal unconstrained choice satisfies $s^* \geq 0$, then it remains optimal.
- If optimal unconstrained choice satisfies $s^* < 0$, then it is optimal to set

$$\begin{aligned} c_1 &= y_1 + A \\ c_2 &= y_2 \\ s &= 0 \end{aligned}$$

- Welfare loss from inability to borrow.

- In the presence of borrowing constraints has a kink at $(y_1 + A, y_2)$.
- For $c_1 < y_1 + A$ we have the usual budget constraint, as here $s > 0$ and the borrowing constraint is not binding.
- But with borrowing constraint any consumption $c_1 > y_1 + A$ is unaffordable, so the budget constraint has a vertical segment at $y_1 + A$



Borrowing Constraints

- Effects of income changes on consumption choices are potentially more extreme in the presence of borrowing constraints, which may give the government's fiscal policy extra power.
- Change in second period income y_2. With borrowing constraints optimal choice satisfies

$$\begin{aligned} c_1 &= y_1 + A \\ c_2 &= y_2 \\ s &= 0 \end{aligned}$$

- Increase in y_2 does not affect consumption in the first period of her life and increases consumption in the second period of his life one-for-one with income.
- Increase in y_1 on the other hand, has strong effects on c_1. If, after the increase it is still optimal to set $s = 0$ (which will be the case if the increase in y_1 is small), then c_1 increases one-for-one with the increase in current income and c_2 remains unchanged.

- Objective: endogenize income (y_1, y_2, A) and interest rate r. Landmark paper by Peter Diamond (1965).

- Households maximize

$$u(c_1, c_2) = \log(c_1) + \log(c_2)$$

- Budget constraint: $A = y_2 = 0$ (retired when old). Income when young equals wage: $y_1 = w$. Thus

$$c_1 + \frac{c_2}{1+r} = w$$

- Optimal consumption and savings decisions

$$
\begin{aligned}
c_1 &= \frac{1}{2} w \\
c_2 &= \frac{1}{2} w(1+r) \\
s &= \frac{1}{2} w
\end{aligned}
$$

- Firms hire l workers, pay wages w, lease capital k at rate ρ, produce consumption goods according to production function $y = k^\alpha l^{1-\alpha}$.
- Takes (w, ρ) as given, and chooses (l, k) to maximize profits

$$\max_{(k,l)} k^\alpha l^{1-\alpha} - wl - \rho k$$

- First order conditions

$$\begin{aligned}(1-\alpha)k^\alpha l^{-\alpha} &= w \\ \alpha k^{\alpha-1} l^{1-\alpha} &= \rho.\end{aligned}$$

- Capital stock k_1 in period 1 given.
- Labor market clearing:

$$l_1 = 1$$

- Thus wages given by

$$w = (1 - \alpha)k_1^{\alpha}$$

- Only asset is physical capital stock. Thus savings have to equal k_2. Asset market clearing condition

$$s = k_2$$

- Plugging in for $s = \frac{1}{2}w$ and using equilibrium wage function gives:

$$\frac{1}{2}(1-\alpha)k_1^{\alpha} = k_2.$$

- Steady state: level of capital that remains constant over time, $k_1 = k_2 = k$.

- Steady state satisfies

$$\begin{aligned} \frac{1}{2}(1-\alpha)k^{\alpha} &= k \\ k^* &= \left[\frac{1}{2}(1-\alpha)\right]^{\frac{1}{1-\alpha}} \end{aligned}$$

- Steady state wages are given by

$$w = (1-\alpha)\left(k^*\right)^{\alpha} = (1-\alpha)\left[\frac{1}{2}(1-\alpha)\right]^{\frac{\alpha}{1-\alpha}}$$

- Steady state interest rate r? When households save in period 1, they purchase capital k_2 which is used in production and earns rental rate ρ.

- Rental rate given by:

$$\rho = \alpha k^{\alpha-1} l^{1-\alpha} = \alpha \left(\left[\frac{1}{2}(1-\alpha) \right]^{\frac{1}{1-\alpha}} \right)^{\alpha-1} = \frac{2\alpha}{1-\alpha}$$

- If we assume that capital completely depreciates after production, then

$$1 + r = \rho = \frac{2\alpha}{1-\alpha}$$

- Time extends from $t = 0$ forever.
- Each period t a total number N_t of new young households are born that live for two periods.
- Assume population grows at a constant rate n:

$$N_t = (1+n)^t N_0 = (1+n)^t$$

- Household problem:

$$\max_{c_{1t}, c_{2t+1}, s_t} \{\log(c_{1t}) + \beta \log(c_{2t+1})\}$$

$$\begin{aligned} c_{1t} + s_t &= w_t \\ c_{2t+1} &= (1 + r_{t+1}) s_t. \end{aligned}$$

with solution:

$$\begin{aligned} c_{1t} &= \frac{1}{1+\beta} w_t \\ s_t &= \frac{\beta}{1+\beta} w_t \end{aligned}$$

Complete Analysis: Production

- Aggregate output Y_t given by

$$Y_t = K_t^{\alpha} L_t^{1-\alpha}$$

- Wages

$$w_t = (1-\alpha)\left(\frac{K_t}{L_t}\right)^{\alpha}$$

- Labor market clearing condition:

$$L_t = N_t$$

- Thus (with $k_t = \frac{K_t}{N_t}$)

$$w_t = (1-\alpha)\left(\frac{K_t}{N_t}\right)^{\alpha} = (1-\alpha)k_t^{\alpha}$$

- Capital market

$$s_t N_t = K_{t+1}$$

- Rewriting:

$$s_t = \frac{K_{t+1}}{N_t} = \frac{K_{t+1}}{N_{t+1}} * \frac{N_{t+1}}{N_t} = k_{t+1}(1+n)$$

- Plugging in from the saving function

$$s_t = \frac{\beta}{1+\beta} w_t = \frac{\beta}{1+\beta}(1-\alpha)k_t^{\alpha} = k_{t+1}(1+n)$$

- Thus

$$k_{t+1} = \frac{\beta(1-\alpha)}{(1+\beta)(1+n)} k_t^{\alpha}$$

- Aggregate population in period t is $N_{t-1} + N_t$.

- Per capita output is

$$y_t = \frac{Y_t}{N_{t-1} + N_t} = \frac{K_t^{\alpha} N_t^{1-\alpha}}{N_{t-1} + N_t}$$

- Steady state: situation in which the per capita capital stock k_t is constant over time thus and $k_{t+1} = k_t$

- Steady state satisfies

$$k = \frac{\beta(1-\alpha)}{(1+\beta)(1+n)} k^{\alpha}$$

or

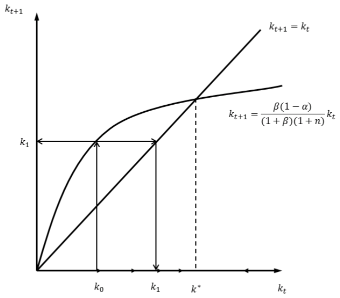
$$k^* = \left[\frac{\beta(1-\alpha)}{(1+\beta)(1+n)} \right]^{\frac{1}{1-\alpha}}$$

- Plotting k_{t+1} against k_t (together with 45^0-line) we can determine steady states, entire dynamics of model.

$$k_{t+1} = \frac{\beta(1-\alpha)}{(1+\beta)(1+n)} k_t^{\alpha}$$

- If $k_t = 0$, then $k_{t+1} = 0$. Since $\alpha < 1$, the curve $\frac{\beta(1-\alpha)}{(1+\beta)(1+n)} k_t^{\alpha}$ is strictly concave, initially above 45^0-line, but eventually intersects it.

- Unique positive steady state k^*. This steady state is globally asymptotically stable.


k_{t+1}
$k_{t+1} = k_t$
$k_{t+1} = \frac{\beta(1-\alpha)}{(1+\beta)(1+n)} k_t$
k_1
k_0
k_1
k^*
k_t

Labor income Taxes and first period transfers when $u(c_1) + \beta u(c_2)$

- Consider the budget constraint to be

$$\begin{aligned} c_1 + s &= w(1-\tau) + T \\ c_2 &= (1+r)s \end{aligned}$$

- The first order condition (after substituting c_2 and s) is

$$u'(c_1) = (1+r)\,\beta\,u'\left[\left(w(1-\tau) + T - c_1\right)(1+r)\right]$$

- But if there is no net collection by the government of any revenue, i.e. if $\tau w = T$ we have the same allocation as if there were no taxes

$$u'(c_1) = (1+r)\beta u'\left[(w - c_1)(1+r)\right]$$

- No net wealth-income or substitution effects

- Consider the budget constraint to be

$$\begin{aligned}(1+\tau^c)c_1 + s &= w + T \\ c_2 &= (1+r)s\end{aligned}$$

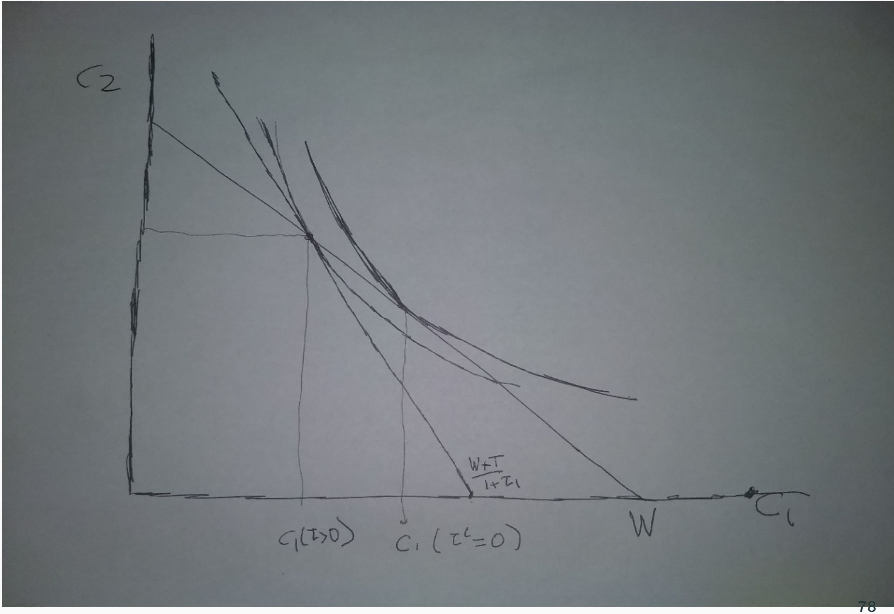
- The first order condition (after substituting c_2 and s) is

$$u'(c_1) = (1+r)(1+\tau^c)\,\beta\, u'\left[(w + T - c_1(1+\tau^c))\,(1+r)\right]$$

- If there is no collection by the government of any revenue, i.e. if $\tau^c c_1 = T$ (note that the household cannot take this into account) things ARE different

$$u'(c_1) = (1+r)(1+\tau^c)\,\beta\, u'\left[(w - c_1)\,(1+r)\right]$$

- No net wealth-income effect but a substitution effect. Now c_1 is lower.


C2
W+T
1+τ1
W
C1
c1(τ>0)
c1 (τ1=0)

Part II

The Life Cycle Model

- Generalization of the two-period model to multiple periods
- Modigliani-Ando life cycle hypothesis focuses on consumption and savings profiles as well as wealth accumulation over a household's lifetime
- Friedman's permanent income hypothesis focuses on impact of timing and characteristics of uncertain income on consumption choices.

Model Description

- Household lives for T periods. We allow that $T = \infty$
- In each period t household earns after-tax income y_t and consumes c_t.
- May have initial wealth $A \geq 0$

- Period budget constraint

$$c_t + s_t = y_t + (1 + r)s_{t-1}$$

Here r denotes interest rate, s_t denotes financial assets carried over from period t to period $t + 1$ and s_{t-1} denotes assets from period $t - 1$ carried to period t.

- Net Saving in period t is defined as the difference between total income $y_t + rs_{t-1}$ and consumption c_t.

$$s_t - s_{t-1} \quad = \quad y_t + rs_{t-1} - c_t$$

- Period 1 budget constraint

$$c_1 + s_1 = A + y_1.$$

$$U(c_1, c_2, \ldots, c_T) = u(c_1) + \beta u(c_2) + \beta^2 u(c_3) + \ldots + \beta^{T-1} u(c_T)$$

or

$$U(c) = \sum_{t=1}^{T} \beta^{t-1} u(c_t)$$

where $c = (c_1, c_2, \ldots, c_T)$ denotes the lifetime consumption profile

Rewrite the period-by-period budget constraints as a single intertemporal budget constraint: note that

$$\begin{aligned} c_1 + s_1 &= A + y_1 \\ c_2 + s_2 &= y_2 + (1+r)s_1 \end{aligned}$$

- Solve second equation for s_1

$$s_1 = \frac{c_2 + s_2 - y_2}{1+r}$$

- and plug into first equation, to obtain

$$c_1 + \frac{c_2 + s_2 - y_2}{1+r} = A + y_1$$

- which can be rewritten as

$$c_1 + \frac{c_2}{1+r} + \frac{s_2}{1+r} = A + y_1 + \frac{y_2}{1+r}$$

- Repeat this procedure: from third period budget constraint

$$c_3 + s_3 = y_3 + (1 + r)s_2$$

we can solve for

$$s_2 = \frac{c_3 + s_3 - y_3}{1 + r}$$

and plug in to obtain

$$c_1 + \frac{c_2}{1 + r} + \frac{c_3}{(1 + r)^2} + \frac{s_3}{(1 + r)^2} = A + y_1 + \frac{y_2}{1 + r} + \frac{y_3}{(1 + r)^2}$$

- Continue the process T times, to arrive at the intertemporal budget constraint

$$\begin{aligned} & c_1 + \frac{c_2}{1 + r} + \frac{c_3}{(1 + r)^2} + \ldots + \frac{c_T}{(1 + r)^{T-1}} + \frac{s_T}{(1 + r)^{T-1}} \\ = \quad & A + y_1 + \frac{y_2}{1 + r} + \frac{y_3}{(1 + r)^2} \ldots + \frac{y_T}{(1 + r)^{T-1}} \end{aligned}$$

- s_T denotes saving from period T to $T+1$. Household lives only for T periods, so she has no use for saving in period $T+1$. We don't allow $s_T < 0$. Thus $s_T = 0$ and

$$\begin{aligned} & c_1 + \frac{c_2}{1+r} + \frac{c_3}{(1+r)^2} + \ldots + \frac{c_T}{(1+r)^{T-1}} \\ = \quad & A + y_1 + \frac{y_2}{1+r} + \frac{y_3}{(1+r)^2} \ldots + \frac{y_T}{(1+r)^{T-1}} \end{aligned}$$

or

$$\sum_{t=1}^{T} \frac{c_t}{(1+r)^{t-1}} = A + \sum_{t=1}^{T} \frac{y_t}{(1+r)^{t-1}}$$

Present discounted value of lifetime consumption $(c_1, \ldots, c_T)$ equals the present discounted value of lifetime income $(y_1, \ldots, y_T)$ plus initial bequests.

- Household maximizes utility subject to budget constraint

- In order to solve this problem, need to use Lagrangian method.

1. Rewrite all constraints of the problem in the form

$$stuff = 0$$

For our problem

$$\begin{aligned} & A + y_1 + \frac{y_2}{1+r} + \frac{y_3}{(1+r)^2} \dots + \frac{y_T}{(1+r)^{T-1}} \\ & -c_1 - \frac{c_2}{1+r} - \frac{c_3}{(1+r)^2} - \dots - \frac{c_T}{(1+r)^{T-1}} \\ = \quad & 0 \end{aligned}$$

- Write down the Lagrangian: take the objective function and add all constraints, each pre-multiplied by a so-called Lagrange multiplier. This entity λ can be treated as a constant number. Lagrangian becomes

$$
\begin{aligned}
& \mathcal{L}(c_1, \ldots, c_T) \\
= \quad & u(c_1) + \beta u(c_2) + \beta^2 u(c_3) + \ldots + \beta^{T-1} u(c_T) + \\
& \lambda \begin{pmatrix} A + y_1 + \frac{y_2}{1+r} + \frac{y_3}{(1+r)^2} \ldots + \frac{y_T}{(1+r)^{T-1}} \\ -c_1 - \frac{c_2}{1+r} - \frac{c_3}{(1+r)^2} \ldots - \frac{c_T}{(1+r)^{T-1}} \end{pmatrix} \\
= \quad & \sum_{t=1}^{T} \beta^{t-1} u(c_t) + \lambda \left(A + \sum_{t=1}^{T} \frac{y_t}{(1+r)^{t-1}} - \sum_{t=1}^{T} \frac{c_t}{(1+r)^{t-1}} \right)
\end{aligned}
$$

- Take first order conditions with respect to all choice variables and set them equal to 0. For example chose variables are $(c_1, \ldots, c_T)$

$$u'(c_1) - \lambda = 0 \qquad \text{or} \qquad u'(c_1) = \lambda.$$

Doing the same for c_2 yields

$$\beta u'(c_2) - \lambda \frac{1}{1+r} = 0 \qquad \text{or} \qquad (1+r)\beta u'(c_2) = \lambda$$

and for an arbitrary c_t we find $(1+r)^{t-1}\beta^{t-1}u'(c_t) = \lambda$. Combining

$$\begin{aligned} u'(c_1) &= (1+r)\beta u'(c_2) \\ &= \ldots = [(1+r)\beta]^{t-1} u'(c_t) = [(1+r)\beta]^{t} u'(c_{t+1}) \\ &= \ldots = [(1+r)\beta]^{T-1} u'(c_T) \end{aligned}$$

These equations determine relative consumption levels across periods, that is, the ratios $\frac{c_2}{c_1}, \frac{c_3}{c_2}$ and so forth. For absolute consumption levels need to use the budget constraint.

- For $t = 1$, $$u'(c_1) = (1+r)\beta u'(c_2).$$
  - If consume a little less in period 1, and save the amount to consume a bit extra in the second period, then the utility cost is $-u'(c_1)$ and the benefit is $(1+r)\beta u'(c_2)$. Thus entire utility consequences from saving a little more today and eating it tomorrow are
    $$-u'(c_1) + (1+r)\beta u'(c_2) \leq 0$$
    because the household should not be able to improve his lifetime utility from doing so.
  - Similar argument for consuming one unit more today and saving one unit less leads to
    $$-u'(c_1) + (1+r)\beta u'(c_2) \geq 0.$$
  - Combining the two equations leads to the Euler equation.

- Suppose the market discounts income at the same rate $\frac{1}{1+r}$ as the household discounts utility, β. In this case $\beta = \frac{1}{1+r}$ or $\beta(1+r) = 1$. Euler equation becomes

$$u'(c_1) = u'(c_2) = \ldots = u'(c_t) = \ldots = u'(c_T)$$

- Since utility function is strictly concave (i.e. $u''(c) < 0$) we have that

$$c_1 = c_2 = \ldots = c_t = \ldots = c_T = \bar{c}$$

Consumption is constant over a households' lifetime; the timing of income and consumption is completely de-coupled.

- Consumption level: from the intertemporal budget

$$\sum_{t=1}^{T} \frac{c_t}{(1+r)^{t-1}} = I$$

- Since $c_t = \bar{c}$ for all times t we have:

$$\begin{aligned} \bar{c}\sum_{t=1}^{T} \frac{1}{(1+r)^{t-1}} &= I \\ \bar{c} &= \frac{1}{\sum_{t=1}^{T} \frac{1}{(1+r)^{t-1}}} * I \end{aligned}$$

- The term $\sum_{t=1}^{T} \frac{1}{(1+r)^{t-1}}$ annuitizes a constant stream of consumption.

- Note:

$$\sum_{t=1}^{T} \frac{1}{(1+r)^{t-1}} = \begin{cases} \frac{1+r-\frac{1}{(1+r)^{T-1}}}{r} & \text{if } T < \infty \\ \frac{1+r}{r} & \text{if } T = \infty \end{cases}$$

- Thus, if households are infinitely lived:

$$\bar{c} = c_1 = c_t = \frac{r}{1+r} I$$

- Household lives 60 years, from age 1 to age 60
- Household inherits nothing, i.e. $A = 0$.
- In the first 45 years of life, household works and makes annual income of $\$40,000$ per year. For last 15 years of her life household is retired and earns nothing
- We assume that the interest rate is $r = 0$ and $\beta = 1$.

- From previous discussion we know that consumption over the households' lifetime is constant

$$c_1 = c_2 = \ldots = c_{60} = c$$

- Level of consumption? Total discounted lifetime value of income.

$$\begin{aligned} & y_1 + \frac{y_2}{1+r} + \frac{y_3}{(1+r)^2} \cdots + \frac{y_{60}}{(1+r)^{T-1}} \\ = \quad & y_1 + y_2 + y_3 \ldots + y_{60} = y_1 + y_2 + y_3 \ldots + y_{45} \\ = \quad & 45 * \$40,000 = \$1,800,000 \end{aligned}$$

- Total discounted lifetime cost of consumption

$$\begin{aligned} & c_1 + \frac{c_2}{1+r} + \frac{c_3}{(1+r)^2} + \ldots + \frac{c_{60}}{(1+r)^{59}} \\ = \quad & c_1 + c_2 + \ldots + c_{60} = 60 * c \end{aligned}$$

- Equating lifetime income and cost of lifetime consumption yields

$$
\begin{aligned}
c &= \frac{45}{60} * \$40,000 \\
&= \$30,000
\end{aligned}
$$

- In all working years the household consumes $\$10,000$ less than income and puts the money aside for consumption in retirement.

- Savings in all working periods is

$$\begin{aligned} sav_t &= y_t + rs_{t-1} - c_t \\ &= y_t - c_t \\ &= \$40,000 - \$30,000 \\ &= \$10,000 \end{aligned}$$

whereas for all retirement periods

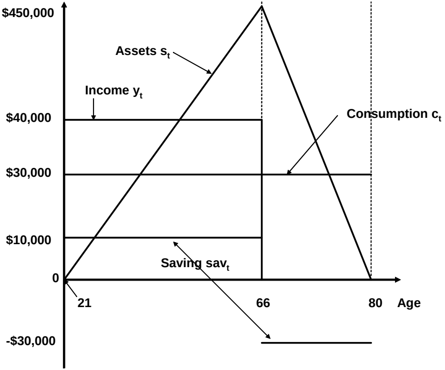
$$\begin{aligned} sav_t &= y_t + rs_{t-1} - c_t \\ &= -c_t \\ &= -\$30,000 \end{aligned}$$

- Asset position of the household. Remember that

$$\begin{aligned} sav_t &= s_t - s_{t-1} \text{ or} \\ s_t &= s_{t-1} + sav_t \end{aligned}$$

Since the household starts with 0 bequests, $s_0 = 0$. Thus

$$\begin{aligned} s_1 &= s_0 + sav_1 \\ &= \$0 + \$10,000 = \$10,000 \\ s_2 &= s_1 + sav_2 = \$10,000 + \$10,000 = \$20,000 \\ s_{45} &= s_{44} + sav_{45} = \$440,000 + \$10,000 = \$450,000 \\ s_{46} &= s_{45} + sav_{46} = \$450,000 - \$30,000 = \$420,000 \\ s_{60} &= s_{59} + sav_{60} = \$30,000 - \$30,000 = \$0 \end{aligned}$$



Life Cycle Profiles, Model

- If $\beta \neq \frac{1}{1+r}$, we need stronger assumptions on the utility function to make more progress.
- Two periods and utility function $u(c) = \log(c)$.
- Euler equation becomes

$$\frac{1}{c_1} = \frac{(1+r)\beta}{c_2} \qquad \text{or} \qquad c_2 = (1+r)\beta c_1$$

- Combining this with the intertemporal budget constraint

$$c_1 + \frac{c_2}{1+r} = A + y_1 + \frac{y_2}{1+r}$$

yields

$$\begin{aligned} c_1 &= \frac{I}{1+\beta} \\ c_2 &= \frac{(1+r)\beta}{1+\beta} I \end{aligned}$$

- If $\beta = \frac{1}{1+r}$, consumption over the life cycle is constant.
- Now suppose $\beta > \frac{1}{1+r}$ or $\beta(1+r) > 1$.
- From Euler equations we have

$$\begin{aligned} u'(c_1) &= (1+r)\beta u'(c_2) \\ &= \ldots = [(1+r)\beta]^{t-1} u'(c_t) = [(1+r)\beta]^t u'(c_{t+1}) \\ &= \ldots = [(1+r)\beta]^{T-1} u'(c_T) \end{aligned}$$

- This implies $\frac{u'(c_1)}{u'(c_2)} = (1+r)\beta.$ and thus

$$\begin{aligned} \frac{u'(c_1)}{u'(c_2)} &> 1 \\ u'(c_1) &> u'(c_2) \end{aligned}$$

- Since $u'(c)$ is a strictly decreasing function we have $c_1 < c_2$.

- Similarly

$$\begin{aligned} [(1+r)\beta]^{t-1} u'(c_t) &= [(1+r)\beta]^{t} u'(c_{t+1}) \\ \frac{u'(c_t)}{u'(c_{t+1})} &= \frac{[(1+r)\beta]^{t}}{[(1+r)\beta]^{t-1}} = (1+r)\beta > 1 \end{aligned}$$

so that $c_{t+1} > c_t$.

- Now suppose that $\beta < \frac{1}{1+r}$ or $\beta(1+r) < 1$.
- Identical argument to the one above shows that now

$$c_1 > c_2 > \ldots > c_t > \ldots > c_T.$$

- Consider the specific CRRA period utility function

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}$$

- Note: for $\sigma = 1$ this utility function becomes

$$u(c) = \log(c)$$

- In this case

$$u'(c) = c^{-\sigma}$$

- Euler equations

$$(c_1)^{-\sigma} = (1+r)\beta(c_2)^{-\sigma} = [(1+r)\beta]^{t-1}(c_t)^{-\sigma} = [(1+r)\beta]^t(c_{t+1})^{-\sigma}$$

- Thus for any period t

$$\begin{aligned} [(1+r)\beta]^{t-1}(c_t)^{-\sigma} &= [(1+r)\beta]^t(c_{t+1})^{-\sigma} \\ (c_t)^{-\sigma} &= [(1+r)\beta](c_{t+1})^{-\sigma} \\ \left(\frac{c_{t+1}}{c_t}\right)^{\sigma} &= (1+r)\beta \\ \frac{c_{t+1}}{c_t} &= [(1+r)\beta]^{\frac{1}{\sigma}} \end{aligned}$$

Explicit Solution for CRRA Utility

- Consumption levels: note that

$$\begin{aligned} c_{t+1} &= [(1+r)\beta]^{\frac{1}{\sigma}} c_t = [(1+r)\beta]^{\frac{2}{\sigma}} c_{t-1} = \ldots [(1+r)\beta]^{\frac{t}{\sigma}} c_1 \text{ or} \\ c_t &= [(1+r)\beta]^{\frac{t-1}{\sigma}} c_1 \end{aligned}$$

- Intertemporal budget constraint

$$\sum_{t=1}^{T} \frac{c_t}{(1+r)^{t-1}} = I$$

- Plugging in for c_t yields

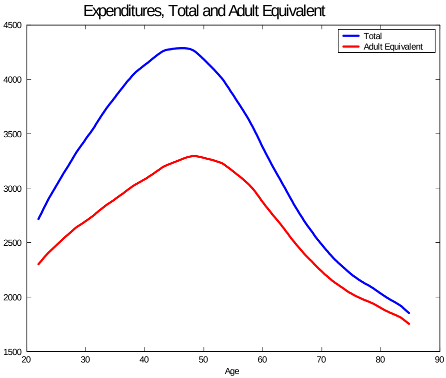
$$\sum_{t=1}^{T} \frac{[(1+r)\beta]^{\frac{t-1}{\sigma}} c_1}{(1+r)^{t-1}} = I$$

- Solving this out for c_1 yields

$$\begin{aligned} c_1 &= \frac{1-\beta^{\frac{1}{\sigma}}(1+r)^{\frac{1}{\sigma}-1}}{1-\left[\beta^{\frac{1}{\sigma}}(1+r)^{\frac{1}{\sigma}-1}\right]^T} * I \\ c_t &= [(1+r)\beta]^{\frac{t-1}{\sigma}} * \frac{1-\beta^{\frac{1}{\sigma}}(1+r)^{\frac{1}{\sigma}-1}}{1-\left[\beta^{\frac{1}{\sigma}}(1+r)^{\frac{1}{\sigma}-1}\right]^T} * I \end{aligned}$$

Empirical Evidence

- Main predictions of the model: consumption should be smooth over the life cycle.
- Assets should display a hump, increasing until retirement and then declining
- In data
  - disposable income follows a hump over the life cycle, with a peak around the age of 45
  - consumption follows a hump over the life cycle



Consumption over the Life Cycle

- Theoretical prediction: consumption is either monotonically upward trending, monotonically downward trending or perfectly flat over the life cycle.
- Data: consumption is hump-shaped over the life cycle (as is income)
- How can we account for the difference?

Potential Explanations

- Changes in household size and household composition
- Family size also is hump-shaped over the life cycle
- Life cycle model only asserts that marginal utility of consumption should be smooth over the life cycle, not necessarily consumption expenditures themselves.
- But: if adjust data by household equivalence scales, still 50% of the hump persists

Potential Explanations: Spend to Work

- Households spend resources to be able to work:
  - Commuting
  - Working Clothes

- Households can either produce a good at home or buy it. When there is more time available they produce it:
  - Household Chores: cooking, cutting grass, shoveling
  - Tax filing

Potential Explanations: Time to Shop

- Retired Households spend more time shopping hence they have more time:
- As a consequence they purchase consumption goods cheaper
- So expenditures may be lower but consumption is actually not lower

- Same predictions as before if consumption and leisure are separable in the utility function,

$$U(c, l) = \sum_{t=1}^{T} \beta^{t-1} \left[u(c_t) + v(l_t) \right]$$

- But if consumption and leisure are substitutes, then if labor supply is hump-shaped over the live cycle (because labor productivity is), then households may find it optimal to have a hump-shaped labor supply and consumption profile over the life cycle.

Potential Explanations: Missing Data at the End of Life

- End of Life Can be very Expensive
- Long term Care: Retirement Home.
- In the absence of Annuities Closeness to Death Makes End of Life Scary.

- Declining consumption profile over the life cycle can be explained by $\beta(1 + r) < 1$.
- If young households can't borrow against their future labor income, then best thing they can do is to consume whatever income whey have when young. Since income is increasing in young ages, so is consumption.
- As households age they want to start saving and the borrowing constraints lose importance. But now the fact that $\beta(1 + r) < 1$ kicks in and induces consumption to fall.

- Uncertain life time acts as additional discount factor, make consumption fall when probability of dying increases.
- Uncertain income induces precautionary savings behavior (as long as $u'''(c) > 0$). As more and more uncertainty is resolved, households start to save less for precautionary reasons and save more.
- Combination of changes in household size and income and lifetime uncertainty can generate a hump in consumption over the life cycle of similar magnitude and timing as in the data (see Attanasio et al., 1999).

- Now assume that incomes $\{y_1, y_2, \ldots, y_T\}$ are risky.
- Only extra concept required is the conditional expectation E_t of an economic variable that is uncertain.
- Thus, $E_t y_{t+1}$ is expectation in period t of income in period $t+1$, $E_t y_{t+2}$ is period t expectation of income in period $t+2$ etc. Timing convention: when expectations are taken in t, y_t is known.

- Assume interest rate r is not random. Also assume lifetime horizon of the household is infinite, $T = \infty$. Generalization of Euler equation

$$u'(c_t) = \beta(1+r)E_t u'(c_{t+1})$$

- Since income in period $t+1$ is risky from the perspective of period t, so is consumption c_{t+1}.

- Main problem for analysis: in general cannot pull the expectation into the marginal utility function, since in general

$$E_t u'(c_{t+1}) \neq u'(E_t c_{t+1})$$

- But now assume that the utility function is quadratic:

$$u(c_t) = -\frac{1}{2}(c_t - \bar{c})^2$$

- $\bar{c}$ is the bliss level of consumption, assumed so large that given the household's lifetime income this consumption level cannot be attained.

- For all consumption levels $c_t < \bar{c}$ we have

$$\begin{aligned} u'(c_t) &= -(c_t - \bar{c}) = \bar{c} - c_t > 0 \\ u''(c_t) &= -1 < 0 \end{aligned}$$

- For all consumption levels $c_t < \bar{c}$ we have

$$\begin{aligned} u'(c_t) &= -(c_t - \bar{c}) = \bar{c} - c_t > 0 \\ u''(c_t) &= -1 < 0 \end{aligned}$$

- Thus this utility function is strictly increasing and strictly concave for all $c_t < \bar{c}$.

- Recall a household with strictly concave utility function is risk averse

- Euler equation becomes

$$-(c_t - \bar{c}) = -E_t(c_{t+1} - \bar{c})$$

Thus

$$E_t c_{t+1} = c_t$$

- Households arrange consumption such that, in expectation, it stays constant between today and tomorrow.

- But: in presence of income risk realized consumption c_{t+1} in period $t+1$ might deviate from this plan.

- In order to determine the level of consumption we need the intertemporal budget constraint:

$$E_t \sum_{s=0}^{\infty} \frac{c_{t+s}}{(1+r)^s} = (1+r)s_{t-1} + E_t \sum_{s=0}^{\infty} \frac{y_{t+s}}{(1+r)^s}$$

- Euler equation implies (by law of iterated expectations) that

$$\begin{aligned} E_t c_{t+1} &= c_t \\ E_t c_{t+2} &= E_t E_{t+1} c_{t+2} = E_t c_{t+1} = c_t \\ E_t c_{t+s} &= c_t \end{aligned}$$

- Left hand side of intertemporal budget constraint:

$$
\begin{aligned}
E_t \sum_{s=0}^{\infty} \frac{c_{t+s}}{(1+r)^s} &= \sum_{s=0}^{\infty} \frac{E_t c_{t+s}}{(1+r)^s} = \\
c_t \sum_{s=0}^{\infty} \frac{1}{(1+r)^s} &= \frac{1}{1-\frac{1}{1+r}} c_t = \frac{1+r}{r} c_t .
\end{aligned}
$$

- Optimal consumption rule:

$$
c_t = \frac{r}{1+r} \left((1+r) s_{t-1} + E_t \sum_{s=0}^{\infty} \frac{y_{t+s}}{(1+r)^s} \right)
$$

- For period 1, thus consumption becomes

$$c_1 = \frac{r}{1+r}\left(A + E_1 \sum_{s=0}^{\infty} \frac{y_{1+s}}{(1+r)^s}\right) = \frac{r}{1+r} E_1 I$$

- Compare this to the certainty case

$$c_1 = \frac{r}{1+r} I$$

- Both expressions: optimal consumption rules are exactly alike: in both cases the household consumes permanent income!

- Surprising result: despite presence of income risk the household makes the same planned consumption choices as in the absence of risk. Called certainty equivalence behavior

- Household do *not* engage in *precautionary savings behavior* by saving more in the presence than in the absence of future income risk: only expected future income matters for planned consumption, not income risk.

- This is true despite household risk aversion.

- *Realized* consumption in period $t+1$ will in general deviate from $E_t c_{t+1} = c_t$

- Realized change in consumption between period t and $t+1$ is given by

$$c_{t+1} - c_t = \frac{r}{1+r} \sum_{s=0}^{\infty} \frac{E_{t+1} y_{t+1+s} - E_t y_{t+1+s}}{(1+r)^s}$$

- Realized change in consumption given by annuity value $\frac{r}{1+r}$ of the sum of discounted *revisions in expectations* about future income in periods $t+1+s$, that is, $E_{t+1} y_{t+1+s} - E_t y_{t+1+s}$.

- How large are realized changes in consumption? Depends crucially on type of income shock the household experiences between the two periods.

- Consider two examples: perfectly permanent shock (unexpected but permanent promotion) and fully transitory shock (unexpected one-time bonus).

- Permanent promotion: extra income p for rest of households' life. Since unexpected in period t, for *all* future periods

$$E_{t+1}y_{t+1+s} - E_t y_{t+1+s} = p.$$

- Thus

$$c_{t+1} - c_t = \frac{r}{1+r}\sum_{s=0}^{\infty}\frac{p}{(1+r)^s} = p\frac{r}{1+r}\frac{1}{1-\frac{1}{1+r}} = p$$

- Consumption goes up by full amount of the unexpected but permanent income increase between period t and $t+1$.

- Now consider a one time unexpected bonus b in period $t+1$. Then

$$E_{t+1} y_{t+1} - E_t y_{t+1} = b$$

and for all future periods beyond $t+1$

$$E_{t+1} y_{t+1+s} - E_t y_{t+1+s} = 0.$$

- Then

$$c_{t+1} - c_t = \frac{r}{1+r} \sum_{s=0}^{\infty} \frac{b}{(1+r)^0} = \frac{r}{1+r} b$$

- Realized consumption change

$$c_{t+1} - c_t = \frac{r}{1+r} b$$

- Increase in consumption is only $\frac{r}{1+r}$ of the bonus (of about 2% if the real interest rate is $r = 2\%$).

- Instead, most of the bonus is saved and used to increase consumption in *all* future periods by a small bit.

- Example: Two periods. Income in first is 1. Income in second is $1+\ell$ with probability .5 and $1-\ell$ with probability .5. Log utility. $1+r=1$.

$$\max_{c_1, s, c_{2g}, c_{2b}} \log c_1 + \frac{1}{2}\log c_{2g} + \frac{1}{2}\log c_{2b}$$

$$\begin{aligned} c_1 + s &= 1 \\ c_{2g} &= s + 1 + \ell \\ c_{2b} &= s + 1 - \ell \end{aligned}$$

- Rewriting after substitution

$$\max_{c_1, s, c_{2g}, c_{2b}} \log(1-s) + \frac{1}{2}\log(s+1+\ell) + \frac{1}{2}\log(s+1-\ell)$$

What if preferences are not quadratic, but like logs?

- First order conditions (absent algebra errors)

$$\frac{-1}{1-s} + \frac{1}{2}\frac{1}{s+1+\ell} + \frac{1}{2}\frac{1}{s+1-\ell} = 0$$

- Simplifying

$$\frac{1}{1-s} = \frac{1+s}{s^2+2s+1-\ell^2}$$
$$(s+1)^2 - \ell^2 = (1+s)(1-s)$$
$$2s^2 + 2s - \ell^2 = 0$$
$$s = \frac{-2+\sqrt{4+8\ell^2}}{4}$$

- Note that if $\ell = 0$ then $s = 0$ while if $\ell = 1$ then $s = .36$.
- The higher the variance (here ℓ) the higher the savings

What if preferences are not quadratic, but like logs?

- In $u'''(c) > 0$ then agents have precautionary savings, this is they save more the higher the risk that they save.
- When $\beta(1 + r) = 1$ then $c_t < E[c_{t+1}]$
- People save extra not so much for a rainy day but for wild weather.
- People are more like this than like quadratic preferences.

Part III

Positive Theory of Government Activity

Positive Theory of Government Activity

- So far: analysis of individual household behavior
- Now: introduction of government activity: taxation, transfers, government spending, issuing and repaying debt
- Question 1: What are the constraints the government faces?
- Question 2: How do government policies affect private household decisions?

2011 Federal Budget (in billion $)	
Receipts	2,303.5
Individual Income Taxes	1,091.5
Corporate Income Taxes	181.1
Social Insurance Receipts	818.8
Other	212.1
Outlays	3,603.1
National Defense	705.6
International Affairs	45.7
Health	372.5
Medicare	485.7
Income Security	597.4
Social Security	730.8
Net Interest	230.0
Other	435.5
Surplus	–1,299.6

- Government Expenditures

$$G_t = \text{Defense} + \text{International Affairs} + \text{Health} + \text{Other Outlays}$$

- Net Taxes

$$\begin{aligned} T_t &= \text{Taxes} + \text{Social Insurance Receipts} + \text{Other Receipts} \\ &\quad \text{- Medicare - Social Security - Income Security} \end{aligned}$$

- Interest on government debt: rB_{t-1} =Net Interest

- Denote by $t = 1$ the first period a country exists. Budget constraint of the government reads as
$$G_1 = T_1 + B_1$$

- For an arbitrary period t, the government budget constraint reads as
$$G_t + (1 + r)B_{t-1} = T_t + B_t$$

- For simplicity we assume that all government bonds have a maturity of one period.

- Rewrite budget constraint as

$$G_t - T_t + rB_{t-1} = B_t - B_{t-1}$$

- Primary government deficit: $G_t - T_t$
- Total government deficit: $def_t = G_t - T_t + rB_{t-1}$.
- Note that

$$def_t = B_t - B_{t-1}$$

- For $t = 2$, budget constraint reads as

$$G_2 + (1+r)B_1 = T_2 + B_2 \qquad \text{or} \qquad B_1 = \frac{T_2 + B_2 - G_2}{1+r}$$

- Plug this into budget constraint for period 1 to get

$$\begin{aligned} G_1 &= T_1 + \frac{T_2 + B_2 - G_2}{1+r} \\ G_1 + \frac{G_2}{1+r} &= T_1 + \frac{T_2}{1+r} + \frac{B_2}{1+r} \end{aligned}$$

- Continue this process to

$$\begin{aligned} & G_1 + \frac{G_2}{1+r} + \frac{G_3}{(1+r)^2} + \ldots + \frac{G_T}{(1+r)^{T-1}} \\ = \quad & T_1 + \frac{T_2}{1+r} + \frac{T_3}{(1+r)^2} + \ldots + \frac{T_T}{(1+r)^{T-1}} + \frac{B_T}{(1+r)^{T-1}} \end{aligned}$$

- Assume that even the government cannot die in debt:

$$
\begin{aligned}
& G_1 + \frac{G_2}{1+r} + \frac{G_3}{(1+r)^2} + \ldots + \frac{G_T}{(1+r)^{T-1}} \\
= \quad & T_1 + \frac{T_2}{1+r} + \frac{T_3}{(1+r)^2} + \ldots + \frac{T_T}{(1+r)^{T-1}}
\end{aligned}
$$

or more compactly

$$\sum_{t=1}^{T} \frac{G_t}{(1+r)^{t-1}} = \sum_{t=1}^{T} \frac{T_t}{(1+r)^{t-1}}$$

- If country lives forever, government budget constraint becomes

$$\sum_{t=1}^{\infty} \frac{G_t}{(1+r)^{t-1}} = \sum_{t=1}^{\infty} \frac{T_t}{(1+r)^{t-1}}$$

- Present discounted value of total government expenditures equals present discounted value of total taxes.

- Question: How should the government finance a war?
- Two principal ways to levy revenues for a government
    - Tax in the current period
    - Issue government debt, the interest and principal of which has to be paid via taxes in the future.
- What are the macroeconomic consequences of using these different instruments, and which instrument is to be preferred from a normative point of view?

- Ricardian Equivalence: it makes no difference. A switch from taxing today to issuing debt and taxing tomorrow does not change real allocations and prices in the economy.

- Origin: David Ricardo (1772-1823).

- His question: how to finance a war with annual expenditures of \$20 millions. Asked whether it makes difference to finance the \$20 millions via current taxes or to issue government bonds with infinite maturity (so-called consols) and finance the annual interest payments of \$1 million in all future years by future taxes (at an assumed interest rate of 5%).

- His conclusion was (in “Funding System”) that

  > *in the point of the economy, there is no real difference in either of the modes; for twenty millions in one payment [or] one million per annum for ever … are precisely of the same value*

- Ricardo formulates and explains the equivalence hypothesis, but is sceptical about its empirical validity

...but the people who pay the taxes never so estimate them, and therefore do not manage their affairs accordingly. We are too apt to think, that the war is burdensome only in proportion to what we are at the moment called to pay for it in taxes, without reflecting on the probable duration of such taxes. It would be difficult to convince a man possessed of $20,000, *or any other sum, that a perpetual payment of* $50 *per annum was equally burdensome with a single tax of* $1,000.

- Ricardo doubts that agents are as rational as they should, according to "in the point of the economy", or that they rationally believe not to live forever and hence do not have to bear part of the burden of the debt. Since Ricardo didn't believe in the empirical validity of the theorem, he has a strong opinion about which financing instrument ought to be used to finance the war

  war-taxes, then, are more economical; for when they are paid, an effort is made to save to the amount of the whole expenditure of the war; in the other case, an effort is only made to save to the amount of the interest of such expenditure.

- Suppose the world only lasts for two periods
- Government has to finance a war in the first period. The war costs G_1 pounds. Assume that government does not do any spending in the second period, so that $G_2 = 0$.
- Question: does it makes a difference whether the government collects taxes for the war in period 1 or issues debt and repays the debt in period 2?

- Budget constraints for the government

$$\begin{aligned} G_1 &= T_1 + B_1 \\ (1+r)B_1 &= T_2 \end{aligned}$$

where we used the fact that $G_2 = 0$ and $B_2 = 0$

- Policy 1: Immediate taxation: $T_1 = G_1$ and $B_1 = T_2 = 0$

- Policy 2: Debt issue, to be repaid tomorrow: $T_1 = 0$ and $B_1 = G_1, T_2 = (1+r)B_1 = (1+r)G_1$.

- Note that both policies satisfy the intertemporal government budget constraint

$$G_1 = T_1 + \frac{T_2}{1+r}$$

- Individual behavior: Household maximizes utility

$$u(c_1) + \beta u(c_2)$$

- subject to the lifetime budget constraint

$$c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r} + A$$

where y_1 and y_2 are the after-tax incomes in the first and second period of the households' life.

- Let

$$\begin{aligned} y_1 &= e_1 - T_1 \\ y_2 &= e_2 - T_2 \end{aligned}$$

where e_1, e_2 are the pre-tax earnings of the household and T_1, T_2 are taxes paid by the household.

- Government policies only affect after tax incomes. But

$$
\begin{aligned}
c_1 + \frac{c_2}{1+r} &= e_1 - T_1 + \frac{e_2 - T_2}{1+r} + A \\
c_1 + \frac{c_2}{1+r} + T_1 + \frac{T_2}{1+r} &= e_1 + \frac{e_2}{1+r} + A
\end{aligned}
$$

- Household spends present discounted value of pre-tax income $e_1 + \frac{e_2}{1+r} + A$ on present discounted value of consumption $c_1 + \frac{c_2}{1+r}$ and present discounted value of income taxes.

- Two tax-debt policies that imply the same present discounted value of lifetime taxes therefore lead to exactly the same lifetime budget constraint and thus exactly the same individual consumption choices.

- For immediate taxation we have $T_1 = G_1$ and $T_2 = 0$, and thus $T_1 + \frac{T_2}{1+r} = G_1$
- For debt issue we have $T_1 = 0$ and $T_2 = (1+r)G_1$, and thus $T_1 + \frac{T_2}{1+r} = G_1$
- Both policies imply the same present discounted value of lifetime taxes for the household. Present discounted value of taxes is not changed.
- Consumption choices do not change, but savings choices do.
- Period by period budget constraints

$$
\begin{aligned}
c_1 + s &= e_1 - T_1 \\
c_2 &= e_2 - T_2 + (1+r)s
\end{aligned}
$$

- Let (c_1^*, c_2^*) be the optimal consumption choices in the two periods and let s^* denote the optimal saving.
- New savings choice $\tilde{s}$ denote the new saving policy. Thus

$$\begin{aligned} c_1^* &= e_1 - T_1 - s^* \\ &= e_1 - \tilde{s} \end{aligned}$$

- Thus

$$\begin{aligned} e_1 - T_1 - s^* &= e_1 - \tilde{s} \\ \tilde{s} &= s^* + T_1. \end{aligned}$$

- Under second policy the household saves exactly T_1 more than under the first policy, the full extent of the tax reduction from the second policy. This extra saving T_1 yields $(1+r)T_1$ extra income in the second period, exactly enough to pay the taxes levied in the second period by the government to repay its debt.

Theorem

(Ricardian Equivalence) A policy reform that does not change government spending $(G_1, \ldots, G_T)$*, and only changes the timing of taxes, but leaves the present discounted value of taxes paid by each household in the economy has no effect on aggregate consumption in any time period.*

- Key Assumption 1: No Borrowing Constraint
- Key Assumption 2: No Redistribution of the Burden of Taxes
- Key Assumption 3: Lump Sum Taxation

- Binding borrowing constraints can lead a household to change her consumption choices, even if a change in the timing of taxes does not change her discounted lifetime income.
- Proof by example: French British war; costs \$100 per person.
- Utility function

$$\log(c_1) + \log(c_2)$$

  and pre-tax income of $\$1,000$ in both periods of their life.
- For simplicity $r = 0$.
- Policy 1: tax \$100 in the first period
- Policy 2: incur \$100 in government debt, to be repaid in the second period. Since $r = 0$, government has to repay \$100 in the second period

- Without borrowing constraints we know from general theorem that the two policies have identical consequences. Under both policies discounted lifetime income is \$1,900 and

$$c_1 = c_2 = \frac{1,900}{2} = 950$$

- With borrowing constraints: policy 1

$$c_1 = y_1 = 900 \text{ and } c_2 = y_2 = 1000$$

- Second policy

$$c_1 = c_2 = 950$$

- If households are borrowing constrained, current taxes have stronger effects on current consumption than the issuing of debt, since postponing taxes to the future relaxes borrowing constraints.

No Redistribution of Tax Burden

- If change in timing of taxes involves redistribution of the tax burden across generations, then, unless these generations are linked together by operative, altruistically motivated bequest motives Ricardian equivalence fails.

- Example: as before, but now interest rate of 5%

- Policy 1: levy the \$100 cost per person by taxing everybody \$100 in period 1

- Policy 2: issue government debt of \$100 and to repay simply the interest on that debt. Under that households face taxes of $T_2 = \$5$, $T_3 = \$5$ and so forth.

- For person born in period 1: under policy 1, his present discounted value of lifetime income is

$$I = \$1000 - \$100 + \frac{\$1000}{1.05} = 1852.38$$

and under policy 2 it is

$$I = \$1000 + \frac{\$995}{1.05} = 1947.6$$

- Under policy 1 consumption equals

$$
\begin{aligned}
c_1 &= 926.2 \\
c_2 &= 972.5
\end{aligned}
$$

  and under policy 2 it equals

$$
\begin{aligned}
c_1 &= 973.8 \\
c_2 &= 1022.5
\end{aligned}
$$

- Under policy 2, part of the cost of the war is borne by future generations that inherit the debt from the war, at least the interest on which has to be financed via taxation.

- Ricardian equivalence was thought to be an empirically irrelevant theorem because timing of taxes always shifts tax burden across generations.
- Robert Barro (1974) resurrected debate.
- Step 1: if households live forever, Ricardian equivalence holds.
- Consider two arbitrary government tax policies. Since we keep G_t fixed in every period, the intertemporal budget constraint

$$\sum_{t=1}^{\infty} \frac{G_t}{(1+r)^{t-1}} = \sum_{t=1}^{\infty} \frac{T_t}{(1+r)^{t-1}}$$

requires that the two tax policies have the same present discounted value.

- Without borrowing constraints only the present discounted value of lifetime after-tax income matters for a household's consumption choice. But since the present discounted value of taxes is the same under the two policies it follows that present discounted value of after-tax income is unaffected by the switch from one tax policy to the other. Private decisions thus remain unaffected, therefore all other economic variables in the economy remain unchanged by the tax change. Ricardian equivalence holds.

- Step 2: argue that households live forever. Key: bequests.
- Suppose that people live for one period and have utility function

$$U(c_1) + \beta V(b_1)$$

where V is the maximal lifetime utility of children with bequests b.

- Now parameter β measures intergenerational altruism. A value of $\beta > 0$ indicates that you are altruistic, a value of $\beta < 1$ indicates that you love your children not as much as you love yourself.
- Budget constraint

$$c_1 + b_1 = y_1$$

- Bequests are constrained to be non-negative, that is $b_1 \geq 0$.

- Utility function of child is given by

$$U(c_2) + \beta V(b_2)$$

and the budget constraint is

$$c_2 + b_2 = y_2 + (1 + r)b_1$$

- Note that $V(b_1)$ equals the maximized value of $U(c_2) + \beta V(b_2)$
- Economy with one-period lived people that are linked by altruism and bequests is identical to economy with people that live forever and face borrowing constraints (since we have that bequests $b_1 \geq 0$, $b_2 \geq 0$ and so forth).
- But: binding borrowing constraints invalidate Ricardian equivalence.
- Conclusion: in Barro model with one-period lived individuals Ricardian equivalence holds if a) individuals are altruistic ($\beta > 0$) and bequest motives are operative.

- A lump-sum tax is a tax that does not change the relative price between two goods that are chosen by private households.
- Demonstrate that timing of taxes is not irrelevant if the government does not have access to lump-sum taxes by example
- Utility function

$$\log(c_1) + \log(c_2)$$

- Income before taxes of $\$1000$ in each period and $r = 0$. The war costs $\$100$.
- Policy 1: levy a $\$100$ tax on first period labor income.
- Policy 2: issue $\$100$ in debt, repaid in the second period with proportional consumption taxes at rate τ.

- Under first policy optimal consumption choice is

$$\begin{aligned} c_1 &= c_2 = \$950 \\ s &= \$900 - \$950 = -\$50 \end{aligned}$$

- The two budget constraints under policy 2 read as

$$\begin{aligned} c_1 + s &= \$1000 \\ c_2(1+\tau) &= \$1000 + s \end{aligned}$$

which can be consolidated to $c_1 + (1+\tau)c_2 = \$2000$

- Maximizing utility subject to the lifetime budget constraint yields

$$\begin{aligned} c_1 &= \$1000 \\ c_2 &= \frac{\$1000}{1+\tau} \end{aligned}$$

- Under second policy the households consumes strictly more than under the first policy. Reason: tax on second period consumption makes consumption in the second period more expensive, relative to consumption in the first period. Households substitute away from the now more expensive good.

- Fact that the tax changes the effective relative price between the two goods qualifies this tax as a non-lump-sum tax.

- Government must levy \$100 in taxes. Tax revenues are given by

$$\tau c_2 = \frac{\tau 1000}{1+\tau} = 100$$

- Thus

$$\begin{aligned} \tau &= \frac{0.1}{0.9} = 0.1111 \\ c_2 &= 900 \\ s &= 0 \end{aligned}$$

- Households prefer the lump-sum way of financing the war to the distortionary way:

$$\log(950) + \log(950) > \log(1000) + \log(900).$$

- Report by Jagadeesh Gokhale from 2013, *Spending Beyond our Means: How We are Bankrupting Future Generations*

https://object.cato.org/sites/cato.org/files/pubs/pdf/spending-beyond-our-means.pdf

https://www.cato.org/sites/cato.org/files/articles/gokhale-generational_accounting.pdf

- Fiscal Imbalance:

$$FI_t = PVE_t^{cfp} + B_t - PVR_t^{cfp}$$

where PVE_t^{cfp} is the present discounted value of projected expenditures under current fiscal policy, PVR_t^{cfp} is present discounted value of all projected receipts and B_t is government debt at the end of period t.

- In terms of our previous notation

$$PVE_t = \sum_{\tau=t+1}^{\infty} \frac{G_\tau}{(1+r)^{\tau-t}}$$

and

$$PVR_t = \sum_{\tau=t+1}^{\infty} \frac{T_\tau}{(1+r)^{\tau-t}}$$

as well as

$$B_t = \sum_{\tau=1}^{t} \frac{G_\tau}{(1+r)^{\tau-t}} - \sum_{\tau=1}^{t} \frac{T_\tau}{(1+r)^{\tau-t}}$$

- Intertemporal budget constraint suggests that feasible fiscal policy must have $FI_t = 0$.

- But

$$PVE_t^{cfp} \neq PVE_t$$

$$PVR_t^{cfp} \neq PVR_t$$

- Which means that either PVE_t^{cfp} or PVR_t^{cfp} will change to adjust to reality.

- In order to assess which generations bear what burden of the total fiscal imbalance, an additional concept is needed.

- Generational imbalance

$$GI_t = PVE_t^{cfp\,L} + B_t - PVR_t^{cfp\,L}$$

  where $PVE_t^{cfp\,L}$ is the present discounted value of outlays paid to generations currently alive, with $PVR_t^{cfp\,L}$ defined correspondingly.

- GI_t is that part of the fiscal imbalance FI_t that results from transactions of the government with past (through B_t) and living generations.

- Difference $FI_t - GI_t$ denotes the projected part of fiscal imbalance due to future generations.

Main Assumptions

- Real interest rate (discount rate for the present value calculations) of 3.68% per annum (average yield on a 30 year Treasury bond in recent years).
- Annual growth rate of real wages of between 1% and 2%, based on future projections of CBO.
- Growth of health care costs? Account for fact that the expenditure (in per capita terms) growth rate in Medicare is projected to be significantly above the projections from growth rates of wages for immediate future. Beyond 2035 this gap is assumed to gradually shrink to zero.

1. Baseline policy scenario corresponds to current fiscal policy

$$PVE_t^{cfp},\ PVR_t^{cfp}$$

2. Alternative policy scenario factors in likely policy changes.

$$PVE_t^{as},\ PVR_t^{as}$$

- In order to compute GI, one needs to break down taxes paid and outlays received by generations.

Fiscal Imbalance, Baseline (Billion of 2012 Dollars) Current Fiscal Projections

Part of the Budget	2012	2017	2022
FI in Social Insurance	64,853	70,961	82,564
FI in Rest of Federal Government	−10,502	−10,687	−11,742
Total *FI*	54,675	60,274	70,822

Fiscal Imbalance, Alternative Fiscal Scenario (Billion of 2012 Dollars)

Part of the Budget	2012	2017	2022
FI in Social Insurance (SS+Med.)	65,934	72,036	83,606
FI in Social Security	20,077	22,272	26,660
FI in Medicare	45,857	49,764	56,946
FI in Rest of Federal Government	25,457	29,826	36,660
Total *FI*	91,391	101,862	120,266

Fiscal Imbal., Alternative Fiscal Scenario (% of Pres. Val. GDP)

Part of the Budget	2012	2017	2022
FI in Social Insurance (SS+Med.)	6.5%	6.5%	6.8%
FI in Rest of Federal Government	2.5%	2.7%	3.0%
Total *FI*	9.0%	9.1%	9.8%

1. *FI* is huge: requires the confiscation of 9% of GDP in perpetuity to close this imbalance from the perspective of 2012. Required increase in payroll taxes about 20% points
2. This is before the 2017 TCJA that introduced a further tax reduction.
3. *FI* grows over time at a gross rate of $(1 + r) = 1.0368$ per year.
4. Largest part (about 1/2) of *FI* is due to Medicare. Comes from a) fast increases of medical goods prices and b) population aging.
5. *FI* dwarfs official government debt by a factor of 5.

Generational Imbalance, Alternative (Bill of 2012 Dollars)

Part of the Budget	2012	2017	2022
FI in Social Insurance	65,934	72,036	83,606
FI in Social Security	20,077	22,272	26,660
GI in Social Security (incl. Trust Fund)	19,586	21,726	26,032
FI − *GI* in Social Security	491	546	628
FI in Medicare	45,857	49,764	56,946
GI in Medicare	34,487	38,311	44,693
FI − *GI* in Medicare (incl. Trust Fund)	11,370	11,453	12,253

1. 3/4 of Medicare *FI* is due to generations currently alive. But even future generations have benefits exceeding contributions (mainly because of Medicare prescription drug benefits).

2. *FI* in social security is due *entirely* to past and current generations.

3. Magnitude of numbers depends on: growth rate of wages, discount rate applied to future revenues and outlays, temporary differential between expenditure growth in Medicare and the economy.

4. **But** conclusion robust: large spending cuts or tax increases required to restore fiscal balance. Medicare and Social Security key.

The U.S. Federal Income Tax Code: Brief History

- Early U.S. history: few commodity taxes, on alcohol, tobacco and snuff, real estate sold at auctions, corporate bonds and slaves.
- British-American War in 1812: added sales taxes on gold, silverware and other jewelry
- In 1817 all internal taxes were abolished. Government relies exclusively on tariffs on imported goods.
- Civil war from 1861-1865 required increased funds for the federal government.
- In 1862, office of Commissioner of Internal Revenue was established. Right to assess, levy and collect taxes, and to enforce the tax laws though seizure of property and income and through prosecution.
- Individuals with earnings between \$600 – \$10000 had to pay an income tax of 3%; higher rates for people with income above \$10000.

- Additional sales and excise taxes were introduced. For the first time an inheritance tax was introduced.
- Total tax collections reached $310 million in 1866, highest amount in U.S. history to that point, an amount not reached again until 1911.
- General income tax was scrapped in 1872, with other taxes besides excise taxes on alcohol and tobacco.
- Re-introduced in 1894, but declared unconstitutional in 1895, because it did not levy taxes and distribute funds among states in accordance with the constitution.
- Modern federal income tax was permanently introduced in the U.S. in 1913 through the 16-th Amendment to the Constitution. Gave Congress legal authority to tax income of both individuals and corporations.

- By 1920 IRS collected $5.4 billion dollars, rising to $7.3 billion dollars at the eve of WWII. Still, income tax was still largely a tax on corporations and very high income individuals, since exemption levels were high.
- In 1943 the government introduced a withholding tax on wages By 1945 number of income taxpayers increased to 60 million and tax revenues increased to $43 billion, a six-fold increase from the revenues in 1939.
- Most far-reaching tax reforms in recent history: President Reagan in 1981 and 1986, President Clinton's tax reform of 1993 and the tax reforms of President George W. Bush in 2001-2003. Also the 2017 TCJA. Still too early to assess.
- The Reagan tax reforms reduced income tax rates by individuals drastically (with a total reduction amounting to the order of $500 – 600 billion), partially offset by an increase in tax rates for corporations and moderate increases of taxes for the very wealthy.

- Mounting budget deficits: President Clinton partially reversed Reagan's tax cuts in 1993.
- Further tax reforms under the Clinton presidency included tax cuts for capital gains, the introduction of a $500 tax credit per child and tax incentives for education expenses.
- Large tax cuts in 2003 by President Bush temporarily reduced dividend and capital gains taxes as well as increase child tax credits and lower marginal tax rates for most Americans. Were set to expire in 2012. Did partially expire in 2013.
- Further tax cut in 2017 with the TCJA.

Tax Cuts and Jobs Act of 2017

- Too early to assess its role properly
- Reduces tax rates for businesses and individuals;
- Seems to be a regressive change that reduces revenue
- Simplifies personal taxes and
    - Increases the standard deduction and family tax credits,
    - Eliminates personal exemptions and making it less beneficial to itemize deductions;
    - Limits deductions for state and local income taxes (SALT) and property taxes;
    - Limits the mortgage interest deduction; reduces the alternative minimum tax for individuals and eliminates it for corporations;
    - Reduces the number of estates impacted by the estate tax;
    - Repeals the individual mandate of the Affordable Care Act (ACA)

- Let y denote taxable income. If we model a deduction d explicity, then taxable income is $y - d$.

- A tax code is defined by a tax function $T(y)$, which for each possible taxable income y gives the amount of taxes that are due to be paid.

- Example: if $y = \$100,000$ and $T(y) = \$25,000$, then every person with taxable income of $\$100,000$ in 2013 owes the government $\$25,000$ in taxes.

- For a given tax code T we define as
  1. Average tax rate of individual with taxable income y as
  $$t(y) = \frac{T(y)}{y}$$
  for all $y > 0$.
  2. Marginal tax rate of individual with taxable income y as
  $$\tau(y) = T'(y)$$
  whenever $T'(y)$ is well-defined (that is, whenever $T'(y)$ is differentiable.
- Interpretation: average tax rate $t(y)$ indicates what *fraction* of her taxable income a person with income y has to deliver to the government as tax. Marginal tax rate $\tau(y)$ measures how high the tax rate is on the last dollar earned, for a total taxable income of y.

- Equivalent definitions of tax code: can define tax code by

1. Average tax rate schedule, since

$$T(y) = y * t(y)$$

2. Marginal tax rate schedule (and the tax for $y = 0$), since

$$T(y) = T(0) + \int_0^y T'(y)dy$$

where the equality follows from the fundamental theorem of calculus.

3. Current U.S. federal personal income tax code is defined by a collection of marginal tax rates.

- A tax code is progressive if the function $t(y)$ is strictly increasing in y for all income levels y . It is progressive over an income interval (y_l, y_h) if $t(y)$ is strictly increasing for all income levels $y \in (y_l, y_h)$. It is also progressive if it is proportional over all income intervals but the proportion is higher in each successive interval.

- A tax code is regressive if the function $t(y)$ is strictly decreasing in y for all income levels y. It is regressive over an income interval (y_l, y_h) if $t(y)$ is strictly decreasing for all income levels $y \in (y_l, y_h)$.

- A tax code is proportional if the function $t(y)$ is constant y for all income levels y. It is proportional over an income interval (y_l, y_h) if $t(y)$ is constant for all income levels $y \in (y_l, y_h)$.

- Head tax or poll tax

$$T(y) = T$$

where $T > 0$ is a number. This tax is regressive since

$$t(y) = \frac{T}{y}$$

is a strictly decreasing function of y. Also note that the marginal tax $\tau(y) = 0$ for all income levels.

Flat tax or proportional tax

$$T(y) = \tau * y$$

where $\tau \in [0, 1)$ is a parameter.

Note that

$$t(y) = \tau(y) = \tau$$

that is, average and marginal tax rates are constant in income and equal to the tax rate τ. This tax system is proportional.

$$T(y) = \begin{cases} 0 & \text{if } y < d \\ \tau(y - d) & \text{if } y \geq d \end{cases}$$

x where $d, \tau \geq 0$ are parameters. Household pays no taxes if her income does not exceed the exemption level d, and then pays a fraction τ in taxes on every dollar earned above d. Average tax rates

$$t(y) = \begin{cases} 0 & \text{if } y < d \\ \tau\left(1 - \frac{d}{y}\right) & \text{if } y \geq d \end{cases}$$

Marginal tax rates

$$\tau(y) = \begin{cases} 0 & \text{if } y < d \\ \tau & \text{if } y \geq d \end{cases}$$

Tax system is progressive for all income levels above d; for all income levels below it is proportional.

Such a tax code is defined by its marginal tax rates and the income brackets for which these taxes apply.

Example with three brackets

$$\tau(y) = \begin{cases} \tau_1 & \text{if } 0 \leq y < b_1 \\ \tau_2 & \text{if } b_1 \leq y < b_2 \\ \tau_3 & \text{if } b_2 \leq y < \infty \end{cases}$$

The tax code is characterized by the three marginal rates (τ_1, τ_2, τ_3) and income cutoffs (b_1, b_2) that define the income tax brackets.

- Compute tax schedule
  - For $0 \leq y < b_1$

$$T(y) = \int_0^y \tau(y) dy = \int_0^y \tau_1 \, dy = \tau_1 \int_0^y dy = \tau_1 y,$$

  - For $b_1 \leq y < b_2$

$$T(y) = \int_0^y \tau(y) dy = \int_0^{b_1} \tau_1 \, dy + \int_{b_1}^y \tau_2 \, dy = \tau_1 b_1 + \tau_2 (y - b_1)$$

  - For $y \geq b_2$

$$\begin{aligned} T(y) &= \int_0^{b_1} \tau_1 \, dy + \int_{b_1}^{b_2} \tau_2 \, dy + \int_{b_2}^y \tau_3 \, dy \\ &= \tau_1 b_1 + \tau_2 (b_2 - b_1) + \tau_3 (y - b_2) \end{aligned}$$

- Average tax rates are given by

$$t(y) = \begin{cases} \tau_1 & \text{if } 0 \leq y < b_1 \\ \frac{\tau_1 b_1}{y} + \tau_2 \left(1 - \frac{b_1}{y}\right) & \text{if } b_1 \leq y < b_2 \\ \frac{\tau_1 b_1 + \tau_2 (b_2 - b_1)}{y} + \tau_3 \left(1 - \frac{b_2}{y}\right) & \text{if } b_2 \leq y < \infty \end{cases}$$

- If $\tau_1 < \tau_2 < \tau_3$ then this tax system is proportional for $y \in [0, b_1]$ and progressive for $y > b_1$.

- With just two brackets we get back a flat tax with deduction, if $\tau_1 = 0$.

- Current U.S. tax code resembles the last example closely, but consists of seven marginal tax rates and six income cut-offs that define the income tax brackets. The income cut-offs vary with family structure.

Theorem

A differentiable tax code $T(y)$ is progressive, that is, $t(y)$ is strictly increasing in y (i.e. $t'(y) > 0$ for all y) if and only if the marginal tax rate $T'(y)$ is higher than the average tax rate $t(y)$ for all income levels $y > 0$, that is

$$T'(y) > t(y)$$

Proof: By definition

$$t(y) = \frac{T(y)}{y}$$

Using the definition the rule for differentiating a ratio of two functions we obtain

$$t'(y) = \frac{yT'(y) - T(y)}{y^2}$$

This expression is positive if and only if

$$yT'(y) - T(y) > 0$$

or

$$T'(y) > \frac{T(y)}{y} = t(y)$$

QED.

- Intuition: for average tax rates to increase with income requires that the tax rate you pay on the last dollar earned is higher than the average tax rate you paid on all previous dollars.

- This result provides us with another, equivalent, way to characterize a progressive tax system.

- Differentiability of $T(y)$ not needed for the argument.

- A similar result can be stated and proved for a regressive or proportional tax system.

$$\begin{aligned} \text{Gross Income} &= \text{Wages and Salaries} \\ &+ \text{Interest Income and Dividends} \\ &+ \text{Net Business Income} \\ &+ \text{Net Rental Income} \\ &+ \text{Other Income} \end{aligned}$$

- Other income includes unemployment insurance benefits, alimony, income from gambling, income from illegal activities. Not included: child support, gifts below a certain threshold, interest income from state and local bonds (so-called Muni's), welfare and veterans benefits, employer contributions for health insurance and retirement accounts.

Adjusted Gross Income and Taxable Income

$$\begin{aligned}\text{Adjusted Gross Income } (AGI) &= \text{Gross Income} \\ &\quad -\text{contributions to IRA's} \\ &\quad -\text{alimony not after 2017 law and new divorces} \\ &\quad -\text{health insurance of self-employed}\end{aligned}$$

$$\begin{aligned}\text{Taxable Income} &= AGI \\ &\quad -\text{Deductions (Standard or Itemized)} \\ &\quad -\text{Exemptions} \\ &= y\end{aligned}$$

Note

$$\begin{aligned}\text{Taxes due upon filing} &= T(y) \\ &\quad -\text{Tax witholdings} \\ &\quad -\text{Tax credits}\end{aligned}$$

The Marriage Penalty: Before 2017

Tax Rates for 2013, Singles

Income	$T'(y)$	$T(y)$
$0 \leq y < \$8,925$	10%	$0.1y$
$\$8,925 \leq y < \$36,250$	15%	$\$892 + 0.15(y - 8,925)$
$\$36,250 \leq y < \$87,850$	25%	$\$4,991 + 0.25(y - 36,250)$
$\$87,850 \leq y < \$183,250$	28%	$\$17,891 + 0.28(y - 87,850)$
$\$183,250 \leq y < \$398,350$	33%	$\$44,603 + 0.33(y - 183,250)$
$\$398,350 \leq y < \$400,000$	35%	$\$115,586 + 0.35(y - 398,350)$
$\$400,000 \leq y < \infty$	39.6%	$\$116,164 + 0.396(y - 400,000)$

Tax Rates for 2013, Married Filing Jointly

Income	$T'(y)$	$T(y)$
$0 \leq y < \$17,850$	10%	$0.1y$
$\$17,850 \leq y < \$72,500$	15%	$\$1,785 + 0.15(y - 17,850)$
$\$72,500 \leq y < \$146,400$	25%	$\$9,982 + 0.25(y - 72,500)$
$\$146,400 \leq y < \$223,050$	28%	$\$28,457 + 0.28(y - 146,400)$
$\$223,050 \leq y < \$398,350$	33%	$\$49,919 + 0.33(y - 223,050)$
$\$398,350 \leq y < \$450,000$	35%	$\$107,768 + 0.35(y - 398,350)$
$\$450,000 \leq y < \infty$	39.6%	$\$125,846 + 0.396(y - 450,000)$

Large Marriage Penalty for incomes above $\$146,000$ in 2013

Tax Brackets and Tax Rates, 2019 by Family Type

Tax Rate	For Unmarried Individuals	For Married Individuals Filing Joint Returns	For Heads of Households
	Taxable Income Over		
10%	$0	$0	$0
12%	$9,700	$19,400	$13,850
22%	$39,475	$78,950	$52,850
24%	$84,200	$168,400	$84,200
32%	$160,725	$321,450	$160,700
35%	$204,100	$408,200	$204,100
37%	$510,300	$612,350	$510,300

2019 Standard Deduction and Personal Exemption

Filing Status	Deduction Amount
Single	$12,200
Married Filing Jointly	$24,400
Head of Household	$18,350

Marriage Penalty Almost Gone

One can show that is is *impossible* to design a tax system that simultaneously is:

1. Progressive, as defined above
2. Satisfies across family equity: families with equal household incomes pay equal taxes (independent of how much of that income is earned by different members of each household)
3. Marriage-neutral: a given family pays the same taxes independent of whether the partners of the family are married or not.

So why did people get married?

- Pension Benefits
- Health Care Benefits
- Swift Legal System
- Tradition
- No privately designed substitute (prenups are not)

Under the new law is a good deal.

Alternative Minimum Tax (AMT) I

- Created in the 1960s to prevent high-income taxpayers from avoiding the individual income tax: high-income taxpayers calculate their tax bill twice: once under the ordinary income tax system and again under the AMT, then it needs to pay the higher of the two.
- The AMT uses an alternative definition of taxable income, Alternative Minimum Taxable Income). Exemptions are significant but phase out for high-income taxpayers. The AMT is levied at two rates: 26% and 28%.
- The 2019 exemption amount is $71,700 for singles and $111,700 for married couples
- The 28 percent AMT rate applies to excess AMTI of $194,800 for all taxpayers.
- AMT exemptions phase out at 25 cents per dollar earned once taxpayer AMTI hits a certain threshold. In 2019, the exemption will start phasing out at $510,300 in AMTI for singles and $1,020,600 for married

Alternative Minimum Tax (AMT) II

- Before the 2017 the individual alternative minimum tax (AMT) primarily affected well-off households, but not those with the very highest incomes.
- It was also more likely to hit taxpayers with large families, those who were married, and those who lived in high-tax states.
- TCJA shields almost all upper-middle and high-income taxpayers from the reach of the AMT.
- The AMT is now most likely to hit those at the top of the income scale who are engaged in certain sheltering activities.
- It has quite less bite than before.

- It is a refundable tax credit for low- to moderate-income working individuals and couples, particularly those with children.

- The amount of EITC benefit depends on a recipient's income and number of children. For a person or couple to claim one or more persons as their qualifying child, requirements such as relationship, age, and shared residency must be met.

- The maximum Earned Income Tax Credit in 2019 for single and joint filers is $529, if the filer has no children (Table 5). The maximum credit is $3,526 for one child, $5,828 for two children, and $6,557 for three or more children. All these are relatively small increases from 2018.

- Small Potatoes

- Product of selling an asset at higher price (often a business)
- Long-term capital gains are taxed at 15% up to \$434,550 for single and \$488,850 for married and then at 20%
- There is a Qualified Business Income Deduction. The TCJA includes a 20% deduction for pass-through businesses against up to \$160,700 (singles)
- **Annual Exclusion for Gifts** In 2019, the first $\$15,000$ of gifts to any person are excluded from tax.

- Simple example: two households in the economy
  - Household 1, taxable income $y = 100,000$ and household 2 with $y = 20,000$.
  - Lifetime utility $\log(c)$ only depends on their current after-tax income $c = y - T(y)$.
  - Their Consumption is Separate
  - Compare social welfare under progressive tax system with a proportional tax system.

- Hypothetical progressive tax system

$$\tau(y) = \begin{cases} 0\% & \text{if } 0 \leq y < 15000 \\ 10\% & \text{if } 15000 \leq y < 50000 \\ 20\% & \text{if } 50000 \leq y < \infty \end{cases}$$

- Under this tax system total tax revenues from the two agents are

$$\begin{aligned} & T(15,000) + T(100,000) \\ = \quad & 0.1 * (20000 - 15000) \\ & +0.1 * 35000 + 0.2(100000 - 50000) \\ = \quad & \$500 + \$13500 \\ = \quad & \$14000 \end{aligned}$$

and consumption for the households are

$$\begin{aligned} c_1 &= 20000 - 500 = 19500 \\ c_2 &= 100000 - 13500 = 86500 \end{aligned}$$

- Determine proportional tax rate τ such that revenues are same under hypothetical proportional tax system as under progressive system:

$$\begin{aligned} 14000 &= \tau * 20,000 + \tau * 100,000 = \tau * 120,000 \\ \tau &= \frac{14,000}{120,000} = 11.67\% \end{aligned}$$

- Under proportional tax system consumption of both households equals

$$\begin{aligned} c_1 &= (1 - 0.1167) * 20000 = 17667 \\ c_2 &= (1 - 0.1167) * 100000 = 88333 \end{aligned}$$

- Which tax system is better? Hard question! Use social welfare function

$$W(u(c_1), \ldots, u(c_N))$$

Examples of Social Welfare Functions: Dictator

- Household i is a "dictator"

$$W(u(c_1), \ldots, u(c_N)) = u(c_i)$$

- If dictator is $i = 1$, prefer U.S. system. If dictator is $i = 2$, prefer proportional tax system.

- Utilitarian social welfare function given by

$$W(u(c_1), \ldots, u(c_N)) = u(c_1) + \ldots + u(c_N)$$

- Posits that everybody's utility should be counted equally.
- Basis: Jeremy Bentham (1748-1832) & John Stuart Mill's (1806-1873) "Utilitarism" (published in 1863). Principle of Utility.

  Actions are right in proportion as they tend to promote happiness; wrong as they tend to produce the reverse of happiness

- Comparison for example

$$\begin{aligned} W^{\text{prog}}(u(c_1), u(c_2)) &= \log(19500) + \log(86500) = 21.2461 \\ W^{\text{prop}}(u(c_1), u(c_2)) &= \log(17667) + \log(88333) = 21.1683 \end{aligned}$$

- Interpersonal Comparisons are difficult (need same utility)

- Rawlsian social welfare function

$$W(u(c_1), \ldots, u(c_N)) = \min_i \{u(c_1), \ldots, u(c_N)\}$$

- Idea: veil of ignorance plus extreme risk aversion

- For example

$$\begin{aligned} W^{\text{prog}}(u(c_1), u(c_2)) &= \min\{\log(c_1), \log(c_2)\} = \log(19500) \\ W^{\text{prop}}(u(c_1), u(c_2)) &= \min\{\log(c_1), \log(c_2)\} = \log(17667) \\ &< W^{\text{prog}}(u(c_1), u(c_2)) \end{aligned}$$

Suppose that taxable incomes are not affected by the tax code and suppose that u is strictly concave and the same for every household. Then under Rawlsian and Utilitarian social welfare function it is optimal to have complete income redistribution:

$$c_1 = c_2 = \ldots = c_N = \frac{y_1 + y_2 + \ldots + y_N - G}{N} = \frac{Y - G}{N}$$

where G is total required tax revenue and $Y = y_1 + y_2 + \ldots + y_N$ Tax code that achieves this is given by

$$T(y_i) = y_i - \frac{Y - G}{N}$$

i.e. tax income at a 100% and then rebate $\frac{Y-G}{N}$ back to everybody.

- Suppose that $N = 2$ and $c_2 > c_1$ as the result of tax code. This cannot be optimal!
- Take way a little from household 2 and give it to household 1
- Under Rawlsian social welfare function this improves societal welfare since the poorest person has been made better off.
- Under Utilitarian social welfare function, loss of agent 2, $u'(c_2)$ is smaller than the gain of agent 1, $u'(c_1)$, since by concavity $c_2 > c_1$ implies

$$u'(c_1) > u'(c_2).$$

- But: assumption that changes in the tax system do not change a households' incentive to work, save and thus generate income is a very strong one. Therefore now want to analyze how income and consumption taxes change the economic incentives of households to work, consume and save.

An Assessment of Welfare Functions

- Utilitarism takes utilities more seriously than it should.
  - Monotonic transformations of Utilities are yield the same allocations but not necessarily the same welfare.

- Rawlsian has the issue of only caring about the worst. But this cannot be taking literally: All societies are in terrible shape as long as there is any infant mortality.

- The idea of the veil of ignorance is an excellent one. It separates our circumstances from our assessment.

- It allows us to pick some particular utility function. It ends up yielding very egalitarian results.

- Still it is difficult to use to assess changes as the veil of ignorance does not apply. We know where we were before the policy change

- Household problem

$$\max_{c_1 c_2, s, \ell} \log(c_1) + \theta \log(1-\ell) \quad + \quad \beta \log(c_2)$$

$$\begin{aligned} s.t. \qquad (1+\tau_{c_1})c_1 + s &= (1-\tau_\ell)w\ell \\ (1+\tau_{c_2})c_2 &= (1+r(1-\tau_s))s + b \end{aligned}$$

- Parameter θ measures how much households value leisure, relative to consumption.

- Intertemporal budget constraint. Solving second budget constraint yields

$$s = \frac{(1+\tau_{c_2})c_2 - b}{(1+r(1-\tau_s))}$$

and thus

$$(1+\tau_{c_1})c_1 + \frac{(1+\tau_{c_2})c_2}{(1+r(1-\tau_s))} = (1-\tau_\ell)w\ell + \frac{b}{(1+r(1-\tau_s))}$$

- Rewrite this. Note that $\ell = 1 - (1 - \ell)$. Then

$$\begin{aligned} & (1+\tau_{c_1})c_1 + \frac{(1+\tau_{c_2})c_2}{(1+r(1-\tau_s))} = \\ = \quad & (1-\tau_\ell)w * (1-(1-\ell)) + \frac{b}{(1+r(1-\tau_s))} \end{aligned}$$

$$\begin{aligned} & (1+\tau_{c_1})c_1 + \frac{(1+\tau_{c_2})c_2}{(1+r(1-\tau_s))} + (1-\ell)(1-\tau_\ell)w = \\ = \quad & (1-\tau_\ell)w + \frac{b}{(1+r(1-\tau_s))} \end{aligned}$$

- Interpretation: household has potential income from social security $\frac{b}{(1+r(1-\tau_s))}$ and from supplying *all* her time to the labor market, $(1-\tau_\ell)w$.

- Buys three goods

    - Consumption c_1 in first period, at effective price $(1+\tau_{c_1})$

    - Consumption c_2 in second period, at effective price $\frac{(1+\tau_{c_2})}{(1+r(1-\tau_s))}$

    - Leisure $1-\ell$ at effective price $(1-\tau_\ell)w$, equal to the opportunity cost of not working.

- Lagrangian

$$
\begin{aligned}
L \quad = \quad & \log(c_1) + \theta \log(1-\ell) + \beta \log(c_2) \\
& + \lambda \left\{ (1-\tau_\ell) w + \frac{b}{(1+r(1-\tau_s))} \right. \\
& \left. -(1+\tau_{c_1}) c_1 - \frac{(1+\tau_{c_2}) c_2}{(1+r(1-\tau_s))} - (1-\ell)(1-\tau_\ell) w \right\}
\end{aligned}
$$

- First order conditions:

$$
\begin{aligned}
\frac{1}{c_1} - \lambda(1+\tau_{c_1}) &= 0 \\
\frac{\beta}{c_2} - \lambda\frac{(1+\tau_{c_2})}{(1+r(1-\tau_s))} &= 0 \\
\frac{-\theta}{1-\ell} + \lambda(1-\tau_\ell)w &= 0
\end{aligned}
$$

- Rewriting

$$
\begin{aligned}
\frac{1}{c_1} &= \lambda(1+\tau_{c_1}) \\
\frac{\beta}{c_2} &= \lambda\frac{(1+\tau_{c_2})}{(1+r(1-\tau_s))} \\
\frac{\theta}{1-\ell} &= \lambda(1-\tau_\ell)w
\end{aligned}
$$

- Intertemporal optimality condition

$$\frac{\beta c_1}{c_2} = \frac{(1+\tau_{c_2})}{(1+\tau_{c_1})} * \frac{1}{(1+r(1-\tau_s))}$$

Interpretation: marginal rate of substitution

$$\frac{\beta u'(c_2)}{u'(c_1)} = \frac{\beta c_1}{c_2}$$

should equal relative price between consumption in the second to consumption in the first period, $\frac{1}{(1+r(1-\tau_s))}$. With differential consumption taxes, the relative price has to be adjusted by relative taxes $\frac{(1+\tau_{c_2})}{(1+\tau_{c_1})}$.

- Comparative statics

    1. Increase in capital income tax rate τ_s reduces after-tax interest rate $1 + r(1 - \tau_s)$ and induces households to consume more in first period, relative to second period (ratio $\frac{c_1}{c_2}$ increases).

    2. Increase in consumption taxes in first period τ_{c_1} induces households to consume less in first period, relative to second period (ratio $\frac{c_1}{c_2}$ decreases).

    3. Increase in consumption taxes in second period τ_{c_2} induces households to consume more in first period, relative to second period (ratio $\frac{c_1}{c_2}$ increases).

- Intratemporal optimality condition

$$\frac{\theta c_1}{1-\ell} = \frac{(1-\tau_\ell)w}{(1+\tau_{c_1})}.$$

- Interpretation: marginal rate of substitution between current period leisure and current period consumption,

$$\frac{\theta u'(1-\ell)}{u'(c_1)} = \frac{\theta c_1}{1-\ell}$$

should equal after-tax wage, adjusted by first period consumption taxes $\frac{(1-\tau_\ell)w}{(1+\tau_{c_1})}$.

- Comparative statics

  1. Increase in labor income taxes τ_ℓ reduces after-tax wage and reduces consumption, relative to leisure, that is $\frac{c_1}{1-\ell}$ falls.

  2. Increase in consumption taxes τ_{c_1} reduces consumption, relative to leisure, that is $\frac{c_1}{1-\ell}$ falls.

Proposition

Proposition: *Suppose we start with tax system with no labor income taxes, $\tau_\ell = 0$ and uniform consumption taxes $\tau_{c_1} = \tau_{c_2} = \tau_c$. Denote by c_1, c_2, ℓ, s the optimal consumption, savings and labor supply decision. Then there exists a labor income tax τ_ℓ and a lump sum tax T such that for $\tau_c = 0$ households find it optimal to make exactly the same consumption choices as before.*

Proof: If consumption tax is uniform, it drops out of the intertemporal optimality condition. Rewrite intratemporal optimality condition as

$$\frac{\theta c_1}{(1-\ell)w} = \frac{(1-\tau_\ell)}{(1+\tau_c)}$$

Right hand side, for $\tau_\ell = 0$, is equal to

$$\frac{1}{(1+\tau_c)}$$

Set $\widehat{\tau}_\ell = \frac{\tau_c}{1+\tau_c}$ and $\widehat{\tau}_c = 0$. Then

$$\frac{(1-\widehat{\tau}_\ell)}{(1+\widehat{\tau}_c)} = 1 - \frac{\tau_c}{1+\tau_c} = \frac{1}{(1+\tau_c)},$$

and household faces the same intratemporal optimality condition as before. Appropriate lump-sum tax T guarantees that tax payments remain the same.

Analytical Solution

- Intratemporal optimality condition yields

$$c_1 = \frac{(1-\tau_\ell)(1-\ell)w}{(1+\tau_{c_1})\theta}$$

- Intertemporal optimality condition yields

$$c_2 = \beta c_1(1+r(1-\tau_s))\frac{(1+\tau_{c_1})}{(1+\tau_{c_2})} = \frac{(1-\tau_\ell)(1-\ell)w}{\theta}\frac{\beta(1+r(1-\tau_s))}{(1+\tau_{c_2})}$$

- Plugging into budget constraint yields

$$\begin{aligned} \frac{(1-\tau_\ell)(1-\ell)w}{\theta} + \beta\frac{(1-\tau_\ell)(1-\ell)w}{\theta} &= (1-\tau_\ell)w\ell + \frac{b}{(1+r(1-\tau_s))} \\ (1+\beta)\frac{(1-\tau_\ell)(1-\ell)w}{\theta} &= (1-\tau_\ell)w\ell + \frac{b}{(1+r(1-\tau_s))} \end{aligned}$$

- Solve for ℓ to obtain

$$\ell^* = \frac{1+\beta}{1+\beta+\theta} - \frac{b}{(1+r(1-\tau_s))\theta w(1-\tau_\ell)(1+\beta+\theta)}$$

- If $b = 0$, then
  - $\ell^* = \frac{1+\beta}{1+\beta+\theta} \in (0, 1)$
  - Interpretation: the more the household values leisure (the higher is θ), the less she finds it optimal to work. With $b > 0$, higher social security benefits in retirement reduce labor supply in the working period If b gets really big, then the optimal $\ell^* = 0$.
  - Rest of solution

$$\begin{aligned}
c_1 &= \frac{(1-\tau_\ell)}{(1+\tau_{c_1})(1+\beta+\theta)} w \\
c_2 &= \frac{\beta(1-\tau_\ell)(1+r(1-\tau_s))}{(1+\beta+\theta)(1+\tau_{c_2})} w \\
s &= \frac{\beta(1-\tau_\ell) w}{1+\beta+\theta}
\end{aligned}$$

- Let's consider countries i that differ in their
  - tax rates on labor $\tau_{\ell i}$, and consumption τ_{ci}
  - perhaps in their wages w_i
  - and in their use of government revenues: Which fraction ξ_i of $\tau_{\ell i}\ell_i w_i + \tau_{ci} c_i$ is used for consumption:

$$T_i = \xi_i \left(\tau_{\ell i}\ell_i w_i + \tau_{ci} c_i\right)$$

- Households live only one period and maximize

$$\log c + \theta \ \log(1 - \ell_i)$$

subject to:

$$(1 + \tau_{ci})c = \ell \ w_i(1 - \tau_{\ell i}) + T_i$$

- $\max_{\ell} \quad \log \frac{\ell\ w_i(1-\tau_{\ell i})+T_i}{(1+\tau_{ci})} + \theta\ \log(1-\ell)$

- $\max_{\ell} \quad \log\left[\ell\ w_i(1-\tau_{\ell i}) + T_i\right] - \log(1+\tau_{ci}) + \theta\ \log(1-\ell)$

- The FOC

$$\frac{w_i\ (1-\tau_{\ell i})}{\ell_i\ w_i(1-\tau_{\ell i}) + T_i} = \frac{\theta}{1-\ell_i}$$

- Getting rid of the denominators

$$(1-\ell_i)\ w_i\ (1-\tau_{\ell i}) = \theta[\ell_i\ w_i(1-\tau_{\ell i}) + T_i]$$

- Isolating the term with labor

$$w_i\ (1-\tau_{\ell i}) - \theta T_i = (1+\theta)[\ell_i\ w_i\ (1-\tau_{\ell i})]$$

Without Taxes

- Which yields

$$\ell_i = \frac{1}{1+\theta} \; \frac{w_i(1-\tau_{\ell i}) - \theta \; T_i}{w_i \; (1-\tau_{\ell i})}$$

- Note that without transfers, labor is independent of wages: $\ell_i = \frac{1}{1+\theta}$

- A very important feature: We work as much as our great grandparents despite having wages that are much higher (this is a straight implication of the preferences that we have posed)

- It is not historically true, but almost.

With Taxes we had

$$\ell_i = \frac{1}{1+\theta} \; \frac{w_i(1-\tau_{\ell i}) - \theta \; T_i}{w_i \; (1-\tau_{\ell i})} = \frac{1}{1+\theta} \; \frac{w_i(1-\tau_{\ell i}) - \theta \; \xi_i \; (\tau_{\ell i}\ell_i w_i + \tau_{ci} c_i)}{w_i \; (1-\tau_{\ell i})}$$

- Because of $$T_i = \xi_i \; (\tau_{\ell i}\ell_i w_i + \tau_{ci} c_i)$$
- With $\tau_{ci} = 0$, $$\ell_i = \frac{1}{1+\theta} \; \frac{(1-\tau_{\ell i}) - \tau_{\ell i} \; \theta \; \xi_i \ell_i}{(1-\tau_{\ell i})}$$
- $$\ell_i = \frac{1-\tau_{\ell i}}{(1+\theta)(1-\tau_{\ell i}) + \tau_{\ell i}\theta \; \xi_i} < \frac{1}{1+\theta}, \quad \text{if } \theta\xi_i \text{ is not far from 2}$$

Taking Stock

- Hours Worked Depends on Taxes

- With Consumption Taxes the Expressions get a bit more Complicated, but the same logic follows.

- We will use such an expression to actually compare across countries.

- Key for labor supply: tax wedge $\frac{(1-\tau_{\ell i})}{(1+\tau_{ci})}$ in the intratemporal optimality condition

$$\frac{\theta c_i}{1-\ell_i} = \frac{(1-\tau_{\ell i})}{(1+\tau_{ci})} w_i$$

- Wages w_i? Recall neoclassical production function operated by typical firm in the economy.

$$Y_i = A_i K_i^{\alpha} L_i^{1-\alpha}$$

- Profit maximization

$$\max_{(K_i, L_i)} A_i K_i^{\alpha} L_i^{1-\alpha} - w_i L_i - \rho_i K_i.$$

- Taking FOC with respect to L and setting it equal to 0 yields

$$\begin{aligned}(1-\alpha)A_i K_i^{\alpha} L_i^{-\alpha} &= w_i \\ \frac{(1-\alpha)A_i K_i^{\alpha} L_i^{1-\alpha}}{L_i} &= w_i \\ (1-\alpha)\frac{Y_i}{L_i} &= w_i\end{aligned}$$

- Labor share equals $1-\alpha$, capital share equals α.
- Use $w_i = (1-\alpha)\frac{Y_i}{L_i}$ and equilibrium in labor market $L_i = \ell_i$ to obtain

$$\frac{\theta c_i}{1-\ell_i} = \frac{(1-\tau_{\ell i})}{(1+\tau_{ci})}(1-\alpha)\frac{Y_i}{\ell_i}$$

- Solving relates hours to taxes and Consumption to Output ratios.

$$\ell_i = \frac{1-\alpha}{1-\alpha+\frac{\theta(1+\tau_{ci})}{(1-\tau_{\ell i})}\frac{c_i}{Y_i}} \quad \in (0,1)$$

DATA: 1990's

Country	GDP p.p.	Hours	GDP p.h.
Germany	74	75	99
France	74	68	110
Italy	57	64	90
Canada	79	88	89
United Kingdom	67	88	76
Japan	78	104	74
United States	100	100	100

Data: 1970's

Country	GDP p.p.	Hours	GDP p.h.
Germany	75	105	72
France	77	105	74
Italy	53	82	65
Canada	86	94	91
United Kingdom	68	110	62
Japan	62	127	49
United States	100	100	100

Main Observations

- GDP per capita, relative to the U.S. in Germany, France and Italy lags the U.S. by $25 - 40\%$, both in early 70's and mid 90's
- Early 70's due to lower productivity.
- In mid 90's: not due to lower productivity, but rather due to lower hours worked.

- Why do Europeans now work so much less than Americans? Proposed answer by Prescott (2004): taxes.

- Use

$$\ell_{it} = \frac{1-\alpha}{1-\alpha+\frac{\theta(1+\tau_{cit})}{(1-\tau_{\ell it})}\frac{c_{it}}{Y_{it}}}$$

  to assess whether answer makes quantitative sense.

- $\frac{c_{it}}{Y_{it}}$ from NIPA accounts.
  - Assume that all but military government spending is yielding private consumption.
  - Indirect consumption taxes part of NIPA consumption, but not part of c in model.
  - This is what we meant by ξ.
- τ_{cit} is set to ratio between total indirect consumption taxes and total consumption expenditures in data.

- Labor income taxes

$$\tau_\ell = \tau_{ss} + \tau_{inc}$$

For τ_{ss} take payroll tax rates (currently 15.3%, shared by employers and employees). To compute marginal income tax rate τ_{inc}, compute average income taxes. by dividing total direct taxes by national income. Multiply by 1.6, to capture progressivity of tax code.

- Specify parameter values, θ and α.
  - Since α equals the capital share, set $\alpha = 0.3224$, the average across countries and time.
  - Parameter θ determines fraction of time worked. Choose θ such that in model number of hours spent working equals the average hours (across countries) in the data, which requires 1.54.

- Combined labor income and consumption tax rate relevant for the labor supply decision.
$$\frac{(1-\tau_\ell)}{(1+\tau_c)} = 1-\tau$$
where $\tau = \frac{\tau_\ell+\tau_c}{1+\tau_c}$.

- A person wanting to spend one dollar on consumption needs to earn x dollars as labor income, where x solves
$$\begin{aligned} x(1-\tau) &= 1 \text{ or} \\ x &= \frac{1}{1-\tau} \end{aligned}$$

Country	Tax Rate τ	$\frac{c}{Y}$	Hours per Person per Week	
			Actual	Predicted
Germany	0.59	0.74	19.3	19.5
France	0.59	0.74	17.5	19.5
Italy	0.64	0.69	16.5	18.8
Canada	0.52	0.77	22.9	21.3
United Kingdom	0.44	0.83	22.8	22.8
Japan	0.37	0.68	27.0	29.0
United States	0.40	0.81	25.9	24.6

Country	Tax Rate τ	$\frac{c}{Y}$	Hours per Person per Week	
			Actual	Predicted
Germany	0.52	0.66	24.6	24.6
France	0.49	0.66	24.4	25.4
Italy	0.41	0.66	19.2	28.3
Canada	0.44	0.72	22.2	25.6
United Kingdom	0.45	0.77	25.9	24.0
Japan	0.25	0.60	29.8	35.8
United States	0.40	0.74	23.5	26.4

Main Findings I

- Measured effective tax rates differ substantially by countries.
- Model does very well in explaining the cross-country differences in hours worked for the 90's.,
- Large part of the difference in hours worked between the U.S. and Europe (but not all of it) is explained by tax differences.

Main Findings II

- Model is not quite as successful matching all countries for early 70's.
- Does predict that in the early 70's Germans and French did not work so much less than Americans, precisely because tax rates on labor were lower then than in the 90's in these countries.
- Two big failures of the model: Japan and Italy. What other than taxes depressed labor supply in these countries in this time period.

- History of U.S. system
- Taxes and Benefits under the current law
- Theoretical Analysis: a) Effect of private savings, b) welfare consequences

History of the U.S. Social Security System

- Historical ancestor: system in Germany, introduced in 1889 by Bismarck. Benefits started at age 70.
- Response to rapid industrialization that transformed a largely agrarian society into a modern industrialized economy.
- Also a response to the growing popularity of the socialist movement and their demands for basic, publicly provided social insurance.
- 23 states in U.S. had introduced some public pension systems for needy elders in the early 1930's.
- A national old-age social insurance system started in 1935
- Various major forces responsible for the introduction of social security at that time.

- U.S. economy had undergone a dramatic transition for an agrarian to an industrialized economy.
- Share of employment in agriculture dropped from more than 50% in 1880 to less than 20% in 1935.
- Why was farm life less likely to leave the elders impoverished? Elders could perform less physically demanding tasks on family farms. Also, elders tended to own the farms. Second, employment opportunities in agriculture were less volatile than in the rest of the economy.

II: The Great Depression

- Great depression in 1929-1932, the most severe recession in U.S. economic history, reduced unemployment opportunities of the elderly.
- Destroyed most of retirement wealth: September 1, 1929, value of stocks listed at NYSE was $89.7 billion; in middle of 1932 it was $15.6 billion, a decline of over 80%. In 1930 and 1931 over $3,000$ banks suspended operations, deposits being lost were more than $2 billion. Prices of wheat and cotton dropped by 66% and 75%, respectively, with it incomes and asset values in agricultural sector.
- Consequently, the great depression left an entire generation impoverished.

- Franklin D. Roosevelt's "New Deal" "It was an idea that all the political and practical forces of the community should and could be directed to making life better for ordinary people." (Francis Perkins)

- Several public programs arose out of this idea, one of which was social security. Designed to deal with the specific problems of the impoverished elders.

- The Elderly Population had started to grow as a result of increased life expectancy
- It is hard to coexist with large numbers of very poor elderly.

- Social Security Act passed in 1935
- Original plan: use the 2% payroll tax for the accumulation of financial assets for retirement.
- Why special tax (rather than general tax revenues) to finance benefits?

  > *These taxes were never a problem of economics. They are politics all the way through. We put those payroll contributions there so as to give the contributors a legal, moral, and political right to collect their pensions. With these taxes in there, no damn politician can ever scrap my social security program. [Franklin D. Roosevelt]*

- By 1939 it become clear that the widespread poverty of the old could needed more than a funded system: It was changed to pay-as-you go.

- Basically a pay-as-you-go system. Taxes paid by current workers are immediately used for paying benefits of current retirees.

- Fully funded system would save taxes of current workers, invest them in some assets and uses the returns to pay benefits when these current workers are old.

- U.S. social security system has accumulated the so-called trust fund, but with the expressed purpose of handling the retirement of the massive baby boom generation without having to increase payroll taxes.

Current system is defined by

- Payroll tax rate τ,
- A maximum amount of earnings $\bar{y}$ for which this payroll tax applies
- A benefit formula that calculates social security benefits as a function of the labor earnings over your lifetime.

- Currently, both employers and employees currently pay a proportional tax on labor income of $\tau = 6.2\%$, for a total of 12.4% of wages and salaries.

- Applies to all income below a threshold of $\$132,900$. (2019)
- Maximum amount an employee has to pay in 2019 is

$$0.062 * 132,900 = \$8,239.80$$

Year	Max. Taxable Ear.	Tax Rate
1937	$3,000	2.00%
1950	$3,000	3.00%
1960	$4,800	6.00%
1970	$7,800	8.40%
1980	$29,700	10.16%
1990	$51,300	12.40%
1998	$68,400	12.40%
2007	$97,500	12.40%
2012	$110,100	12.40%
2017	$127,200	12.40%
2019	$132,900	12.40%

- Consider a person that just turned 66 and retires in 2019
- Two steps.
  - Compute average indexed monthly earnings (AIME). Basically average monthly salary, where salaries early in life are adjusted by inflation and average wage growth.
  - Apply benefit formula

$$b = f(AIME)$$

- https://www.ssa.gov/news/press/factsheets/colafacts2019.pdf
- https://www.ssa.gov/oact/cola/Benefits.html

- Suppose household worked for 45 years, from age 21 to age 65, starting in 1975
- Let income in year t be denoted by y_t, for $t = 1975, 1976, \ldots 2019$.
- Denote maximal taxable earnings in year t by $\bar{y}_t$

1. For each year t define qualified earnings as

$$\hat{y}_t = \min\{y_t, \bar{y}_t\}$$

2. Adjust for inflation. Let P_{1975} denote CPI in 1975 and P_{2019} CPI in 2019. Then $\frac{P_{2019}}{P_{1975}}$ is the relative price of a typical basket of consumption goods in 2019, relative to 1975. Thus we take

$$\begin{aligned} \tilde{y}_{1975} &= \hat{y}_{1975} * \frac{P_{2019}}{P_{1975}} \\ \tilde{y}_t &= \hat{y}_t * \frac{P_{2019}}{P_t} \end{aligned}$$

1. 3 Adjust by average wage growth. Define as the gross growth rate of average wages between 1975 and 2019

$$G_{1975,2019} = \frac{\bar{w}_{2019}}{\bar{w}_{1975}}$$
$$G_{t,2019} = \frac{\bar{w}_{2019}}{\bar{w}_{t}}$$

In addition to inflation earnings in early years of a persons's life are therefore adjusted in the following fashion

$$Y_t = \tilde{y}_t * G_{t,2019}$$

2. 4 We arrive at 45 numbers, $\{Y_{1975}, Y_{1976}, \ldots, Y_{2019}\}$. AIME equals the average of the 35 highest entries from the list (so if worked less than 35 years the AIME will be calculated with zero earnings for those years).

From AIME to Benefits: Primary insurance amount (PIA).

- Benefit formula (monthly)

$$b = \begin{cases} 0.9AIME & \text{if } AIME \leq \$926 \\ 833 + 0.32(AIME - 926) & \text{if } \$926 < AIME \leq \$5,583 \\ 2,323 + 0.15(AIME - 5,583) & \text{if } \$5,583 < AIME \end{cases}$$

- With a maximum of \$3,030.50 (maximum AIME \$10,296).

- Coefficients $\{0.9, 0.32, 0.15\}$ are determined by law. *Bend points* are adjusted every year.

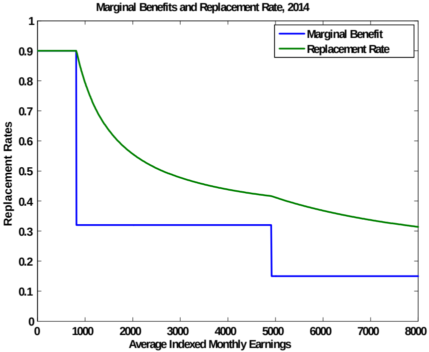
- This gives household's benefits in 2020. From that point on benefits are indexed by inflation. Benefits are paid until death.

- Social security benefits are perfectly determined by average indexed monthly earnings, that is, by the best 35 working years.

- Rational forward-looking household understand that working more today will increase social security benefits, although the link becomes weaker the higher is income.

- Define the replacement rate as

$$rr(AIME) = \frac{b(AIME)}{AIME}$$


Marginal Benefits and Replacement Rate, 2014
Marginal Benefit
Replacement Rate
Replacement Rates
Average Indexed Monthly Earnings
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
1000
2000
3000
4000
5000
6000
7000
8000

Need for Reforms

- Tax rate now stands at 12.4%
- In the 1990's the situation and especially the future outlook of social security system deteriorated, due to demographic changes.
  - Life expectancy increased
  - Fertility rates decreased
  - Higher (predicted) dependency ratio (the ratio of people above 65 to the population aged 16-65)

- Increase social security tax rates further
- Reduce benefits (e.g. increase the retirement age). Retirement age will gradually increase to age 67.
- Limit the scope of the program by reducing benefits and giving incentives to complement public pensions by private retirement accounts.

- Does a Pay-As-You-Go social security system reduce private savings rates?

- Under what conditions is the introduction of a Pay-As-You-Go social security system a good idea.

- Social Security as Insurance against Longevity Risk

- Household maximizes

$$\begin{aligned} \max_{c_1, c_2, s} \quad & \log(c_1) + \beta \log(c_2) \quad \text{s.t.} \\ c_1 + s &= (1-\tau)y \\ c_2 &= (1+r)s + b \end{aligned}$$

- Population grows at rate n, technical progress at rate g
- Social security system balances its budget

$$b = (1+n)(1+g)\tau y$$

- Rewrite the budget constraints of the household as

$$\begin{aligned} c_1 + s &= (1-\tau)y \\ c_2 &= (1+r)s + (1+n)(1+g)\tau y \end{aligned}$$

- Consolidate

$$c_1 + \frac{c_2}{1+r} = (1-\tau)y + \frac{(1+n)(1+g)\tau y}{1+r} = I(\tau)$$

Solution of the maximization problem

$$c_1 = \frac{I}{1+\beta}$$

$$c_2 = \frac{\beta}{1+\beta}(1+r)I$$

$$s = (1-\tau)y - \frac{I}{1+\beta}$$

$$
\begin{aligned}
s &= (1-\tau)y - \frac{I}{1+\beta} \\
&= \frac{\beta y}{1+\beta} - \frac{(1+n)(1+g)+\beta(1+r)}{(1+r)(1+\beta)} * \tau y
\end{aligned}
$$

- which is obviously decreasing in τ. The larger the public pay-as-you-go system, the smaller are private savings.

- Because of its pay-as-you go nature of the system the social security system itself does not save, so total savings in the economy unambiguously decline with an increase in the size of the system as measured by τ.

- Is the introduction of PAYGO social security system good for households being born into the system?
- Social security tax rate only appears in $I(\tau)$, which is given as

$$I(\tau) = (1-\tau)y + \frac{(1+g)(1+n)\tau y}{1+r}.$$

- Under which $I(\tau)$ is strictly increasing in τ?

$$\begin{aligned} I(\tau) &= y - \tau y + \frac{(1+g)(1+n)\tau y}{1+r} \\ &= y + \left[\frac{(1+g)(1+n)}{1+r} - 1\right]\tau y \end{aligned}$$

- Pay-as-you go social security system is welfare improving if and only if $(1+n)(1+g) > 1+r$.
- As good approximation

$$n + g > r$$

- If people save by themselves for their retirement, the return on their savings equals $1 + r$. If they save via a social security system (are forced to do so), their return to this forced saving consists of $(1+n)(1+g)$.
- May help to understand why in some countries the reform away from a PAYGO system is underway, in others not.
- But transition problem: there is one missing generation (since initial generation received benefits without paying taxes). If we abolish the system, either the currently young pay double, or we just default on the promises for the old.

- Modern social security systems provide some form of insurance to individuals, namely insurance against the risk of living longer than expected.
- Why: social security benefits paid as long as the person lives.
- People that live (unexpectedly) longer receive more over their lifetime than those that die prematurely.
- But: could also be done by private annuities.

- Household lives up to two periods, but die after the first period with probability $1 - p$. Normalize the utility of being dead to 0
- Household problem

$$\begin{aligned} \max_{c_1, c_2, s} \quad & \log(c_1) + p \log(c_2) \\ \text{s.t.} \quad & \\ c_1 + s &= y \\ c_2 &= (1+r)s \end{aligned}$$

- Intertemporal budget constraint: $c_1 + \frac{c_2}{1+r} = y$
- Solution

$$\begin{aligned} c_1 &= \frac{1}{1+p} y \\ c_2 &= \frac{p(1+r)}{1+p} y \end{aligned}$$

- With social security: budget constraints

$$\begin{aligned} c_1 + s &= (1-\tau)y \\ c_2 &= (1+r)s + b \end{aligned}$$

- Budget constraint of the social security administration

$$pb = (1+n)(1+g)\tau y$$

- Consolidating household budget constraints and substituting for b yields

$$c_1 + \frac{c_2}{1+r} = y + \tau y \left(\frac{(1+n)(1+g)}{p(1+r)} - 1 \right)$$

- Two reasons for social security
  - If $(1+n)(1+g) > 1+r$, the implicit return on social security is higher than the return on private assets, even absent the insurance aspect.
  - As long as $p < 1$, even if $(1+n)(1+g) \leq 1+r$ social security may be good, since the surviving individuals are implicitly insured by their dead brethren: the implicit return on social security is $\frac{(1+n)(1+g)}{p} > (1+n)(1+g)$.

- Focus on the insurance aspect and assume

$$(1+n)(1+g) = 1+r$$

- Implicit return on social security is $\frac{(1+n)(1+g)}{p} = \frac{1+r}{p}$.

- Private insurance via annuities. An annuity is a contract where the household pays \$1 today, for the promise of the insurance company to pay you $\$(1+r_a)$ as long as you live, from tomorrow on.

- If perfect competition among insurance companies, then zero profits.

- Insurance company takes \$1 today, which it can invest at the market interest rate $1 + r$

- Tomorrow it has to pay out with probability p. It has to pay out $1 + r_a$ per \$ of insurance contract. Thus zero profits imply

$$
\begin{aligned}
1 + r &= p(1 + r_a) \\
1 + r_a &= \frac{1 + r}{p}
\end{aligned}
$$

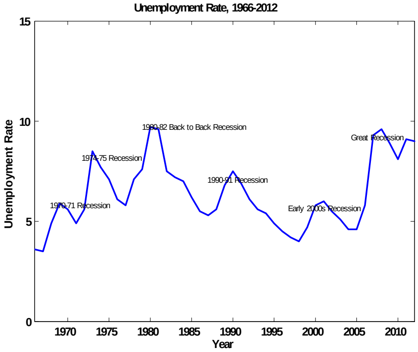
- Return on the annuity equals return via social security, as long as $(1 + n)(1 + g) = 1 + r$. Insurance against longevity can equally be provided by a social security system or by private annuity markets.

Public or Private Insurance?

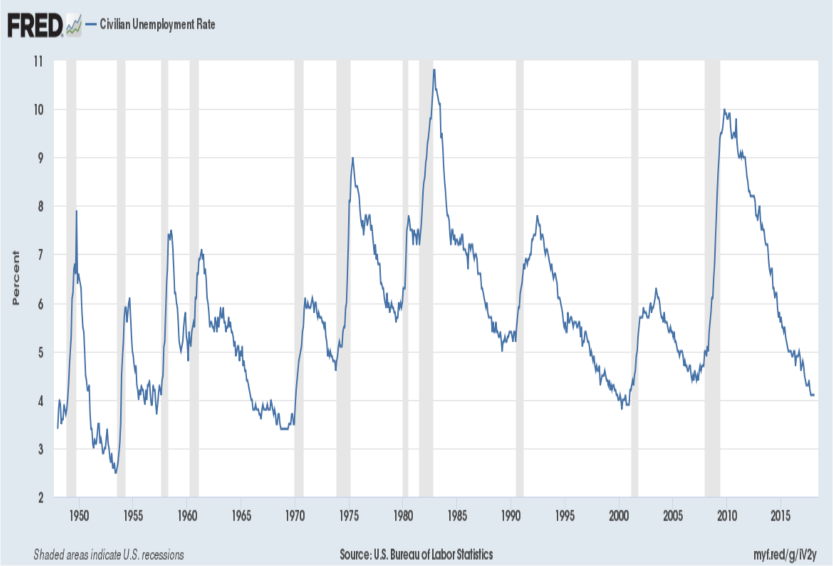
- Most countries provide this insurance publicly, with social security system? Why?
  - If there is already a public system in place (for whatever reason), there are no strong incentives to purchase additional private insurance.
  - Adverse selection: individuals have better information about their life expectancy than insurance companies

Social Insurance

- A variety of public insurance programs
- Goal: insuring citizens against the major risks of life
- Examples: unemployment insurance, welfare, food stamps, social security, public health insurance



The U.S. Unemployment Rate


FRED
Civilian Unemployment Rate
Percent
11
10
9
8
7
6
5
4
3
2
1950
1955
1960
1965
1970
1975
1980
1985
1990
1995
2000
2005
2010
2015
Shaded areas indicate U.S. recessions
Source: U.S. Bureau of Labor Statistics
myf.red/g/iV2y

Length of Unemployment Spells over the Cycle

Unemployment Spell	2006	2010
$<$ 5 weeks	37%	19%
5 - 14 weeks	30%	22%
15 - 26 weeks	15%	16%
$>$ 26 weeks	18%	43%

	Unemployment (%)				$\geq$ 1 Year		
	2000	2008	2011	2017	1999	2006	2011
France	9.0	7.8	9.7	9.0	38.7	41.9	41.4
Germany	8.0	7.5	5.9	3.7	51.7	56.4	48.0
Spain	11.7	11.3	21.6	15.6	46.3	21.7	41.6
Italy	10.1	6.7	8.4	11.0	61.4	49.6	51.9
Greece	11.2	7.7	17.7	20.7	55.3	54.3	49.6
Portugal	4.0	7.7	12.9	9.0	41.2	50.2	48.2
Sweden	5.6	6.2	7.5	6.3	30.1	13.0	17.2
UK	5.4	5.7	8.0	4.4	29.6	22.3	33.4
US	4.0	5.8	9.7	4.1	6.8	10.0	31.7
Tot. OECD	6.1	6.0	5.6	8.0	32.2	31.4	33.6

	Single			With Dependent Spouse		
	1. Y.	2.-3. Y.	4.-5. Y.	1. Y.	2.-3. Y.	4.-5. Y.
Belgium	79	55	55	70	64	64
France	79	63	61	80	62	60
Germany	66	63	63	74	72	72
Netherlands	79	78	73	90	88	85
Spain	69	54	32	70	55	39
Sweden	81	76	75	81	100	101
UK	64	64	64	75	74	74
US	34	9	9	38	14	14

European Unemployment Dilemma of 1980's and 1990's: A Potential Explanation

- Ljungqvist & Sargent: unemployment benefits & increased turbulence

- Increased turbulence in the 80's: laid-off workers faced a higher risk of losing their skills when becoming unemployed.

- Newly laid off worker in Europe has access to high and long-lasting unemployment compensation; on other hand, he may have lost his skill and thus is not offered new jobs that are attractive enough. Decides to stay unemployed, rather than accept a bad job. European unemployment dilemma.

Recent U.S. Unemployment Dilemma: A Potential Explanation

- Why U.S. unemployment dilemma in/after Great Recession: Mitman and Rabinovich (2014) point to extension of unemployment benefits.

- Reduces incentives for workers to search for new job.

- Reduces incentives of firms to post new vacancies

- Europeization of U.S. labor market.

- Household lives for two periods. First period: job for sure, wage y_1.
- Second period: with probability p he has a job and earns y_2^e; with probability $1-p$ he is unemployed and earns y_2^u.
- Interest rate $r = 0$.

- Utility function

$$\log(c_1) + p\log(c_2^e) + (1-p)\log(c_2^u)$$

- Budget constraints are

$$\begin{aligned} c_1 + s &= y_1 \\ c_2^e &= y_2^e + s \\ c_2^u &= y_2^u + s \end{aligned}$$

- Suppose that $y_1 = y$ and $y_2^e = y_2^u = y_1 = y$.
- Maximization problem

$$\max_{c_1, c_2^e, c_2^u, s} \quad \log(c_1) + p\log(c_2^e) + (1-p)\log(c_2^u)$$

s.t.

$$\begin{aligned} c_1 + s &= y \\ c_2^e &= y + s \\ c_2^u &= y + s \end{aligned}$$

- Solution

$$\begin{aligned} c_1 &= c_2^e = c_2^u = y \\ s &= 0 \end{aligned}$$

- Income is perfectly smooth and $\beta(1+r) = 1$, so consumption simply equals income in every period.

- Let $y_1 = y$ and $p = 0.5$ and $y_2 = 2y_1 = 2y$. Mean-preserving spread, since

$$0.5 * 2y + 0.5 * 0 = y$$

- Maximization problem

$$\max \log(c_1) + 0.5 \log(c_2^e) + 0.5 \log(c_2^u)$$

$$\begin{aligned} c_1 + s &= y \\ c_2^e &= 2y + s \\ c_2^u &= s \end{aligned}$$

- Lagrangian

$$
\begin{aligned}
L &= \log(c_1) + 0.5\log(c_2^e) + 0.5\log(c_2^u) + \lambda_1\left(y - c_1 - s\right) \\
&\quad + \lambda_2\left(2y + s - c_2^e\right) + \lambda_3\left(s - c_2^u\right)
\end{aligned}
$$

- First order conditions with respect to (c_1, c_2^e, c_2^u, s) yields

$$
\begin{aligned}
\frac{1}{c_1} - \lambda_1 &= 0 \\
\frac{0.5}{c_2^e} - \lambda_2 &= 0 \\
\frac{0.5}{c_2^u} - \lambda_3 &= 0 \\
-\lambda_1 + \lambda_2 + \lambda_3 &= 0
\end{aligned}
$$

- Rewriting

$$\begin{aligned}\frac{1}{c_1} &= \lambda_1\\ \frac{0.5}{c_2^e} &= \lambda_2\\ \frac{0.5}{c_2^u} &= \lambda_3\\ \lambda_2+\lambda_3 &= \lambda_1\end{aligned}$$

- Substituting the first three equations into the last yields

$$\frac{0.5}{c_2^e}+\frac{0.5}{c_2^u}=\frac{1}{c_1}$$

- Use the budget constraints to obtain

$$\frac{0.5}{2y+s}+\frac{0.5}{s}=\frac{1}{(y-s)}$$

- Bringing the equation to one common denominator, $s*(2y+s)*(y-s)$, yields

$$\frac{0.5s(y-s)}{s\left(2y+s\right)\left(y-s\right)}+\frac{0.5\left(2y+s\right)\left(y-s\right)}{s\left(2y+s\right)\left(y-s\right)}=\frac{s\left(2y+s\right)}{s\left(2y+s\right)\left(y-s\right)}$$

or

$$\frac{0.5s(y-s)+0.5\left(2y+s\right)\left(y-s\right)-s\left(2y+s\right)}{s\left(2y+s\right)\left(y-s\right)}=0$$

- But this can only be 0 if the numerator is 0, or

$$0.5s(y-s)+0.5\left(2y+s\right)\left(y-s\right)-s\left(2y+s\right)=0$$

- Multiplying things out and simplifying a bit yields

$$s^2+ys-\frac{1}{2}y^2=0$$

- Quadratic equation, has two solutions:

$$
\begin{aligned}
s_1 &= -\frac{y}{2} - \left[\left(\frac{3}{4}\right) y^2\right]^{0.5} = -\frac{1}{2} y \left(1 + 3^{0.5}\right) < 0 \\
s_2 &= -\frac{y}{2} + \left[\left(\frac{3}{4}\right) y^2\right]^{0.5} = \frac{1}{2} y \left(3^{0.5} - 1\right) > 0
\end{aligned}
$$

- Discard first solution on economic grounds, since

$$
c_2^u = s = -\frac{1}{2} y \left(1 + 3^{0.5}\right) < 0
$$

- Optimal consumption and savings choices with uncertainty satisfy

$$
\begin{aligned}
\hat{s} &= \frac{1}{2} y \left(3^{0.5} - 1\right) > 0 \\
\hat{c}_1 &= y - \frac{1}{2} y \left(3^{0.5} - 1\right) = \frac{1}{2} y \left(3 - 3^{0.5}\right) < y \\
\hat{c}_2^e &= 2y + \hat{s} = \frac{1}{2} y \left(3 + 3^{0.5}\right) \\
\hat{c}_2^u &= \frac{1}{2} y \left(3^{0.5} - 1\right)
\end{aligned}
$$

- Note:

$$\hat{c}_1 = \frac{1}{2}y\left(3 - 3^{0.5}\right) < y = c_1$$
$$\hat{s} = \frac{1}{2}y\left(3^{0.5} - 1\right) > 0 = s$$

- Even though income in the first period and expected income in the second period have not changed, households increase their savings, compared to situation without uncertainty.
- This effect is called precautionary savings. Households save more with increased uncertainty.
- Precautionary saving behavior arises whenever $u'''(c) > 0$.
- Strict concavity of u (that is, risk-aversion, $u'' < 0$) is not enough for this result. If utility quadratic, $u(c) = -\frac{1}{2}(c - 100,000)^2$ then consumption and savings choice in first period identical to no uncertainty case.

- Suppose

$$u(c) = -\frac{1}{2}(c - 100,000)^2$$

- In this case the first order conditions become

$$\begin{aligned}
-(c_1 - 100,000) &= \lambda_1 \\
-0.5(c_2^e - 100,000) &= \lambda_2 \\
-0.5(c_2^u - 100,000) &= \lambda_3 \\
\lambda_2 + \lambda_3 &= \lambda_1
\end{aligned}$$

- Inserting the first three equations into the fourth yields

$$-(c_1 - 100,000) = -0.5(c_2^e - 100,000) - 0.5(c_2^u - 100,000)$$

or

$$c_1 = 0.5(c_2^e + c_2^u)$$

- Now using the budget constraints one obtains

$$\begin{aligned} y - s &= 0.5(2y + s + s) \\ y - s &= y + s \\ 2s &= 0 \end{aligned}$$

and thus the optimal savings choice with quadratic utility is $s = 0$, as in the case with no uncertainty.

- Optimal consumption choices exhibit "certainty equivalence", that is, even with risk households make exactly the same choices as without uncertainty.
- But: realized consumption in period differs with and without uncertainty. With uncertainty one consumes $2y$ with probability 0.5 and 0 with probability 0.5, whereas under certainty one consumes y for sure.
- Even with quadratic utility households dislike risk, but they optimally don't change their saving behavior to hedge against it.
- It is easy to verify that with quadratic utility $u''' = 0$.

- Now the government levies unemployment insurance taxes on employed people in the second period at rate τ and pays benefits b to unemployed people
- Budget of the unemployment insurance system is balanced. There are many identical people, so fraction of employed in the second period is $p = 0.5$ and the fraction of unemployed is $1 - p = 0.5$
- Budget constraint of the unemployment administration reads as

$$\begin{aligned} 0.5\tau y_2 &= 0.5b \\ \tau y_2 &= b \end{aligned}$$

- For concreteness $\tau = 0.5$ and $y_2 = 2y_1 = 2y$ as before.
- Budget constraints in the second period of life become

$$\begin{aligned} c_2^e &= y + s \\ c_2^u &= y + s \end{aligned}$$

- Unemployment system perfectly insures the unemployed: unemployment benefits are exactly as large as after tax income when being employed.
- Optimal consumption and savings choices: It immediately follows that

$$c_2^e = c_2^u = c_2$$

- Then maximization problem of household becomes

$$\begin{aligned} & \max \log(c_1) + 0.5 \log(c_2) + 0.5 \log(c_2) \\ = & \max \log(c_1) + \log(c_2) \\ & \text{s.t.} \\ c_1 + s = & \; y \\ c_2 = & \; y + s \end{aligned}$$

with obvious solution

$$\begin{aligned} c_1 &= c_2 = y \\ s &= 0. \end{aligned}$$

- Exactly as without income uncertainty. When the government completely insures unemployment risk, private households make exactly the same choices as if there was no income uncertainty.

- With perfect unemployment insurance lifetime utility equals

$$V^{ins} = \log(y) + \log(y)$$

which exactly equals the lifetime utility without income uncertainty.

- Without unemployment insurance lifetime utility is $V^{no} =$

$$\log\left(\frac{1}{2}y\left(3 - 3^{0.5}\right)\right) + 0.5\log\left(\frac{1}{2}y\left(3 + 3^{0.5}\right)\right) + 0.5\log\left(\frac{1}{2}y\left(3^{0.5} - 1\right)\right)$$

- Easy to calculate that $V^{ins} > V^{no}$.

- Even with partial insurance, that is $0 < \tau < 0.5$, household would be better of with that partial insurance than without any insurance.

- Risk-averse individuals always benefit from public (or private) provision of actuarially fair insurance.

- But they prefer more insurance to less, absent any adverse selection or moral hazard problem.

- No country provides full insurance against being unemployed. Why not?
- In the real world with perfect insurance a strong moral hazard problem arises. Why work if one get's the same money by not working.
- Trade-off between insurance and economic incentives. If the government could perfectly monitor individuals things would be easy: simply condition payment of benefits on good behavior. But with private information the complicated trade-off between efficiency and insurance arises.