Economics 702

Suggested Solutions for the Midterm[1]

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1 Part I: Growth Models

1.1 question 1

Notice that the shock (hence, its history) is common for both types of agents, since it affects the wages for the whole economy. As Victor said, firms pay only one wage, which is a payment for *efficient units of labor*.[2] Given the usual definition of a history of shocks h_t we can define

The commodity space:

$$L = \{(l_{1t}(h_t), l_{2t}(h_t), l_{3t}(h_t)) : \ \sup|l| < \infty \forall i, t, h_t\}$$

where we use the usual order of output, capital services and labor services.

The consumption possibility set for type $i = 1, 2$:

$$X^i = \{x \in L : \exists \left\{c_t^i(h_t), k_{t+1}^i(h_t)\right\}_{\substack{t=0..\infty \\ h_t}} \geq 0 \text{ such that } \forall t, h_t$$

$$\begin{aligned} x_{1t}^i(h_t) + (1-\delta) k_t^i(h_{t-1}) &= c_t^i(h_t) + k_{t+1}^i(h_t) \\ x_{2t}^i(h_t) &\in \left[-k_t^i(h_{t-1}), 0\right] \\ x_{3t}^i(h_t) &\in [-1, 0] \\ & k_0 \text{ given}\} \end{aligned}$$

Notice that the household chooses her amount of work in the interval $[0,1]$ since she chooses the amount of time dedicated to work and not the efficient units of labor.

Now, the production possibility set is

$$Y_t = \{(y_{1t}(h_t), y_{2t}(h_t), y_{3t}(h_t)) : 0 \leq y_{1t}(h_t) \leq F(-y_{2t}(h_t), -y_{3t}(h_t))\}$$

The problem of the household of type $i = 1, 2$ is[3]

[1] *Disclaimer*: these solutions were prepared by both TAs, with enough time so what it's written here is fairly complete, but not what is expected in a real exam situation; there, is understood, you should try to economize time, notation and be as direct as possible.

[2] You could also have defined a wage per units of time, which would have changed slightly the problem.

[3] assumming dot-product representation of prices.

$$\max_{x\in X}\sum_{t=0}^{\infty}\beta^t\sum_{h_t}\pi\left(h_t\right)u\left[c_t^i\left(h_t\right),-x_{3t}^i\left(h_t\right)\right]$$

subject to

$$\sum_{t=0}^{\infty}\sum_{h_t}\left(p_{1t}\left(h_t\right)x_{1t}^i\left(h_t\right)+p_{2t}\left(h_t\right)x_{2t}^i\left(h_t\right)+p_{3t}\left(h_t\right)\epsilon^i z_t^{d_i}x_{3t}^i\left(h_t\right)\right)\leq 0$$

where $d_i=1$ if the agent is of type 2 and zero otherwise. Here the budget constraint is different since the household is choosing how much to work, but will get paid according to how many efficient units of labor that time will provide.

On the other hand, the problem of the firm given h_t

$$\max_{y\in Y_t}\sum_{j=1}^{3}p_{jt}\left(h_t\right)y_{jt}\left(h_t\right)$$

Definition 1 (Arrow-Debreu Equilibrium) *Is an allocation $\{x^{1*},x^{2*},y^*\}$ and a price p^* such that*

1. *Given p^*, x^{i*} solves the problem for household of type i*
2. *Given p^*, y^* solves the problem of the firm*
3. *Markets clear, i.e.*

$$\begin{aligned}\frac{1}{2}x_{it}^{1*}\left(h_t\right)+\frac{1}{2}x_{it}^{2*}\left(h_t\right) &= y_{it}^*\left(h_t\right) \qquad\qquad i=1,2\\ \frac{1}{2}\epsilon^1 x_{3t}^{1*}\left(h_t\right)+\frac{1}{2}z_t\epsilon^2 x_{3t}^{2*}\left(h_t\right) &= y_{3t}^*\left(h_t\right)\end{aligned}$$

1.2 question 2

In order to define a SMEE (the additional "E" for "easy"), we need to include additional markets for the household to be able to choose the same allocations as in the AD framework. This additional market is the market for state-contingent securities.

The problem of the household of type $i=1,2$ is

$$\max_{\left\{c_t^i(h_t),k_{t+1}^i(h_t),n_t^i(h_t),b_{t+1}^i(h_{t+1})\right\}_{t,h_t}}\sum_t\beta^t\sum_{h_t}u\left[c_t^i\left(h_t\right),n_t^i\left(h_t\right)\right]$$

subject to:

$$c_t^i(h_t) + k_{t+1}^i(h_t) + \sum_{z_t} q(h_t, z_t)\, b_{t+1}^i(h_t, z_t) =$$

$$R_t(h_t)\, k_t^i(h_{t-1}) + (1-\delta)\, k_t^i(h_{t-1}) + w(h_t)\, \epsilon^i z_t^{d_i} n_t^i(h_t) + b_t^i(h_{t-1}, z_t)$$

The problem of the firm is

$$\max_{K_t(h_t), L_t(h_t)} F(K_t(h_t), L(h_t)) - R_t(h_t) K_t(h_t) - w(h_t) L_t(h_t)$$

Definition 2 (SMEE) *is a list* $\{q^*, R^*, w^*, c_t^{i*}, k_{t+1}^{i*}, n_t^{i*}, b_{t+1}^i\}$ *such that*

1. *given* $\{q^*, R^*, w^*\}$, $\{c_t^{i*}, k_{t+1}^{i*}, n_t^{i*}, b_{t+1}^i\}$ *solves the problem of household* $i = 1, 2$
2. $K_t^* \equiv \frac{1}{2} k_t^{1*}(h_{t-1}) + \frac{1}{2} k_t^{2*}(h_{t-1})$ *and* $L_t^* \equiv \frac{1}{2}\epsilon^1 n_t^1(h_t) + \frac{1}{2} z_t \epsilon^2 n_t^2(h_t)$ *solve the problem of the firm; in other words*

$$\begin{aligned} R_t^*(h_t) &= F_1(K_t^*, L_t^*) \\ w_t^* &= F_2(K_t^*, L_t^*) \end{aligned}$$

3. *Market for state-contingent securities clears, i.e., prices of state contingent assets* $q^*(h_t, z_t)$ *are such that*

$$b_{t+1}^1(h_t, z_t) + b_{t+1}^2(h_t, z_t) = 0 \qquad \forall\ h_t,\ z_t$$

1.3 question 3

The question asks only about the allocation of the ADE. Hence, start with any ADE $\{x^{1*}, x^{2*}, y^*, p^*\}$. We have to prove that $\{x^{1*}, x^{2*}, y^*\}$ can be supported as a SME. The ADE lacks $q_t^*(h_t, z_t)$ and $b_{t+1}^{i*}(h_t, z_t)$ from the SME, but the earlier is easily constructed by way of first order conditions and non-arbitrage. The tricky part has to do with $b_{t+1}^{i*}(h_t, z_t)$, since we have two different types of agents, hence, we cannot use $b_{t+1}^{i*}(h_t, z_t) = 0\ \forall z_t$ but rather we have to construct the bonds for a SME.

Take the allocation x^{i*} for agent i. From the definition of ADE, we know that the following holds (with equality, to make things easier)

$$\sum_{t=0}^{\infty}\sum_{h_t}\left(p_{1t}^{*}(h_t)\,x_{1t}^{*}(h_t)+p_{2t}^{*}(h_t)\,x_{2t}^{*}(h_t)+p_{3t}^{*}(h_t)\,\epsilon^{i}z_t^{d_i}x_{3t}^{*}(h_t)\right)=0$$

Now, replace x for their counterparts of the SME (to ease notation, drop h_t)

$$\sum_{t=0}^{\infty}\sum_{h_t}\left(p_{1t}^{*}\left(c_t^{i*}+k_{t+1}^{i*}-(1-\delta)\,k_t^{i*}\right)-p_{2t}^{*}k_t^{i*}-p_{3t}^{*}\epsilon^{i}z_t^{d_i}n_t^{i*}\right)=0$$

this condition is the general budget constraint, which has to hold after we add all periods and all history nodes. Take a particular t and h_t and express the budget in terms of p_{1t}

$$c_t^{i*}+k_{t+1}^{i*}-(1-\delta)\,k_t^{i*}-\hat{p}_{2t}^{*}k_t^{i*}-\hat{p}_{3t}^{*}\epsilon^{i}z_t^{d_i}n_t^{i*} \tag{1}$$

where $\hat{p}_{2t}^{*}=F_k(.)+1-\delta$ and $\hat{p}_{3t}^{*}=F_L(.)$, conditions derived from FOCs. Equation (1) doesn't neccessarily hold for any particular t nor history h_t. What we do know from the definition of ADE, is that at each period the firm is solving a static problem and therefore there must be aggregate market clearing, i.e.,

$$\begin{aligned}\frac{1}{2}\left(c_t^{1*}+c_t^{2*}\right)+\frac{1}{2}\left(k_{t+1}^{1*}+k_{t+1}^{2*}\right) &= F\left(\frac{1}{2}\left(k_t^{1*}+k_t^{2*}\right),\frac{1}{2}\epsilon^{1}n_t^{1*}+\frac{1}{2}z_t\epsilon^{2}n_t^{2*}\right)\\ &\quad+\frac{1}{2}(1-\delta)\left(k_t^{1*}+k_t^{2*}\right)\end{aligned}$$

In other words, if (1) is not equal to zero for a particular agent at time t and history h_t, it means that the constraint for the other agent is not equal to zero also. Hence, we have a simple algorithm to calculate the missing bonds for all time and histories. Start at $t=0$ (no histories at this point): we know that $b_o^{i*}=0$ for $i=1,2$. Then

$$\sum_{z_t}q^{*}(h_0=\emptyset,z_t)\,b_1^{i*}(h_0,z_t)=\hat{p}_{21}^{*}k_0^{i*}+\hat{p}_{31}^{*}\epsilon^{i}z_0^{d_i}n_0^{i*}+(1-\delta)\,k_0^{i*}-c_0^{i*}-k_1^{i*}$$

with $b_1^{1*}(h_0,z_t)=-b_1^{2*}(h_0,z_t)\ \forall z_t$. At $t=1$, for each node h_1 we have already pinned down $b_1^{i*}(h_0,z_t)$ for each possible realization of the shock z_t; therefore we can procceed in the same way to calculate $b_2^{1*}(h_0,z_t)=-b_2^{2*}(h_0,z_t)\ \forall z_t$ and so on ad infinity.

1.4 question 4

Notice that capital is owned by firms. Hence, the decisions of the households are only how much to work and how much state-contingent assets to buy. For agent of type $i=1,2$ the recursive problem is

$$V^{i}\left(z,b_z^{i},K\right)=\max_{c^i,n^i,b^{i\prime}}\left\{u\left[c^{i},n^{i}\right]+\beta\sum_{z'}\Gamma_{zz'}V^{i}\left(z',b_{z'}^{i},K'\right)\right\}$$

subject to

$$
\begin{aligned}
c^i + \sum_{z'} \tilde{q}(z', K') b^i_{z'} &= w(z,K)\epsilon^i z^{d_i} n^i + b^i_z \\
K' &= G(z,K)
\end{aligned}
$$

The solutions for this problem are optimal policies for assets and supplied labor:

$$
\begin{aligned}
b^i_{z'} &= \varphi^i\left(z, b^i_z, K\right) \\
n^i &= h^i\left(z, b^i_z, K\right)
\end{aligned}
$$

Now, the firm has a dynamic problem also because it owns capital (although it's problem is NOT shock dependent: the firm doesn't care the composition of its labor force. Therefore, the law of motion of capital is not shock dependent either)

$$
\Omega(k,K) = \max_{k',L} \left\{ \begin{array}{c} F(K,L) - w(K)L - k' + (1-\delta)k \\ +q(K')\Omega(k',K') \end{array} \right\}
$$

subject to

$$
K' = G(K)
$$

And the solutions are

$$
\begin{aligned}
k' &= g(k,K) \\
L &= H(k,K)
\end{aligned}
$$

Definition 3 (RCE1) *is a list of functions* $\{V^i, \varphi^i, h^i, \Omega, g, H, G, \tilde{q}, q\}$ *for* $i = 1, 2$ *such that*

1. *Given* $\{G, \tilde{q}\}$, $\{V^i, \varphi^i, h^i\}$ *solve the problem of agent* i
2. *Given* $\{G\}$, $\{\Omega, g, H\}$ *solve the problem of the firm*
3. *Representative agent conditions are satisfied, i.e.*

$$
\begin{aligned}
g(K,K) &= G(K) \\
H(K,K) &= \frac{1}{2}\epsilon^1 h^1\left(z, b^1_z, K\right) + \frac{1}{2} z \epsilon^2 h^2\left(z, b^2_z, K\right)
\end{aligned}
$$

4. *Market of contingent claims clears*

$$\varphi^1\left(z,b^1_z,K\right)+\varphi^2\left(z,b^2_z,K\right)=0$$

5. *Condition on prices*

$$\begin{aligned}
w\left(K\right) &= F_L\left(.\right)\\
q^{-1}\left(G\left(.\right)\right) &= F_K\left(K,H\left(.\right)\right)\\
\sum_{z'}\tilde{q}\left(z',K'\right) &= 1
\end{aligned}$$

1.5 question 5

Here, the problem of the firm is the same as in q4, but the household now owns the firm through shares. Hence, the recursive formulation for agent $i=1,2$ is

$$V^i\left(z,a,b^i_z,K\right)=\max_{c^i,a',n^i,b^i_{z'}}\left\{u\left[c^i,n^i\right]+\beta\sum_{z'}\Gamma_{zz'}V^i\left(z',a',b^i_{z'},K'\right)\right\}$$

subject to

$$\begin{aligned}
c^i+a^iq\left(z',K'\right)+\sum_{z'}\tilde{q}\left(z',K'\right)b^i_{z'} &= a+w\left(K\right)\epsilon^iz^{d_i}n^i+b^i_z\\
K' &= G\left(K\right)
\end{aligned}$$

with solutions

$$\begin{aligned}
a^{i\prime} &= \phi^i\left(z,a^i,b^i_{z'},K\right)\\
b^i_{z'} &= \varphi^i\left(z,a^i,b^i_{z'},K\right)\\
n^i &= h^i\left(z,a^i,b^i_{z'},K\right)
\end{aligned}$$

Definition 4 (RCE2) *is a list of functions* $\left\{V^i,\phi^i,\varphi^i,h^i,\Omega,g,H,G,\tilde{q},q\right\}$ *for* $i=1,2$ *such that*

1. *Given* $\{G,\tilde{q},q\}$, $\left\{V^i,\phi^i,\varphi^i,h^i\right\}$ *solve the problem of agent* i

2. *Given $\{G\}$, $\{\Omega, g, H\}$ solve the problem of the firm*

3. *Representative agent conditions are satisfied, i.e.*

$$
\begin{aligned}
g(K,K) &= G(K) \\
H(K,K) &= \frac{1}{2}\epsilon^1 h^1\left(z,\Omega,b_z^1,K\right)+\frac{1}{2}z\epsilon^2 h^2\left(z,\Omega,b_z^2,K\right) \\
\Omega(G(K),G(K)) &= \frac{1}{2}\left[\phi^1\left(z,\Omega,b_z^1,K\right)+\phi^2\left(z,\Omega,b_z^2,K\right)\right]
\end{aligned}
$$

4. *Market clears*

   goods:

$$
\begin{aligned}
\Omega(z,K,K) &= F(K,H(.))-F_L(.)\,H(.)-G(.)+(1-\delta)K \\
&\quad +q(G(.))\,\Omega(G(.),G(.))
\end{aligned}
$$

   state-contingent assets:

$$
\varphi^1\left(z,b_z^1,K\right)+\varphi^2\left(z,b_z^2,K\right)=0
$$

5. *Condition on prices*

$$
\begin{aligned}
q^{-1}(G(.)) &= F_K(K,H(.)) \\
\sum_{z'}\tilde{q}(z',K') &= 1
\end{aligned}
$$

1.6 question 6

The assumptions for the SBWT are:

i) Let X be convex.

ii) Let Y be convex.

iii) Let preferences be convex (so if $u()$ is quasi concave this is satisfied).

iv) Let $u : X \rightarrow \mathcal{R}$ be continuous.

v) Let Y have a interior point.

Then, if (x^*, y^*) is a allocation (and x^* is a non satiation point), there exist a linear functional $p^* : \mathcal{L} \rightarrow \mathcal{R}$, such that (x^*, y^*, p^*) is a quasi equilibrium with transfers. If, moreover, there exists a cheaper point, then (x^*, y^*, p^*) is an ADE.

Notice that in this model there is a distortion. For example agents of type 1 get paid:

$$w_t^1 = (1-\theta)\, k_t^\theta \left[1/2\varepsilon^1 N_t^1 + 1/2\ \varepsilon^2 z_t N_t^2\right]^{-\theta} (1/2\varepsilon^1)$$

Clearly, each type's wage is affected by how much the other type of agents work. Hence, the SBWT will not hold in this economy, because the equilibrium outcome cannot be Pareto Optimal.

1.7 question 7

Now the state contigent claims are outlawed. Note that you can use any specification that you like, that is, agents can own or not own capital, or they can buy shares of the firm. What is really important, and most of the students failed to point out, is that without state contigent claims the agents cannot buy insurance against the shock any more (and this will harm them no matter what the assumptions for the enviroment are)

We will stick to the assumptions of part 5 (firms own capital and agents buy shares of the firm, but now they cannot buy state contigent claims). The recursive formulation of the problem for firms is:

$$\Omega(k,K) = \max_{k',L} \left\{ \begin{array}{c} F(K,L) - w(K)L - k' + (1-\delta)k \\ +q(K')\,\Omega(k',K') \end{array} \right\}$$

subject to

$$K' = G(K)$$

And the solutions are

$$\begin{aligned} k' &= g(k,K) \\ L &= H(k,K) \end{aligned}$$

For agent of type $i = 1,2$ the recursive problem is

$$V^i(z,a,K) = \max_{c^i,a',n^i} \left\{ u\left[c^i, n^i\right] + \beta \sum_{z'} \Gamma_{zz'} V^i(z',a',K') \right\}$$

subject to

$$\begin{aligned} c^i + a^i q\left(z', K'\right) &= a + w\left(K\right)\epsilon^i z^{d_i} n^i \\ K' &= G\left(K\right) \end{aligned}$$

with solutions

$$\begin{aligned} a^{i\prime} &= \phi^i\left(z, a^i, b^i_{z'}, K\right) \\ n^i &= h\left(z, a^i, b^i_{z'}, K\right) \end{aligned}$$

Definition 5 *Now a RCE is a list of functions* $\left\{V^i, \phi^i, h^i, \Omega, g, H, G, q\right\}$ *for* $i = 1, 2$ *such that*

1. *Given* $\{G, q\}$, $\left\{V^i, \phi^i, h^i\right\}$ *solve the problem of agent* i
2. *Given* $\{G\}$, $\{\Omega, g, H\}$ *solve the problem of the firm*
3. *Representative agent conditions are satisfied, i.e.*

$$\begin{aligned} g\left(K, K\right) &= G\left(z, K\right) \\ H\left(K, K\right) &= \frac{1}{2}\epsilon^1 h^1\left(z, \Omega, b^1_z, K\right) + \frac{1}{2} z\epsilon^2 h^2\left(z, \Omega, b^2_z, K\right) \\ \Omega\left(G\left(K\right), G\left(K\right)\right) &= \frac{1}{2}\left[\phi^1\left(z, \Omega, b^1_z, K\right) + \phi^2\left(z, \Omega, b^2_z, K\right)\right] \end{aligned}$$

4. *Market of goods clears*

$$\begin{aligned} \Omega\left(K, K\right) &= F\left(K, H\left(.\right)\right) - F_L\left(.\right) H\left(.\right) - G\left(.\right) + \left(1-\delta\right) K \\ &\quad + q\left(G\left(.\right)\right)\Omega\left(G\left(.\right), G\left(.\right)\right) \end{aligned}$$

5. *Condition on prices:*

$$q^{-1}\left(z', G\left(.\right)\right) = F_K\left(K, H\left(.\right)\right)$$

Once again, the math is not very important in this question. What you had to mention is that now agents cannot buy insurance against the shocks.

1.8 Question 8

Questions 8,9 are in the same spirit as 7. Without state contingencies, the agents cannot buy insurance. We have to come up with a tax transfer scheme that may solve exactly this problem.

Note that the shocks affect the two types in a different way. By the problem of the firm, the wage of the two types are:

$$w_1 = (1-\theta)\, k^{\theta} \left[1/2\varepsilon^1 N^1 + 1/2\ \varepsilon^2 z N^2\right]^{-\theta} (1/2\varepsilon^1) \text{ and}$$
$$w_2 = (1-\theta)\, k_t^{\theta} \left[1/2\varepsilon^1 N^1 + 1/2\ \varepsilon^2 z N^2\right]^{-\theta} (1/2\varepsilon^1 z)$$

Notice that high shocks are always bad for type 1 agents (just take the derivatie of w_1 with respect to z; you'll see it's always negative), but it's uncertain whether they are good or bad for type 2's wage. More precisely,

$$\frac{dw_2}{dz} = (1-\theta)\, k_t^{\theta} \left[1/2\varepsilon^1 N^1 + 1/2\ \varepsilon^2 z N^2\right]^{-\theta} (1/2\varepsilon^2) \left[1 - \theta \left(1/2\varepsilon^1 N^1 + 1/2\ \varepsilon^2 z N^2\right)^{-1} N^2\right]$$

and the sign of this derivative is ambiguous.

The tax scheme that we propose depends on whether we are in the case $\frac{dw_2}{dz} \geq 0$, or < 0.

Suppose first that $\frac{dw_2}{dz} \geq 0$. Then type 1 agents dislike high shocks, while type 2 agents like them. In this case we take advantage of the different way in which the two types are affected by the shock, and propose a lump sum transfer between the two types, depending on the realization of z. More precisely, let $\bar{z}$ be the unconditional mean of the shock realization. The transfer scheme could say that $T^1(z) \geq 0$, if and only if $z \geq \bar{z}$. That is, for high states the goverment transfers resources from agents of type 2 to type 1 agents, and viceversa. The (additional) equilibrium condition here is $T^1(z) = -T^2(z)$. Note also that for this proposal to make sense, there should be no absorbing states (this, in turn, requires the transition matrix to have some "nice" properties about which we will talk soon in class).

If, on the other hand, $\frac{dw_2}{dz} < 0$, then both types are negatively affected by higher shocks. For this case we propose a tax rate $\tau(z)$ over the wage income, which has to be dicreasing in the shock. The tax revenues are returned to the agents through lump sum transfers. The reason for this specification, is that now both agents dislike high shocks because this will lower their wages. So a higher shock means a lower wage, but at the same time it leads to lower tax payment, hence consumption is smoother.

Assume that the specification of the model is the same as in question 5 (firms own capital and agents buy shares of the firms). The only difference from the problem of that question is that now the budget constraint is, for $i = 1, 2$:

$$c^i + a^i q(z', K') = a^i + \left[w(z,K)\epsilon^i z^{d_i} n^i\right](1-\tau(z)) + T^i(z)$$

and there is an additional equilibrium condition (balanced goverment budget constraint) given by:

$$T^i(z) = \tau(z)\left(w(z,K)\epsilon^i z^{d_i} h^i(z,K)\right)$$

Suppose that $z^i \in (0,1]$, all i (this is just a sufficient condition to ensure that $L = 1/2\varepsilon^1 N^1 + 1/2\, \varepsilon^2 z N^2$ will be in $[0,1]$- given that also $\varepsilon^i \in [0,1]$, $i = 1,2$). A very simple tax rate that works is $\tau(z) = 1 - z \in (0,1]$. Notice that whenever the shock is high both agents will have a lower wage, but in these periods the tax payment will also be lower. Consumption will be smoother as a result.

2 PART II : Lucas Trees

We begin with a few observations about the enviroment. Note that the state of the world regarding, say, bananas, depends not only on the realization of bananas in the last period but also appricots. We assume that the state space regarding the two fruits has the same dimension, I. Then, if we let i, j be the typical state realization of appricots and bananas respectively, the transition matrix is given by Γ, where the typical element is $\Gamma_{ij,kl}$ and has the following interpretation: $\Gamma_{ij,kl} = \Pr\left[a_{t+1} = a^k, b_{t+1} = b^l \mid a_t = a^i, b_t = b^j\right]$.

Moreover, one should not forget that in this economy there are two dinstinct commodities that will have their own prices. Since there is no such thing as money (or something like that) in this economy whenever we are talking about (any) price, this should be in terms of either of these two commodities. It doesn't matter which one as long as you make it clear in the text.

2.1 Question 10

Some notation: Let

c^a = consumption of apricots

c^b = consumption of bananas

$p^{R,ij}$ = the relative price of apricots to bananas (in some certain period)

p^{ij} = the price of a share of the tree in units of apricots and given that current state is (i,j), i.e, $a_t = a^i, b_t = b^j$.

Then the recursive problem of the representative agent is:

$$V_{ij}(s) = \max_{c^a, c^b, s'} \left[u\left(c^a, c^b\right) + \beta \sum_{k,l} \Gamma_{ij,kl}\ V_{kl}(s') \right]$$

$$\text{s.t: } c^a + p^{R,ij} c^b + p^{ij} s' = s\left(p^{ij} + a^i + p^{R,ij} b^j\right)$$

Note that all terms in the budget constraint are in units of apricots.

Replace c^a from the BC into the objective to get:

$$V_{ij}(s) = \max_{c^a, c^b, s'} \left[u\left[s\left(p^{ij} + a^i + p^{R,ij} b^j\right) - p^{R,ij} c^b - p^{ij} s, c^b\right] + \beta \sum_{k,l} \Gamma_{ij,kl}\ V_{kl}(s') \right]$$

The first order conditions are:

$$\{c^b\}: \ -u_a\left(c^a, c^b\right) p^{R,ij} + -u_b\left(c^a, c^b\right) = 0 \qquad (1),$$

$$\{s'\}: \ -p^{ij}\ u_a\left(c^a, c^b\right) + \beta \sum_{k,l} \Gamma_{ij,kl}\ V'_{kl}(s') \qquad (2).$$

The first one (not surprisingly) says that the relative price of the two fruits should be equal to the marginal rate of substitution between them.

To obtain a more useful expression for the second condition we use the enevelpe theorem. Note that

$$V'_{ij}(s) = u_a\left(c^a, c^b\right)\ \left(p^{ij} + a^i + p^{R,ij} b^j\right),$$

and using that into (2) we have:

$$p^{ij}\ u_a\left(c^a, c^b\right) = \beta \sum_{k,l} \Gamma_{ij,kl}\ \left[u_a\left(c^{a\prime}, c^{b\prime}\right)\ \left(p^{kl} + a^k + p^{R,kl} b^l\right)\right] \qquad (2')$$

Definition 6 *A RCE is a list of functions* $\{V_{ij}(s), g_{ij}(s), \chi_{ij}(s), \psi_{ij}(s)\}$, *such that (here* $\chi()$ *is the pricing function for the relative price of appricots to bananas, and* $\psi()$ *is the pricing function for the shares of the tree in terms of appricots) :*

1. *Given prices* $\{\chi_{ij}(), \psi_{ij}()\}$, $V_{ij}()$ *and* $g_{ij}()$ *solve the agents maximization problem, i.e, they satisfy* (1) *and* (2′) *above, with* $c^a = a^i$ *and* $c^b = b^j$ *(also* $\left(c^{a\prime}, c^{b\prime}\right) = \left(a^k, b^l\right)$*).*
2. *Prices* $\chi_{ij}()$ *satisfy* $\chi_{ij}() = \frac{u_a\left(a^i, b^j\right)}{u_b\left(a^i, b^j\right)}$.
3. *(Representative agent condition)* $g_{ij}(1) = 1$

2.2 Question 11

In this question many of you gave a correct formula, but failed to explain what the state contingent claims are here. It was sufficient to say that the claims pay in terms of one of the two commodities. More precisely, let $q_{ij,kl}$ be the price of a claim that will pay for sure in terms of apricots, only if state (k,l) occurs tomorrow (given that current state is (i,j)).

Let's start by getting an expression for the second part of the option. This is the option to buy shares tomorrow at p_1. The value is:

$$p_{kl}^{1}(p_1) = \sum_{m,n} q_{kl,mn} \ \max\{0, p^{mn} - p_1\} \qquad (3)$$

Again, p^{mn} is the price of a share of the tree in terms of units of apricots, if the current state is (m,n).

Now let's find the price for the whole option. It is clear from the text that even if you exercise your option tomorrow you can exercise it again the day after tomorrow. In other words the price of the second part of the option (found above), will be added to the price of the first part. Hence, we have:

$$p_{ij}^{1,2}(p_1) = \sum_{k,l} q_{ij,kl} \quad \left[\max\{0, p^{kl} - p_1\} + p_{kl}^{1}(p_1)\right]$$

or using (3)

$$p_{ij}^{1,2}(p_1) = \sum_{k,l} q_{ij,kl} \quad \left[\max\{0, p^{kl} - p_1\} + \sum_{m,n} q_{kl,mn} \ \max\{0, p^{mn} - p_1\}\right]$$

3 Appendix

If you assume that the firm pays two different wages to the different agents, then we must think of wages as payments per unit of time worked. Taking that into consideration, let's rewrite questions 1 and 2.

3.1 question 1 (alt.)

The commodity space:

$$L = \{(l_{1t}(h_t), l_{2t}(h_t), l_{3t}(h_t), l_{4t}(h_t)) : \ \sup |l| < \infty \forall i, t, h_t\}$$

order: output, capital services, labor services of agent 1, labor services of agent 2. Since we have two types of labor with different prices, we need an additional commodity in the set.

The consumption possibility set for type $i = 1, 2$:

$$X^i = \{x \in L : \exists \left\{c_t^i(h_t), k_{t+1}^i(h_t)\right\}_{\substack{t=0..\infty \\ h_t}} \geq 0 \text{ such that } \forall t, h_t$$

$$\begin{aligned} x_{1t}^i(h_t) + (1-\delta) k_t^i(h_{t-1}) &= c_t^i(h_t) + k_{t+1}^i(h_t) \\ x_{2t}^i(h_t) &\in \left[-k_t^i(h_{t-1}), 0\right] \\ x_{3t}^i(h_t), x_{4t}^i(h_t) &\in [-1, 0] \\ & \quad k_0 \text{ given}\} \end{aligned}$$

We need both types of labor in the CPS for each type of agent, since the equilibrium conditions state that there is only one price system in the economy, used by both the households and the firm

Now, the production possibility set is

$$Y_t = \left\{ \begin{array}{c} (y_{1t}(h_t), y_{2t}(h_t), y_{3t}(h_t), y_{4t}(h_t)) : \\ 0 \leq y_{1t}(h_t) \leq F(-y_{2t}(h_t), -y_{3t}(h_t), -y_{4t}(h_t)) \end{array} \right\}$$

or, using the definition of the production function:

$$Y_t = \left\{ \begin{array}{c} (y_{1t}(h_t), y_{2t}(h_t), y_{3t}(h_t), y_{4t}(h_t)) : \\ 0 \leq y_{1t}(h_t) \leq \left((-y_{2t}(h_t))^\theta \left(-\frac{1}{2}\epsilon^1 y_{3t}(h_t) - \frac{1}{2} z_t \epsilon^2 y_{4t}(h_t)\right)^{1-\theta}\right) \end{array} \right\}$$

The problem of the household of type $i = 1, 2$ is

$$\max_{x\in X}\sum_{t=0}^{\infty}\beta^t\sum_{h_t}\pi\left(h_t\right)u\left[c_t^i\left(h_t\right),x_{3t}^i\left(h_t\right)\right]$$

subject to

$$\sum_{t=0}^{\infty}\sum_{h_t}\sum_{j=1}^{4}p_{jt}\left(h_t\right)x_{1t}^i\left(h_t\right)\leq 0$$

On the other hand, the problem of the firm, given h_t

$$\max_{y\in Y_t}\sum_{j=1}^{4}p_{jt}\left(h_t\right)y_{jt}\left(h_t\right)$$

Definition 7 (Alternative Arrow-Debreu Equilibrium) *Is an allocation* $\{x^{1*},x^{2*},y^*\}$ *and a price* p^* *such that*

1. *Given* p^*, $x^{1*}=\left(x_{1t}^{1*}\left(h_t\right),x_{2t}^{1*}\left(h_t\right),x_{3t}^{1*}\left(h_t\right),x_{4t}^{1*}\left(h_t\right)=0\right)$ *and* $x^{2*}=\left(x_{1t}^{2*}\left(h_t\right),x_{2t}^{2*}\left(h_t\right),x_{3t}^{2*}\left(h_t\right)=0,x_{4t}^{1*}\left(h_t\right)\right)$ *solve the problem for agent 1 and 2 respectively*

2. *Given* p^*, y^* *solves the problem of the firm*

3. *Markets clear, i.e.*

$$\begin{aligned}\frac{1}{2}x_{jt}^{1*}\left(h_t\right)+\frac{1}{2}x_{jt}^{2*}\left(h_t\right) &= y_{jt}^*\left(h_t\right) && j=1,2\\ x_{3t}^{1*}\left(h_t\right)+x_{3t}^{2*}\left(h_t\right) &= y_{3t}^*\left(h_t\right)\\ x_{4t}^{1*}\left(h_t\right)+x_{4t}^{2*}\left(h_t\right) &= y_{4t}^*\left(h_t\right)\end{aligned}$$

3.2 question 2 (alt.)

The problem of the household of type $i=1,2$ is

$$\max_{\left\{c_t^i(h_t),k_{t+1}^i(h_t),n_t^i(h_t),b_{t+1}^i(h_{t+1})\right\}_{t,h_t}}\sum_t\beta^t\sum_{h_t}u\left[c_t^i\left(h_t\right),n_t^i\left(h_t\right)\right]$$

subject to:

$$c_t^i(h_t) + k_{t+1}^i(h_t) + \sum_{z_t} q(h_t, z_t)\, b_{t+1}^i(h_t, z_t) =$$

$$R_t(h_t)\, k_t^i(h_{t-1}) + (1-\delta)\, k_t^i(h_{t-1}) + w^i(h_t)\, n_t^i(h_t) + b_t^i(h_{t-1}, z_t)$$

The problem of the firm is

$$\max_{K_t(h_t), n_t^1(h_t), n_t^2(h_t)} F\left(K_t(h_t), n_t^1(h_t), n_t^2(h_t)\right) - R_t(h_t)\, K_t(h_t) - \sum_{j=1,2} w^i(h_t)\, n_t^i(h_t)$$

Definition 8 (Alternative SMEE) *is a list* $\left\{q^*, R^*, w^{i*}, c_t^{i*}, k_{t+1}^{i*}, n_t^{i*}\right\}$ *such that*

1. *given* $\{q^*, R^*, w^{i*}\}$, $\left\{c_t^{i*}, k_{t+1}^{i*}, n_t^{i*}\right\}$ *solves the problem of household* $i = 1, 2$
2. $K_t^* \equiv \frac{1}{2} k_t^{1*}(h_{t-1}) + \frac{1}{2} k_t^{2*}(h_{t-1})$, n_t^{i*} *and* n_t^{2*} *solve the problem of the firm; in other words*

$$\begin{aligned} R_t^*(h_t) &= F_1\left(K_t^*, n_t^{1*}, n_t^{2*}\right) \\ w_t^{1*} &= F_2\left(K^*, n_t^{1*}, n_t^{2*}\right) \\ w_t^{2*} &= F_3\left(K_t^*, n_t^{1*}, n_t^{2*}\right) \end{aligned}$$

3. *Market for state-contingent securities clears, i.e.*

$$b_{t+1}^{1*}(h_t, z_t) + b_{t+1}^{2*}(h_t, z_t) = 0 \qquad \forall z_t$$