Econ 702, Spring 2006

Problem Set 12 - Suggested Answers

PROBLEM 1

The probelm here is given by

$$\Phi(V) = \max_{\{c_s,\omega_s\}_{s=1}^{S}} \sum_s \Pi_s[(y_s - c_s) + \beta\Phi(\omega_s)]$$
$$\text{subject to: } \sum_s \Pi_s[u(c_s) + \beta\omega_s] = V$$

$$u(c_s) + \beta\omega_s \geq u(s) + \beta V^A \qquad \forall s.$$

Set up the Langrangian as:

$$L = \max_{\{c_s,\omega_s\}_{s=1}^{S}} \sum_s \Pi_s[(y_s - c_s) + \beta\Phi(\omega_s)] - \lambda\left[V - \sum_s \Pi_s[u(c_s) + \beta\omega_s]\right] +$$
$$+ \sum_s \theta_s \left[u(c_s) + \beta\omega_s - u(s) - \beta V^A\right]$$

The FOCs with respect to c_s and w_s are:

$$\{c_s\}: \ \Pi_s = (\theta_s + \lambda\Pi_s)\ u'(c_s) \qquad (1)$$
$$\{w_s\}: \ -\Pi_s\Phi'(\omega_s) = \lambda\Pi_s + \theta_s \qquad (2)$$

Moreover, the Envelope condition is :

$$\Phi'(V) = -\lambda \qquad (EC)$$

Observe the following facts:

1) Φ is decreasing and concave in V. The monotonicity is obvious. For the concavity property, notice that $\Phi''(V) = -\lambda'(V)$. But $\lambda(V)$ is the cost of promising utility V. Hence, $\lambda'(V)$ is increasing and so $\Phi''(V) < 0$.

2) Dividing (1) by (2) yields:

$$-\Phi'(\omega_s) = \frac{1}{u'(c_s)} \qquad (3)$$

Consider $w_2 > w_1$. Then $\Phi'(\omega_1) > \Phi'(\omega_2) \Rightarrow \frac{1}{u'(c_1)} < \frac{1}{u'(c_2)} \Rightarrow c_2 > c_1$. This means that bigger future promises go together with current consumption at the optimal. Let's call this the $(*)$ fact.

Now notice that we can always find a V such that there exists $s^* \in int\{S\}$ and $\theta_{s^*} > 0$, and $\theta_{s^*-1} = 0$.That is, the IC contraint is binding in state s^* and not binding in $s^* - 1$. Compare states $s^*, s^* + 1$. At s^* we know that $u(c_{s^*}) + \beta\omega_{s^*} = u(s^*) + \beta V^A$. Assume by way of contradiction that $w_{s^*} \geq w_{s^*+1}$. Then

$$u(s^*) + \beta V^A = u(c_{s^*}) + \beta\omega_{s^*} \geq u(c_{s^*+1}) + \beta\omega_{s^*+1} \geq u(s^*+1) + \beta V^A,$$

where the first inequality follows from the $(*)$ fact, and the second represents the IC constraint under state s^*+1. The above inequality says that

$$u(s^*) \geq u(s^*+1),$$

which is a contradiction. Hence, we conclude that $w_{s^*} < w_{s^*+1}$. If the constraint in s^*+1 is also binding, then we can repeat the analysis above. We will thus have shown that for every $s_2 > s_1 \geq s^*$, $w_{s_2} > w_{s_1}$. We show that $\theta_{s^*+1} > 0$ by contradiction: assume that $\theta_{s^*+1} = 0$. Then (1) implies that

$$\lambda = \frac{1}{u'\left(c_{s^*+1}\right)} \qquad (4)$$

Again from (1) but for the state s^*, we have

$$\theta_{s^*} = \pi_s \left[\frac{1}{u'(c_{s^*})} - \lambda\right]$$

and so from (4)

$$\theta_{s^*} = \pi_s \left[\frac{1}{u'(c_{s^*})} - \frac{1}{u'\left(c_{s^*+1}\right)}\right] \qquad (5)$$

We have already seen that $w_{s^*} < w_{s^*+1}$. Then the $(*)$ fact implies $c_{s^*} < c_{s^*+1}$, which in turn implies that $\frac{1}{u'(c_{s^*})} < \frac{1}{u'\left(c_{s^*+1}\right)}$. Hence, from (5) $\theta_{s^*} < 0$, a contradiction. We conclude that $\theta_{s^*+1} > 0$. Then we can repeat the first step above to show that $w_{s^*+1} < w_{s^*+2}$. After that we can repeat the second step above and show that $\theta_{s^*+2} > 0$. etc, etc. We have thus shown that for every $s_2 > s_1 \geq s^*$, $w_{s_2} > w_{s_1}$.

It remains to show that for every $s < s^*$, w_s is constant. To that end, notice that for every $s < s^*$, the constraint is not binding. To see why this is true, suppose that for some $s \in \{1, 2, ...s^* - 2\}$, $\theta_s > 0$. Then by the proof above, for all states above the one under consideration the ICC will also be binding, a contradiction (since we fixed V so that $\theta_{s^*-1} = 0$). Given this result, consider any $s < s^*$. Equation (1) implies $-\Phi'(\omega_s) = \lambda$, and using the (EC) it follows that

$$\Phi'(\omega_s) = \Phi'(V)$$

Since Φ' is monotone, we conclude that for every $s < s^*$, $w_s = V$ and hence constant.This concludes the proof.

PROBLEM 2

To have a better understanding of concavity let's re-write the problem so that the distribution of the state has a continuous support, S. The problem is now

$$\Phi(V) = \max_{\{c(s),\omega_{(s)}\}} \int_S [(s - c(s)) + \beta\Phi(\omega(s))] f(s)\, ds$$
$$\text{subject to: } \int_S [u(c(s)) + \beta\omega(s)] f(s)\, ds = V$$

$$u(c(s)) + \beta\omega(s) \geq u(s) + \beta V^A \quad \forall s.$$

The Langrangian is

$$L = \max_{\{c(s),\omega_{(s)}\}} \int_S [(s - c(s)) + \beta\Phi(\omega(s))] f(s)\, ds -$$
$$-\lambda \left[\int_S [u(c(s)) + \beta\omega(s)] f(s)\, ds = V\right] +$$
$$+ \int_S \theta(s) \left[u(c(s)) + \beta\omega(s) - u(s) - \beta V^A\right] f(s)\, ds.$$

The FOCs are now given by

$$\{c(s)\}: \; f(s) = (\theta(s) + \lambda f(s))\; u'(c(s)) \qquad (1)$$
$$\{w(s)\}: \; -f(s)\, \Phi'(\omega(s)) = \lambda f(s) + \theta(s) \qquad (2)$$

To make things even more simple let's assume that the shock is uniformly distributed on $[0, 1]$. Then $f(s) = 1$ for $s \in [0, 1]$, and the FOCs simplify to

$$\{c(s)\}: \; 1 = (\theta(s) + \lambda)\; u'(c(s)) \qquad (1')$$
$$\{w(s)\}: \; -\Phi'(\omega(s)) = \lambda + \theta(s) \qquad (2')$$

After a little algebra one can show that the second derivative of the opimal choise of $c(s)$ is given by:

$$c''(s) = u''(c(s)) - c'(s) \left[u''(s) - \frac{u'''\Phi'' + u''\Phi'''}{(u''\Phi'')^2}\right]$$

Therefore, even for this simple example concavity is not guaranteed, unless we impose some restrictions on the third derivative of u.

PROBLEM 3

Clearly $\overline{c}$ is related to the highest shock realization, i.e. $\overline{c} = c_{\overline{s}}$ and $\bar{w} = w_{\overline{s}}$. We know that at state $\overline{s}$, the ICC will be binding. Therefore,

$$u(\overline{c}) + \beta\bar{w} \ \geq u(\overline{s}) + \beta V^A \qquad (1)$$

Also recall that once the agent gets the highest shock current consumption and future promise never change again (the agent is already getting the highest possible values). In other words once the agent gets $\overline{s}$, the planner promises $\bar{w}$ for every future period, and gives consumption $\overline{c}$ in every future period. Under these facts, the promise keeping constraint can be "translated" as:

$$V = \sum_s \Pi_s[u(c_s) + \beta\omega_s] \Rightarrow \ \bar{w} \ = \sum_s \Pi_s[u(\overline{c}) + \beta\bar{w}] \Rightarrow \bar{w}\,(1-\beta) \ = u(\overline{c})$$

$$\Rightarrow \ \bar{w} = \tfrac{u(\overline{c})}{1-\beta}. \qquad (2)$$

Equations $(1)\,,(2)$ can be used to solve for the two unknowns $\overline{c}, \bar{w}$.