Macro 704 Lectures 4+

Preliminary

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The Lucas Tree

- The Purpose: To Price Assets so they do the right thing
- The Environment:
  - Goods: A measure one of trees that give fruit, z, that follows a Markov Process with transition matrix $\Gamma_{zz'}$.
  - Preferences: $E \sum_t \beta^t u(c_t)$.
  - Markets: Hholds buy shares s' of trees in stock markets at price $p(z)$, and consume fruit. They receive dividends $d(z)$ and have shares.
- State Variables
  - Aggregate z
  - Individual s

$$V(z,s) = \max_{c,s'} \quad u(c) + \beta \sum_{z'} \Gamma_{zz'} V(z', s')$$

$$s.t. \quad c + p(z)\, s' = s\,[p(z) + d(z)]\,,$$

Definition

A Rational Expectations Recursive Competitive Equilibrium is a set of functions, V, g, d, and p, such that

1. V and g solves the household's problem given prices,
2. $d(z) = z$, and,
3. $g(z, 1) = 1$, for all z.

$$u_c\left(c\left(z,1\right)\right) = \beta \sum_{z'} \Gamma_{zz'} \left[\frac{p\left(z'\right)+d\left(z'\right)}{p\left(z\right)}\right] \ u_c\left(c\left(z',1\right)\right).$$

where we have $u_c\left(z\right) := u_c\left(c\left(z,1\right)\right)$. Then this simplifies to

$$p\left(z\right) u_c\left(z\right) = \beta \sum_{z'} \Gamma_{zz'} u_c\left(z'\right)\left[p\left(z'\right)+z'\right] \qquad \forall z.$$

A system of n_z equations. Denote $\mathbf{p} := \left[p\left(z_1\right) \vdots p\left(z_n\right)\right]_{(n_z \times 1)}$ and

$$\mathbf{u}_c := \begin{bmatrix} u_c\left(z_1\right) & & 0 \\ & \ddots & \\ 0 & & u_c\left(z_n\right) \end{bmatrix}_{(n_z \times n_z)} .$$

Then

$$\mathbf{u}_c.\mathbf{p} = \begin{bmatrix} p(z_1)\, u_c(z_1) \\ \vdots \\ p(z_n)\, u_c(z_n) \end{bmatrix}_{(n_z \times 1)},$$

Now, rewrite the system above as

$$\mathbf{u}_c\mathbf{p} = \beta\Gamma\mathbf{u}_c\mathbf{z} + \beta\Gamma\mathbf{u}_c\mathbf{p},$$

where Γ is the transition matrix for z, as before. Hence, the price for the shares is given by

$$(\mathbf{I}_{\mathbf{n}_\mathbf{z}} - \beta\Gamma)\, \mathbf{u}_c\mathbf{p} = \beta\Gamma\mathbf{u}_c\mathbf{z},$$

or

$$\mathbf{p} = \left([\mathbf{I}_{\mathbf{n}_\mathbf{z}} - \beta\Gamma]\, \mathbf{u}_c\right)^{-1} \beta\Gamma\mathbf{u}_c\mathbf{z},$$

where **p** is the vector of prices that clears the market.

An asset is "a claim to a chunk of fruit, sometime in the future."

An asset that promises $m_t(z^t)$ after history $z^t = (z_0, z_1, \ldots, z_t) \in H^t$. The price of such an asset is the price of what it entitles its owner to.

This follows from a no-arbitrage argument.

$$p^m(z_0) = \sum_t \sum_{z^t \in H^t} q_t^0(z^t) \; a_t(z^t),$$

$q_t^0(z^t)$ is the price of one unit of fruit after z^t in time zero's goods.

Given the $\{q_t^0(z^t)\}$, we can *replicate any possible asset by a set of state-contingent claims* and use this formula to price that asset.

To find those q^0 consider a world where agents solve

$$\max_{c_t(z^t)} \quad \sum_{t=0}^{\infty} \beta^t \sum_{z^t} \pi_t\left(z^t\right) u\left(c_t\left(z^t\right)\right)$$

$$s.t. \quad \sum_{t=0}^{\infty} \sum_{z^t} q_t^0\left(z^t\right) c_t\left(z^t\right) \leq \sum_{t=0}^{\infty} \sum_{h^t} q_t^0\left(z^t\right) z_t.$$

The $\pi(z^t)$ are the prob and can be constructed recursively with Γ.

(note that this is the familiar Arrow-Debreu market structure, where the household owns a tree, and the tree yields $z \in Z$ amount of fruit in each period). The FOC for this problem imply:

$$q_t^0\left(z^t\right) = \beta^t \pi_t\left(z^t\right) \frac{u_c\left(z_t\right)}{u_c\left(z_0\right)}.$$

This enables us to price the good in each history of the world and price any asset accordingly.

Add state-contingent shares b to the Lucas tree

$$
\begin{aligned}
V(z,s,b) = \max_{c,s',b'(z')} \quad & u(c) + \beta \sum_{z'} \Gamma_{zz'} V(z', s', b'(z')) \\
s.t. \quad & c + p(z) s' + \sum_{z'} q(z,z') b'(z') = s\left[p(z) + z\right] + b.
\end{aligned}
$$

A characterization of q can be obtained by the FOC, evaluated at the equilibrium, and thus written as:

$$
q(z,z') u_c(z) = \beta \Gamma_{zz'} u_c(z').
$$

We can thus price *all types* of securities using p and q in this economy.

To sell the tree tomorrow at price P

$$\widehat{q}\left(z,P\right)=\sum_{z'}q\left(z,z'\right)\max\left\{P-p\left(z'\right),0\right\},$$

The (American) option to sell either tomorrow or the day after

$$\tilde{q}\left(z,P\right)=\sum_{z'}q\left(z,z'\right)\ \max\left\{P-p\left(z'\right),\widehat{q}\left(z',P\right)\right\}.$$

The European option to buy the day after tomorrow is

$$\bar{q}\left(z,P\right)=\sum_{z'}\sum_{z''}\max\left\{p\left(z''\right)-P,0\right\}q\left(z',z''\right)q\left(z,z'\right).$$

Rates of Return

If today's shock is z, the gross risk free rate

$$R(z) = \left[\sum_{z'} q(z, z') \right]^{-1}$$

The unconditional gross risk free rate is

$$R^f = \sum_{z} \mu_z^* R(z)$$

where μ^* is the steady-state distribution of the shocks.

Stock Market and Risk Premium

The average gross rate of return on the stock market is

$$\sum_{z} \mu_z^* \sum_{z'} \Gamma_{zz'} \left[\frac{p(z') + z'}{p(z)} \right]$$

The Risk Premia is

$$\sum_{z} \mu_z^* \left(\sum_{z'} \Gamma_{zz'} \left[\frac{p(z') + z'}{p(z)} \right] - R(z) \right).$$

Use the expressions for p and q and the properties of the utility function to show that risk premium is positive.

Taste Shocks

- The fruit is constant over time (normalized to 1)
- The agent is subject to preference shocks for the fruit each period given by $\theta \in \Theta$ with transition Γ^{θ}.

$$\begin{aligned} V(\theta, s) = \max_{c, s'} & \quad \theta u(c) + \beta \sum_{\theta'} \Gamma_{\theta\theta'} V(\theta', s') \\ s.t. & \quad c + p(\theta) s' = s \left[p(\theta) + d(\theta) \right]. \end{aligned}$$

The equilibrium is defined as before.
In Eq $d(\theta) = 1$
Discussion of Demand vs Supply Shocks and what RBC vs Lucas trees are.

Endogenous Productivity in a Product Search Model

A Twist on the Lucas Tree Model

- So far
  - Hholds own the tree
  - Purchase Shares
  - To access the fruit they JUST have to Purchase it.
- Now They also have to FIND the fruit

A Slightly Different Environment

- There is matching function $M(T, D)$: Trees and Search Effort.
  - Constant Returns to Scale, e.g. $D^{\varphi} T^{1-\varphi}$. Let $\frac{1}{Q} := \frac{D}{T}$, i.e. the ratio of shoppers per trees, *the market tightness* .

- The probability that a unit of shopping effort finds a tree is

$$= \Psi^{h}(Q) := \frac{M(T, D)}{D} = Q^{1-\varphi}$$

- The probability that a tree finds a shopper is

$$\Psi^{f}(Q) := \frac{M(T, D)}{T} = Q^{-\varphi}$$

- Here $T = 1$

Shocks

- A hunger (demand) shock θ with transition matrix $\Gamma_{\theta\theta'}$
- A Productvity (TFP, supply) shock z with transition matrix $\Gamma_{zz'}$
- We look for a Lucas tree *type* Equilibrium
- State Variables
  - Aggregate θ, z
  - Individual s

Hhould solves

$$V(\theta, z, s) = \max_{c,d,s'} \quad u(c, d, \theta) + \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} \; V(\theta', z', s')$$

$$s.t. \quad c + P(\theta, z) s' = P(\theta, z) \left[s \left(1 + \widehat{R}(\theta, z) \right) \right]$$

$$c = d \; \Psi^{h}(Q(\theta, z)) z$$

- $P(\theta, z)$ is the price of the tree relative to that of consumption
- $\widehat{R}(\theta, z)$ is the dividend income (in units of the tree).

Strategy to characterize equilibrium

- Substitute the constraints into the objective, solve for d and get the Euler equation for the household.
- Using THEN the market clearing condition in equilibrium, the problem is reduced to one equation and two unknowns, $P(\theta, z)$ and $Q(\theta, z)$
- Still need another functional equationi.e. we need to specify the search protocol.
- HWK: Derive the Euler equation of the household from the problem defined above.

Competitive Search

- It is a particular search protocol of what is called non-random (or directed) search.
- Ex-ante Commitment to the terms of trade (other search protocols it is not the case)
- Consider a world consisting of a large number of islands. Each island has a sign that displays two numbers, $P(\theta, z)$ and $Q(\theta, z)$. (price and market tightness) in
- Searchers and choose which island to go to. They have different trade-offs of price versus tightness
- Equilibrium determines which island (Optimal so unique)

$$V(\theta, z, s) = \max_{c,d,s',P,Q} \quad u(\theta\ c, d) + \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} V(\theta', z', s') \qquad (1)$$

$$s.t. \quad c + Ps' = P\left[s\left(1 + \widehat{R}(\theta, z)\right)\right], \qquad (2)$$

$$c = d\ \Psi^h(Q)\, z \qquad (3)$$

$$\frac{z\Psi^f(Q)}{P} \geq \widehat{R}(\theta, z) \qquad (4)$$

- The last constraint states that for a market to exist firms have to be guaranteed $\widehat{R}(\theta, z)$.

FOC: How much to search given the Island d

Plug the first two constraints into the objective function (c and s' as functions of d) and (recall that $\Psi^h = Q^{1-\varphi}$) :

$$\theta Q^{1-\varphi} z u_c(\theta d Q^{1-\varphi} z, d) + u_d(\theta d Q^{1-\varphi} z, d) = \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} V_3 \left(\theta', z', s(1+\widehat{R}(\theta, z)) - \frac{d Q^{1-\varphi} z}{P} \right) \frac{Q^{1-\varphi} z}{P} \quad (5)$$

Get rid of V_3 using original problem and use the envelope theorem

$$V_3(\theta, z, s) = \left[\theta u_c(\theta d Q^{1-\varphi} z, d) + \frac{u_d(\theta d Q^{1-\varphi} z, d)}{Q^{1-\varphi} z} \right] P(1+\widehat{R}(\theta, z))$$

Combining these two gives the Euler equation:

$$\theta u_c(\theta d Q^{1-\varphi} z, d) + \frac{u_d(\theta d Q^{1-\varphi} z, d)}{Q^{1-\varphi} z} = \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} \frac{P'(1+\widehat{R}(\theta', z'))}{P} \left[\theta' u_c(\theta' d' Q'^{1-\varphi} z', d') + \frac{u_d(\theta' d' Q'^{1-\varphi} z', d')}{Q'^{1-\varphi} z'} \right] \quad (6)$$

FOC with respect to Q and P.

λ: Lagrange multiplier on the firm's participation constraint, then

$$\theta d(1-\varphi)Q^{-\varphi} z u_c(\theta d Q^{1-\varphi} z, d) = \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} V_3 \left(\theta', z', s(1+\widehat{R}(\theta, z)) - \frac{dQ^{1-\varphi} z}{P}\right) \frac{d(1-\varphi)Q^{-\varphi} z}{P} - \lambda \frac{\varphi Q^{-\varphi-1} z}{P} \quad (7)$$

and

$$\beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} V_3 \left(\theta', z', s(1+\widehat{R}(\theta, z)) - \frac{dQ^{1-\varphi} z}{P}\right) dQ = -\lambda \quad (8)$$

Combining these two equation gives us:

$$\theta u_c(\theta d Q^{1-\varphi} z, d) = \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} \\ V_3\left(\theta', z', s(1+\widehat{R}(\theta, z)) - \frac{dQ^{1-\varphi} z}{P}\right) \left[\frac{1}{(1-\varphi)P}\right] \quad (9)$$

Recall $V_3(\cdot,\cdot,\cdot)$ so

$$(1-\varphi)\theta u_c(\theta d Q^{1-\varphi} z, d) = \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} \\ \frac{P'(1+\widehat{R}(\theta', z'))}{P} \left[\theta' u_c(\theta' d' {Q'}^{1-\varphi} z', d') + \frac{u_d(\theta' d' {Q'}^{1-\varphi} z', d')}{{Q'}^{1-\varphi} z'}\right] \quad (10)$$

Equilibrium

Definition

An Eq with competitive search is functions $\{V, c, d, s', P, Q, \widehat{R}\}$ that:

1. Household's budget constraint, (condition 2)
2. Household's shopping constraint, (condition 3)
3. Household's Euler equation, (condition 6)
4. Market condition, (condition 10)
5. Firm's participation constraint, (condition 4), which gives us that the dividend payment is the profit of the firm, $\widehat{R}(\theta, z) = \frac{zQ^{-\varphi}}{P}$,
6. Market clearing, i.e. $s' = 1$ and $Q = 1/d$.

Firms' Problem

Firms maximize returns by choosing market, Q, P. Numeraire is price of trees: $\widehat{P}(Q) = 1/P$ is price of consumption. We define implicitly the set of available markets for firms as $\widehat{P}(Q)$

$$\pi = \max_{Q} \widehat{P}(Q)\Psi^{f}(Q)z \qquad \text{s.t.}$$

$$\widehat{P}'(Q)\Psi^{f}(Q) + \widehat{P}(Q)\Psi^{f'}(Q) = 0,$$

which then determines $\widehat{P}(Q)$ as

$$\frac{\widehat{P}'(Q)}{\widehat{P}(Q)} = -\frac{\Psi^{f'}(Q)}{\Psi^{f}(Q)}.$$

Measure Theory

Measure theory is a tool that helps us aggregate.

Definition

For a set S, $\mathcal{S}$ is a family of subsets of S, if $B \in \mathcal{S}$ implies $B \subseteq S$ (but not the other way around).

Remark

Note that in this section we will assume the following convention

1. *small letters (e.g. s) are for elements,*
2. *capital letters (e.g. S) are for sets, and*
3. *fancy letters (e.g. $\mathcal{S}$) are for a set of subsets (or families of subsets).*

Definition

A family of subsets of S, $\mathcal{S}$, is called a σ-algebra in S if

1. $S, \emptyset \in \mathcal{S}$;
2. if $A \in \mathcal{S} \Rightarrow A^c \in \mathcal{S}$ (i.e. $\mathcal{S}$ is closed with respect to complements and $A^c = S \backslash A$); and,
3. for $\{B_i\}_{i \in \mathbb{N}}$, if $B_i \in \mathcal{S}$ for all $i \Rightarrow \bigcap_{i \in \mathbb{N}} B_i \in \mathcal{S}$ (i.e. $\mathcal{S}$ is closed with respect to countable intersections.

Example

1. The power set of S and $\{\emptyset, S\}$ are σ-algebras in S.
2. $\left\{\emptyset, S, S_{1/2}, S_{2/2}\right\}$, where $S_{1/2}$ means the lower half of S (imagine S as an closed interval in $\mathbb{R}$), is a σ-algebra in S.
3. If $S = [0, 1]$, then $\mathcal{S} = \left\{\emptyset, \left[0, \frac{1}{2}\right), \left\{\frac{1}{2}\right\}, \left[\frac{1}{2}, 1\right], S\right\}$ is *not* a σ-algebra in S. But $\mathcal{S} = \left\{\emptyset, \left\{\frac{1}{2}\right\}, \left\{\left[0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right]\right\}, S\right\}$ is.

It allows us to define sets where things happen and we can *weigh* those sets (avoiding math troubles)

Definition

Suppose $\mathcal{S}$ is a σ-algebra in S. A measure is a real-valued function $x : \mathcal{S} \to \mathbb{R}_+$, that satisfies

1. $x(\emptyset) = 0$;
2. if $B_1, B_2 \in \mathcal{S}$ and $B_1 \cap B_2 = \emptyset \Rightarrow x(B_1 \cup B_2) = x(B_1) + x(B_2)$ (additivity); and,
3. if $\{B_i\}_{i \in \mathbb{N}} \in \mathcal{S}$ and $B_i \cap B_j = \emptyset$ for all $i \neq j \Rightarrow x(\cup_i B_i) = \sum_i x(B_i)$ (countable additivity).

A set S, a σ-algebra in it ($\mathcal{S}$), and a measure on $\mathcal{S}$ x, define a measurable space, $(S, \mathcal{S}, x)$.

Definition

A Borel σ-algebra is a σ-algebra generated by the family of all open sets $\mathfrak{B}$ (generated by a topology). A Borel set is any set in $\mathfrak{B}$.

A Borel σ-algebra corresponds to complete information.

Definition

A probability measure is measure where $x(S) = 1$. $(S, \mathcal{S}, x)$ is a probab space. The probab of an event is then given by $x(A)$, where $A \in \mathcal{S}$.

Definition

Given a m'able space $(S, \mathcal{S}, x)$, a real-valued function $f : S \to \mathbb{R}$ is m'able (with respect to the m'able space) if, for all $a \in \mathbb{R}$, we have

$$\{b \in S \mid f(b) \leq a\} \in \mathcal{S}.$$

Interpret σ-algebras as describing available information.

Similarly, a function is m'able wrt a σ-algebra $\mathcal{S}$, if it can be evaluated

Example

Suppose $S = \{1, 2, 3, 4, 5, 6\}$. Consider a function f that maps the element 6 to the number 1 (i.e. $f(6) = 1$) and any other elements to -100. Then f is NOT measurable with respect to $\mathcal{S} = \{\emptyset, \{1, 2, 3\}, \{4, 5, 6\}, S\}$. Why? Consider $a = 0$, then $\{b \in S \mid f(b) \leq a\} = \{1, 2, 3, 4, 5\}$. But this set is not in $\mathcal{S}$.

Extend the notion of Markov stuff to any measurable space

Definition

Given a measurable space $(S, \mathcal{S}, x)$, a function $Q : S \times \mathcal{S} \to [0, 1]$ is a transition probability if

1. $Q(s, \cdot)$ is a probability measure for all $s \in S$; and,
2. $Q(\cdot, B)$ is a measurable function for all $B \in \mathcal{S}$.

Intuitively, for $B \in \mathcal{S}$ and $s \in S$, $Q(s, B)$ gives the probability of being in set B tomorrow, given that the state is s today.

1. A *Markov chain* with transition matrix given by

$$\Gamma = \begin{bmatrix} 0.2 & 0.2 & 0.6 \\ 0.1 & 0.1 & 0.8 \\ 0.3 & 0.5 & 0.2 \end{bmatrix},$$

   on $S = \{1, 2, 3\}$, with the the power set being the σ-algebra of S).

$$Q\left(3, \{1, 2\}\right) = \Gamma_{31} + \Gamma_{32} = 0.3 + 0.5 \ .$$

2. Consider a measure x on $\mathcal{S}$. x_i is the fraction of type i. Then

$$\begin{aligned} x_1' &= x_1\Gamma_{11} + x_2\Gamma_{21} + x_3\Gamma_{31}, \\ x_2' &= x_1\Gamma_{12} + x_2\Gamma_{22} + x_3\Gamma_{32}, \\ x_3' &= x_1\Gamma_{13} + x_2\Gamma_{23} + x_3\Gamma_{33}. \end{aligned}$$

   In other words: $\mathbf{x}' = \Gamma^T \mathbf{x}$, where $\mathbf{x}^T = (x_1, x_2, x_3)$.

From a measure x today to one tomorrow x'

$$\begin{aligned} x'(B) &= T(x, Q)(B) \\ &= \int_S Q(s, B)\, x(ds), \quad \forall B \in \mathcal{S}, \end{aligned}$$

we integrated over all $s \in S$ to get the prob of B tomorrow.

A stationary distribution is a fixed point of T, that is x^* such that

$$x^*(B) = T(x^*, Q)(B), \quad \forall B \in \mathcal{S}.$$

Theorem

If Q has nice properties (American Dream and Nightmare) then $\exists$ a unique stationary distribution x^* and

$$x^* = \lim_{n\to\infty} T^n(x_0, Q), \qquad \text{for any } x_0.$$

Exercise

Consider unemployment in a very simple economy (in which the transition matrix is exogenous). There are two states of the world: being employed and being unemployed. The transition matrix is given by

$$\Gamma = \begin{pmatrix} 0.95 & 0.05 \\ 0.50 & 0.50 \end{pmatrix}.$$

Compute the stationary distribution corresponding to this Markov transition matrix.

Industry Equilibrium

- Study the dynamics of the distribution of firms in partial equilibrium
- A single firm produces a good using labor:
- Output is $sf(n)$ (f increasing, strictly concave, $f(0) = 0$, s is productivity.
- Markets are competitive, (p and $w = 1$) as given.
- A firm solves

$$\pi(s, p) = \max_{n \geq 0} \{psf(n) - wn\}. \quad (11)$$

- With FOC

$$psf_n(n^*) = 1. \quad (12)$$

  Solution is $n^*(s, p)$.
- n^* is an increasing function of both arguments. Prove it.

A Static Predetermined Industry

- A mass of firms in the industry, indexed by $s \in S \subset \mathbb{R}_+$, $S := [\underline{s}, \bar{s}]$.
- $\mathcal{S}$ is a σ-algebra on S (a Borel σ-algebra, for instance).
- x is a measure on $(S, \mathcal{S})$, which describes the cross-sectional distribution of productivity among firms.
- Use x to define statistics of the industry: Since individual supply is $sf\left(n^*\left(s,p\right)\right)$, then the aggregate supply

$$Y^S\left(p\right) = \int_S sf\left(n^*\left(s,p\right)\right) x\left(ds\right). \tag{13}$$

Y^S is a function of the price p only.

- Let Demand $Y^D\left(p\right)$. Then p^* clears the market:

$$Y^D\left(p^*\right) = Y^S\left(p^*\right). \tag{14}$$

Where does x come from?

- Price p and output Y are constant over time.
- Firms face the problem above every period and discount profits at exogenous r.
- Each firm faces a probability $1-\delta$ of disappearing in each period.
- The choice is static. The value of an s firm is

$$V(s;p) = \sum_{t=0}^{\infty} \left(\frac{\delta}{1+r}\right)^t \pi(s,p) = \quad \left(\frac{1+r}{1+r-\delta}\right) \pi(s,p)$$

- Every period a mass of firms die. To achieve a stationary equilibrium we need firms entry: assume that there is a constant flow of firms entering the economy in each as well, so that entry equals exit.
- x is the measure of firms. Firms that die are $(1-\delta)x(S)$.
- Entrants draw s from probability measure γ over $(S, \mathcal{S})$.

- What keeps other firms out of the market in the first place?
- (if $\pi(s;p) = psf(n^*(s;p)) - wn^*(s;p) > 0$, then any firm with $s \in S$ would enter.
- Assume a fixed entry cost, c^E before learning s. Value of an entrant

$$V^E(p) = \int_S V(s;p)\gamma(ds) - c^E. \tag{15}$$

If $V^E > 0$ there will be entry.

- Equilibrium requires $V^E = 0$

The distribution of firms in the market

- x_t : cross-sectional distribution of firms. For any $B \subset S$, fraction $1-\delta$ of firms with $s \in B$ die and mass m of newcomers enter. Next period's measure of firms on set B is

$$x_{t+1}(B) = \delta x_t(B) + m\gamma(B). \tag{16}$$

- Mass m of firms would enter $t+1$, with fraction $\gamma(B)$ having $s \in B$, $\forall B \in \mathcal{S}$.
- Cross-sectional distribution of firms completely determined by γ.
- Consider an updating operator T

$$Tx(B) = \delta x(B) + m\gamma(B), \quad \forall B \in \mathcal{S}, \tag{17}$$

a stationary dbon is a fixed point, i.e. x^* such that $Tx^* = x^*$, so

$$x^*(B; m) = \frac{m}{1-\delta}\gamma(B), \quad \forall B \in \mathcal{S}. \tag{18}$$

- Demand and supply condition in equation (14) is

$$Y^D\left(p^*\left(m\right)\right) = \int_S s\ f\left[n^*\left(s;p\right)\right]\ dx^*\left(s;m\right), \tag{19}$$

whose solution $p^*\left(m\right)$ is a continuous function

- We have two equations, (15) and (19), and two unknowns, p and m.

Definition

A stationary distribution for this environment consists of functions V, π^*, n^*, p^*, x^*, and m^*, that satisfy:

1. Given prices, V, π^*, and n^* solve the incumbent firm's problem;
2. $Y^D\left(p^*\left(m\right)\right) = \int_S s\ f\left[n^*\left(s;p\right)\right]\ dx^*\left(s;m\right)$;
3. $\int_s V\left(s;p\right)\gamma\left(ds\right) - c^E = 0$; and,
4. $x^*\left(B\right) = \delta x^*\left(B\right) + m^*\gamma\left(B\right), \quad \forall B \in \mathcal{S}$.

More Economics: Introducing Exit Decisions

- Assume s follows a Markov process with transition Γ. This would change the mapping T in Equation (17) to

$$Tx(B) = \delta \int_S \Gamma(s, B)\, x(ds) + m\gamma(B), \quad \forall B \in \mathcal{S}. \tag{20}$$

But no firm exits (c^E is a sunk cost). Still not much Econ.

- Suppose now an operating cost c^v each period.
    - when s is low, firm's profits maybe negative and firm exits
    - But it is not enough. Assume Γ satisfies stochastic dominance: $s^1 > s^2$ implies $\sum_{s'=1}^{\widehat{s}} \Gamma_{s^1,s'} < \sum_{s'=1}^{\widehat{s}} \Gamma_{s^2,s'}$.
    - Then $\exists$ a threshold, $s^* \in S$, below which firms exit and above stay.

$$V(s;p) = \max \left\{ 0, \pi(s;p) + \frac{1}{(1+r)} \int_S V(s';p)\, \Gamma(s, ds') - c^v \right\}. \tag{21}$$

Stationary Equilibrium with Exit

- Updating operator becomes

$$x'(B) = \int_{s^*}^{\bar{s}} \Gamma(s, B \cap [s^*, \bar{s}])\, x(ds) + m\gamma(B \cap [s^*, \bar{s}]), \quad \forall B \in \mathcal{S}. \tag{22}$$

  A stationary distribution of the firms in this economy, x^*, is the fixed point of this equation.

- With x^* we get all class of statistics:
  - Threshold for being in top 10% by size? Solve for $\widehat{s}$

$$\frac{\int_{\hat{s}}^{\bar{s}} x^*(ds)}{\int_{s^*}^{\bar{s}} x^*(ds)} = 0.1,$$

  - Fraction of workers in largest top 10% of firms

$$\frac{\int_{\hat{s}}^{\bar{s}} n^*(s,p)\, x^*(ds)}{\int_{s^*}^{\bar{s}} n^*(s,p)\, x^* \leq ft\,(ds)}.$$

Do

Exercise

Compute the average growth rate of the smallest one third of the firms.

Exercise

What would be the fraction of firms in the top 10% largest firms in the economy that remain in the top 10% in next period?

Exercise

What is the fraction of firms younger than five uears?

Stationary Equilibrium

Definition

π^*, n^*, d^*, s^*, V, a price p^*, a measure x^*, and mass m^* such that

1. Given p^*, the functions V, π^*, n^*, d^* solve the firm's
2. The reservation productivity s^* satisfies $d^*(s; p^*) = \begin{cases} 1 & \text{if } s \geq s^* \\ 0 & \text{otherwise} \end{cases}$.
3. Free-entry condition: $V^E(p^*) = 0.$
4. For any $B \in \mathcal{S}$

$$x^*(B) = m^* \gamma (B \cap [s^*, \bar{s}]) + \int_{s^*}^{\bar{s}} \Gamma(s, B \cap [s^*, \bar{s}]) \, x^*(ds)$$

5. Market clearing:

$$Y^d(p^\star) = \int_{s^\star}^{\bar{s}} s \; f(n^\star(s; p^\star)) x^\star(ds)$$

- Average output of the firm is given by

$$\frac{Y}{N} = \frac{\int_{s^\star}^{\bar{s}} sf(n^*(s))x^*(ds)}{\int_{s^\star}^{\bar{s}} x^*(ds)}$$

- Share of output produced by the top 1% of firms. Need to find $\tilde{s}$

$$\frac{\int_{\tilde{s}}^{\bar{s}} x^*(ds)}{\int_S x^*(ds)} = .01$$

$$\frac{\int_{\tilde{s}}^{\bar{s}} sf(n^*(s))x^*(ds)}{\int_{s^\star}^{\bar{s}} sf(n^*(s))x^*(ds)}$$

- Fraction of firms in the top 1% two periods in a row (s continuous)

$$\int_{s \geq \tilde{s}} \int_{s' \geq \tilde{s}} \Gamma_{ss'} x^*(ds)$$

- Gini coefficient.

Consider adjustment costs to labor $c\left(n^{-}, n\right)$ due to hiring n units of labor in t as

- *Convex Adjustment Costs*: if the firm wants to vary the units of labor, it has to pay $\alpha\left(n_t - n_{t-1}\right)^2$ units of the numeraire good. The cost here depends on the size of the adjustment.
- *Training Costs or Hiring Costs*: if the firm wants to increase labor, it has to pay $\alpha\left[n_t - (1-\delta)\, n_{t-1}\right]^2$ units of the numeraire good only if $n_t > n_{t-1}$. We can write this as

$$\mathbf{1}_{\{n_t > n_{t-\mathbf{1}}\}} \alpha\left[n_t - (1-\delta)\, n_{t-1}\right]^2,$$

  where $\mathbf{1}$ is the indicator function and δ measures the exogenous attrition of workers in each period.
- *Firing Costs*: the firm has to pay if it wants to reduce the number of workers.

Recursive formulation of the problem

$$V\left(s, n^{-}; p\right) = \max\left\{0, \max_{n\geq 0}\ sf\left(n\right) - wn - c^{v} - c\left(n^{-}, n\right) + \frac{1}{(1+r)}\int_{s'\in S} V\left(s', n, p\right)\Gamma(s, ds')\right\},$$

$c(\cdot,\cdot)$ isi cost function (note limited liability: exit value is 0)

Note $n = g(s, n^{-}; p) < \bar{N}$. Let $\mathcal{N}$ be a σ-algebra on $[0, \bar{N}]$.

$$x'\left(B^{S}, B^{N}\right) = m\gamma\left(B^{S}\cap[s^{*},\bar{s}]\right)\mathbf{1}_{\{0\in B^{N}\}} + \int_{s^{*}}^{\bar{s}}\int_{0}^{\bar{N}}\mathbf{1}_{\{g(s,n_{-};p)\in B^{N}\}}\ \Gamma\left(s, B^{S}\cap[s^{*},\bar{s}]\right)x\left(ds, dn_{-}\right),$$

$$\forall\ B^{S}\in\mathcal{S}, \forall\ B^{N}\in\mathcal{N}.$$

- Write the first order conditions.
- Define the recursive competitive equilibrium for this economy.
- Another example of labor adjustment costs is when the firm has to post vacancies to attract labor. As an example of such case, suppose the firm faces a firing cost according to function c. The firm also pays a cost κ to post vacancies and after posting vacancies, it takes one period for the workers to be hired. How can we write the problem of firms in this environment?
- Add Adjustment Costs to Capital
- Add R& D

Non-stationary Equilibrium

- So far *stationary industry equilibria* (invariant distribution of firms).
- If p were constant, the firm distribution would converge to the stationary equilibrium distribution x^*.
- What is an alternative?
- Prices are changing over time and so is the distribution of firms.
- There are two ways of modeling non-stationary equilibria
    - In Sequence Space (or stochastic process state)
    - Recursively
- What is best depends on the purpose. They should give the same answer. It is an issue of computation.
- We will look at both ways (for now deterministic).
- Given the convergence that we talked about we need a rationale for the non stationarity.
- Consider demand shifters z_t so that $D(P, z_t)$ where $z_{t+1} = \phi(z_t)$ so we can choose to represent it as a sequence or recursively.

Sequentially: Perfect foresight equilibrium

- Note the need for an initial condition. Then objects are relatively simple.
- Given a path $\{z_t\}_{t=0}^{\infty}$ and an initial x_0, an equilibrium defined in term of sequences is: Sequences $\{p_t m_t, s_t^*\}$ of numbers, a sequence of meausres x_t, and sequences $\{V_t(s), n_t(s)\}_{t=0}^{\infty}$ of functions:

1. **Optimality**: Given $\{p_t\}$, $\{V_t, s_t^*, n_t\}$ sole

$$V_t(s) = \max \left\{ 0, \max p_t s f(n) - wn - c^v + \frac{\int_S V_{t+1}(s') \Gamma(s, ds')}{1+r} \right\}$$

2. **Free-entry**: $\int V_t(s) \gamma(ds) \leq c^e$, with strict equality if $m_t > 0$.
3. **Law of motion**:
   $x_{t+1}(B) = m_{t+1} \gamma(B \cap [s_{t+1}^*, \bar{s}]) + \int_{s_t^*}^{\bar{s}} \Gamma(s, B \cap [s_{t+1}^*, \bar{s}]) x_t(ds)$, $\forall B \in \mathcal{S}$.
4. **Market clearing**: $D[p_t, z_t) = \int_{s_t^*}^{\bar{s}} p_t s \ f[n_t(s)] \ x_t(ds)$.

Recursively: Perfect foresight equilibrium

- Only from today to tomorrow: need objects that given the state today, $\{z, x\}$, give us the state tomorrow $\{\phi, G\}$.
- Given ϕ, an equil defined recursively is functions $G(z, x)$, $m(z, x)$, $p(z, x)$, values and decisions $\{V(s, z, x), n(s, z, x), s^*(s, z, x)\}$ s.t.

1. **Optimality**: $\{V(s, z, x), s^*(s, z, x), n(s, z, x)\}$ solve

$$V(s, z, x) = \max_n \left\{0, \max p(s, z, x)s\ f(n) - wn - c^v + \frac{1}{1+r}\int_S V[s', \phi(z), G(z, x)]\ \Gamma(s, ds')\right\}$$

2. **Free-entry**: $\int V(s, z, x)\gamma(ds) \leq c^e$, ($=$ if $m(z, x) > 0$).
3. **Law of motion**: $\forall B \in \mathcal{S}$, we have $G(z, x)(B) = m(z, x)\gamma(B \cap [s^*(s, z, x), \bar{s}]) + \int_{s^*(s,z,x)}^{\bar{s}} \Gamma(s, B \cap [s^*(s, z, x), \bar{s}])x(ds)$,
4. **Market clearing**: $D(p(z, x), z) = \int_{s^*(s,z,x)}^{\bar{s}} p(z, x)\ s\ f[n(s, z, x)]\ x(ds)$.

- It is the same but in Stochastic Processes Language
- They extend the same for sequences and for the Recursive
- Obviously You have to add the Expectations to the terms of one period later.

Linear Approximation to a Stochastic Equilibrium

- There is a new (Boppart, Mitman & Krusell (2017)) way of thinking of Stochastic Equilibria that w is NOT recursive.
- It is based on a linear approximation to a completely unanticipated (MIT) shock.
- It requires to compute a transition as a Perfect Foresight Equilibrium
- Then do linear approximations in sequence space.

- Consider the social planner's problem (with full depreciation)

$$\begin{aligned} V(k_t) &= \max_{c_t, k_{t+1}} u(c_t) + \beta V(k_{t+1}) \\ & \text{s.t. } c_t + k_{t+1} \leq f(k_t), \quad \forall\ t \geq 0 \\ & c_t, k_{t+1} \geq 0,\ \forall \quad t \geq 0 \\ & k_0 > 0 \text{ given.} \end{aligned}$$

- The solution $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ satisfies

$$u_c(c_t) = \beta u_c(c_{t+1}) f_k(k_{t+1}), \forall\ t \geq 0$$

$$c_t + k_{t+1} = f(k_t), \forall\ t \geq 0$$

$$\lim_{t \to \infty} \beta^t u_c(c_t) k_{t+1} = 0$$

- Derive the above equilibrium conditions.

Computing a Transition in the Simplest Growth Model

- Look at the a steady state k^*
- Rewrite solution as

$$\psi(k_t, k_{t+1}, k_{t+2}) = u_c[f(k_t - k_{t+1})] - \beta u_c[f(k_{t+1} - k_{t+2})] f_k(k_{t+1}) = 0,$$

  a second order difference equation with exactly two boundary conditions, k_0 and $k_\infty = k^*$.
- It is solvable:
  1. guess k_1, use k_0 and $\psi(k_t, k_{t+1}, k_{t+2}) = 0$ to get $k_2, k_3, \ldots$ forward up until some T, and solve $k_T^\psi(k_1) = k^*$.
  2. Or guess k_{T-1} solve backward using ψ to find $k_0^\psi(k_{t-1}) = k_0$
  3. Solve for the whole sequence as a system of equations (almost diagonal
  4. Use dynare.
- Either way you get a numerical solution starting from any k_0

- We can compute any transition. Also one with time varying ψ.
- Consider this model with $c_t + k_{t+1} = e^{z_t}\ f(k_t),\ z_{t+1} = \rho z_t,\ z_0 = 1.$

$$\psi_t(k_t, k_{t+1}, k_{t+2}) = u_c[\rho^t f(k_t - k_{t+1})] - \beta u_c[\rho^{t+1} f(k_{t+1} - k_{t+2})] f_k(k_{t+1}),$$

- Let now $\widehat{k}_t = \log k_t - \log k^*$, (log st st deviation) (in fact it is like an impulse response function)
- Want: linearly approximate using $\{\widehat{k}_t\}_{t=0}^{\infty}$ the equilibrium given any sequence of innovations (we think of $z_{t+1} = \rho^t z_t + \epsilon_{t+1}$.). So we want a $\widetilde{k}_t(k_0, \epsilon^{t-1})$ (whole set of histories)

$$\begin{aligned}
\widetilde{k}_1(k_0, \epsilon_0) &= \epsilon_0\ \widehat{k}_1 \\
\widetilde{k}_2(k_0, \epsilon_0, \epsilon_1) &= \epsilon_0\ \widehat{k}_2 + \epsilon_1\ \widehat{k}_1, \\
&\vdots \\
\widetilde{k}_{t+1}(k_0, \epsilon^t) &= \sum_{\tau=0}^{t} \epsilon_t\ \widehat{k}_{t-\tau+1} \qquad \text{exact if } \epsilon_0 = 1,\ \epsilon_t = 0, \forall t \neq 0,
\end{aligned}$$

Uses

- This can be done for all Economies.
- Including industry equilibria.
- For all Statistics of all Economies.
- The computational costs is linear not exponential in the number of shocks.
- We do not know how to use it for asymmetric shocks (e.g. downward rigid wages)

- What happens if demand suddenly doubles starting from a stationary equilibrium?
- Define Formally the stochastic counterparts (sequentially and recursivrly) to the ones that we wrote above?
- Sketch an algorithm to find the equilibrium prices.
- Describe a way to compute the evolution of the Gini Index or the Herfindahl Index of the industry over the first fifteen periods.
- Imagine now that the industry is subject to demand shocks that follow an AR(1). Describe an algorithm to approximate it.

Incomplete Market Models

Consider the problem of a farmer with storage possibilities

$$V(s,a) = \max_{c,a'\geq 0} \quad u(c) + \beta \sum_{s'} \Gamma_{ss'} \; V(s',a') \qquad s.t.$$

$$c + qa' = a + s$$

a assets, c consu, and $s \in \{s^1, \cdots, s^{N^s}\} = S$ has transition Γ. q units today yield 1 unit tomorrow. Only nonnegative storage.

The Problem with certainty

- If s constant, then

$$V(a) = \max_{c, a' \geq 0} \left\{ u(a + s - qa') + \beta V(a') \right\}.$$

- with FOC $q\, u_c \geq \beta u_c'$
- With equality if $a' > 0$. Then
  - if $q > \beta$ (i.e. nature is more stingy, or the farmer is less patient),
    - Either $c' < c$ and the farmer dis-saves
    - Or $c = s$ and $a' = 0$.
  - If $q < \beta$, $c' > c$ and consumption grows without bound.
  - If $q = \beta$, $c' = c$ (with noise and $u_{ccc} > 0$ grows without bound).
- So we assume $\beta / q < 1$

Back to Uncertainty

- Assuming $\beta/q < 1$, allows us to bound asset holdings.
- They also save in best states when a is low.
- The FOC is

$$u_c\left[c\left(s,a\right)\right] \geq \frac{\beta}{q}\sum_{s'}\Gamma_{ss'}\; u_c\left(c\left[s',g\left(s,a\right)\right]\right),$$

  with equality when $a'\left(s,a\right) > 0$
- Note: $a' = g\left(s,a\right) = 0,\;\; \forall s$ for large $\bar{a}$. So $a' \in A = [0,\bar{a}]$
- We can construct a prob distribution over states $S \times A$. Define $\mathcal{B}$ as all subsets of S times Borel-σ-algebra sets in A.
- For any such prob measure x its evolution is

$$x'\left(B\right) = \widetilde{T}(B,x;\Gamma,g) = \sum_{s}\int_0^{\bar{a}}\sum_{s'\in B_s}\Gamma_{ss'}\;\mathbf{1}_{\{g(s,a)\in B_a\}}x\left(s,da\right), \quad \forall B \in \mathcal{B}$$

where B_s and B_a are projections of B on S and A,

Unique Stationary Distribution (and we get there)

Theorem

With a well behaved Γ, there is a unique stationary probability x^*, so that:

$$\begin{aligned} x^*(B) &= \widetilde{T}(B, x^*; \Gamma, g)(B), \quad \forall B \in \mathcal{B}, \\ x^*(B) &= \lim_{n\to\infty} \widetilde{T}^n(B, x_0; \Gamma, g)(B), \quad \forall B \in \mathcal{B}, \end{aligned}$$

for all initial probability measures X_0 on $(E, \mathcal{B})$.

We use compactness of $[0, \overline{A}]$.

Two Interpretations of x

1. Our ignorance of what is going on with the farmer or fisherman.
    - Even if we know at $t = 0$ s, a, no news lead us to x^*.
    - We can use x^* to compute the statistics of what happens to the fisherman: Average wealth is $\int_{S \times A} a \, dx^*$.
2. A description of a large number of fishermen (an archipelago). Notice how even if there is no contact between them. Stationarity arises

- There is a unique distribution of wealth.

Huggett (1993) Economy

- How can $a < 0$? Because of borrowing.
- Consider now an economy with many farmers and NO storage.

$$
\begin{aligned}
V(s,a) = \max_{c \geq 0, a'} \quad & u(c) + \beta \sum_{s'} \Gamma_{ss'} V(s', a') \\
s.t. \quad & c + q\, a' = a + s \\
& a' \geq \underline{a},
\end{aligned}
$$

where $\underline{a} < 0$ and $\beta / q < 1$. With solution $a' = g(s,a)$. Again

- One possibility for $\underline{a}$ is the natural borrowing limit: the agent can pay back his debt with certainty, no matter what:

$$
a^n := -\frac{s_{\min}}{\left(\frac{1}{q} - 1\right)}. \tag{23}
$$

- Or it could be tighter which makes it likely to bind $0 > \underline{a} > a^n$.

Huggett (1993) Economy II

- To determine q in general equilibrium, consider this function of q:

$$\int_{A\times S} a\,dx^*(q) \qquad \text{Aggregate asset holdings}$$

- A Stationary Equilibrium requires two things

$$\begin{aligned}\int_{A\times S} a\,dx^*(q) &= 0,\\ x^*(q) &= \widetilde{T}^n(B, x^*(q); \Gamma, g)(B).\end{aligned}$$

- It exists in $q \in (\beta, \infty]$ (intermediate value thm). Need to ensure:
  1. $\int_{A\times S} a\,dX^*(q)$ is a continuous function of q;
  2. $\lim_{q\to\beta} \int_{A\times S} a\,dX^*(q) \to \infty$; (As $q \to \beta$, the interest rate $R = 1/q$ increases up to $1/\beta$, (steady state interest rate in deterministic Econ. With $u_{ccc} > 0$ we have precautionary savings
  3. $\lim_{q\to\infty} \int_{A\times S} a\,dX^*(q) < 0$. As $q \to \infty$, arbitrary large consumption is achievable by borrowing.

Aiyagari (1994) Economy

- Workhorse models of modern macroeconomics.
- An Environment like the ones before
- On top of a growth model with $f(K, L)$ that yield factor prices.

$$K = \int_{A \times S} a\, dx,$$
$$N = \int_{A \times S} s\, dx.$$

- s fluctuations in the employment status (either efficiency units of labor or actual employment).
- Now we need $\beta(1+r) < 1$. We write

$$V(s, a) = \max_{c, a' \geq 0} \quad u(c) + \beta \int_{s'} V(s', a') \;\; \Gamma(s, d\, s') \qquad s.t.$$
$$c + a' = (1+r)\, a + ws$$

where r is the return on savings and w is the wage rate.

- Factor prices depend on the capital-labor ratio: $x^*\left(\frac{K}{L}\right)$. Equilibrium requires

$$\frac{K^*}{L^*} = \frac{\int_{A\times S} a \;\, dX^*\left(\frac{K^*}{L^*}\right)}{\int_{A\times S} s \, dX^*\left(\frac{K^*}{L^*}\right)}.$$

Exercise

Show that aggregate capital is higher in the stationary equilibrium of the Aiyagari economy than it is the standard representative agent economy.

Exercise

Rewrite the economy when households like leisure

Policy Changes and Welfare

- Let the Economy's parameters be summarized by $\theta = \{u, \beta, s, \Gamma, F\}$.
- $V(s, a; \theta)$ and $x^*(\theta)$ are functions of those parameters.
- Suppose an unexpected policy change that shifts θ to $\hat{\theta} = \{u, \beta, s, \hat{\Gamma}, F\}$.
- Consider $V\left(s, a; \hat{\theta}\right)$ and $x^*\left(\hat{\theta}\right)$.
- Define $\eta(s, a)$ by

$$V\left(s, a + \eta(s, a); \hat{\theta}\right) = V(s, a; \theta),$$

- Transfer necessary to make the (a, s) agent indifferent between living in the old environment and in the new.
- Total transfer needed to compensate all agents to live in $\hat{\theta}$ is

$$\int_{A \times S} \eta(s, a)\, dX^*(\theta).$$

- This is NOT a Welfare Comparison.
- This compares being parachuted in the stationary distribution of θ versus $\hat{\theta}$.
- Welfare computing the transition from the SAME initial conditions.
- Otherwise the best tax policy in the Rep agent (which is Pareto Optimal) would be to subsidize capital to maximize steady state consumption.

- What if aggregate shocks as in e.g. $z\ F\left(K, \bar{N}\right)$.
- Without leisure aggregate capital is a sufficient statistic for factor prices.
- Will aggregate capital be $K' = G\left(z, K\right)$ or $K' = G\left(z, x\right)$?
- The latter. Decision rules are not usually linear. But then $x' = G\left(z, x\right)$

$$
\begin{aligned}
V\left(z, X, s, a\right) = \max_{c, a' \geq 0} \quad & u\left(c\right) + \beta \sum_{z', s'} \Pi_{zz'} \Gamma_{ss'}^{z'} V\left(z', X', s', a'\right) \\
s.t. \quad & c + a' = azf_k\left(K, \bar{N}\right) + szf_n\left(K, \bar{N}\right) \\
& K = \int adX\left(s, a\right) \\
& X' = G\left(z, X\right)
\end{aligned}
$$

(replaced factor prices with marginal productivities)

- Computationally, this problem is a beast! So, what then?

Consider an economy with *dumb/approximating* agents!

- They people believe tomorrow's capital depends only on K and not on x. This, obviously, is not an economy with rational expectations. The agent's problem in such a setting is

$$\widetilde{V}(z, K, s, a) = \max_{c, a'} \quad u(c) + \beta \sum_{z', s'} \Pi_{zz'} \Gamma_{ss'}^{z'} \widetilde{V}(z', K', s', a')$$

$$s.t. \quad c + a' = a\ z\ f_k\left(K, \bar{N}\right) + szf_n\left(K, \bar{N}\right)$$

$$K' = \widetilde{G}(z, K)$$

- We could approximate the equilibrium in the computer by posing a linear approximation to $\widetilde{G}$. A pain but doable. Krusell Smith (1997).

- They found it works well in boring settings (things are pretty linear)

Linear Approximation Revisited

- We can use the same linear approx in sequences as before for any shocks:

    1. Find the steady state
    2. Obtain the the impulse response (the perfect foresight equilibium) given an MIT shock that is treated as an innovation.
    3. Use these responses to approximate the behavior of any aggregate.

- Valuable for SMALL shocks like all linear approximations.

- Consider an Aiyagari economy with an AR(1) TFP shock z.
  - Choose an initial size innovation $\bar{\epsilon}_0$ (does not have to be 1) and compute the perfect foresight Equilibria of this MIT shock.
    - This involves a fixed point in the space of sequence of capital labor ratios.
    - But can be done with some effort:
    - To evaluate it, given prices solve the household's problem backwards from the final steady state.
    - Then update the distribution forward from the initial steady state obtaining new prices.
    - We look for a fixed point of this (not necessarily iterating mechanically but as solution of a system of equations)

Seeing the light at the end of the tunnel

- We have now the sequence of x_t and any prices that we care for.
- Compute the sequence of all statistics $\{d_t\}_t^T$ of that economy that you care for.
- Get a random draw $\{\epsilon_t\}_{t=0}^T$.
- Linearly approximate those statistic like we did before the same way that we approximated

$$
\begin{aligned}
\widetilde{d}_1(x_0, \epsilon_0) &= \frac{\epsilon_0}{\bar{\epsilon}_0}\ \widehat{d}_1 \\
\widetilde{d}_2(x_0, \epsilon_0, \epsilon_1) &= \frac{\epsilon_0}{\bar{\epsilon}_0}\ \widehat{d}_2 + \frac{\epsilon_1}{\bar{\epsilon}_0}\ \widehat{d}_1, \\
&\vdots \\
\widetilde{d}_{t+1}(x_0, \epsilon^t) &= \sum_{\tau=0}^{t} \frac{\epsilon_t}{\bar{\epsilon}_0}\ \widehat{d}_{t-\tau+1} \qquad \text{exact if } \epsilon_0 = \widetilde{\epsilon}_0,\ \epsilon_t = 0, \forall t \neq 0.
\end{aligned}
$$

- Agents can either not work or work: $\varepsilon = \{0, 1\}$,
- Agents can exert painful effort h to search for a job increasing the probability $\phi(h)$ (with $\phi' > 0$) of finding it.
- An employed worker, does not search for a job so $h = 0$, but its job can be destroyed with some exogenous probability δ.
- s is Markovian (Γ) labor labor productivity. Then the unemployed

$$V(s, 0, a) = \max_{c,h,a' \geq 0} u(c, h) + \beta \sum_{s'} \Gamma_{ss'} \left[\phi(h) V(s', 1, a') + (1 - \phi(h)) V(s', 0, a')\right]$$

$$s.t. \quad c + a' = h + (1 + r)a$$

the employed

$$V(s, 1, a) = \max_{c,a' \geq 0} u(c) + \beta \sum_{s'} \Gamma_{ss'} \left[\delta V(s', 0, a') + (1 - \delta) V(s', 1, a')\right]$$

$$s.t. \quad c + a' = sw + (1 + r)a$$

- Suppose every period agents choose an occupation: entrepreneur or a worker.
- Entrepreneurs run their own business: manage a project that combines entrepreneurial ability (η), capital (k), and labor(n); while workers work for somebody else.
- If worker

$$V^w(s,\eta,a)$$

$$= \max_{c,a'\geq 0, d\in\{0,1\}} u(c) + \beta$$

$$\sum_{s',\eta'} \Gamma_{ss'}\Gamma_{\eta\eta'} \left[dV^w(s',\eta',a') + (1-d)V^e(s',\eta',a')\right]$$

$$s.t.$$

$$c + a' = ws + (1+r)a$$

- Similarly, the entrepreneur's problem can be formulated as follows

$$V^e(s,\eta,a) = \max_{c,a'\geq 0, d\in\{0,1\}} u(c) + \beta \sum_{s',\eta'} \Gamma_{ss'}\Gamma_{\eta\eta'} \left[d\ V^w(s',\eta',a') + (1-d)V^e(s',\eta',a')\right]$$

$$s.t. \quad c + a' = \pi(s,\eta,a)$$

Income is from profits $\pi(a,s,\eta)$ not wages. Assume entrepreneurs have a DRS technology f. Profits are

$$\pi(s,\eta,a) = \max_{k,n} \eta f(k,n) + (1-\delta)k - (1+r)(k-a) - w\max\{n-s,0\}$$

$$s.t. \quad k - a \leq \phi a$$

The constraint here reflects the fact that entrepreneurs can only make loans up to a fraction ϕ of his total wealth.

Aiyagari Economy with Entrepreneurs III

- Entrepreneurs never make an operating loss within a period, (can always choose $k = n = 0$ and earn the risk free rate on savings).
- Agents with high entrepreneurial ability η have access to an investment technology f that provides higher returns than workers so become richer.
- Even the prospects (high η) low wealth suffice to induce high savings? (Γ)
- Who becomes an entrepreneur in this economy? Without financial constraints, wealth will play no role. $\exists\eta^*$ above which it becomes an entrepreneur.
- With financial constraints wealth matters. Wealthy agents with high h will while the poor with low η will not.
- For the rest, it depends. If η is persistent, poor individuals with high entrepreneurial ability will save to one day become entrepreneurs, while rich agents with low entrepreneurial ability will lend their assets and become workers.

- The price of lending incorporates the possibility of default.
- Assume upon default punished to autarky forever after (no borrowing or lending)
- If no default the budget constraint is $c + q(a')a' = a + ws$,

$$V(s,a) = \max \Big\{ u(ws) + \beta \sum_{s'} \Gamma_{ss'} \bar{V}(s'), \\ \max_{c,a'} u[ws + a - q(a')\, a'] + \beta \sum_{s'} \Gamma_{ss'} V(s', a') \Big\}$$

where $\bar{V}(s') = \frac{1}{1-\beta} u(ws')$ is the value of autarky.

- What determines $q(a')$? A zero profit on lenders: Probability of default

Monopolistic Competition

- Models with Nominal Prices.
- Price/Wage Rigidity.
- Firms are sufficiently “different” to set prices.
- Small in the Context of the Aggregate Economy. Hence Monopolistic Competition.

- Consumers have a taste for variety
  - The consumer's utility function has constant elasticity of substitution (CES)

$$u\left(\{c(i)\}_{i\in[0,n]}\right) = \left(\int_0^n c(i)^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{\sigma}{\sigma-1}}$$

  where σ is the elasticity of substitution, and $c(i)$ is the quantity consumed of variety i. For simplicity, we will rename $c(i) = c_i$.

- Assume the agents receive exogenous *nominal* income I
- They are endowed with one unit of time.

$$\max_{\{c_i\}_{i\in[0,n]}} \left(\int_0^n c_i^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{\sigma}{\sigma-1}}$$

$$s.t. \quad \int_0^n p_i \; c_i \; di \leq I$$

- Deriving the FOC, and relating the demand for varieties i and j

$$c_i = c_j \left(\frac{p_i}{p_j}\right)^{-\sigma}$$

- Multiplying both sides by p_i and integrating over i, yields

$$c_i^* = \frac{I}{\int_0^n p_j^{1-\sigma} dj} \, p_i^{-\sigma}$$

- Here c_i^* depends on the price of i and an aggregate price

- Convenient to define the aggregate price index P as

$$P = \left(\int_0^n p_j^{1-\sigma} dj \right)^{\frac{1}{1-\sigma}}$$

- and thus

$$c_i^* = \frac{I}{P} \left(\frac{p_i}{P} \right)^{-\sigma}$$

real income and the second times a measure of the relative price of i.

Exercise

Show the following within this monopolistic competition framework

1. *σ is the elasticity of substitution between varieties.*
2. *Price index P is the expenditure to purchase a unit-level utility.*
3. *Consumer utility is increasing in the number of varieties n.*
4. *Is there a missing n?*

Characterizing the firm's problem

- Assume linear production technology: $f(\ell_j) = \ell_j$.
- Nominal wage rate is given by W.
- The firm solves

$$\max_{p_j} \pi(p_j) = p_j \; c_j^*(p_j) - W \; c_j^*(p_j)$$
$$s.t. \quad c_j^* = \frac{I}{P}\left(\frac{p_j}{P}\right)^{-\sigma}$$

- Firms do not affect P. Solve for the FOC:

$$p_j^* = \frac{\sigma}{\sigma - 1} W \qquad \forall j$$

- $\frac{\sigma}{\sigma-1}$ is a constant mark-up over the marginal cost,
- When varieties are close substitutes ($\sigma \to \infty$), prices converge to W. Not that all

Equilibrium

Set the wage as numeraire. An Eq is prices $\{p_i^*\}_{i\in[0,n]}$, the aggregate price index P, household's consumption, $\{c_i^*\}_{i\in[0,n]}$, income I, firm's labor demand $\{\ell_i^*\}_{i\in[0,n]}$ and profits $\{\pi_i^*\}_{i\in[0,n]}$, such that

1. Given prices, $\{c_i^*\}_{i\in[0,n]}$ solves the household's problem
2. Given P and I, p_i^* and π_i^* solve the firm's problem $\forall i \in [0,n]$
3. Price Aggregation

$$P = \left(\int_0^n (p_j^*)^{1-\sigma}\ dj\right)^{\frac{1}{1-\sigma}}$$

4. Markets clear

$$\begin{aligned} \int_0^n \ell_i^*\ di &= 1 \\ 1 + \int \pi_i^* di &= I \end{aligned}$$

A symmetric equilibria: $c_i^* = \bar{c}$, $p_i^* = \bar{p}$, $\ell_i^* = \bar{\ell}$, $\pi_i^* = \bar{\pi}$ for all i.

Price Rigidity

- To study inflation, (meaningful interactions between output and inflation) needs

    1. A dynamic model
    2. Some source of nominal frictions so nominal variables (things measured in dollars) can affect real variables.

- Most popular friction is *price rigidity*. (firms cannot adjust their prices freely)

    1. *Rotemberg pricing* (menu costs)
    2. *Calvo pricing* (some (randomly set) firms can change prices, others cannot).

Rotemberg pricing

- Firms face adjustment cost $\phi(p_j, p_j^-)$ when changing their prices p_j each period.
- Let the Agg State be S, and let $I(S)$, $W(S)$, $P(S)$. Then firm's per period profit under Rotemberg pricing in a dynamic setup as follows:

$$\Omega(S, p_j^-) = \max_{p_j} \; p_j c_j^* - W(S) c_j^* - \phi(p_j, p_j^-)$$

$$+ E\{R^{-1}(G(S))\, \Omega(G(S), p_j)\}$$

$$\text{where} \quad c_j^* = \left(\frac{p_j}{P(S)}\right)^{-\sigma} \frac{I(S)}{P(S)}$$

- easy algebra when quadratic price adjustment cost.
- Without capital $S = P^-$ and Aggregate Shocks.

- Firms can adjust their prices each period with probability θ.
- A firm that can change its price

$$\Omega^1(S, p_j^-) = \max_{p_j} p_j c_j^* - W(S) c_j^* + (1-\theta) E\{R^{-1}(S')\, \Omega^0(S', p_j)\} + \theta\, E\{R^{-1}(S')\, \Omega^1(S', p_j)\}$$

$$\text{where} \quad c_j^* = \left(\frac{p_j}{P(S)}\right)^{-\sigma} \frac{I(S)}{P(S)} \quad \text{and} \quad S' = G(S)$$

- A firm that cannot

$$\Omega^0(S, p_j^-) = [p_j^- - W(S)] c_j^* + (1-\theta) E\{R^{-1}(S')\, \Omega^0(S', p_j^-)\} + \theta\, E\{R^{-1}(S')\, \Omega^1(S', p_j^-)\}$$

Exercise

Derive the following for the dynamic model with Calvo pricing

1. *Solve the firm's problem in sequence space and write the firm's equilibrium pricing* $p_{j,t}$ *as a function of present and future aggregate prices, wages, and endowments:* $\{P_t, W_t, I_t\}_{t=0}^{\infty}$.

2. *Show that under flexible pricing (*$\theta = 1$*), the firm's pricing strategy is identical to the static model.*

3. *Show that with price rigidity (*$\theta < 1$*), the firm's pricing strategy is identical to the static model in a steady state with zero inflation.*

Extreme Value Shocks

Written with the help of Jinfeng Luo

A very useful tool in both Macro and Micro

- Mostly used to makes sense of Models with discrete choices.
    1. Agents grouped in bins
    2. Fractions of Agents in Each bin take one choice the rest another. How to make sense of this?
- In Macro Models we have decision rules (i.e. functions). The state is a sufficient statistic for your choice. This is too strong. Sometimes we want to have decision *densities*
- In certain private information environments, sometimes there are issues with *pooling vs separating* equilibria. These shocks make all equilibria pooling.
- Gives a natural way to deal (in the cross section) with *off-equilibrium* behavior as all behavior is now *on* equilibrium

A discrete choice setting

- A world with finitely many (ranked or not ranked) states $s \in S$.
- Many agents $i \in I$
- Two choices $d \in \{0, 1\}$.
- Standard way to model is $u(s, d)$ and to $\max_d \; u(s, d)$. What if we see fractions $x_{s,d}$?
- Consider now an idyosincratic shock ϵ^{id} to an s agent. Now choice is

$$\max_d \; u(s, d) + \epsilon^{id}$$

- The prob of choice d depends on the distribution of the ϵ and the difference between $u(s, 0)$ and $u(c, 1)$. Thresholds.
- Hard Problem. Except for when ϵ is extreme value

- Extreme value (Gumbel) distributed shocks $\epsilon \sim G(\mu, \alpha)$ with cdf $F(x) = e^{e^{-\frac{x-\mu}{\alpha}}}$
  - α is the shape parameter
  - larger $\alpha \Rightarrow$ smaller variance ($\frac{\pi^2}{6\alpha^2}$)
- In many occasions we deal with the maximum and especially the expected maximum of (several or a continuum of) these shocks
- Consider the discrete case

$$X^N = \max\left\{\epsilon^1, \epsilon^2, \cdots, \epsilon^N\right\}$$

$$M^N = E\left[X^N\right]$$

Expected max: mode zero ($\mu = 0$)

- It can be proved that if all ϵ follow i.i.d. $G(0, \alpha)$, then we have

$$X^N \sim G\left(\frac{1}{\alpha}\ln N, \frac{1}{\alpha}\right)$$

$$M^N = \frac{1}{\alpha}\ln N + \frac{\gamma}{\alpha}$$

where $\gamma \approx 0.5772$ is the Euler–Mascheroni constant.

- Magic. Just a formula.
- If we want M^N independent of the number of choices, we can require

$$M^N = \bar{M} \Rightarrow \alpha(N) = \frac{\gamma + \ln N}{\bar{M}}$$

Choice Probability: mode zero ($\mu = 0$)

- Moreover how likely it is that $d = 1$ is chosen?
- Again there is em magic (no need to compute thresholds of indifference like with other cdf's)
- It only depends on the difference of fundamental utilities and in the parameter α (inversely related to the variance), a measure of *fickleness*

$$q^1(s) = \frac{1}{1 + e^{\alpha[u(s,0) - u(s,1)]}}$$

Expected max: mode non-zero ($\mu \neq 0$)

- If all ϵ follow i.i.d. $G(\mu, \alpha)$, thne

$$X^N \sim G\left(\frac{1}{\alpha} \ln N \; e^{\alpha\mu}, \frac{1}{\alpha}\right)$$

$$M^N = \frac{1}{\alpha} \ln N \; e^{\alpha\mu} + \frac{\gamma}{\alpha}$$

- Agai, to make M^N independent of N

$$M^N = \bar{M} \Rightarrow \alpha(N) = \frac{\gamma + \ln N}{\bar{M} - \mu}$$

- Still closed-form solution

- In real problems choices often worth differently ex-ante
- If ϵ^i follow $G\left(\mu^i, \alpha\right)$, we have

$$X^N \sim G\left(\frac{1}{\alpha}\ln\sum_i e^{\alpha\mu^i}, \frac{1}{\alpha}\right)$$

$$M^N = \frac{1}{\alpha}\ln\sum_i e^{\alpha\mu^i} + \frac{\gamma}{\alpha}$$

- Again if we want M^N independent of the number of choices, we can require

$$M^N = \bar{M} \Rightarrow \alpha\left(N\right) = \frac{\gamma + \ln\sum_i e^{\alpha(N)\mu^i}}{\bar{M}}$$

- No closed-form solution

It also works with continuum support. Expected max:

- Now we turn to the case with a continuum of choices $S = [w, \bar{w}]$

$$X^S = \max\{\epsilon^w | w \in [w, \bar{w}]\}$$

$$M^S = \mathbb{E}[X^S]$$

- Thus again if we want M^S independent of S, we can similarly find α by

$$M^S = \bar{M} \Rightarrow \alpha(S) = \frac{\gamma + \ln \int e^{\alpha(S)\mu^w} dw}{\bar{M}}$$

Expected max: a formal treatment

- The way we define continuous choice above is not rigorous: the choice density is well defined, but the expected max M^S may not.
- A formal treatment: there are finite many (N) "opportunities" of drawing ϵ, distributed on $S = [w, \bar{w}]$, with density function $f(w)$.
- X^S and M^S defined as above, and we have

$$X^S \sim G\left(\frac{1}{\alpha}\ln\int Nf(w)\, e^{\alpha\mu^w} dw, \frac{1}{\alpha}\right)$$

$$M^S = \frac{1}{\alpha}\ln\int Nf(w)\, e^{\alpha\mu^w} dw + \frac{\gamma}{\alpha}$$

- Regulating $Nf(w) = 1$ for all w brings us back to the previous slide.

An alternative way of adjustment

- It can be seen that with mode heterogeneity, adjusting α may not be easy.
- An alternative (easier but less precise) way is to try to remove the part varying with N or S as much as possible from M
- In discrete case

$$\tilde{M}^N = M^N - \frac{1}{\alpha} \ln \sum_i e^{\alpha 0} = M^N - \frac{1}{\alpha} \ln N$$

- In continuum case

$$\tilde{M}^S = M^S - \frac{1}{\alpha} \ln \int e^{\alpha 0} dw = M^N - \frac{1}{\alpha} \ln \left(\bar{w} - w \right)$$

Macro and COVID-19

Embody A Macro Model With An Epidemiological one

- Short Horizons (No investment)
- Choose what Issues to Worry About
    1. *Mitigation Policy and Heterogeneity Age/Sector*
- Choose wich Allocation Mechanism to Model (large externality)
    1. All Econ choices are Government choices

The Basic SIR Model: Fundamentals

- All variables are shares of a measure 1 population
- Three health states, $j \in \{s, i, r\}$ susceptible, infected, recovered or dead, with associated population shares S, I, R. Initial conditions $S(0), I(0), R(0)$.
- Two parameters: β governs rate of infection, κ the rate of recovery (or death)
- System of differential Equations

$$\begin{aligned} \dot{S}(t) &= -\beta S(t) I(t) \\ \dot{I}(t) &= \beta S(t) I(t) - \kappa I(t) \\ \dot{R}(t) &= \kappa I(t) \end{aligned}$$

- Basic Reproduction Number: define $R_0 = \frac{\beta}{\kappa}$

The Basic SIR Model: The Beginning of a Pandemic

- Growth rate of infections given by $\frac{\dot{I}(t)}{I(t)} = \beta S(t) - \kappa$
- Let $I(0) = \epsilon, S(0) = 1 - I(0)$, when $\epsilon > 0$ is very small, $S(0) \approx 1$.
- Since $\dot{S}(t) = -\beta S(t) I(t)$ and for t close to zero,

  $I(t) \approx 0, S(t) \approx 1$, then $\dot{I}(t)/I(t)$ is roughly constant and equal to

  $$\dot{S}(t) = -\beta S(0) I(0) \qquad \text{So}$$

  $$I(t) = I(0) e^{\kappa\left(\frac{\beta}{\kappa} S(0) - 1\right)} \approx I(0) e^{\kappa\left(\frac{\beta}{\kappa} - 1\right)}$$

- If $R_0 = \frac{\beta}{\kappa} > 1$ exponential growth early (if $I(0) > 0$).
- If $R_0 = \frac{\beta}{\kappa} < 1$ then infections fall to zero and epidemic disappears immediately.

The Basic SIR Model: Long Run

- The Ratio of differential equations: $\frac{\dot{I}(t)}{\dot{S}(t)} = -1 + \frac{1}{R_0 S(t)}$
- Integrating yields $\quad I(t) = -S(t) + \frac{\ln(S(t))}{R_0} + q$

  where q is a constant of integration that does not depend on time.
- Evaluating at $t = 0$ yields (using $R(0) = 0$, thus $S(0) + I(0) = 1$

$$q = 1 - \frac{\ln(S(0))}{R_0}$$

- What is $S(\infty) = S^\star$? share of the population never to get infected
- Evaluating at $t = \infty$ and using the fact that $I(\infty) = 0$ yields

$$S^\star = 1 + \frac{\ln\left[S^\star / S(0)\right]}{R_0}$$

The Basic SIR Model: Properties of Steady State

- Steady state satisfies the trascendental equation:

$$S^\star = 1 + \frac{\ln\left[S^\star / S(0)\right]}{R_0}$$

and $R^\star = 1 - S^\star, I^\star = 0$.

- If $R_0 > 1$ and $S(0) < 1$, $\exists$ a unique long-run S^*.

  Strictly decreasing in R_0 and strictly increasing in $S(0)$.

- For $R_0 \approx 1$ (but > 1), $S^\star = \frac{1}{R_0}$ and $R^\star = \frac{R_0 - 1}{R_0}$

  This approximation (a first good rule of thumb) uses $S(0) \approx 1$ and

$$\ln(1/R_0) = -\ln(R_0) = -\ln(1 + R_0 - 1) \approx 1 - R_0.$$

An epidemiological/economic model with heterogenous agents

- With **costly** transfers across agents
- To Assess combination of two policies
  - Shutdown / mitigation (less output but also less contagion)
  - Redistribution toward those whose jobs are shuttered
- Characterize optimal policy
- Key interaction:
  - Mitigation creates the need for more redistribution
  - But if redistribution is costly, want less mitigation
  - Need heterogeneous-agent model to analyze this

The SAFER SIR Model

- Stage of the disease
  - **S**usceptible
  - Infected **A**symptomatic
  - Infected with **F**lu-like symptoms
  - Infected and needing **E**mergency hospital care
  - **R**ecovered (or dead)

- Worst case disease progression: $S \to A \to F \to E \to D$

- But *R*ecovery is possible at each stage

- Three infected types spread virus in different ways:
  - A at work, while consuming, at home
  - F at home
  - E to health-care workers

Heterogeneity by Age and Sector

- Age $i \in \{y, o\}$
  - Only young work
  - Old have more adverse outcomes conditional on contagion
  - But young more prone to contagion (they work)
- Sector of production $\{b, \ell\}$
  - Basic (health care / food production etc.)
    - Will never want shut-downs in this sector
    - Workers in this sector care for the hospitalized
  - Luxury (restaurants, entertainment etc.)
    - Workers in this sector face shutdown unemployment risk
    - But they are less likely to get infected

- Mitigation
  - Reduces contagion
  - Reduces risk of hospital overload
  - Reduces average consumption
  - Increases inequality (more unemployment in shuttered sectors)
- Redistribution
  - Helps the unemployed $\Rightarrow$ makes mitigation more palatable
  - But redistribution is costly $\Rightarrow$ makes mitigation more expensive
- What policy time paths do different types prefer? When (and how much) to shut down, when to open up? Size of Coronavirus check?
- How does the utilitarian optimal policy vary with the cost of redistribution?

- Lifetime utility for *o*ld

$$E\left\{\int e^{-\rho_o t}\;\left[u^o(c_t^o)+\bar{u}+\widehat{u}_t^j\right]\;dt\right\}$$

  - ρ_o: time discount rate
  - $u^o(c_t^o)$ instantaneous utility from old age consumption c_t^o
  - $\bar{u}$: value of life
  - $\widehat{u}_t^j$: intrinsic utility from health status j (zero for $j \in \{s, a, r\}$)

- Similar lifetime utility for *y*oung.
- Differences in expected longevity through $\rho_y \neq \rho_o$ (no aging)

- Young permanently assigned to b or ℓ
- Linear production: output equals number of workers
- Only workers with $j \in \{s, a, r\}$ work
- Output in basic sector:

$$y^b = x^{ybs} + x^{yba} + x^{ybr}$$

- Output in luxury sector is

$$y^\ell = [1 - m] \left(x^{y\ell s} + x^{y\ell a} + x^{y\ell r} \right)$$

- Total output given by
$$y = y^b + y^\ell.$$
- Fixed amount of output $\eta\Theta$ spent on emergency health care
- Θ measures capacity of emergency health system, η its unit cost

- Types of transmission
  - Work: young S workers infected by A workers, prob $\beta_w(m)$
  - Consumption: young & old S infected by A, prob $\beta_c(m) \times y(m)$
  - Home: young & old S infected by A and F, prob β_h
  - ER: basic S workers infected by E, prob β_e
- Shutdowns (mitigation) help by:
  - Reducing workers $\Rightarrow$ less workplace transmission
  - Reducing output $y(m) \Rightarrow$ less consumption transmission
  - Reducing infection-generating rates $\beta_w(m)$ & $\beta_c(m)$

$$\beta_w(m) = \frac{y^b}{y(m)}\alpha_w + \frac{y^\ell(m)}{y(m)}\alpha_w(1-m)$$

  - Similar for $\beta_c(m)$
  - Micro-founded via sectoral heterogeneity in social contact rates
  - Smart mitigation shutters most contact-intensive sub-sectors first

Flow into asymptomatic (out of susceptible)

$$
\begin{aligned}
\dot{x}^{ybs} &= -\beta_w(m)\left[x^{yba} + (1-m)x^{y\ell a}\right]x^{ybs} \\
&\quad - \left[\beta_c(m)x^a y(m) + \beta_h\left(x^a + x^f\right) + \beta_e x^e\right]x^{ybs} \\
\dot{x}^{y\ell s} &= -\left[\beta_w(m)\left[x^{yba} + (1-m)x^{y\ell a}\right](1-m)x^{y\ell s}\right] \\
&\quad - \left[\beta_c(m)x^a y(m) + \beta_h\left(x^a + x^f\right)\right]x^{y\ell s} \\
\dot{x}^{os} &= -\left[\beta_c(m)x^a y(m) + \beta_h\left(x^a + x^f\right)\right]x^{os}
\end{aligned}
$$

- For each type $j \in \{yb, y\ell, o\}$

$$
\begin{aligned}
\dot{x}^{ja} &= -\dot{x}^{js} - \left(\sigma^{jaf} + \sigma^{jar}\right)\ x^{ja} \\
\dot{x}^{jf} &= \sigma^{jaf}\ x^{ja} - \left(\sigma^{jfe} + \sigma^{jfr}\right)\ x^{jf} \\
\dot{x}^{je} &= \sigma^{jfe}\ x^{jf} - \left(\sigma^{jed} + \sigma^{jer}\right)\ x^{je} \\
\dot{x}^{jr} &= \sigma^{jar} x^{ja} + \sigma^{jfr} x^{jf} + (\sigma^{jer} - \varphi) x^{je} \\
\varphi &= \lambda_o \max\{x^e - \Theta, 0\}.
\end{aligned}
$$

- All flow rates σ vary by age

- $x^e - \Theta$ measures excess demand for emergency health care. Reduces flow of recovered (Increases flow into death)

Redistribution

- Costly transfers between workers, non-workers (old, sick, unemployed)
- Utilitarian planner (or taxes / transfers that cannot depend on age, sector, health)
  - $\Rightarrow$ Workers share common consumption level c^w
  - $\Rightarrow$ Non-workers share common consumption level c^n
- Define measures of non-working and working as

$$\begin{aligned}
\mu^n &= x^{y\ell f} + x^{y\ell e} + x^{ybf} + x^{ybe} + m\left(x^{y\ell s} + x^{y\ell a} + x^{y\ell r}\right) + x^o \\
\mu^w &= x^{ybs} + x^{yba} + x^{ybr} + [1-m]\left(x^{y\ell s} + x^{y\ell a} + x^{y\ell r}\right) \\
\nu^w &= \frac{\mu^w}{\mu^w + \mu^n}
\end{aligned}$$

- Aggregate resource constraint

$$\mu^w c^w + \mu^n c^n + \mu^n T(c^n) = \mu^w - \eta\Theta$$

where $T(c^n)$ is per-capita cost of transferring c^n to non-workers

Instantaneous Social Welfare Function

- Consumption allocation does not affect disease dynamics $\Rightarrow$ optimal redistribution is a static problem
- With log-utility and equal weights, period social welfare given by

$$W(x,m) = \max_{c^n, c^w} [\mu^w \log(c^w) + \mu^n \log(c^n)] + (\mu^w + \mu^n)\bar{u} + \sum_{i,j \in \{f,e\}} x^{ij} \widehat{u}^j$$

- Maximization subject to resource constraint gives $\frac{c^w}{c^n} = 1 + T'(c^n)$.
- Period welfare

$$\begin{aligned} W(x,m) &= [\mu^w + \mu^n] \, w(x,m) \\ w(x,m) &= \log(c^n) + \nu \log(1 + T'(c^n)) + \bar{u} + \sum_{i,j \in \{f,e\}} \frac{x^{ij}}{\mu^w + \mu^w} \widehat{u}^j \end{aligned}$$

Instantaneous Social Welfare Function

- Assume $\mu^n T(c^n) = \mu^w \frac{\tau}{2} \left(\frac{\mu^n c^n}{\mu^w}\right)^2$
- Optimal allocation

$$\begin{aligned} c^n &= \frac{\sqrt{1 + 2\tau \frac{1-\nu^2}{\nu} \tilde{y}} - 1}{\tau \frac{1-\nu^2}{\nu}} \\ c^w &= c^n(1 + T'(c^n))) = c^n \left(1 + \tau \frac{1-\nu}{\nu} c^n\right) \end{aligned}$$

  where $\tilde{y} = \nu - \frac{\eta\Theta}{\mu^w + \mu^n}$.
- $\left(1 + \tau \frac{1-\nu}{\nu} c^n\right)$ is the effective marginal cost (MC) of transfers.
- It increases with c^n and τ, decreases with share of workers ν
- Higher mitigation m reduces ν, thus increases MC
- $\Rightarrow$ policy interaction between m, τ.