Econ 704 Macroeconomic Theory Spring 2018*

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*This is the evolution of class notes by many students over the years, both from Penn and Minnesota including Makoto Nakajima (2002), Vivian Zhanwei Yue (2002-3), Ahu Gemici (2003-4), Kagan (Omer Parmaksiz) (2004-5), Thanasis Geromichalos (2005-6), Se Kyu Choi (2006-7), Serdar Ozkan (2007), Ali Shourideh (2008), Manuel Macera (2009), Tayyar Buyukbasaran (2010), Bernabe Lopez-Martin (2011), Rishabh Kirpalani (2012), Zhifeng Cai (2013), Alexandra (Sasha) Solovyeva (2014), Keyvan Eslami (2015), Sumedh Ambokar (2016), Ömer Faruk Koru (2017) and Jinfeng Luo (2018).

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1 Introduction

A model is an artificial economy. Description of a model's environment may include specifying the agents' preferences and endowment, technology available, information structure as well as property rights. Neoclassical Growth Model becomes one of the workhorses of modern macroeconomics because it delivers some fundamental properties of modern economy, summarized by, among others, Kaldor:

1. Output per capita has grown at a roughly constant rate (2%).
2. The capital-output ratio (where capital is measured using the perpetual inventory method based on past consumption foregone) has remained roughly constant.
3. The capital-labor ratio has grown at a roughly constant rate equal to the growth rate of output.
4. The wage rate has grown at a roughly constant rate equal to the growth rate of output.
5. The real interest rate has been stationary and, during long periods, roughly constant.
6. Labor income as a share of output has remained roughly constant (0.66).
7. Hours worked per capita have been roughly constant.

Equilibrium can be defined as a prediction of what will happen and therefore it is a mapping from environments to outcomes (allocations, prices, etc.). One equilibrium concept that we will deal with is Competitive Equilibrium[1] Characterizing the equilibrium, however, usually involves finding solutions to a system of infinite number of equations. There are generally two ways of getting around this. First, invoke the welfare theorem to solve for the allocation first and then find the equilibrium prices associated with it. The first way sometimes may not work due to, say, presence of externality. So the second way is to look at Recursive Competitive equilibrium, where equilibrium objects are functions instead of variables.

[1] Arrow-Debreu or Valuation Equilibrium.

2 Review: Neoclassical Growth Model

We review briefly the basic neoclassical growth model.

2.1 The Neoclassical Growth Model Without Uncertainty

The commodity space is

$$\mathcal{L} = \{(l_1, l_2, l_3) : l_i = (l_{it})_{t=0}^{\infty} \; l_{it} \in \mathbb{R}, \; \sup_t |l_{it}| < \infty, \; i = 1, 2, 3\}.$$

The consumption possibility set is

$$\begin{aligned} X(\bar{k}_0) = \{x \in \mathcal{L} : \; & \exists (c_t, k_{t+1})_{t=0}^{\infty} \text{ s.th. } \forall t = 0, 1, \ldots \\ & c_t, k_{t+1} \geq 0, \; x_{1t} + (1-\delta) k_t = c_t + k_{t+1}, \; -k_t \leq x_{2t} \leq 0, \; -1 \leq x_{3t} \leq 0, \; k_0 = \bar{k}_0\}. \end{aligned}$$

The production possibility set is $Y = \prod_t Y_t$, where

$$Y_t = \{(y_{1t}, y_{2t}, y_{3t}) \in \mathbb{R}^3 : \; 0 \leq y_{1t} \leq F(-y_{2t}, -y_{3t})\}.$$

Definition 1 *An Arrow-Debreu equilibrium is $(x^*, y^*) \in X \times Y$, and a continuous linear functional ν^* such that*

1. $x^* \in \arg\max_{x \in X, \nu^*(x) \leq 0} \sum_{t=0}^{\infty} \beta^t u(c_t(x), -x_{3t})$,
2. $y^* \in \arg\max_{y \in Y} \nu^*(y)$,
3. *and* $x^* = y^*$.

Note that in this definition, we have added leisure. Now, let's look at the one-sector growth model's

Social Planner's Problem:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, -x_{3t}) \qquad (SPP)$$

s.t.

$$c_t + k_{t+1} - (1-\delta)k_t = x_{1t}$$

$$-k_t \leq x_{2t} \leq 0$$

$$-1 \leq x_{3t} \leq 0$$

$$0 \leq y_{1t} \leq F(-y_{2t}, -y_{3t})$$

$$x = y$$

$$k_0 \text{ given.}$$

Suppose we know that a solution in sequence form exists for (SPP) and is unique.

Exercise 1 *Clearly stating sufficient assumptions on utility and production function, show that (SPP) has a unique solution.*

Two important theorems show the relationship between CE allocations and Pareto optimal allocations:

Theorem 1 *Suppose that for all $x \in X$ there exists a sequence $(x_k)_{k=0}^{\infty}$, such that for all $k \geq 0$, $x_k \in X$ and $U(x_k) > U(x)$. If (x^*, y^*, ν^*) is an Arrow-Debreu equilibrium then (x^*, y^*) is Pareto efficient allocation.*

Theorem 2 *If X is convex, preferences are convex, U is continuous, Y is convex and has an interior point, then for any Pareto efficient allocation (x^*, y^*) there exists a continuous linear functional ν such that (x^*, y^*, ν) is a quasiequilibrium, that is (a) for all $x \in X$ such that $U(x) \geq U(x^*)$ it implies $\nu(x) \geq \nu(x^*)$ and (b) for all $y \in Y$, $\nu(y) \leq \nu(y^*)$.*

Note that at the very basis of the CE definition and welfare theorems there is an implicit assumption of perfect commitment and perfect enforcement. Note also that the FWT implicitly assumes there is no externality or public goods (achieves this implicit assumption by defining a consumer's utility function only on his own consumption set but no other points in the commodity space).

From the First Welfare Theorem, we know that if a Competitive Equilibrium exits, it is Pareto Optimal. Moreover, if the assumptions of the Second Welfare Theorem are satisfied and if the SPP has a unique solution then the competitive equilibrium allocation is unique and they are the same as the PO allocations. Prices can be constructed using this allocation and first order conditions.

Exercise 2 *Show that*

$$\frac{v_{2t}}{v_{1t}} = F_k(k_t, l_t) \text{ and } \frac{v_{3t}}{v_{1t}} = F_l(k_t, l_t).$$

One shortcoming of the AD equilibrium is that all trade occurs at the beginning of time. This assumption is unrealistic. Modern Economics is based on sequential markets. Therefore we define another equilibrium concept, Sequence of Markets Equilibrium (SME). We can easily show that SME is equivalent to ADE. Therefore all of our results still hold and SME is the right problem to solve.

Exercise 3 *Define a Sequential Markets Equilibrium (SME) for this economy. Prove that the objects we get from the AD equilibrium satisfy SME conditions and that the converse is also true. We should first show that a CE exists and therefore coincides with the unique solution of (SPP).*

Note that the (SPP) problem is hard to solve, since we are dealing with infinite number of choice variables. We have already established the fact that this SPP problem is equivalent to the following dynamic problem (removing leisure from now on):

$$\begin{aligned} v(k) = \max_{c,k'} \quad & u(c) + \beta v(k') \qquad (RSPP) \\ & \text{s.t. } c + k' = f(k). \end{aligned}$$

We have seen that this problem is easier to solve.

2.2 A Comment on the Welfare Theorems

Situations in which the welfare theorems would not hold include externalities, public goods, situations in which agents are not price takers (e.g. monopolies), some legal systems or lacking of markets which rule out certain contracts which appears complete contract or search frictions. So, What happens in such situations then? In this case the solutions to the social planners problem and the CE do not coincide and so we cannot use the theorems we have developed for dynamic programming to solve the problem. As we will see in this course, in this case we can work with Recursive Competitive Equilibria. In general, we can prove that the solution to the RCE coincides with a sequential markets problem but not the other way around (for example when we have multiple equilibria). However, in all the models we see in this course, this equivalence will hold.

3 Recursive Competitive Equilibrium

3.1 A Simple Example

What we have so far is that we have established the equivalence between allocation of the SPP problem which gives the unique Pareto optima (which is same as allocation of AD competitive equilibrium and allocation of SME). Therefore we can solve for the very complicated equilibrium allocation by solving the relatively easier Dynamic Programming problem of the social planner. One handicap of this approach is that in a lot of environments, the equilibrium is not Pareto Optimal and hence, not a solution of a social planner's problem, e.g. when you have taxes or externalities. Therefore, the above recursive problem would not be the right problem to solve. In some of these situations we can still write the problem in sequence form. However, we would lose the powerful computational techniques of dynamic programming. In order to resolve this issue we will define Recursive Competitive Equilibrium equivalent to SME that we can always solve for.

In order to write the decentralized household problem recursively, we need to use some equilibrium conditions so that the household knows what prices are as a function of some economy-wide aggregate state variable. We know that if capital is K_t and there is 1 unit of labor, then $w(K) = F_n(K, 1)$ and $R(K) = F_k(K, 1)$. Therefore, for the households to know prices they need to know aggregate capital. Now, a household who is deciding about how much to consume and how much to work has to know the whole sequence of future prices, in order to make his decision. This means that he needs to know the path of aggregate capital. Therefore, if he believes that aggregate capital changes according to $K' = G(K)$, knowing aggregate capital today, he would be able to project aggregate capital path for the future and therefore the path for prices. So, we can write the household problem given function $G(\cdot)$ as follows:

$$\begin{aligned} \Omega(K, a; G) = \max_{c,a'} \quad & u(c) + \beta\Omega(K', a'; G) \qquad\qquad (RCE) \\ \text{s.t.} \quad & c + a' = w(K) + R(K)a \\ & K' = G(K), \\ & c \geq 0 \end{aligned}$$

The above problem is the problem of a household that sees K in the economy, has a belief G, and carries a units of assets from past. The solution of this problem yields policy functions $c(K, a; G), a'(K, a; G)$ and a value function $\Omega(z, K, a; G)$. The functions $w(K), R(K)$ are obtained from the firm's FOCs (below).

$$\begin{aligned} & u_c[c(K, a; G)] = \beta\Omega_a[G(K), a'(K, a; G); G] \\ & \Omega_a[K, a; G] = (1 + r)u_c[c(K, a; G)] \end{aligned}$$

Now we can define the Recursive Competitive Equilibrium.

Definition 2 *A Recursive Competitive Equilibrium with arbitrary expectations G is a set of functions*[2] Ω, $g : \mathcal{K} \times \mathcal{A} \to \mathbb{R}$, R, w, $H : \mathcal{K} \to \mathbb{R}_+$ *such that:*

[2] We could add the policy function for consumption $g_c(K, a; G)$.

1. *given* G*;* Ω, g *solves the household problem in (RCE),*

2. $K' = H(K;G) = g(K,K;G)$ *(representative agent condition),*

3. $w(K) = F_n(K,1)$,

4. *and* $R(K) = F_k(K,1)$.

We define another notion of equilibrium where the expectations of the households are consistent with what happens in the economy:

Definition 3 *A Rational Expectations (Recursive) Equilibrium is a set of functions* Ω*,* g*,* R*,* w*,* G^**, such that:*

1. $\Omega(K,a;G^*), g(K,a;G^*)$ *solves HH problem in (RCE),*

2. $G^*(K) = g(K,K;G^*) = K'$*,*

3. $w(K) = F_n(K,1)$*,*

4. *and* $R(K) = F_k(K,1)$.

What this means is that in a REE, households optimize given what they believe is going to happen in the future and what happens in the aggregate is consistent with the household's decision. The proof that every REE can be used to construct a SME is left as an exercise. The reverse turns out not to be true. Notice that in REE, function G projects next period's capital. In fact, if we construct an equilibrium path based on REE, once a level of capital is reached in some period, next period capital is uniquely pinned down by the transition function. If we have multiplicity of SME, this would imply that we cannot construct the function G since one value of capital today could imply more than one value for capital tomorrow. We will focus on REE unless expressed otherwise.

Remark 1 *Note that unless otherwise stated, we will assume that depreciation rate* δ *is 1.* $R(K)$ *is the gross return on capital which is* $F_k(K,1) + 1 - \delta$*. Net return on capital is* $r(K) = F_k(K,1) - \delta$*.*

3.1.1 A Note on Envelope Theorem

To solve for RCE, we use envelope theorem. This method is valid because of time consistency of consumption choice.

By taking first order condition, we can get:

$$-u_c(c) + \beta V_2(K', a') = 0 \tag{1}$$

Let $a'^{\star}(K, a)$ be the solution for HH problem. Then we can write value function as:

$$V(K, a) = u(w(K) + R(K)a - a'^{\star}(K, a)) + \beta V(K', a'^{\star}(K, a))$$

Using implicit function theorem we can get the derivative with respect to a:

$$V_2(K, a) = R(K)u_c + \frac{\partial a'^{\star}(K, a)}{\partial a}\left[-u_c + \beta V_2(K', a'^{\star}(K, a))\right]$$

Term in the bracket in the right hand side is first order condition, hence it is zero. So we can get $V_2(K, a) = R(K)u_c$. Though, we need $V_2(K', a')$ to find optimal allocation. So we need to do the same for $V(K', a')$. But since tomorrow's incentive will not change we know that equation 1 will hold tomorrow. Hence $-u_c' + \beta V_2\Big[G(K), a'^{\star}(G(K), a'^{\star}(K, a))\Big] = 0$, which implies $V_2(K', a') = R(K')u_c'$.

To illustrate this point, consider an individual who wants to loose weight and decides whether to start diet or not. But he would like to eat well today and start diet tomorrow. Let 1 denotes that he obeys the diet restrictions, 0 no. So his preference ordering is:

1. $(0, 1, 1, 1...)$

2. $(1, 1, 1, 1...)$

3. $(0, 0, 0, 0, ...)$

Even though he promises himself that he will start diet tomorrow and chooses first option today, tomorrow he will face the same problem. So he will choose the same option tomorrow. This way he will never start diet and will end up with least preferred one: $(0, 0, 0, 0, ...)$.

However, in our model this is not the case. Agents preferences are time consistent, so what she promises today is optimal for her to do tomorrow. This way we can use envelope theorem.

3.2 Adding Uncertainty

3.2.1 Markov Processes

In this part, we want to focus on stochastic economies where there is a productivity shock affecting the economy. The stochastic process for productivity that we are assuming is a first order Markov Process that takes on finite number of values in the set $Z = \{z^1 < \cdots < z^{n_z}\}$. A first order Markov process implies

$$\Pr(z_{t+1} = z^j | h_t) = \Gamma_{ij}, \qquad z_t(h_t) = z^i$$

where h_t is the history of previous shocks. Γ is a Markov matrix with the property that the elements of its rows sum to 1.

Let μ be a probability distribution over initial states, i.e.

$$\sum_i \mu_i = 1$$

and $\mu_i \geq 0 \ \forall i = 1, ..., n_z$.

Next periods the probability distribution can be found by the formula: $\mu' = \Gamma^T \mu$.

If Γ is "nice" then $\exists$ a unique μ^* s.t. $\mu^* = \Gamma^T\mu^*$ and $\mu^* = \lim_{m\to\infty}(\Gamma^T)^m\mu_0$, $\forall \mu_0 \in \Delta^i$.

Γ induces the following probability distribution conditional on z_0 on $h_t = \{z^0, z^1, ..., z^t\}$:

$\Pi(\{z^0, z_1\}) = \Gamma_{i,.}$ for $z^0 = z_i$.

$\Pi(\{z^0, z_1, z_2\}) = \Gamma^T\Gamma_{i,.}$ for $z^0 = z_i$.

Then, $\Pi(h_t)$ is the probability of history h_t conditional on z^0. The expected value of z' is $\sum_{z'}\Gamma_{zz'}z'$ and $\sum_{z'}\Gamma_{zz'} = 1$.

3.2.2 Problem of the Social Planner

Let productivity affect the production function in a multiplicative fashion; i.e. technology is $zF(K, N)$, where z is the shock that follows a Markov chain on a finite state-space. The problem of the social planner problem (SPP) in sequence form is

$$\begin{aligned}
\max_{\{c_t(z^t), k_{t+1}(z^t)\}\in X(z^t)} \quad & \sum_{t=0}^{\infty}\sum_{z^t}\beta^t\pi(z^t)u(c_t(z^t)) \\
s.t. \quad & c_t(z^t) + k_{t+1}(z^t) = z_tF(k_t(z^{t-1}), 1),
\end{aligned}$$

where z_t is the realization of shock in period t, and z^t is the history of shocks up to (and including) time t. $X(z^t)$ is similar to the consumption possibility set defined earlier but this is after history z^t has occurred and is for consumption and capital.

Therefore, we can formulate the stochastic SPP in a recursive fashion as

$$\begin{aligned}
V(z_i, K) = \max_{c, K'} \quad & \left\{u(c) + \beta\sum_j\Gamma_{ij}V(z_j, K')\right\} \\
s.t. \quad & c + K' = z_iF(K, 1),
\end{aligned}$$

where Γ is the Markov transition matrix. The solution to this problem gives us a policy function of the form $K' = G(z, K)$.

In a decentralized economy, Arrow-Debreu equilibrium can be defined by:

$$\max_{\{c_t(z^t),k_{t+1}(z^t),x_{1t}(z^t),x_{2t}(z^t),x_{3t}(z^t)\}\in X(z^t)} \quad \sum_{t=0}^{\infty}\sum_{z^t}\beta^t\pi(z^t)u(c_t(z^t))$$
$$s.t. \quad \sum_{t=0}^{\infty}\sum_{z^t}p_t(z^t).x_t(z^t) \leq 0,$$

where $X(z^t)$ is again a variant of the consumption possibility set after history z^t has occurred. Ignore the overloading of notation. Note that we are assuming the markets are dynamically complete; i.e. there is complete set of securities for every possible history that can appear.

By the same procedure as before, SME can be written in the following way:

$$\max_{\{c_t(z^t),b_{t+1}(z^t,z_{t+1}),k_{t+1}(z^t)\}} \quad \sum_{t=0}^{\infty}\sum_{z^t}\beta^t\pi(z^t)u(c_t(z^t))$$
$$s.t. \quad c_t(z^t) + k_{t+1}(z^t) + \sum_{z_{t+1}} b_{t+1}(z^t, z_{t+1})q_t(z^t, z_{t+1})$$
$$= k_t(z^{t-1})R_t(z^t) + w_t(z^t) + b_t(z^{t-1}, z_t)$$
$$b_{t+1}\left(z^t, z_{t+1}\right) \geq -B.$$

To replicate the AD equilibrium, here, we have introduced Arrow securities to allow agents to trade with each other against possible future shocks.

However, in equilibrium and when there is no heterogeneity, there will be no trade, i.e $b_{t+1}(z^t, z_{t+1}) = 0$ for any z^t, z_{t+1}. Moreover, we have two ways of delivering the goods specified in an Arrow security contract: after production and before production. In an after production setting, the goods will be delivered after production takes place and can only be consumed or saved for the next period. This is the above setting. It is also possible to allow the consumer to rent the Arrow security income as capital to firms, which will be the before production setting.

An important condition which must hold true in the *before production setting* is the no-arbitrage condition:

$$\sum_{z_{t+1}} q_t(z^t, z_{t+1}) = 1$$

Exercise 4 *Describe the AD problem, in particular the consumption possibility set X and the production set Y.*

Exercise 5 *Every equilibrium achieved in AD settings can also be achieved in a SM setting, by the relation where*

$$q_t(z^t, z_{t+1}) = p_{1t+1}(z^t, z_{t+1})/p_{1t}(z^t),$$

$$R_t(z^t) = p_{2t}(z^t)/p_{1t}(z^t),$$

and

$$w_t(z^t) = p_{3t}(z^t)/p_{1t}(z^t).$$

Check that from the FOC's, the same allocations result in the two settings.

Exercise 6 *The problem above state contingent goods are delivered in terms of consumption goods. Instead of this assume they are delivered in terms of capital goods. Show that the same allocation would be achieved in both settings.*

3.2.3 Recursive Competitive Equilibrium

Assume that households can trade state contingent assets, as in the sequential market case. We can write a household's problem in recursive form as:

$$
\begin{aligned}
V(K,z,a) = \max_{c,k',d(z')} \quad & \left\{ u(c) + \beta \sum_{z'} \Gamma_{zz'} V(K', z', a'(z')) \right\} \\
s.t. \quad & c + k' + \sum_{z'} d(z')\, q_{z'}(K,z) = w(K,z) + aR(K,z) \\
& K' = G(K,z) \\
& a'(z') = k' + d(z').
\end{aligned}
$$

Exercise 7 *Write the first order conditions for this problem, given prices and the law of motion for aggregate capital.*

Solving this problem gives policy functions. So, a RCE in this case is a collection of functions V, c, k', d, G, w, and R, so that

1. given G, w, and R, V solves household's functional equation, with c, k' and d as the associated policy function,
2. $d(K,z,K,z') = 0$, for all z',
3. $k'(K,z,K) = G(K,z)$,
4. $w(K,z) = zF_n(K,1)$ and $R(K,z) = zF_k(K,1)$,
5. and $\sum_{z'} q_{z'}(K,z) = 1$.

The last condition is known as the no-arbitrage condition (recall that we had this equation in the case of sequential markets as well). To see why this is a necessary equation in the equilibrium, note that an agent can either save in the form of capital, or Arrow securities. However, these two choices must cost the same. This implies Condition 5 above.

Remark 2 *Note that in a sequence version of the household problem in SME, in order for households not to achieve infinite consumption, we need a no-Ponzi condition; a condition that prevents Ponzi schemes is*

$$\lim_{t\to\infty} \frac{a_t}{\prod_{s=0}^{t} R_s} < \infty.$$

This is the weakest condition that imposes no restrictions on the first order conditions of the household's problem. It is harder to come up with its analogue for the recursive case. One possibility is to assume that a' lies in a compact set $\mathcal{A}$, or a set that is bounded from below.[3]

3.3 Economy with Government Expenditures

3.3.1 Lump Sum Tax

The government levies each period T units of goods in a lump sum way and spends it in a public good, say medals. Assume consumers do not care about medals. The household's problem becomes:

$$\begin{aligned} V(K,a) = \max_{c,a'} \quad & \{u(c) + \beta V(K',a')\} \\ s.t. \quad & c + a' + T = w(K) + aR(K) \\ & K' = G(K). \end{aligned}$$

A solution of this problem are functions $g_a^*(K,a;G,M,T)$ and $\Omega(K,a;G)$ and the equilibrium can be characterized by $G^*(K,M,T) = g_a^*(K,K;G^*,M,T)$ and $M^* = T$ (the government budget constraint is balanced period by period). We will write a complete definition of equilibrium for a version with government debt (below).

Note that the equilibrium will be optimal. But if consumers cared about medals, the equilibrium will not be optimal in general.

[3] We must specify $\mathcal{A}$ such that the borrowing constraint implicit in $\mathcal{A}$ is never binding.

Exercise 8 *Define* $\hat{f}(K,1) = f(K,1) - M$ *for the planner. Show that the equilibrium is optimal when consumers do not care about medals.*

3.3.2 Labor Income Tax

We have an economy in which the government levies tax on labor in order to purchase medals. Medals are goods which provide utility to the consumers.

$$
\begin{aligned}
V(K,a) = \max_{c,a'} \quad & \{u(c,M) + \beta V(K',a')\} \\
s.t. \quad & c + a' = (1-\tau(K))\,w(K) + aR(K) \\
& K' = G(K),
\end{aligned}
$$

given $M = \tau(K)w(K)$.

Since leisure is not valued, the labor decision stays trivial. Hence, there is no distortion due to taxes and CE is Pareto optimal. This will also hold when medals do not provide any utility to the consumers.

Exercise 9 *Is there any change in the above implications of optimality if the tax rate is a function of aggregate capital?*

Exercise 10 *Suppose medals do not provide utility to agents but leisure does. Is CE optimal now? Is it distorted? What if medals also provide utility?*

3.3.3 Capital Income Tax

Now let us look at an economy in which the government levies tax on capital in order to purchase medals. Medals are goods which provide utility to the consumers.

$$\begin{aligned} V(K,a) = \max_{c,a'} \quad & \{u(c,M) + \beta V(K',a')\} \\ s.t. \quad & c + a' = w(K) + a[1 + r(K)(1-\tau(K))] \\ & K' = G(K), \end{aligned}$$

given $M = \tau(K) r(K) K$ and $R(K) = 1 + r(K)$. Now, the First Welfare Theorem is no longer applicable; the CE will not be Pareto optimal anymore (if $\tau > 0$ there will be a wedge, and the efficiency conditions will not be satisfied).

Exercise 11 *Derive the first order conditions in the above problem to see the wedge introduced by taxes.*

3.3.4 Taxes and Debt

Assume that government can issue debt and use taxes to finance its expenditures, and these expenditures do affect the utility.

A government policy consists of capital taxes, spending (medals) as well as bond issuance. When the aggregate states are K and B, as you will see why, then a government policy (in a recursive world!) is

$$\tau(K,B),\ M(K,B) \text{ and } B'(K,B).$$

For now, we shall assume these values are chosen so that the equilibrium exists. In this environment, debt issued is relevant for the household because it permits him to correctly infer the amount of taxes. Therefore the household needs to form expectations about the future level of debt from the government.

The government budget constraint now satisfies (with taxes on labor income):

$$B + M(K, B) = KR(K)\tau(K, B) + q(K, B)B'(K, B)$$

Notice that the household does not care about the composition of his portfolio as long as assets have the same rate of return which is true because of the no arbitrage condition. Therefore, the problem of a household with assets equal to a is given by:

$$\begin{aligned} V(K, B, a) = \max_{c,a'} \quad & \{u(c, M(K, B)) + \beta V(K', B', a')\} \\ s.t. \quad & c + a' = aR(K)(1 - \tau(K, B)) + w(K) \\ & K' = G(K, B) \\ & B' = H(K, B). \end{aligned}$$

Let $g(K, B, a)$ be the policy function associated with this problem. Then, we can define a RCE as follows.

Definition 4 *A* Rational Expectations Recursive Competitive Equilibrium, *given policies $M(K, B)$ and $\tau(K, B)$ is a set of functions V, g, G, H, w, and R, such that*

1. *V and g solve the household's problem,*
2. *$w(K) = F_2(K, 1)$ and $R(K) = F_1(K, 1)$,*
3. *$g(K, B, K + q(K^-, B^-)B) = G(K, B) + q(K, B)H(K, B)$,*
4. *No arbitrage condition*

$$[1 - \tau(G(K, B), H(K, B))]\, R(G(K)) = \frac{1}{q(K, B)}$$

5. *Government's budget constraint is satisfied*

$$B + M(K, B) = KR(K)\tau(K, B) + q(K, B)H(K, B)$$

6. *and, government debt is bounded; i.e. there exists some* $\bar{B}$ *so that for all* $K \in \left[0, \widetilde{k}\right)$, $H(K, B) \leq \bar{B}$.

4 Some Other Examples

4.1 A Few Popular Utility Functions

Consider the following three utility forms:

1. $u(c, c^-)$: this function is called *habit formation* utility function; utility is increasing in consumption today, but, decreasing in the deviations from past consumption (e.g. $u(c, c^-) = v(c) - (c - c^-)^2$). In this case, the aggregate states in a standard growth model are K and C^-, and individual states are a and c^-. Is the equilibrium optimum in this case?

2. $u(c, C^-)$; this form is called *catching up with Jones*; there is an externality from the aggregate consumption to the payoff of the agents. Intuitively, in this case, agents care about what their neighbors consume. Aggregate states in this case are K and C^-. But, c^- is no longer an individual state.

3. $u(c, C)$: the last function is called keeping up with Jones. Here, the aggregate state is K; C is no longer a pre-determined variable to appear as a state.

4.2 An Economy with Capital and Land

Consider an economy with capital and land but without labor; a firm in this economy buys and installs capital. They also own one unit of land, that they use in production, according to the production function $F(K, L)$. In other words, a firm is a "chunk of land of are one", in which firm installs its capital. Share of these firms are traded in a stock market.

A household's problem in this economy is given by:

$$
\begin{aligned}
V(K,a) = \max_{c,a'} \quad & \{u(c) + \beta V(K',a')\} \\
s.t. \quad & c + a' = R(K)\, a. \\
& K' = G(K)
\end{aligned}
$$

On the other hand, a firm's problem is

$$
\begin{aligned}
\Omega(K,k) = \max_{k'} \quad & \{F(k,1) - k' + q(K')\, \Omega(K',k')\} \\
s.t. \quad & K' = G(K).
\end{aligned}
$$

Ω here is the *value of the firm,* measured in units of output, today. Therefore, the value of the firm, tomorrow, must be discounted into units of output today. This is done by a discount factor $q(K')$.

A Recursive Competitive Equilibrium consists of functions, V, Ω, g, h, q, G, and R, so that:

1. V and g solve household's problem,
2. Ω and h solve firm's problem,
3. $G(K) = h(K,K)$, and,
4. $q(G(K))\, \Omega(G(K), G(K)) = g\left(K, \frac{\Omega(K,K)}{R(K)}\right)$.

Exercise 12 *One condition is missing in the definition of the RCE above. Find it!* [Hint: it relates the rate of return on the household's assets to the discount rate of firm's value.]

5 Adding Heterogeneity

In the previous section we looked at situations in which recursive competitive equilibria (RCE) were useful. In particular these were situations in which the welfare theorems failed and so we could not use the standard dynamic programming techniques learned earlier. In this section we look at another way in which RCE are helpful, in models with heterogeneous agents.

5.1 Heterogeneity in Wealth

First, let us consider a model in which we have two types of households that differ only in the amount of wealth they own. Say there are two types of agents, labeled type 1 and 2, of equal measure of $1/2$. Agents are identical other than their initial wealth position and there is no uncertainty in the model. The problem of an agent with wealth a is given by

$$
\begin{aligned}
V\left(K^1, K^2, a\right) = \max_{c,a'} \quad & \left\{u\left(c\right) + \beta V\left(K'^1, K'^2, a'\right)\right\} \\
s.t. \quad & c + a' = R\left(\frac{K^1 + K^2}{2}\right) a + W\left(\frac{K^1 + K^2}{2}\right) \\
& K'^i = G^i\left(K^1, K^2\right), \quad i = 1, 2.
\end{aligned}
$$

Remark 3 *Note that (in general) the decision rules of the two types of agents are not linear (even though they might be* almost *linear); therefore, we cannot add the two states, K^1 and K^2, to write the problem with one aggregate state, in the recursive form.*

Definition 5 *A* Rational Expectations Recursive Competitive Equilibrium *is a set of functions V, g, R, w, G^1, and G^2, so that:*

1. *V solves the household's functional equation, with g as the associated policy function,*

2. *w and R are the marginal products of labor and capital, respectively (watch out for arguments!),*

3. *representative agent conditions are satisfied; i.e.*

$$g(K^1, K^2, K^1) = G^1(K^1, K^2),$$

and

$$g(K^1, K^2, K^2) = G^2(K^1, K^2).$$

Remark 4 *Note that $G^1(K^1, K^2) = G^2(K^2, K^1)$ (why?).*

Remark 5 *This is a variation of the simple neoclassical growth model; what does the growth model say about inequality?*

In the steady state of a neoclassical growth model, Euler equations for the two types simplify to

$$u'\left(c^1\right) = \beta R u'\left(c^1\right), \text{ and } u'\left(c^2\right) = \beta R u'\left(c^2\right).$$

Therefore, we must have $\beta R = 1$, where

$$R = F_K\left(\frac{K^1 + K^2}{2}, 1\right).$$

Finally, by the household's budget constraint, we must have:

$$k^i R + W = c^i + k^i,$$

where $k^i = K^i$, by representative agent's condition. Therefore, we have three equation, with four unknowns (k^i and c^i's). This means, this theory is silent about the distribution of wealth in the steady state!

5.2 Heterogeneity in Skills

Now, consider a slightly different economy where type i has labor skill ϵ_i. Measures of agents' types, μ^1 and μ^2, satisfy $\mu^1\epsilon_1 + \mu^2\epsilon_2 = 1$ (below we will consider the case where $\mu^1 = \mu^2 = 1/2$).

The question we have to ask ourselves is, would the value functions of two types remain to be the same, as in the previous subsection? The answer turns out to be no!

The problem of the household $i \in \{1, 2\}$ can be written as follows:

$$
\begin{aligned}
V^i\left(K^1, K^2, a\right) = \max_{c,a'} \quad & \left\{u\left(c\right) + \beta V^i\left(K'^1, K'^2, a'\right)\right\} \\
s.t. \quad & c + a' = R\left(\frac{K^1 + K^2}{2}\right) a + W\left(\frac{K^1 + K^2}{2}\right) \epsilon_i \\
& K'^j = G^j\left(K^1, K^2\right), \quad j = 1, 2.
\end{aligned}
$$

Notice that we have indexed the value function by the agent's type; the reason is that the marginal product of the labor supplied by each of these types is different.

Remark 6 *We can rewrite this problem as*

$$
\begin{aligned}
V^i\left(K, \lambda, a\right) = \max_{c,a'} \quad & \left\{u\left(c\right) + \beta V^i\left(K', \lambda', a'\right)\right\} \\
s.t. \quad & c + a' = R\left(K\right) a + W\left(K\right) \epsilon_i \\
& K = G\left(K, \lambda\right) \\
& \lambda' = H\left(K, \lambda\right),
\end{aligned}
$$

where K is the total capital in this economy, and λ is the share of one type in this wealth (e.g. type 1).

Then, if g^i is the policy function of type i, in the equilibrium, we must have:

$$
G\left(K, \lambda\right) = g^1\left(K, \lambda, \lambda K\right) + g^2\left(K, \lambda, \left(1 - \lambda\right) K\right),
$$

and

$$
H\left(K, \lambda\right) = g^1\left(K, \lambda, \lambda K\right) / G\left(K, \lambda\right).
$$

5.3 An International Economy Model

In an international economy model the specifications which determine the definition of country is an important one; we can introduce the idea of different locations or geography; countries can be victims of different policies; trade across countries maybe more difficult due to different restrictions.

Here we will see a model with two countries, 1 and 2, such that labor is not mobile between the countries, but with perfect capital markets. However, in order to use a product in production, it must have been installed in advanced. So the resource constraint is:

$$C^1 + C^2 + K'^1 + K'^2 = F(K^1, 1) + F(K^2, 1)$$

Suppose that there is a mutual fund that owns the firms in each country and its shares are traded in the market. So, as in the economy with capital and land, individuals own shares of this mutual fund.

The first question to ask, as usual, is *what are the appropriate states in this world?* As it is apparent from the resource constraint and production functions, we need the capital in each country. Moreover, we need to know total wealth in each country. Therefore, we need an additional variable as the aggregate state; shares owned by country 1.

As a result, we can write the country i's problem as:

$$\begin{aligned} V^i\left(z_1, z_2, K_1, K_2, A, a\right) = \max_{c, a'(z)} \quad & \left\{ u\left(c\right) + \beta \sum_{\vec{z}'} \Gamma_{\vec{z}\vec{z}'} V^i\left(\vec{z}_{\prime}, \vec{K}', A'(\vec{z}'), a'(\vec{z}')\right) \right\} \\ s.t. \quad & c + \sum_{\vec{z}'} q(\vec{z}, \vec{K}, A, \vec{z}')a'(\vec{z}') = w^i\left(z_i, K_i\right) + a\Phi(\vec{z}, \vec{K}) \\ & K_i' = G_i\left(\vec{z}, \vec{K}, A\right), i = 1, 2 \\ & A(\vec{z}') = H\left(\vec{z}, \vec{K}, A, \vec{z}'\right) \end{aligned}$$

where A is the total amount of shares that individuals in country 1 own and a is the share that an individual owns in country i.

Exercise 13 *Write this economy with state contingent consumption good claims in own country.*

Exercise 14 *Write this economy where individuals can move in advanced freely, but there is incomplete market.*

Since labor is immobile and capital installed in advanced and wage is simply marginal return to production it only depends on country level shock and capital: $w^i(z_i, K_i) = z_i F_N(K_i, 1)$

Let h^i be the optimal choice of share holding. By market clearing condition sum of shares must be 1:

$$h^1\left(\vec{z}, \vec{K}, A, A, \vec{z}'\right) + h^2\left(\vec{z}, \vec{K}, A, 1-A, \vec{z}'\right) = 1 \quad \forall \vec{z}'$$

In rational expectations equilibrium, representative agent condition must hold:

$$H\left(\vec{z}, \vec{K}, A, A, \vec{z}'\right) = h^1\left(\vec{z}, \vec{K}, A, A, \vec{z}'\right)$$

Now let's look at the net present value of the mutual fund:

$$\Phi(\vec{z}, \vec{K}, A) = \sum_{z_i}\left[z_i F(K_i, 1) - w^i(z_i, K_i)\right] - \sum_i G_i(\vec{z}, \vec{K}, A) + \sum_{z_i} \Gamma_{\vec{z}\vec{z}'} Q(\vec{z}', G(\vec{K}), H(.))\Phi(\vec{z}', G(\vec{K}), H(.)) \tag{2}$$

where Q is inter temporal prices. In the equilibrium, this should satisfy:

$$q(\vec{z}, \vec{K}, A, \vec{z}') = Q(\vec{z}', G(\vec{K}), H(.))\Phi(\vec{z}', G(\vec{K}), H(.))$$

We also need feasibility condition for consumption and investment:

$$\sum_i \left[z_i F(K_i, 1) - G_i(\vec{z}, \vec{K}, A) - c_i \right] = 0$$

Exercise 15 *There is one more condition for G_i that equates expected return in each country. Can you write it?*

Definition 6 *A Recursive Competitive Equilibrium for the (world's) economy is a set of functions, V^i, h^i, w^i, and Φ for $i \in \{1, 2\}$, and q, G,H and Q, such that the following conditions hold:*

1. *V^i and h^i solve the household's problem in country i ($i \in \{1, 2\}$),*
2. *$h^1 \left(\vec{z}, \vec{K}, A, A, \vec{z}'\right) + h^2 \left(\vec{z}, \vec{K}, A, 1 - A, \vec{z}'\right) = 1 \ \forall \vec{z}'$,*
3. *$H \left(\vec{z}, \vec{K}, A, A, \vec{z}'\right) = h^1 \left(\vec{z}, \vec{K}, A, A, \vec{z}'\right)$,*
4. *w^i is equated to the marginal products of labor in each country,*
5. *$q(\vec{z}, \vec{K}, A, \vec{z}') = Q(\vec{z}', G(\vec{K}), H(.))\Phi(\vec{z}', G(\vec{K}), H(.))$,*
6. *Φ satisfies 2*
7. *Resource constraint,*
8. *Expected rate of return on capital is same across countries*

5.4 Heterogeneity in Wealth and Skills with Complete Markets

Now, let us consider a model in which we have two types of households that care about leisure differ in the amount of wealth they own and labor skill. Also, there is uncertainty and Arrow securities like we have seen before.

Let A^1 and A^2 be the aggregate asset holdings of the two types of agents. These will now be state variables for the same reason that K^1 and K^2 were state variables earlier. The problem of an agent

$i \in \{1,2\}$ with wealth a is given by

$$
\begin{aligned}
V^i\left(z, A^1, A^2, a\right) = \max_{c,n,a'(z')} \quad & \left\{ u\left(c,n\right) + \beta \sum_{z'} \Gamma_{zz'} V\left(z', A'^1, A'^2, a'(z')\right) \right\} \\
s.t. \quad & c + \sum_{z'} a'\left(z'\right) q_{z'}\left(z, A^1, A^2\right) = R\left(z,K,N\right) a + W\left(z,K,N\right) n\epsilon_i \\
& A'^i(z') = G^i\left(z, A^1, A^2, z'\right), \quad i = 1,2, \forall z' \\
& N = H\left(z, A^1, A^2\right) \\
& K = \frac{A^1 + A^2}{2}.
\end{aligned}
$$

Let g^i and h^i be the asset and labor policy functions be the solution to this problem. Then, we can define the RCE as below.

Definition 7 *A* Recursive Competitive Equilibrium with Complete Markets *is a set of functions V^i, g^i, h^i, G^i, R, w, H, and q, so that:*

1. *V^i, g^i and h^i solve the problem of household i ($i \in \{1,2\}$),*
2. *$H\left(z, A^1, A^2\right) = \epsilon_1 h^1\left(z, A^1, A^2, A^1\right) + \epsilon_2 h^2\left(z, A^1, A^2, A^2\right)$,*
3. *$G^i\left(z, A^1, A^2, z'\right) = g^i\left(z, A^1, A^2, A^i, z'\right) \quad i = 1,2, \forall z'$*
4. *$\sum_{z'} q_{z'}\left(z, A^1, A^2, z'\right) = 1$,*
5. *$G^1\left(z, A^1, A^2, z'\right) + G^2\left(z, A^1, A^2, z'\right)$ is independent of z' (due to market clearing).*
6. *R and W are the marginal products of capital and labor.*

Exercise 16 *Write down the household problem and the definition of RCE with non-contingent claims instead of complete markets.*

6 Asset Pricing: Lucas Tree Model

We now turn to the simplest of all models in term of allocations as they are completely exogenous, *the Lucas tree model*. We want to characterize the properties of prices that are capable of inducing households to consume the endowment.

6.1 The Lucas Tree with Random Endowments

Consider an economy in which the only asset is a tree that gives fruit. The agents problem is

$$V(z,s) = \max_{c,s'} \quad \left\{ u(c) + \beta \sum_{z'} \Gamma_{zz'} V(z', s') \right\}$$
$$s.t. \quad c + p(z)\, s' = s\left[p(z) + d(z)\right],$$

where $p(z)$ is the price of the shares (to the tree), in state z, and $d(z)$ is the dividend associated with state z.

Definition 8 *A Rational Expectations Recursive Competitive Equilibrium is a set of functions, V, g, d, and p, such that*

1. *V and g solves the household's problem,*
2. *$d(z) = z$, and,*
3. *$g(z,1) = 1$, for all z.*

To explore the problem more, note that the first order conditions for the household's problem imply:

$$u_c(c(z,1)) = \beta \sum_{z'} \Gamma_{zz'} \left[\frac{p(z') + d(z')}{p(z)} \right] u_c(c(z',1)).$$

As a result, if we let $u_c(z) := u_c(c(z,1))$, we get:

$$p(z)\,u_c(z) = \beta \sum_{z'} \Gamma_{zz'} u_c(z')\,[p(z') + z'].$$

Exercise 17 *Derive the Euler equation for household's problem.*

Notice that this is just a system of n equations with unknowns $\{p(z_i)\}_{i=1}^n$. We can use the power of matrix algebra to solve it. To do so, let:

$$\mathbf{p} := \begin{bmatrix} p(z_1) \\ \vdots \\ p(z_n) \end{bmatrix},$$

and

$$\mathbf{u}_c := \begin{bmatrix} u_c(z_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & u_c(z_n) \end{bmatrix}.$$

Then

$$\mathbf{u}_c.\mathbf{p} = \begin{bmatrix} p(z_1)\,u_c(z_1) \\ \vdots \\ p(z_n)\,u_c(z_n) \end{bmatrix},$$

and

$$\mathbf{u}_c.\mathbf{z} = \begin{bmatrix} z_1 u_c(z_1) \\ \vdots \\ z_n u_c(z_n) \end{bmatrix},$$

Now, rewrite the system above as

$$\mathbf{u}_c\mathbf{p} = \beta\Gamma\mathbf{u}_c\mathbf{z} + \beta\Gamma\mathbf{u}_c\mathbf{p},$$

where Γ is the transition matrix for z, as before. Hence, the price for the shares is given by

$$\mathbf{u}_c\mathbf{p} = (\mathbf{I} - \beta\Gamma)^{-1}\beta\Gamma\mathbf{u}_c\mathbf{z},$$

or

$$\mathbf{p} = \begin{bmatrix} u_c(z_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & u_c(z_n) \end{bmatrix}^{-1} (\mathbf{I} - \beta\Gamma)^{-1}\beta\Gamma\mathbf{u}_c\mathbf{z}.$$

6.2 Asset Pricing

Consider our simple model of Lucas tree with fluctuating output; what is the definition of an asset in this economy? It is "a claim to a chunk of fruit, sometime in the future".

If an asset, a, promises an amount of fruit equal to $a_t(z^t)$ after history $z^t = (z_0, z_1, \ldots, z_t)$ of shocks, after a set of (possible) histories in H, the price of such an entitlement in date $t = 0$ is given by:

$$p(a) = \sum_t \sum_{z^t \in H} q_t^0(z^t)\, a_t(z^t),$$

where $q_t^0(z^t)$ is the price of one unit of fruit after history z^t, in today's "dollars"; this follows from a no-arbitrage argument. If we have the date $t = 0$ prices, $\{q_t\}$, as functions of histories, we can *replicate any possible asset by a set of state-contingent claims*, and use this formula to price that asset.

To see how we can find prices at date $t = 0$, consider a world in which the agent wants to solve

$$\max_{c_t(z^t)} \quad \left\{ \sum_{t=0}^{\infty} \beta^t \sum_{z^t} \pi_t\left(z^t\right) u\left(c_t\left(z^t\right)\right) \right\}$$
$$s.t. \quad \sum_{t=0}^{\infty} \sum_{z^t} q_t^0\left(z^t\right) c_t\left(z^t\right) \leq \sum_{t=0}^{\infty} \sum_{h^t} q_t^0\left(z^t\right) z_t.$$

This is the familiar Arrow-Debreu market structure, where the household owns a tree, and the tree can yield $z \in Z$ amount of fruit in each period. The first order condition for this problem implies:

$$q_t^0\left(z^t\right) = \beta^t \pi_t\left(z^t\right) \frac{u_c\left(z_t\right)}{u_c\left(z_0\right)}.$$

This enables us to price the good in each history of the world, and price any asset accordingly.

Comment 1 *What happens if we add state contingent shares into our recursive model? Then the agent's problem becomes:*

$$V(z, s, b) = \max_{c, s', b'(z')} \quad \left\{ u(c) + \beta \sum_{z'} \Gamma_{zz'} V\left(z', s', b'\left(z'\right)\right) \right\}$$
$$s.t. \quad c + p(z) s' + \sum_{z'} q\left(z, z'\right) b'\left(z'\right) = s\left[p(z) + z\right] + b.$$

A characterization of q can be written as:

$$q\left(z, z'\right) u_c(z) = \beta \Gamma_{zz'} u_c\left(z'\right).$$

We can price all types *of securities using p and q in this economy.*

To see how we can price an asset, consider *the option to sell tomorrow* at price P, if today's shock is z, as an example; the price of such an asset today is

$$\hat{q}(z, P) = \sum_{z'} \max\left\{P - p\left(z'\right), 0\right\} q\left(z, z'\right).$$

The American option to sell at price P either tomorrow or the day after tomorrow is priced as:

$$\widetilde{q}(z,P)=\sum_{z'}\max\left\{P-p\left(z'\right),\hat{q}\left(z',P\right)\right\}q\left(z,z'\right).$$

Similarly, a European option to buy at price P the day after tomorrow is priced as:

$$\bar{q}(z,P)=\sum_{z'}\sum_{z''}\max\left\{p\left(z''\right)-P,0\right\}q\left(z',z''\right)q\left(z,z'\right).$$

Note that $R(z)=\left[\sum_{z'}q\left(z,z'\right)\right]^{-1}$ is the gross risk free rate, given today's shock is z. The unconditional gross risk free rate is then given as $R^f=\sum_z \mu_z^* R(z)$ where μ^* is the steady-state distribution of the shocks defined earlier.

The average gross rate of return on the stock market is $\sum_z \mu_z^* \sum_{z'} \Gamma_{zz'}\left[\frac{p(z')+z'}{p(z)}\right]$ and the risk premium is the difference between this rate and the unconditional gross risk free rate.

Exercise 18 *Use the expressions for p and q and the properties of the utility function to show that risk premium is positive.*

6.3 Taste Shocks

Consider an economy in which the only asset is a tree that gives fruits. The fruit is constant over time (normalized to 1) but the agent is subject to preference shocks for the fruit each period, $\theta\in\Theta$. The agent's problem in this economy is

$$\begin{aligned}V(\theta,s)=\max_{c,s'}&\quad\left\{\theta u(c)+\beta\sum_{\theta'}\Gamma_{\theta\theta'}V\left(\theta',s'\right)\right\}\\ s.t.&\quad c+p(\theta)s'=s\left[p(\theta)+d(\theta)\right].\end{aligned}$$

The equilibrium is defined as before; the only difference is that, now, we must have $d(\theta) = 1$. What does it mean that the output of the economy is constant, and fixed at one, but, the tastes in this output changes? In this settings, the function of the price is to convince agents keep their consumption constant.

All the analysis follows through, once we write the FOC's characterizing price, $p(\theta)$, and state contingent prices $q(\theta, \theta')$.

This is a simple model, in a sense that household does not have a real choice. Due to market clearing, household consumes what nature provides her. In each period, according to state productivity z and taste θ, price adjusts such that household would like to consume z, amount of fruit that nature provides. Hence output is equal to z. If we look at the business cycle in this economy, only fluctuation for the output would be caused by nature. Here everything is determined by the supply side, demand side has no impact.

In next section, we are going to introduce search friction to incorporate demand side into the model.

7 Endogenous Productivity in a Product Search Model

Let's model the situation where households need to find the fruit before consuming it; [4] assume that households have to find the tree in order to consume the fruit. Finding trees is characterized by a constant returns to scale (increasing) matching function $M(T, D)$,[5] where T is the *number of trees* and D is the *shopping effort,* exerted by households when searching. Thus, the probability that a tree finds a shopper is $M(T, D)/T$; total number of matches, divided by the number of trees. And, the probability that a unit of shopping effort finds a tree is $M(T, D)/D$.

We further assume that M takes the form $D^{\varphi}T^{1-\varphi}$, and denote the probability of finding a tree

[4] Think of fields in *The Land of Apples*, full of apples, that ,are owned by firms; agents have to buy the apples. In addition, they have to search for them as well!

[5] What does the fact that M is constant returns to scale imply?

by $\Psi^h(Q) := Q^{1-\varphi}$, where $1/Q := D/T$ is the ratio of shoppers per trees, capturing *the market tightness*; the more the number of people searching, the smaller the probability of finding a tree. Then, $\Psi^f(Q) := Q^{-\varphi}$. Note that, in this economy, the number of trees is constant, and equal to one.[6]

Let us assume households face a demand side shock θ and a supply side shock z. They are independent Markov process with transitional probability $\Gamma_{\theta\theta'}$ and $\Gamma_{zz'}$, respectively. Households choose the consumption level c, the search effort d, and the shares of the tree to hold next period s'. The household's problem can be written as:

$$V(\theta, z, s) = \max_{c,d,s'} \quad \left\{ u(c, d, \theta) + \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} V(\theta', z', s') \right\} \tag{3}$$

$$s.t. \quad c = d\Psi^h(Q(\theta, z))\, z \tag{4}$$

$$c + P(\theta, z)\, s' = P(\theta) \left[s\left(1 + \hat{R}(\theta, z)\right) \right], \tag{5}$$

where P is the price of tree relative to that of consumption, and $\hat{R}$ is the dividend income (in units of tree). d is the amount of search the individual household exerts to acquire fruit.

Note some notation conventions here. $P(\theta, z)$ is in terms of consumption goods while $\hat{R}(\theta, z)$ is in terms of shares of the tree (that's why we equip it with a hat). We could also write the household budget constraint in terms of the price of consumption relative to that of the tree. Let's define $\hat{P}(\theta) = \frac{1}{P(\theta)}$: the price of consumption goods in terms of the tree. Then the budget constraint can be put as:

$$c\hat{P}(\theta, z) + s' = s\left(1 + \hat{R}(\theta, z)\right)$$

Let's maintain our notation with $P(\theta, z)$ and $\hat{R}(\theta, z)$ since now. If we substitute the constraints into the objective and solve for d, we get the Euler equation for the household. Using the market clearing

[6] It is easy to find the statements for Ψ^h and Ψ^f, given the Cobb-Douglas matching function:

$$\Psi^h(Q) = \frac{D^\varphi T^{1-\varphi}}{D} = \left(\frac{T}{D}\right)^{1-\varphi} = Q^{1-\varphi},$$

$$\Psi^f(Q) = \frac{D^\varphi T^{1-\varphi}}{T} = \left(\frac{T}{D}\right)^{-\varphi} = Q^{-\varphi}.$$

The question is, is Cobb-Douglas an appropriate choice for the matching function, or its choice is a matter of simplicity?

condition in equilibrium, the problem reduces to one equation and two unknowns, P and Q (other objects, C,D and $\hat{R}$, are known functions of P and Q). We still need another functional equation to find the equilibrium. We now turn to one way of doing so.

Exercise 19 *Derive the Euler equation of the household from the problem defined above.*

7.1 Competitive Search

Competitive search is a particular search protocol of what is called non-random (or directed) search. To understand this protocol, consider a world consisting a large number of islands. Each island has a sign that reads two number, W and Q. W is the wage rate on the island, and Q is a measure of market tightness in that island, or the number of workers on the island divided by the number of job opportunities. Both workers and firms have to decide to go to one island. In an island with higher wage, the worker might be happier, conditioned on finding a job. However, the probability of finding a job might be low on the island, depending on the tightness of the labor market on that island. The same story holds for the job owners, who are searching to hire workers.

In our economy, both firms and workers search for specific markets indexed by price P and market tightness Q. A pair of P, Q is operational if there exist some consumer and firm that choose that market. Therefore, an agent should choose P and Q such that it gives sufficient profit to firm, which will be determined in the equilibrium. Competitive search is magic. It does not presuppose a particular pricing protocol (wage posting, bargaining) that other search protocols need.

Maintain the demand shock θ and supply side shock z we just introduced, we can then define the household problem with competitive search as following:

$$V(\theta,z,s)=\max_{c,d,s',P,Q}\quad\left\{u(c,d,\theta)+\beta\sum_{\theta',z'}\Gamma_{\theta\theta'}\Gamma_{zz'}V(\theta',z',s')\right\}\tag{6}$$

$$s.t.\quad c=d\Psi^{h}(Q)\,z\tag{7}$$

$$c+Ps'=P\left[s\left(1+\hat{R}(\theta,z)\right)\right],\tag{8}$$

$$\frac{z\Psi^{f}(Q)}{P}\geq\hat{R}(\theta,z)\tag{9}$$

Let $u(c,d,\theta)=u(\theta c,d)$ from here on. Last constraint provides that firms would prefer this market to other markets, in which they would get $\hat{R}(\theta,z)$.

To solve the problem, let's take the first order conditions. To do this, first plug first two constraints in to the objective function and then take derivative with respect to d (recall that $\Psi^{h}=Q^{1-\varphi}$):

$$\begin{aligned}&\theta Q^{1-\varphi}zu_c(\theta dQ^{1-\varphi}z,d)+u_d(\theta dQ^{1-\varphi}z,d)=\\&\qquad\beta\sum_{\theta',z'}\Gamma_{\theta\theta'}\Gamma_{zz'}V_3\left(\theta',z',s(1+\hat{R}(\theta,z))-\frac{dQ^{1-\varphi}z}{P}\right)\frac{Q^{1-\varphi}z}{P}\end{aligned}\tag{10}$$

To find V_3 consider the original problem where constraints are not plugged into the objective function. Using envelope theorem we can get:

$$V_3(\theta,z,s)=\left[\theta u_c(\theta dQ^{1-\varphi}z,d)+\frac{u_d(\theta dQ^{1-\varphi}z,d)}{Q^{1-\varphi}z}\right]P(1+\hat{R}(\theta,z))$$

Combining these two would lead to Euler equation:

$$\begin{aligned}&\theta zu_c(\theta dQ^{1-\varphi}z,d)+\frac{u_d(\theta dQ^{1-\varphi}z,d)}{Q^{1-\varphi}}=\\&\qquad\beta\sum_{\theta',z'}\Gamma_{\theta\theta'}\Gamma_{zz'}\frac{P'(1+\hat{R}(\theta',z))}{P}\left[\theta'u_c(\theta'd'Q'^{1-\varphi}z',d'))+\frac{u_d(\theta'd'Q'^{1-\varphi}z',d')}{Q'^{1-\varphi}z'}\right]\end{aligned}\tag{11}$$

Observe that this equation is same with the Euler equation of random search model. This equation gives us the optimal search and saving behaviour for a given market tightness Q and price level P. To understand which market to search we need to look at FOC's regarding to Q and P. Let's λ denote the Lagrange multiplier for profit constraint, then FOC's with respect to Q and P are:

$$\theta d(1-\varphi)Q^{-\varphi} z u_c(\theta d Q^{1-\varphi} z, d) = \\ \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} V_3 \left(\theta', z', s(1+\hat{R}(\theta, z)) - \frac{dQ^{1-\varphi} z}{P}\right) \frac{d(1-\varphi)Q^{-\varphi} z}{P} + \lambda \frac{\varphi Q^{-\varphi-1} z}{P} \quad (12)$$

$$\beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} V_3 \left(\theta', z', s(1+\hat{R}(\theta, z)) - \frac{dQ^{1-\varphi} z}{P}\right) dQ = \lambda \quad (13)$$

Combining these two equation would give us:

$$\theta u_c(\theta d Q^{1-\varphi} z, d) = \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} V_3 \left(\theta', z', s(1+\hat{R}(\theta, z)) - \frac{dQ^{1-\varphi} z}{P}\right) \left[\frac{1}{(1-\varphi)P}\right] \quad (14)$$

Now we can define the equilibrium:

Definition 9 *An equilibrium with competitive search consists of V,c, d, s, P, Q and R that satisfy:*

1. *household's shopping constraint, (condition 7)*
2. *budget constraint, (condition 8)*
3. *Euler equation, (condition 11)*
4. *Market condition, (condition 14)*
5. *Firm's participation constraint, (condition 9)*
6. *Market clearing,* $s = 1$, $Q = 1/d$
7. *Dividend payment is the profit of the firm,* $\hat{R}(\theta, z) = (zQ^{-\varphi})/P$

8 Measure Theory

This section will be a quick review of measure theory to be able to use it in the subsequent sections. In macroeconomics we encounter the problem of aggregation often and it's crucial that we do it in a reasonable way. Measure theory is a tool that tells us when and how we could do so. Let us start with some definitions on sets.

Definition 10 *For a set S, $\mathcal{S}$ is a family of subsets of S, if $B \in \mathcal{S}$ implies $B \subseteq S$ (but not the other way around).*

Remark 7 *Note that, in this section we will follow the convention of notations as following*

1. *small letters (e.g. s) are for elements,*
2. *capital letters (e.g. S) for sets, and*
3. *fancy letters (e.g. $\mathcal{S}$) are for a set of subsets (or families of subsets).*

Definition 11 *A family of subsets of S, $\mathcal{S}$, is called a σ-algebra in S if*

1. *$S, \emptyset \in \mathcal{S}$;*
2. *$A \in \mathcal{S} \Rightarrow A^c \in \mathcal{S}$ (i.e. $\mathcal{S}$ is closed with respect to complements); and,*
3. *for $\{B_i\}_{i\in\mathbb{N}}$, $B_i \in \mathcal{S}$ for all i implies $\bigcap_{i\in\mathbb{N}} B_i \in \mathcal{S}$ (i.e. $\mathcal{S}$ is closed with respect to countable intersections).*

Example 1

1. *The power set of S (i.e. all the possible subsets of a set S), is a σ-algebra in S.*
2. *$\{\emptyset, S\}$ is a σ-algebra in S.*

3. *$\{\emptyset, S, S_{1/2}, S_{2/2}\}$, where $S_{1/2}$ means the lower half of S (imagine S as an closed interval in $\mathbb{R}$), is a σ-algebra in S.*

4. *If $S = [0,1]$, then*

$$\mathcal{S} = \left\{\emptyset, \left[0, \frac{1}{2}\right), \left\{\frac{1}{2}\right\}, \left[\frac{1}{2}, 1\right], S\right\}$$

is not *a σ-algebra in S. But*

$$\mathcal{S} = \left\{\emptyset, \left\{\frac{1}{2}\right\}, \left\{\left[0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right]\right\}, S\right\}$$

is a σ-algebra in S.

Why do we need this σ-algebra? The answer is it defines which sets may be considered as "events": things that could happen. Elements not in it may have no properly defined measure. Basically, σ-algebra is the "patch" that lets us avoid some pathological behaviors of mathematics, namely non-measurable sets. We are now ready to define a measure.

Definition 12 *Suppose $\mathcal{S}$ is a σ-algebra in S. A measure is a function $x : \mathcal{S} \to \mathbb{R}_+$, that satisfies*

1. *$x(\emptyset) = 0$;*

2. *$B_1, B_2 \in \mathcal{S}$ and $B_1 \cap B_2 = \emptyset$ implies $x(B_1 \cup B_2) = x(B_1) + x(B_2)$ (additivity); and,*

3. *$\{B_i\}_{i \in \mathbb{N}} \in \mathcal{S}$ and $B_i \cap B_j = \emptyset$, for all $i \neq j$, implies $x(\cup_i B_i) = \sum_i x(B_i)$ (countable additivity).*[7]

Put simply, a measure is just a way to assign each possible "event" a non-negative real number. A set S, a σ-algebra in it, $\mathcal{S}$, and a measure on $\mathcal{S}$, define a measure space, $(S, \mathcal{S}, x)$.

Definition 13 *Borel σ-algebra is a σ-algebra generated by the family of all open sets (generated by a topology).*

[7] Countable additivity means that the measure of the union of countable disjoint sets is the sum of the measure of these sets.

Since a Borel σ-algebra contains all the subsets generated by the intervals, you can recognize any subset of a set using Borel σ-algebra. In other words, Borel σ-algebra corresponds to complete information.

Definition 14 *A probability (measure) is a measure with the property that $x(S) = 1$.*

Definition 15 *Given a measure space $(S, \mathcal{S}, x)$, a function $f : S \to \mathbb{R}$ is measurable (with respect to the measure space) if, for all $a \in \mathbb{R}$, we have*

$$\{b \in S \mid f(b) \leq a\} \in \mathcal{S}.$$

One way to interpret a σ-algebra is that it describes the information available based on observations; a structure to organize information, and how fine are the information that we receive. Suppose that S is comprised of possible outcomes of a dice throw. If you have no information regarding the outcome of the dice, the only possible sets in your σ-algebra can be $\emptyset$ and S. If you know that the number is even, then the smallest σ-algebra given that information is $\mathcal{S} = \{\emptyset, \{2,4,6\}, \{1,3,5\}, S\}$. Measurability has a similar interpretation. A function is measurable with respect to a σ-algebra $\mathcal{S}$, if it can be evaluated under the current measure space $(S, \mathcal{S}, x)$.

Example 2 *Suppose $S = \{1,2,3,4,5,6\}$. Consider a function f which maps the element 6 to a number 1 (i.e. $f(6) = 1$) and any other elements to -100. Then f is NOT measurable with respect to $\mathcal{S} = \{\emptyset, \{1,2,3\}, \{4,5,6\}, S\}$. Why? Consider $a = 0$, then $\{b \in S \mid f(b) \leq a\} = \{1,2,3,4,5\}$. But this set is not in $\mathcal{S}$.*

We can also generalize Markov transition matrix to any measurable space. This is what we do next.

Definition 16 *A function $Q : \mathcal{S} \times S \to [0,1]$ is a transition probability if*

1. *$Q(\cdot, s)$ is a probability measure for all $s \in S$; and,*
2. *$Q(B, \cdot)$ is a measurable function for all $B \in \mathcal{S}$.*

Intuitively, given $B \in \mathcal{S}$ and $s \in S$, $Q(B, s)$ gives the probability of being in set B tomorrow, given that the state is s today. Consider the following example: a *Markov chain* with transition matrix given by

$$\Gamma = \begin{bmatrix} 0.2 & 0.2 & 0.6 \\ 0.1 & 0.1 & 0.8 \\ 0.3 & 0.5 & 0.2 \end{bmatrix},$$

on the set $S = \{1, 2, 3\}$, with the σ-algebra $\mathcal{S} = P(S)$ (where $P(S)$ is the power set of S). If Γ_{ij} denotes the probability of state j happening, given a present state i, then

$$Q(\{1, 2\}, 3) = \Gamma_{31} + \Gamma_{32} = 0.3 + 0.5 .$$

As another example, suppose we are given a measure x on $\mathcal{S}$; x_i gives us the fraction of type i, for $i \in S$. Given the previous transition function, we can calculate the fraction of types tomorrow using the following formulas:

$$\begin{aligned} x_1' &= x_1\Gamma_{11} + x_2\Gamma_{21} + x_3\Gamma_{31}, \\ x_2' &= x_1\Gamma_{12} + x_2\Gamma_{22} + x_3\Gamma_{32}, \\ x_3' &= x_1\Gamma_{13} + x_2\Gamma_{23} + x_3\Gamma_{33}. \end{aligned}$$

In other words

$$\mathbf{x}' = \Gamma^T \mathbf{x},$$

where $\mathbf{x}^T = (x_1, x_2, x_3)$.

To extend this idea to a general case with a general transition function, we define an *updating operator* as $T(x,Q)$, which is a measure on S with respect to the σ-algebra $\mathcal{S}$, such that

$$\begin{aligned} x'(B) =& T(x,Q)(B) \\ =& \int_S Q(B,s)\, x(ds), \quad \forall B \in \mathcal{S}. \end{aligned}$$

A stationary distribution is a fixed point of T, that is x^* so that

$$x^*(B) = T(x^*,Q)(B), \quad \forall B \in \mathcal{S}.$$

We know that, if Q has nice properties,[8] then a unique stationary distribution exists (for example, we discard *flipping* from one state to another), and

$$x^* = \lim_{n\to\infty} T^n(x_0, Q),$$

for any x_0 in the space of measures on $\mathcal{S}$.

Exercise 20 *Consider unemployment in a very simple economy (we have an exogenous transition matrix). There are two states; employed and unemployed. The transition matrix is given by*

$$\Gamma = \begin{pmatrix} 0.95 & 0.05 \\ 0.50 & 0.50 \end{pmatrix}.$$

Compute the stationary distribution corresponding to this Markov transition matrix.

[8] See Chapter 11 in **?**.

9 Industry Equilibrium

9.1 Preliminaries

Now we are going to study a type of models initiated by **?**. We will abandon the general equilibrium framework from the previous section to study the dynamics of distribution of firms in a partial equilibrium environment.

To motivate things, let's start with the problem of a single firm that produces a good using labor input according to a technology described by the production function f. Let us assume that this function is increasing, strictly concave, with $f(0) = 0$. A firm that hires n units of labor is able to produce $sf(n)$, where s is a productivity parameter. Markets are competitive, in the sense that a firm takes prices as given and chooses n in order to solve

$$\pi(s, p) = \max_{n \geq 0} \{psf(n) - wn\}.$$

The first order condition implies that in the optimum, n^*,

$$psf_n(n^*) = w.$$

Let us denote the solution to this problem as a function $n^*(s, p)$.[9] Given the above assumptions, n^* is an increasing function of s (i.e. more productive firms have more workers), as well as p.

Suppose now there is a mass of firms in the industry, each associated with a productivity parameter $s \in S \subset \mathbb{R}_+$, where $S := [\underline{s}, \bar{s}]$. Let $\mathcal{S}$ denote a σ-algebra on S (Borel σ-algebra for instance). Let x be a measure defined over the space $(S, \mathcal{S})$ that describes the cross sectional distribution of productivity among firms. Then, for any $B \subset S$ with $B \in \mathcal{S}$, $x(B)$ is the mass of firms having productivities in S.

We will use x to define statistics of the industry. For example, at this point, it is convenient to define

9 As we declared in advance, this is a partial equilibrium analysis. Hence, we ignore the dependence of the solution on w to focus on the determination of p.

the aggregate supply of the industry. Since individual supply is just $sf(n^*(s,p))$, the aggregate supply can be written as[10]

$$Y^S(p) = \int_S sf(n^*(s,p))\, x(ds)\,.$$

Observe that Y^S is a function of the price p; for any price, p, $Y^S(p)$ gives us the supply in this economy.

Exercise 21 *Search Wikipedia for an index of concentration in an industry, and adopt it for our economy.*

Suppose now that the demand of the market is described by some function $Y^D(p)$. Then the equilibrium price, p^*, is determined by the market clearing condition

$$Y^D(p^*) = Y^S(p^*)\,. \tag{15}$$

So far, everything is too simple to be interesting. The ultimate goal here is to understand how the object x is determined by the fundamentals of the industry. Hence, we will be adding tweaks to this basic environment in order to obtain a theory of firms' distribution in a competitive environment. Let's start by allowing firms to die.

9.2 A Simple Dynamic Environment

Consider now a dynamic environment, in which the situation above repeats every period. Firms discount profits at rate r_t, which is exogenously given. In addition, assume that a single firm, in each period, faces a probability $1-\delta$ of disappearing! We will focus on *stationary equilibria*; i.e. equilibria in which the price of the final output p, the rate of return, r, and the productivity of firm, s, stay constant through time.

Notice first that firm's decision problem is still a static problem; we can easily write the value of an

[10] S in Y^S stands for supply.

incumbent firm as

$$
\begin{aligned}
V(s,p) &= \sum_{t=0}^{\infty}\left(\frac{\delta}{1+r}\right)^{t}\pi(s,p) \\
&= \left(\frac{1+r}{1+r-\delta}\right)\pi(s,p)
\end{aligned}
$$

Note that we are considering that p is fixed (therefore we can omit it from the expressions above). Observe that every period there is positive mass of firms that die. Therefore, how can this economy be in a stationary equilibrium? To achieve that, we have to assume that there is a constant flow of firms entering the economy in each period, as well.

As before, let x be the measure describing the distribution of firms within the industry. The mass of firms that die is given by $(1-\delta)x(S)$. We will allow these firms to be replaced by new entrants. These entrants draw a productivity parameter s from a probability measure γ.

One might ask what keeps these firms out of the market in the first place? If

$$
\pi(s,p) = psf(n^{*}(s,p)) - wn^{*}(s,p) > 0,
$$

which is the case for the firms operating in the market, then all the (potential) firms with productivities in S would want to enter the market!

We can fix this flaw by assuming that there is a fixed entry cost that each firm must pay in order to operate in the market, denoted by c^E. Moreover, we will assume that the entrant has to pay this cost before learning s. Hence the value of a new entrant is given by the following function:

$$
V^{E}(p) = \int_{S} V(s,p)\,\gamma(ds) - c^{E}. \tag{16}
$$

Entrants will continue to enter if V^E is greater than 0, and decide not to enter if this value is less than zero. As a result, stationarity occurs when V^E is exactly equal to zero (this is the *free entry* assumption, and we are assuming that there is an infinite number (mass) of prospective firms).

Let's analyze how this environment shapes the distribution of firms in the market. Let x_t be the cross sectional distribution of firms in period t. For any $B \subset S$, portion $1-\delta$ of the firms with productivity $s \in B$ will die, and that will attract some newcomers. Hence, next period's measure of firms on set B will be given by:

$$x_{t+1}(B) = \delta x_t(B) + m\gamma(B).$$

That is, mass m of firms would enter the market in $t+1$, and only fraction $\gamma(B)$ of them will have productivities in the set B. As you might suspect, this relationship must hold for every $B \in \mathcal{S}$. Moreover, since we are interested in stationary equilibria, the previous expression tells us that the cross sectional distribution of firms will be completely determined by γ.

If we let mapping T be defined by

$$Tx(B) = \delta x(B) + m\gamma(B), \quad \forall B \in \mathcal{S}, \tag{17}$$

a stationary distribution of productivity is the fixed point of the mapping T; i.e. x^* with $Tx^* = x^*$, implying:

$$x^*(B; m) = \frac{m}{1-\delta}\gamma(B), \quad \forall B \in \mathcal{S}.$$

Now, note that the demand and supply relation in (15) takes the form:

$$y^d(p^*(m)) = \int_S sf(n^*(s,p))\, dx^*(s; m), \tag{18}$$

whose solution, $p^*(m)$, is continuous function under regularity conditions stated in **?**.

We have two equations, (16) and (18), and two unknowns, p and m. Thus, we can defined the equilibrium as:

Definition 17 *A stationary distribution for this environment consists of functions p^*, x^*, and m^*, that* *satisfy:*

1. $y^d(p^*(m)) = \int_S sf(n^*(s,p))\,dx^*(s;m)$;
2. $\int_s V(s,p)\,\gamma(ds) - c^E = 0$; *and,*
3. $x^*(B) = \delta x^*(B) + m^*\gamma(B), \quad \forall B \in \mathcal{S}$.

9.3 Introducing Exit Decisions

We want to introduce more (economic) content by making the exit of firms endogenous (a decision of the firm). One way to do so is to assume that the productivity of the firms follow a Markov process governed by a transition function, Γ. This would change the mapping T in Equation (17), as:

$$Tx(B) = \delta \int_S \Gamma(s,B)\,x(ds) + m\gamma(B), \quad \forall B \in \mathcal{S}.$$

But, this wouldn't add much economic content to our environment; firms still do not make any (interesting) decision. To change this, let's introduce cost of operation into the model; suppose firms have to pay c^v each period in order to stay in the market. In this case, when s is low, the firm's profit might not cover its cost of operation. So, the firm might decide to leave the market. However, firm has already paid (a sunk cost of) c^E, and, since s changes according to a Markov process, prospects of future profits might deter the firm from quitting. Therefore, negative profit in one period does not imply immediately that the firm's optimal choice is to leave the market.

By adding such a minor change, the solution will have a reservation productivity property under some conditions (to be discussed in the comment below). In words, there will be a minimum productivity, $s^* \in S$, above which it is profitable for the firm to stay in the market.

To see this, note that the value of a firm with productivity $s \in S$ in a period is given by

$$V(s,p) = \max \left\{ 0, \pi(s,p) + \frac{1}{(1+r)} \int_S \Gamma(s, ds') V(s',p) - c^v \right\}.$$

Exercise 22 *Show that the firm's decision takes the form of a reservation productivity strategy, in which, for some $s^* \in S$, $s < s^*$ implies that the firm would leave the market.*

In this case, the transition of the distribution of productivities on S will be:

$$x'(B) = m\gamma(B \cap [s^*, \bar{s}]) + \int_{s^*}^{\bar{s}} \Gamma(s, B \cap [s^*, \bar{s}]) x(ds), \quad \forall B \in \mathcal{S}.$$

A stationary distribution of the firms in this economy, x^*, is the fixed point of this equation.

Example 3 *How productive does a firm have to be, to be in the top 10% largest firms in this economy? The answer to this question is the solution to the following equation, $\hat{s}$:*

$$\frac{\int_{\hat{s}}^{\bar{s}} x^*(ds)}{\int_{s^*}^{\bar{s}} x^*(ds)} = 0.1 \ .$$

Then, the fraction of the labor force in the top 10% largest firms in this economy, is

$$\frac{\int_{\hat{s}}^{\bar{s}} n^*(s,p) x^*(ds)}{\int_{s^*}^{\bar{s}} n^*(s,p) x^*(ds)}.$$

Exercise 23 *Compute the average growth rate of the smallest one third of the firms. What would be the fraction of firms in the top 10% largest firms in the economy that remain in the top 10% in next period?*

Comment 2 *To see that this will be the case you should prove that the profit before variable cost function $\pi(s,p)$ is increasing in s. Hence the productivity threshold is given by the s^* that satisfies the following condition:*

$$\pi(s^*, p) = c_v$$

for an equilibrium price p. Now instead of considering γ as the probability measure describing the distribution of productivities among entrants, you must consider $\widehat{\gamma}$ defined as follows

$$\widehat{\gamma}(B) = \frac{\gamma(B \cap [s^*, \overline{s}])}{\gamma([s^*, \overline{s}])}$$

for any $B \in \mathcal{S}$.

One might suspect that this is an ad hoc *way to introduce the exit decision. To make things more concrete and easier to compute, we will assume that s is a Markov process. To facilitate the exposition, let's make S finite and assume s has transition matrix Γ. Assume further that Γ is regular enough so that it has a stationary distribution γ^*. For the moment we will not put any additional structure on Γ.*

The operation cost c^v is such that the exit decision is meaningful. Let's analyze the problem from the perspective of the firm's manager. He has now two things to decide. First, he asks himself the question "Should I stay or should I go?". Second, conditional on staying, he has to decide how much labor to hire. Importantly, notice that this second decision is still a static decision. Later, we will introduce adjustment cost that will make this decision a dynamic one.

Let $\phi(s,p)$ be the value of the firm before having decided whether to stay or to go. Let $V(s,p)$ be the value of the firm that has already decided to stay. $V(s,p)$ satisfies

$$V(s,p) = \max_n \left\{ spf(n) - n - c^v + \frac{1}{1+r} \sum_{s' \in S} \Gamma_{ss'} \phi(s',p) \right\}$$

Each morning the firm chooses d in order to solve

$$\phi(s,p) = \max_{d \in \{0,1\}} dV(s,p)$$

Let $d^*(s,p)$ be the optimal decision to this problem. Then $d^*(s,p) = 1$ means that the firm stays in the market. One can alternatively write:

$$\phi(s,p) = \max_{d \in \{0,1\}} d \left[\pi(s,p) - c^v + \frac{1}{1+r} \sum_{s' \in S} \Gamma_{ss'} \phi(s',p) \right]$$

or even

$$\phi(s,p) = \max\left[\pi(s,p) - c^v + \frac{1}{1+r}\sum_{s'\in S}\Gamma_{ss'}\phi(s',p), 0\right] \tag{19}$$

All these are valid. Additionally, one can easily add minor changes to make the exit decision more interesting. For example, things like scrap value or liquidation costs will affect the second argument of the max *operator above, which so far is just zero.*

What about $d^*(s,p)$? Given a price, this decision rule can take only finitely many values. Moreover, if we could ensure that this decision is of the form "stay only if the productivity is high enough and go otherwise" then the rule can be summarized by a unique number $s^* \in S$. Without doubt, that would be very convenient, but we don't have enough structure to ensure that such is the case. Because, although the ordering of s (lower s are ordered before higher s) gives us that the value of s today is bigger than value of smaller s', depending on the Markov chain, on the other hand, the value of productivity level s tomorrow may be lower than the value of s' (note $s' < s$) tomorrow. Therefore we need some additional regularity conditions.

In order to get a cutoff rule for the exit decision, we need to add an assumption about the transition matrix Γ. *Let the notation $\Gamma(s)$ indicate the probability distribution over next period state conditional on being on state s today. You can think of it as being just a row of the transition matrix. Take s and $\widehat{s}$. We will say that the matrix Γ displays first order stochastic dominance $(FOSD)$ if $s > \widehat{s}$ implies $\sum_{s'\leq b}\Gamma(s' \mid s) \leq \sum_{s'\leq b}\Gamma(s' \mid \widehat{s})$ for any $b \in S$. It turns out that $FOSD$ is a sufficient condition for having a cutoff rule. You can prove that by using the same kind of dynamic programming tricks that have been used in a different course for obtaining the reservation wage property in search problems. Try it as an exercise. Also note that this is just a sufficient condition.*

Finally, we need to mention something about potential entrants. Since we will assume that they have to pay the cost c^E before learning their s, they can leave the industry even before producing anything. That requires us to be careful when we describe industry dynamics.

Now the law of motion becomes:

$$x'(B) = m\gamma(B \cap [s^*, \bar{s}]) + \int_{s^*}^{\bar{s}} \Gamma(s, B \cap [s^*, \bar{s}])\, x(ds), \quad \forall B \in \mathcal{S}.$$

9.4 Stationary Equilibrium

Now that we have all the ingredients in the table, let's define the equilibrium formally.

Definition 18 *A stationary equilibrium for this environment consists of a list of functions (ϕ, n^*, d^*), a price p^* and a measure x^* such that*

1. *Given p^*, the functions ϕ, n^*, d^* solve the problem of the incumbent firm*
2. $V^E(p^*) = 0$
3. *For any $B \in \mathcal{S}$ (assuming we have a cut-off rule with s^* is cut-off in stationary distribution)*[11]

$$x^*(B) = m\gamma(B \cap [s^*, \bar{s}]) + \int_{s^*}^{\bar{s}} \Gamma(s, B \cap [s^*, \bar{s}])\, x^*(ds).$$

4. *Market clearing:*

$$Y^d(p^\star) = \int_{s^*}^{\bar{s}} s f(n^\star(s, p^\star)) dx^\star(ds)$$

[11] If we do not have such cut-off rule we have to define

$$x^*(B) = \int_S \sum_{s' \in S} \Gamma_{ss'} \mathbf{1}_{\{s' \in B\}} \mathbf{1}_{\{d(s', p^*)=1\}} x^*(ds) + \mu^* \int_S \mathbf{1}_{\{s \in B\}} \mathbf{1}_{\{d(s, p^*)=1\}} \gamma(ds)$$

where

$$\mu^* = \int_S \sum_{s' \in S} \Gamma_{ss'} \mathbf{1}_{\{d(s', p^*)=0\}} x^*(ds)$$

You can think of condition (2) as a "*no money left over the table*" condition, which ensures additional entrants find unprofitable to participate in the industry.

We can use this model to compute interesting statistics. For example the average output of the firm is given by

$$\frac{Y}{N} = \frac{\sum sf(n^*(s))x^*(ds)}{\sum x^*(ds)}$$

Next, suppose that we want to compute the share of output produced by the top 1% of firms. To do this we first need to compute $\widetilde{s}$ such that

$$\frac{\sum_{\widetilde{s}}^{\bar{s}} x^*(ds)}{N} = .01$$

where N is the total measure of firms. Then the share output produced by these firms is given by

$$\frac{\sum_{\widetilde{s}}^{\bar{s}} sf(n^*(s))x^*(ds)}{\sum_{\underline{s}}^{\bar{s}} sf(n^*(s))x^*(ds)}$$

Suppose now that we want to compute the fraction of firms who are in the top 1% two periods in a row. This is given by

$$\sum_{s \geq \widetilde{s}} \sum_{s' \geq \widetilde{s}} \Gamma_{ss'} x^*(ds)$$

We can use this model to compute a variety of other statistics including the Gini coefficient.

9.5 Adjustment Costs

To end with this section it is useful to think about environments in which firm's productive decisions are no longer static. A simple way of introducing dynamics is by adding adjustment costs.

We will consider labor adjustment costs.[12] Consider a firm that enters period t with n_{t-1} units of labor, hired in the previous period. We can consider three specifications for the adjustment costs, due to hiring n_t units of labor in t, $c(n_t, n_{t-1})$:

- *Convex Adjustment Costs*: if the firm wants to vary the units of labor, it has to pay $\alpha(n_t - n_{t-1})^2$ units of the numeraire good. The cost here depends on the size of the adjustment.
- *Training Costs or Hiring Costs*: if the firm wants to increase labor, it has to pay $\alpha[n_t - (1-\delta)n_{t-1}]^2$ units of the numeraire good, only if $n_t > n_{t-1}$; we can write this as

$$\mathbf{1}_{\{n_t > n_{t-1}\}}\alpha[n_t - (1-\delta)n_{t-1}]^2,$$

  where $\mathbf{1}$ is the indicator function, and δ measures the exogenous attrition of workers in each period.
- *Firing Costs.*

The recursive formulation of the firm's problem would be:

$$V(s, n_-, p) = \max\left\{0, \max_{n\geq 0}\ sf(n) - wn - c^v - c(n, n_-) + \frac{1}{(1+r)}\sum_{s'\in S}\Gamma_{ss'}V(s', n, p)\right\}, \quad (20)$$

where c gives the specified cost of adjusting n_- to n. Note that due to limited liability of the firm, the exit value of the firm is 0 and not $-c(0, n_-)$.

Now, a firm is characterized by both its productivity s and labor n_- in the previous period. Note that since the production function f has decreasing returns to scale, there exists an amount of labor $\bar{N}$ such that none of the firms hire labor greater than $\bar{N}$. So, $n_- \in N := [0, \bar{N}]$. Let $\mathcal{N}$ be a σ-algebra on N. If the labor policy function is $n = g(s, n_-)$, then the law of motion now becomes:

$$x'(B^S, B^N) = m\gamma\left(B^S \cap [s^*, \bar{s}]\right)\mathbf{1}_{\{0\in B^N\}} + \int_{s^*}^{\bar{s}}\int_0^{\bar{N}}\mathbf{1}_{\{g(s,n_-)\in B^N\}}\Gamma\left(s, B^S\cap[s^*,\bar{s}]\right)x(ds, dn_-),$$

[12] These costs work pretty much like capital adjustment costs, as one might suspect.

$$\forall B^S \in \mathcal{S}, \quad \forall B^N \in \mathcal{N}.$$

Exercise 24 *Write the first order conditions for the problem in* (20).

Define the recursive competitive equilibrium for this economy.

Exercise 25 *Another example of labor adjustment costs is when the firm has to post vacancies to attract labor. As an example of such case, suppose the firm faces a firing cost according to the function c. The firm also pays a cost κ to post vacancies, and after posting vacancies, it takes one period for the workers to be hired. How can we write the problem of firms in this environment?*

9.6 Non-stationary Equilibrium

Up until now we focus on the *stationary industrial equilibrium*, in which individual firms enter and exit, but the whole distribution of firms stays invariant. A more interesting case is to look at the *non-stationary equilibrium* and examine how the distribution of firms shift across time.

Let's maintain our baseline model (with entry & exit, but no adjustment costs), and think about the economy starting with some (arbitrary) initial distribution of incumbent firms x_0. We can imagine that, without any shocks, the firm distribution would converge to the stationary equilibrium distribution x^* defined in 9.4. And on the transitional path towards the stationary equilibrium, firms would face a sequence of prices $\{p_t\}_{t=0}^{\infty}$. We now feed in shocks. We will maintain that wage is normalized to 1 and prices p_t each period is going to be pinned down by equating the endogenous aggregate supply and ad-hoc aggregate demand which we denote $D(p_t, z_t)$, where z_t is a demand side shock that shifts aggregate demand.

It's important that we make it clear on the nature of this shock. In general, z_t can be *deterministic* or *stochastic*. Deterministic shocks are fully anticipated by agents in the economy. Stochastic shocks, on the other hand, come in a random manner and agents only know the random process that governs them. Solving the model with deterministic shocks are not harder than solving the transitional path of the model with no shocks. But models with stochastic shocks are much harder to solve. We will say for now the z_t shocks are deterministic and thus focus on the notion of perfect foresight equilibrium (PFE).

We are now ready to define the firm's problem. Note now state variables would incorporate both the individual state s (idiosyncratic productivity shock) and aggregate states: z (aggregate demand shock) and x (measure of firms).

$$V(s, z_t, x_t) = \max \left\{ 0, \pi(s, z_t, x_t) + \frac{1}{1+r} \sum_{s'} \Gamma_{ss'} V(s', z_{t+1}, x_{t+1}) \right\} \tag{21}$$
$$s.t. \quad \pi(s, z_t, x_t) = \max_{n \geq 0} \ p_t(z_t, x_t) s f(n) - w n_t - c^v$$

Note that we can maintain the cutoff property of the decision rule given our regularity conditions. Let's denote the exit cutoff s_t^*. Note that in order to solve the problem, firms need to know the measure of firms. So we need to figure out the law of motion of firm measure. For each $B \in \mathcal{S}$, we should have

$$x_{t+1}(B) = m_{t+1} \gamma(B \cap [s_{t+1}^*, \bar{s}]) + \int_{s_t^*}^{\bar{s}} \Gamma(s, B \cap [s_{t+1}^*, \bar{s}]) x_t(ds) \tag{22}$$

where m_{t+1} is the mass of firms that enter at the beginning of period $t+1$, which is pinned down by the free entry condition

$$\int V(s, z_t, x_t) \gamma(ds) \leq c^e \tag{23}$$

with strict equality holds if $m_t > 0$. The distribution of the initial draw γ and entry cost c^e are exogenously given. Finally, the market clearing condition will close the model by pinning down price p_t

$$D(p_t, z_t) = \int_{s_t^*}^{\bar{s}} s p_t f(n^*(s, z_t, x_t)) x_t(ds) \tag{24}$$

Exercise 26 *Figure out the time line behind the above formulation of the firm's problem, the law of motion of firm measure, and the free entry condition.*

We can thus define the perfect foresight equilibrium as following

Definition 19 *For a given path of shock realizations* $\{z_t\}$ *and a initial firm measure* x_0*, a perfect foresight equilibrium (PFE) for this environment consists of sequences of functions* $\{p_t, m_t, s_t^*, x_t\}$*, that satisfy:*

1. **Optimality**: *given* $\{p_t\}$, $\{s_t^*\}$ *solve the firm's problem* (21) *for each period t.*
2. **Free entry**: $\int V(s, z_t, x_t)\gamma(ds) \leq c^e$*, with strict equality holds if* $m_t > 0$.
3. **Law of motion**: $x_{t+1}(B) = m_{t+1}\gamma(B \cap [s_{t+1}^*, \bar{s}]) + \int_{s_t^*}^{\bar{s}} \Gamma(s, B \cap [s_{t+1}^*, \bar{s}])x_t(ds)$, $\forall B \in \mathcal{S}$.
4. **Market clearing**: $D(p_t, z_t) = \int_{s_t^*}^{\bar{s}} sp_t f(n^*(s, z_t, x_t))x_t(ds)$.

Having figured out the equilibrium of the perfect foresight model, the natural next step is thus to solve the fully stochastic equilibrium. It is actually a much harder one. We will resort to some notion of linearization to achieve that. So we will divert a bit in the next subsection to talk about linear approximation.

9.7 Digression: Linear Approximation

To better understand the linearization, let's look at a very basic growth model and approximate the solution linearly. Consider such a social planner's problem (with full depreciation)

$$\begin{aligned} v(k_t) = \max_{c_t, k_{t+1}} & \ u(c_t) + \beta v(k_{t+1}) \\ & \text{s.t. } c_t + k_{t+1} \leq f(k_t), \ \ \forall t \geq 0 \\ & c_t, k_{t+1} \geq 0, \ \ \forall t \geq 0 \\ & k_0 > 0 \text{ given.} \end{aligned} \tag{25}$$

We can show that $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ is a solution to the above social planner's problem if and only if

$$u'(c_t) = \beta u'(c_{t+1}) f'(k_{t+1}), \forall t \geq 0 \tag{26}$$

$$c_t + k_{t+1} = f(k_t), \forall t \geq 0 \tag{27}$$

$$\lim_{t\to\infty} \beta^t u'(c_t) k_{t+1} = 0 \tag{28}$$

Exercise 27 *Prove the above claim.*

We will focus on cases where a steady state k^* exists. Note that the above necessary and sufficient condition give us a second order difference equation system (we can combine the above solution as $\psi(k_t, k_{t+1}, k_{t+2}) = 0$), with exactly two boundary conditions. so the model is totally solvable. The question is how to do that. One obvious option is to find the global solution. For instance, you can guess a k_1, use k_0 and $\psi(k_t, k_{t+1}, k_{t+2}) = 0$ to get $k_2, k_3, \ldots$ forward up until some k_T, and adjust k_1 to make sure k_T is close enough to the steady state k^* (this is called forward shooting). Or you can guess a k_{T-1} and do it backward (which is called backward shooting). Or you can guess and adjust the whole path (which is called the extended path method). All these methods will give you a numerical solution starting from an arbitrary k_0 (that's why we call it a *global* solution).

You can see the above process is time consuming. Linearization is a short cut, that can yield good approximation of the solution *locally*, that is, around the neighborhood of some point. Usually, people will do it around the steady state. The idea is simple. We know the true solution is in the form of $k_{t+1} = g(k_t)$. Let's simply use a linear function to approximate the true solution $g(\cdot)$. Let's say our approximation is $k_{t+1} = \hat{g}(k_t) = a + bk_t$. Then we only need to figure two numbers: a and b. We thus need two conditions. Since we know the steady state is k^*, which means $a + bk^* = k^*$, we get one condition for free (remember we approximate around k^*). Where to find the other one?

We can only find it in ψ and our criteria is that we are going to choose b such that the slope of $\hat{g}$ exactly matches the slope of true decision rule g *at the steady state* k^*. So we take a first order Taylor expansion of $\psi[k, g(k), g(g(k))]$ around k^* and obtain

$$\psi[k, g(k), g(g(k))] \approx \psi(k^*, k^*, k^*) + \psi_k(k^*, k^*, k^*)(k - k^*) \tag{29}$$

We know $\psi[k, g(k), g(g(k))] = 0$, and k is in the neighborhood of k^*, so it must be

$$\psi_k(k^*, k^*, k^*) = \psi_1^* + \psi_2^* g'(k^*) + \psi_3^* g'(k^*) g'(k^*) = 0 \tag{30}$$

Solve this equation gives us $g'(k^*)$ which is exactly what we need (note ψ_1, ψ_2, and ψ_3 may also involve $g'(k^*)$). We can then let $b = g'(k^*)$ and use $\hat{g}(k_t) = a + bk_t$ to approximate the solution near the steady state.

Comment 3 *In practice, it's messy to do the total derivative as above. A cleaner way is to linearize the system directly with k_t, k_{t+1}, k_{t+2}, and then solve the linear system using whatever method you like. Usually, we cast it on a state space and solve it using matrix algebra (here it helps to know some econometrics).*

Exercise 28 *Suppose $f(k_t) = k_t^\alpha$, $u(c_t) = \ln c_t$. Verify that the solution to the social planner's problem is $k_{t+1} = \alpha\beta k_t^\alpha$. Get the linearized solution around the steady state and compare it with the closed form solution. How precise is the linear approximation?*

Exercise 29 *Extend the linearization to the case where we have stochastic productivity shocks z_t.*

10 Incomplete Market Models

10.1 A Farmer's Problem

Consider the following problem of a farmer:

$$
\begin{aligned}
V(s,a) = \max_{c,a'} \quad & \left\{ u(c) + \beta \sum_{s'} \Gamma_{ss'} V(s', a') \right\} \qquad (31) \\
s.t. \quad & c + qa' = a + s \\
& c \geq 0 \\
& a' \geq 0,
\end{aligned}
$$

where a is his holding of coconuts, which can only take positive values, c is his consumption, and s is amount of coconuts that he produces. s follows a Markov chain, taking values in a finite set S. q is the fraction of coconuts that can be stored to be consumed tomorrow. Note that, the constraint on the holding of coconuts tomorrow, is a constraint imposed by nature; nature allows the farmer to store coconuts at rate $1/q$, but, it does not allow him to *transfer coconuts* from tomorrow to today.

We are going to consider this problem in the context of a partial equilibrium, where q is given. The first question is, what can be said about q?

Remark 8 *Assume there are no shocks in the economy, so that s is a fixed number. Then, we could write the problem of the farmer as:*

$$
V(a) = \max_{c,a' \geq 0} \left\{ u(a + s - qa') + \beta V(a') \right\}.
$$

If u is assumed to be logarithmic, the first order condition for this problem implies:

$$
\frac{c'}{c} \geq \frac{\beta}{q},
$$

with equality if $a' > 0$. Therefore, if $q > \beta$ (i.e. nature is more stingy than farmer's patience), then $c' < c$, and the farmer dis-saves (at least, as long as $a' > 0$). But, when $q < \beta$, consumption grows without bound. This is the reason we put this assumption on the model, in what follows.

A crucial assumption for generating a bounded asset space is $\beta/q < 1$, stating that agents are sufficiently impatient so they tend to consume more and decumulate their asset when they get richer and far away from the non-negativity constraint, $a' \geq 0$. However, this does not mean that, when faced with a possibility of very low consumption, agents would not save, even though the rate of return, $1/q$, is smaller than the rate of impatience $1/\beta$.

The first order condition for farmer's problem (31) is given by:

$$u_c\left(c\left(s,a\right)\right) \geq \frac{\beta}{q}\sum_{s'}\Gamma_{ss'}u_c\left(c\left(s',a'\left(s,a\right)\right)\right),$$

with equality when $a'\left(s,a\right) > 0$, where $c\left(\cdot\right)$ and $a'\left(\cdot\right)$ are policy functions from the farmer's problem. Notice that $a'\left(s,a\right) = 0$ is possible, if we assume appropriate shock structure in the economy. Specifically, it depends on the value of $s_{\min} := \min_{s_i \in S}\ s_i$.

The solution to the problem of the farmer, for a given value of q, implies a distribution of coconut holdings in each period. This distribution, together with the Markov chain describing the evolution of s, can be summed together as a single probability measure for the distribution of shocks and assets (coconut holdings) over the product space $E = S \times \mathbb{R}_+$, and its σ-algebra, $\mathcal{B}$, which we denote by X. The evolution of this probability measure is given by:

$$X'\left(B\right) = \sum_{s' \in B_s}\int \Gamma_{ss'}\mathbf{1}_{\{a'(s,a)\in B_a\}}dX\left(s,a\right), \quad \forall B \in \mathcal{B}, \tag{32}$$

where B_s and B_a are the S-section and $\mathbb{R}$-section of B (projections of B on S and $\mathbb{R}_+$), respectively, and $\mathbf{1}$ is the indicator function. Let $\widetilde{T}\left(\Gamma, a', \cdot\right)$ be the mapping associated with (32) (the adjoint operator), so that:

$$X'(B) = \widetilde{T}(\Gamma, a', X)(B), \quad \forall B \in \mathcal{B}.$$

Define $\widetilde{T}^n(\Gamma, a', \cdot)$ as:

$$\widetilde{T}^n(\Gamma, a', X) = \widetilde{T}\left(\Gamma, a', \widetilde{T}^{n-1}(\Gamma, a', X)\right).$$

Then, we have the following theorem.

Theorem 3 *Under some conditions on $\widetilde{T}(\Gamma, a', \cdot)$, there is a unique probability measure X^*, so that:*

$$X^*(B) = \lim_{n\to\infty} \widetilde{T}^n(\Gamma, a', X_0)(B), \quad \forall B \in \mathcal{B},$$

for all initial probability measures X_0 on $(E, \mathcal{B})$.

A condition that makes things considerably easier for this theorem to hold is that E is a compact set. Then, we can use Theorem (12.12) in **?**, to show this result holds. Given that S is finite, this is equivalent to a compact support for the distribution of asset holdings. We discuss this in detail in Appendix A.

10.2 Huggett Economy

Now we modify the farmer's problem in (31) a little bit (**?**):

$$\begin{aligned} V(s,a) = \max_{c,a'} \quad & \left\{ u(c) + \beta \sum_{s'} \Gamma_{ss'} V(s', a') \right\} \qquad (33) \\ s.t. \quad & c + qa' = a + s \\ & c \geq 0 \\ & a' \geq \underline{\mathsf{a}}, \end{aligned}$$

where $\underline{a} < 0$, so now farmers can borrow and lend with each other, but with a borrowing limit. We continue to make the same assumption on q; i.e. $\beta/q < 1$. Solving this problem gives the policy function $a'(s,a)$. It is easy to extend the analysis in the last section to show that there is an upper bound of the asset space, which we denote by $\bar{a}$, so that for any $a \in A := [\underline{a}, \bar{a}]$, $a'(s,a) \in A$, for any $s \in S$.

Remark 9 *One possibility for $\underline{a}$ is what we call the natural borrowing constraint. This is the constraint that ensures the agent can pay back his debt for sure, no matter what the nature reveals (whatever sequence of idiosyncratic shocks is realized). This is given by*

$$a^n := -\frac{s_{\min}}{\left(\frac{1}{q} - 1\right)}.$$

If we impose this constraint on (33)*, even when the farmer receives an infinite sequence of bad shocks, he can pay back his debt by setting his consumption equal to zero, forever.*

But, what makes this problem more interesting is to tighten this borrowing constraint; the natural borrowing constraint is very unlikely to be hit. One way to restrict borrowing is no borrowing at all, as in the previous section. Another case is to choose $0 > \underline{a} > a^n$. This is the case that we consider in this section.

Now suppose there is a (unit) mass of farmers with distribution function $X(\cdot)$, where $X(D,B)$ denotes fraction of people with shock $s \in D$ and $a \in B$, where D in an element of the power set of S , $P(S)$ (which, when S is finite, is the natural σ-algebra over S) and B is a Borel subset of A, $B \in \mathcal{A}$. Then the distribution of farmers tomorrow is given by:

$$X'(S', B') = \int_{A \times S} \mathbf{1}_{\{a'(s,a) \in B'\}} \sum_{s' \in S'} \Gamma_{ss'} dX(s,a), \tag{34}$$

for any $S' \in P(S)$ and $B' \in \mathcal{A}$.

Implicitly this defines an operator T such that $T(X) = X'$. If T is *sufficiently nice*, then there exits a unique X^* such that $X^* = T(X^*)$ and $X^* = \lim_{n\to\infty} T^n(X_0)$, for any initial distribution over the product space $S \times A$, X_0. Note that the decision rule is obtained given q. Hence, the resulting stationary distribution X^* also depends on q. So, let us denote it by $X^*(q)$.

To determine the equilibrium value of q, in a general equilibrium setting, consider the following variable (as a function of q):

$$\int_{A \times S} a dX^*(q).$$

This expression give us the average asset holdings, given the price, q.

We want to determine an endogenous q, so that the asset market clears; we assume that there is no storage technology so that asset supply is 0 in equilibrium. Hence, price q should be such that asset demand equals asset supply; i.e.

$$\int_{A \times S} a dX^*(q) = 0.$$

In this sense, equilibrium price q is the price that generates the stationary distribution that clears the asset market.

We can show that a solution exists by invoking intermediate value theorem, by showing that the following three conditions are satisfied (note that $q \in [\beta, \infty]$):

1. $\int_{A \times S} a dX^*(q)$ is a continuous function of q;
2. $\lim_{q \to \beta} \int_{A \times S} a dX^*(q) \to \infty$; (Intuitively, as $q \to \beta$, interest rate increases. Hence, agents would like to save more. Together with precautionary savings motive, they accumulate unbounded quantities of assets in the stationary equilibrium, in this case.) and,
3. $\lim_{q \to \infty} \int_{A \times S} a dX^*(q) < 0$. (This is also intuitive; as $q \to \infty$, interest rate converges to 0. Hence, everyone would rather borrow.)

10.3 Aiyagari Economy

In the **?** Economy, there is physical capital. In this sense, the average asset holdings in the economy must be equal to the average (physical) capital. So, if we keep denoting the stationary distribution of assets by X^*, we must have:

$$\int_{A\times S} a dX^*(q) = K,$$

where A is the support of the distribution of wealth. (It is not difficult to see that this set is compact.)

On the other hand, the shocks affect the labor income; we can think of these shocks as fluctuations in the employment status of individuals. Thus, the problem of an individual in this economy can be written as:

$$\begin{aligned} V(s,a) = \max_{c,a'} \quad & \left\{ u(c) + \beta \sum_{s'} \Gamma_{ss'} V(s',a') \right\} \\ s.t. \quad & c + a' = (1+r)a + ws \\ & c \geq 0 \\ & a' \geq \underline{\mathsf{a}}, \end{aligned} \tag{35}$$

where, here, r is the return to savings, and w is the wage rate. Therefore,

$$\int_{A\times S} s dX^*(q)$$

gives the average labor in this economy. If we decide to think of agents supplying one unit of labor, then, we may think of the expression as determining the *effective labor supply*.

We assume the standard constant returns to scale production technology for the firm, of the form:

$$F(K,L) = AK^{1-\alpha}L^{\alpha},$$

with the rate of depreciation δ. Hence:

$$\begin{aligned} r =& F_k (K, L) - \delta \\ =& (1-\alpha) A \left(\frac{K}{L}\right)^{-\alpha} - \delta \\ =:& r\left(\frac{K}{L}\right), \end{aligned}$$

and

$$\begin{aligned} w =& F_l (K, L) \\ =& \alpha A \left(\frac{K}{L}\right)^{1-\alpha} \\ =:& w\left(\frac{K}{L}\right). \end{aligned}$$

In terms of Huggett **?**, q, the price of assets, is given by

$$q = \frac{1}{(1+r)} = \frac{1}{[1 + F_k (K, L) - \delta]}.$$

Therefore, prices faced by agents are all functions of the capital-labor ratio. As a result, we may write the stationary distribution of assets as a function of capital-labor ratio as well, $X^*\left(\frac{K}{L}\right)$. Thus, the equilibrium condition becomes:

$$\frac{K}{L} = \frac{\int_{A\times S} a dX^*\left(\frac{K}{L}\right)}{\int_{A\times S} s dX^*\left(\frac{K}{L}\right)}.$$

Using this condition, one can solve for the equilibrium capital-labor ratio.

10.3.1 Policy

In **?** or **?** economy, model parameters can be summarized by $\theta = \{u, \beta, s, \Gamma, F\}$. In stationary equilibrium, value function $v(s, a; \theta)$ as well as $X^*(\theta)$ can be obtained, where $X^*(\theta)$ is a mapping from model parameters to stationary distribution of agent's asset holding and shocks. Suppose now there is a policy change that shifts θ to $\hat{\theta} = \{u, \beta, s, \hat{\Gamma}, F\}$. Associated with this new environment there is a new value function $v\left(s, a; \hat{\theta}\right)$ and $X^*\left(\hat{\theta}\right)$. Define $\eta(s,a)$ to be the solution of:

$$v\left(s, a + \eta(s,a), \hat{\theta}\right) = v(s, a, \theta)$$

which is the transfer payment necessary to the households so that they are indifferent between living in the old environment and in the new one. Hence total payment needed to compensate households for this policy change is given by:

$$\int_{A\times S} \eta(s,a)\, dX^*(\theta)$$

Notice that the changes do not take place when the government is trying to compensate the households. Hence we use the original stationary distribution associated with θ to aggregate the households.

If $\int_{A\times S} v(s,a)\, dX^*\left(\hat{\theta}\right) > \int_{A\times S} v(s,a)\, dX^*(\theta)$, does this necessarily mean that households are willing to accept this policy change? The answer is not necessarily because the economy may well spend a long time in the transition path to the new steady state, during which there may be severe welfare loss.

10.3.2 Aggregate Shocks

In this section, we consider an economy that is subject to both aggregate and idiosyncratic shocks, at the same time; consider **?** economy again, now, with a production function that is subject to an aggregate shock; $zF\left(K, \bar{N}\right)$.

Let X be the distribution of types; then the aggregate capital is given by:

$$K = \int a dX(s,a).$$

$$K' = G(z, K)$$

The question is what are the sufficient statistics for predicting the aggregate capital stock and, consequently, prices tomorrow? Are z and K sufficient determine capital tomorrow? The answer to these questions is no, in general; this is true if, and only if, the decision rules are linear. Therefore, X, the distribution of types becomes a state variable (even in the stationary equilibrium) for this economy.

Then, the problem of an individual becomes:

$$\begin{aligned} V(z, X, s, a) = \max_{a'} \quad & \left\{ u(c) + \beta \sum_{z', s'} \Pi_{zz'} \Gamma_{ss'}^{z'} V(z', X', s', a') \right\} \\ s.t. \quad & c + a' = azf_k\left(K, \bar{N}\right) + szf_n\left(K, \bar{N}\right) \\ & K = \int a dX(s, a) \\ & X' = G(z, X) \\ & c, a' \geq 0. \end{aligned}$$

Computationally, this problem is a beast! So, how can we solve it? To provide some idea, we will first consider an economy with *dumb* agents!

Consider an economy in which people are stupid; people believe tomorrow's capital depends only on K, and not X. This, obviously, is not an economy in which expectations are rational. Nevertheless, people's problem in such settings becomes:

$$\begin{aligned} \widetilde{V}(z, X, s, a) = \max_{a'} \quad & \left\{ u(c) + \beta \sum_{z', s'} \Pi_{zz'} \Gamma_{ss'}^{z'} V(z', X', s', a') \right\} \\ s.t. \quad & c + a' = azf_k\left(K, \bar{N}\right) + szf_n\left(K, \bar{N}\right) \\ & K = \int a dX(s, a) \\ & X' = \widetilde{G}(z, K) \\ & c, a' \geq 0. \end{aligned}$$

Next step is to allow people become slightly smarter; they now can use extra information, like mean and variance of X, to predict X'. Does this economy work better than our *dumb benchmark*? Computationally no! This answer, as stupid as it may sound, has an important message: people actually act linearly in the economy; decision rules are approximately linear. Therefore, we may use **?** results without fear; the approximations are quite reliable!

10.3.3 Linear Approximation Revisited

Let's now continue our discussion of linear approximation in the context of Aiyagari model. As we can see in section 10.3.2, solving the heterogeneous agent model with aggregate shocks is computationally hard. We need to guess a reduced form rule for agents to forecast future prices, and when the model has frictions on several dimensions, little could we say on how to choose such a rule.

We can, however, use linear approximation as a short cut to obtain the solution near the steady state. The idea is as following: firstly, starting from the steady state, obtain the perfect foresight equilibrium (PFE) path given a specified path of small deterministic shocks, and then use the PFE to approximate the behavior of the economy facing small stochastic shocks around the steady state. This method is proposed recently by **?**[13].

To fix the idea, let's consider the above Aiyagari model with a TFP shock z. Consider log(z_t) follows an AR(1) process with a serial correlation parameter ρ. Thus, (the log of) the shock will go up by, say, one unit in period 0 and thus delivers the full sequence of values $(1, \rho, \rho^2, \rho^3, \dots)$. When we solve for our resulting deterministic equilibrium transition path, the individual takes as given a sequence of prices and because it is irrelevant for the individual how these prices are determined, they can be summarized as depending simply on time. Solving the deterministic path is straightforward: we guess on a price path (or the path for an aggregate variable like consumption), solve the household's problem backwards—given that we know that there will be convergence back to the same steady state—and then derive the aggregate implications of the households' behavior and update our guess for the price path. This iterative procedure is also standard and fully nonlinear.

After solving the PFE, we have a sequence of whatever variable we care about. Let's label this sequence:

[13] The description of the method below is from their paper with slight modifications.

$\{x_0, x_1, x_2, \dots\}$. Now consider the same economy subjected to recurring aggregate shocks to z. The key assumption behind this procedure is that we regard the x sequence in response to the one-time shock as well approximated by a linear system. A linear system would precisely mean that the effects of shocks are linearly scalable and additive so that the *level* of x at some future time T, after a sequence of random shocks is given by

$$x_T \approx x_0\epsilon_T + x_1\epsilon_{T-1} + x_2\epsilon_{T-2} + \dots$$

where ϵ_t is the innovation to $\log(z_t)$ at period t. Thus, the model with aggregate shocks can be obtained by mere simulation based on the one deterministic path: the non-linear impulse response function of the PFE.

10.4 Aiyagari Economy with Entrepreneurs

Aiyagari economy now becomes the workhorse of modern macroeconomics. It features incomplete markets and an endogenous wealth distribution, in which we can examine interactions between heterogeneous agents and distributional effects of public polices. Now let's take a quick look on a very simple extension and have a sense on how the model can be used to study a wide variety of macroeconomic issues.

Specifically, we will introduce entrepreneurs into the Aiyagari world. Suppose every period agents choose an occupation: to be an entrepreneur or to be a worker. Entrepreneurs run their own business, and workers supply labor in the market. Entrepreneurs can manage one project which combines her entrepreneur ability (ϵ), capital(k) and labor(n).

Let's denote $V^w(a, s, \epsilon)$ the value of a worker with wealth a, labor productivity s, and entrepreneur ability ϵ. Also denote $V^e(a, s, \epsilon)$ the value of an entrepreneur. The worker's problem is to choose tomorrow's occupation and wealth level, as well as today's consumption, at given wage rate w and interest rate r.

$$
\begin{aligned}
V^w(a,s,\epsilon) = \max_{c,a',i} & \, u(c) + \beta\{i\mathbb{E}\left[V^w(a',s',\epsilon')\right] + (1-i)\mathbb{E}\left[V^e(a',s',\epsilon')\right]\} \\
s.t. \quad & c + a' = ws + (1+r)a \\
& a' \geq 0
\end{aligned}
$$

Similarly, the entrepreneur's problem can be formulated as following

$$
\begin{aligned}
V^e(a,s,\epsilon) = \max_{c,a',i} & \, u(c) + \beta\{i\mathbb{E}\left[V^w(a',s',\epsilon')\right] + (1-i)\mathbb{E}\left[V^e(a',s',\epsilon')\right]\} \\
s.t. \quad & c + a' = \pi(a,s,\epsilon) \\
& a' \geq 0
\end{aligned}
$$

Note the entrepreneur's income is from profits $\pi(a,s,\epsilon)$ rather than wage. We assume entrepreneurs have access to a DRS technology f, that can produce output given (k,n,ϵ). After paying costs of factors and loans, profits $\pi(a,s,\epsilon)$ are given by

$$
\begin{aligned}
\pi(a,s,\epsilon) = \max_{k,n} & \, f(k,n,\epsilon) + (1-\delta)k - (1+r)(k-a) - w\max\{n-s,0\} \\
s.t. \quad & k - a \leq \phi a
\end{aligned}
$$

The constraint here reflects the fact that entrepreneurs can only make loans up to a fraction ϕ of his total wealth. A limit of this model is that entrepreneurs never make an operating loss within a period, as they can always choose $k = n = 0$ and earn the risk free rate on saving. In this model, agents with high entrepreneur ability have access to a investment technology f with higher return than workers, and therefore they accumulate wealth faster.

11 Monopolistic Competition

11.1 Benchmark Monopolistic Competition

The two most important macro economic variables are perhaps output and inflation (movement of aggregate price). In this section we take a first step to build a theory of aggregate price. To achieve this, we need a framework in which firms can choose prices, and the aggregate price is well defined and easy to handle. Monopolistic competition by **?** is such a framework.

In a world with monopolistic competition, firms are sufficiently "different" so that they face a downward sloping demand curve, but also sufficiently small so that they ignore any strategic interactions with their competitors. We thus assume there are infinitely many measure 0 firms, each producing one variety of goods. Varieties span on the $[0, n]$ interval and are imperfect substitutes. Consumers have a "taste for variety" in that they prefer to consume a diversified bundle of goods (this gives firms some market power as we want). We specify consumer's utility function in a CES form as following

$$U = \left(\int_j x_j^{\frac{\sigma-1}{\sigma}} dj \right)^{\frac{\sigma}{\sigma-1}}$$

where σ is the elasticity of substitution, which is a constant (as the name CES (constant elasticity of substitution) correctly suggests), and x_j is the quantity consumed of variety j. Given the utility function above, and an exogenous total *nominal* budget I, we can solve the household problem

$$\begin{aligned} \max_x U &= \left(\int_j x_j^{\frac{\sigma-1}{\sigma}} dj \right)^{\frac{\sigma}{\sigma-1}} \\ s.t. \quad & \int_j p_j x_j dj \leq I \end{aligned}$$

and derive the downward sloping demand curve faced by an individual firm producing variety j

$$x_j^* = \frac{I}{p_j^\sigma \int_0^n p_i^{1-\sigma} di}$$

We can see that the demand for variety j depend both on the price of variety j and some notion of "aggregate price". It is actually convenient to define an aggregate price index P as following (the exercise below builds some intuition of it)

$$P = \left(\int_0^n p_j^{1-\sigma} dj \right)^{\frac{1}{1-\sigma}}$$

and thus the demand curve faced by firm j can be reformulated as

$$x_j^* = (\frac{p_j}{P})^{-\sigma} \frac{I}{P}$$

Exercise 30 *Show the following within the monopolistic competition framework above:*

1. *σ is the elasticity of substitution between varieties.*
2. *Price index P is the expenditure to purchase a unit-level utility for consumers.*
3. *Consumer utility is increasing in the number of varieties n.*

We are now ready to characterize the firm's problem. Let's say the production technology is extremely simple such that the 1 unit of output is produced by 1 unit of labor linearly, i.e., $x_j = n_j$. And nominal wage rate is given by W. So firm j's problem is just

$$\max_{p_j} \pi(p_j) = p_j x_j^*(p_j) - W x_j^*(p_j)$$
$$s.t. \quad x_j^* = (\frac{p_j}{P})^{-\sigma} \frac{I}{P}$$

Recall that we assume firms are sufficiently small so they would ignore the effect of their own pricing strategies on aggregate price index P, which greatly simplify the algebra. In the end, we would get a simple pricing rule

$$p_j = \frac{\sigma}{\sigma - 1} W$$

which basically means all firms would choose a constant mark-up that reflects elasticity of substitution of consumers. When varieties are very close substitutes ($\sigma \to \infty$), price just converge to the factor cost: wage W.

11.2 Price Rigidity

We now have a simple theory of aggregate price P, which is ultimately shaped by elasticity of substitution of consumers. However, we are still silent on inflation. To study inflation, and to have meaningful interactions between output and inflation, we need i) a dynamic model and ii) some nominal frictions.

Nominal frictions mean some devices through which nominal variables (things measured in dollars, say, quantity of money) can affect real variables (say, the number of iPhone produced in 2017). Nowadays the most popular device is called *price rigidity*. It simply means firms can not change price freely. Two commonly used specifications to achieve this in the model are *Rotemberg pricing* and *Calvo pricing*.

In Rotemberg pricing, firms face adjustment cost $\phi(p_j, p_j^-)$ when changing their prices each period. So inheriting everything from the static model above (assuming a exogenous process of nominal endowment I_t), we can specify the firm's per period profit under Rotemberg pricing in a dynamic set-up as following:

$$\pi_{jt} = p_{jt}x_{jt} - W_t x_{jt} - \phi(p_{jt}, p_{j,t-1})I_t$$
$$\text{where} \quad x_{jt} = \left(\frac{p_{jt}}{P_t}\right)^{-\sigma} \frac{I_t}{P_t}$$

Then each period, firms choose price to maximize the expected present discounted value of flow profit. Rotemberg is easy in terms of algebra when we assume a quadratic price adjustment cost. However it generates some fiscal effects that are not so realistic.

A more popular version of price rigidity is the Calvo pricing. It says that, instead of facing some adjustment costs, firms simply, with some probability each period, can not change their price. It's a bit more complicated in terms of algebra. I will set up the problem a bit but leave most as exercise for you.

Let's say each period with probability θ a firm can change its price, and with probability $1-\theta$ changing price is not allowed. So when setting the price, now the firm needs to incorporate its long run effects. The firm's problem at period t becomes:

$$\max_{p_{jt}} \sum_{k=0}^{\infty} (\frac{1-\theta}{1+r})^k \left[p_{jt} x_{j,t+k} - W_{t+k} x_{j,t+k} \right]$$
$$s.t. \quad x_{jt} = (\frac{p_{jt}}{P_t})^{-\sigma} \frac{I_t}{P_t}$$

Exercise 31 *Derive the following for the dynamic model with Calvo pricing*

1. *Solve the firm's problem and write the firm's pricing p_{jt} as a function of present and future aggregate prices, wages, and endowments: $\{P_t, W_t, I_t\}_{t=0}^{\infty}$.*
2. *Show that under flexible pricing ($\theta = 1$), the firm's pricing strategy is identical to the static model.*
3. *Show that with price rigidity ($\theta < 1$), at steady state with zero inflation, the firm's pricing strategy is identical to the static model.*

11.3 Endogenous Growth and R&D

We will now introduce R&D into the model. So far, we have seen the *neoclassical growth model* as our benchmark model, and built on it for the analysis of more interesting economic questions. One peculiar characteristic of our benchmark model, unlike its name suggests, is the lack of growth (after reaching the steady state), whereas, many interesting questions in economics are related to the cross-country differences of growth rates. To see why this is the case, consider the standard neoclassical technology:

$$F(K, N) = AK^{\theta_1} L^{\theta_2},$$

for some $\theta_1, \theta_2 \geq 0$. We already know that the only possible case that is consistent with the notion of competitive equilibrium is that $\theta_1 + \theta_2 = 1$. However, this implies a decreasing marginal rate of product for capital. Given a fixed quantity for labor supply, in the presence of depreciation, this implies a maximum sustainable capital stock, and puts a limit on the sustainable growth; economy converges to some steady-state, without exhibiting any balanced growth.

So if our economy is to experience sustainable growth for a long period of time, we either give up the curvature assumption on our technology, or we have to be able to shift our production function upwards. Given a fixed amount of labor, this shift is possible either by an increasing (total factor) productivity parameter or increasing labor productivity. We will cover a model that will allow for growth, so that we will be able to attempt to answer such questions.

Consider the following economy due to the highly cited model of endogenous growth **?**; there are three sectors in the economy: a final good sector, an intermediate goods sector, and an R&D sector. Final goods are produced using labor (as we will see there is only one wage, since there is only one type of labor) and intermediate goods according to the production function

$$N_{1,t}^{\alpha} \int_0^{A_t} x_t(i)^{1-\alpha} \, di$$

where $x(i)$ denotes the utilization of intermediate good of variety $i \in [0, A_t]$.[14] Note that marginal contribution of each variety is decreasing (since $\alpha < 1$), however, an increase in the number of varieties would increase the output. We will assume that the final good producers operate in a competitive market.

Exercise 32 *If the price of all varieties are the same, what is the optimal choice of input vector for a producer?*

What if they do not have the same amount? Would a firm decide not to use a variety in the production?

[14] The function that aggregates consumption of intermediate goods is often referred as Dixit-Stiglitz aggregator.

Intermediate producers are monopolists that have access to a differentiated technology of the form:

$$x(i) = \frac{k(i)}{\eta}.$$

Therefore, they can end up charging a mark up above the marginal cost for their product. This is the main force behind research and development in this economy; developer of a new variety is the sole proprietor of the blue print that allows him earn profit. It is easy to observe that the aggregate demand of capital from the intermediate sector is $\int_0^{A_t} \eta x(i)\, di$.

The R&D sector in the economy is characterized by a flow of intermediate goods in each period; a new good is a new variety of the intermediate good. The flow of the new intermediate goods is created by using labor, according to the following production technology:

$$\frac{A_{t+1}}{A_t} = 1 + \xi N_{2,t}.$$

Notice that, after some manipulation, one can express growth in the stock of intermediate goods as follows:

$$A_{t+1} - A_t = A_t \xi N_{2,t}. \tag{36}$$

Hence, the flow of new intermediate goods is determined by the current stock of them in the economy. This type of externality in the model is the key propeller in the model. This assumption provides us with a constant returns to scale technology in the R&D sector. In what follows, we will assume that the inventors act as price takers in the economy.

Remark 10 *The reason we see A_t on the right hand side of* (36) *as an externality is that it is indeed used as an input in the process of R&D, while, it is not paid for. Thus, inventors, in a sense, do not pay the investors of the previous varieties, while using their inventions. They only pay for the labor they hire. Perhaps, the basic idea of this production function might be traced back to* Isaac Newton*'s quote:* "If I have seen further, it is only by standing on the shoulders of giants".

The preferences of the consumers are represented by the following utility function:

$$\sum_{t=0}^{\infty} \beta^t u(c_t),$$

and their budget constraint in period t is give by:

$$c_t + k_{t+1} \leq r_t k_t + w_t + (1-\delta) k_t.$$

Remark 11 *In this economy, GDP, in terms of gross product, is given by:*

$$Y_t = W_t + r_t K_t + \pi_t,$$

where π_t is the net profits. On the other hand, in terms of expenditures, GDP is:

$$Y_t = C_t + K_{t+1} - (1-\delta) K_t + \pi_t,$$

where $K_{t+1} - (1-\delta) K_t$ is the investment in physical capital. At last, in terms of value added, it is given by:

$$Y_t = N_t^{\alpha} \int_0^{A_t} x_t(i)^{1-\alpha} \, di + p_t (A_{t+1} - A_t).$$

Certainly, this is not a model that one can map to the data. Instead it has been carefully crafted to deliver what is desired and it provides an interesting insight in thinking about endogenous growth.

Solving the Model Let's first consider the problem of a final good producer; in every period, he chooses $N_{1,t}$ and $x_t(i)$, for every $i \in [0, A_t]$, in order to solve:

$$\max \; N_{1,t}^{\alpha} \int_0^{A_t} x_t(i)^{1-\alpha} \, di - w_t N_{1,t} - \int_0^{A_t} q_t(i) \, x_t(i) \, di,$$

where $q_t(i)$ is the price of variety i in period t. First order conditions for this problem are:

1. $N_{1,t}$: $\alpha N_{1,t}^{\alpha-1} \int_0^{A_t} x_t(i)^{1-\alpha} di = w_t$; and,

2. $x_t(i)$: $(1-\alpha) N_{1,t}^{\alpha} x_t(i)^{-\alpha} = q_t(i)$, for all $i \in [0, A_t]$.

From the second condition, one obtains:

$$x_t(i) = \left(\frac{(1-\alpha)}{q_t(i)}\right)^{\frac{1}{\alpha}} N_{1,t}, \tag{37}$$

which, given N_{1t}, is the *demand function* for variety i, by the final good producer.

Next, let's consider the problem of an intermediate firm; these firms acts as price setters. The reason is the ownership of a differentiated patent, whose sole owner is the intermediate good producer of variety i. In addition, as long as $\alpha < 1$, this variety does not have a perfect substitute, and always demanded in the equilibrium. Therefore, their problem is to choose $q_t(i)$, in order to solve:

$$\begin{aligned} \pi_t(i) = \max \quad & q_t(i) x_t(q_t(i)) - r_t \eta x_t(q_t(i)) \\ s.t. \quad & x_t(q_t(i)) = \left(\frac{(1-\alpha)}{q_t(i)}\right)^{\frac{1}{\alpha}} N_{1,t}, \end{aligned}$$

where $x_t(q_t(i))$ is the demand function, substituted from (37). Notice that we have substituted for the technology of the monopolist, $x(i) = k(i)/\eta$. First order condition for this problem, with respect to $q_t(i)$, is:

$$x_t(q_t(i)) + (q_t(i) - r_t\eta) \frac{\partial x_t(q_t(i))}{\partial q_t(i)} = 0,$$

which can be written as

$$\frac{(1-\alpha)^{\frac{1}{\alpha}}}{q_t(i)^{\frac{1}{\alpha}}} N_{1,t} = \frac{(q_t(i) - r_t\eta)}{\alpha} \frac{(1-\alpha)^{\frac{1}{\alpha}}}{q_t(i)^{\frac{1+\alpha}{\alpha}}} N_{1,t}.$$

Rearranging yields:

$$q_t(i) = \frac{1}{(1-\alpha)} r_t \eta. \tag{38}$$

This is the familiar pricing function of a monopolist; price is marked-up above the marginal cost.

By substituting (38) into (37), we get:

$$x_t(i) = \left[\frac{(1-\alpha)^2}{r_t \eta}\right]^{\frac{1}{\alpha}} N_{1,t}, \tag{39}$$

and the demand for capital services is simply $\eta x_t(i)$. In a symmetric equilibrium, where all the intermediate good producers choose the same pricing rule, we have:

$$\int_0^{A_t} x_t(i)\, di = A_t x_t = \frac{k_t}{\eta},$$

where x_t is the common supply of intermediate goods. Therefore:

$$x_t = \frac{k_t}{\eta A_t}.$$

Moreover, if we let Y_t be the production of the final good; by plugging (39) we get:

$$Y_t = N_{1,t} A_t \left[\frac{(1-\alpha)^2}{r_t \eta}\right]^{\frac{1-\alpha}{\alpha}}. \tag{40}$$

Hence the model displays constant returns to scale in $N_{1,t}$ and A_t.

Let us study the problem of the R&D firms, next; a representative firm in this sector (recall that this is a competitive sector) chooses $N_{2,t}$, in order to solve the following problem:

$$\max_{N_{2,t}} \quad p_t A_t \xi N_{2,t} - w_t N_{2,t}.$$

The first order condition for this problem implies:

$$p_t = \frac{w_t}{A_t \xi}.$$

In summary, there are two equations to be solved form; one relating the choice of consumption versus saving (or capital accumulation), and one dividing labor demand for R&D, and that for final good production. Consumption-investment decisions result from solving household's problem in equilibrium, and the corresponding Euler equation:

$$u'(c_t) = \beta u'(c_{t+1}) \left[r_{t+1} + (1 - \delta) \right].$$

For determining the labor choices $N_{1,t}$ and $N_{2,t}$, first note that the demand for patterns produced by R&D sector, is from the prospect monopolists. As long as there is positive profit from buying demand, the new monopolists would keep entering markets in a given period. This fact derives the profits of prospect monopolists to zero. So, the lifetime profit of the monopolist, must be equal to the price he pays for the blueprints;

$$p_t = \sum_{s=t}^{\infty} \left(\prod_{\tau=t}^{s} \frac{1}{1 + r_\tau - \delta} \right) \pi_s.$$

This completes the solution to the model. Notice that output can grow at the same rate as A_t, from Equation (40). In addition, K_t grows at the same rate. As a result, the rate of growth of A_t would be the important aspect of equilibrium. For instance, if A_t grows at rate γ in the long-run, we have a balanced growth path in equilibrium. This growth comes from the externality in the R&D sector. Without that, we cannot get sustained growth in this model. The nice thing about this model is how neat is is in delivering the balanced growth, with just enough structure imposed on the economy.

A A Farmer's Problem: Revisited

Consider the following problem of a farmer that we studied in class:

$$V(s,a) = \max_{c,a'} \left\{ u(c) + \beta \sum_{s'} \Gamma_{ss'} V(s',a') \right\} \tag{41}$$

$$\begin{aligned} s.t. \quad & c + qa' = a + s \\ & c \geq 0 \\ & a' \geq 0. \end{aligned}$$

As we discussed, we are in particular interested in the case where $\beta/q < 1$. In what follows, we are going to show that, under monotonicity assumption on the Markov chain governing s, the optimal policy associated with (41) implies a finite support for the distribution of asset holding of the farmer, a.[15]

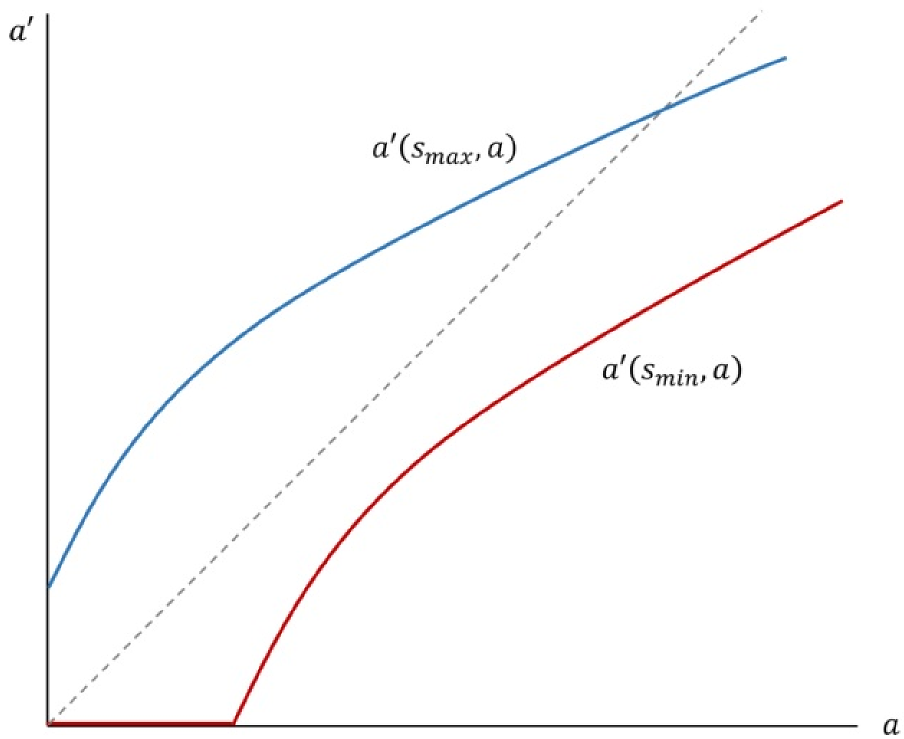
Before we start the formal proof, suppose $s_{\min} = 0$, and $\Gamma_{ss_{\min}} > 0$, for all $s \in S$. Then, the agent will optimally always choose $a' > 0$. Otherwise, there is a strictly positive probability that the agent enters tomorrow into state $s_{\min}$, where he has no *cash in hand* ($a' + s_{\min} = 0$) and is forced to consume 0, which is extremely painful to him (e.g. when Inada conditions hold for the instantaneous utility). Hence he will raise his asset holding a' to insure himself against such risk.

If $s_{\min} > 0$, then the above argument no longer holds, and it is indeed possible for the farmer to choose zero assets for tomorrow.

Notice that the borrowing constraint $a' \geq 0$ is affecting agent's asset accumulation decisions, even if he is away from the zero bound, because he has an incentive to ensure against the risk of getting a series of bad shocks to s and is forced to 0 asset holdings. This is what we call *precautionary savings motive*.

[15] This section was prepared by *Keyvan Eslami*, at the *University of Minnesota*. This section is essentially a slight variation on the proofs found in **?**. However, he accepts the responsibility for the errors.

Figure 1: Policy function associated with farmer's problem.



Next, we are going to prove that the policy function associated with (41), which we denote by $a'(\cdot)$, is similar to that in Figure 1. We are going to do so, under the following assumption.

Assumption 1 *The Markov chain governing the state s is monotone; i.e. for any $s_1, s_2 \in S$, $s_2 > s_1$ implies $E(s|s_2) \geq E(s|s_1)$.*

It is straightforward to show that, the value function for Problem (41) is concave in a, and bounded. Now, we can state our intended result as the following theorem.

Theorem 4 *Under Assumption 1, when $\beta/q < 1$, there exists some $\hat{a} \geq 0$ so that, for any $a \in [0, \hat{a}]$, $a'(s, a) \in [0, \hat{a}]$, for any realization of s.*

To prove this theorem, we proceed in the following steps. In all the following lemmas, we will assume that the hypotheses of Theorem 4 hold.

Lemma 1 *The policy function for consumption is increasing in a and s;*

$$c_a(a,s) \geq 0 \text{ and } c_s(a,s) \geq 0.$$

Proof 1 *By the first order condition, we have:*

$$u'(c(s,a)) \geq \frac{\beta}{q} \sum_{s'} \Gamma_{ss'} V_a\left(s', \frac{a+s-c(s,a)}{q}\right),$$

with equality, when $a+s-c(s,a) > 0$.

For the first part of the lemma, suppose a increases, while $c(s,a)$ decreases. Then, by concavity of u, the left hand side of the above equation increases. By concavity of the value function, V, the right hand side of this equation decreases, which is a contradiction.

For the the second part, we claim that $V_a(s,a)$ is a decreasing function of s. To show this is the case, firs consider the mapping T as follows:

$$\begin{aligned} Tv(s,a) = \max_{c,a'} \quad & \left\{u(c) + \beta \sum_{s'} \Gamma_{ss'} v(s',a')\right\} \\ s.t. \quad & c + qa' = a + s \\ & c \geq 0 \\ & a' \geq 0. \end{aligned}$$

Suppose $v_a^n(s,a)$ is decreasing in its first argument; i.e. $v_a^n(s_2,a) < v_a^n(s_1,a)$, for all $s_2 > s_1$ and $s_1, s_2 \in S$. We claim that, $v^{n+1} = Tv^n$ inherits the same property. To see why, note that for $a^{n+1}(s,a) = a'$ (where a^{n+1} is the policy function associated with n'th iteration) we must have:

$$u'(a+s-qa') \geq \frac{\beta}{q} \sum_{s'} \Gamma_{ss'} v_a^n(s',a'),$$

with strict equality when $a' > 0$. For a fixed value of a', an increase in s leads to a decrease in both sides of this equality, due to the monotonicity assumption of Γ, and the assumption on v_a^n. As a result, *we must have*

$$u'\left(a+s_2-qa^{n+1}\left(s_2,a\right)\right)\leq u'\left(a+s_1-qa^{n+1}\left(s_1,a\right)\right),$$

for all $s_2>s_1$. *By Envelope theorem, then:*

$$v_a^{n+1}\left(s_2,a\right)\leq v_a^{n+1}\left(s_1,a\right).$$

It is straightforward to show that v^n *converges to the value function* V *point-wise. Therefore,*

$$V_a\left(s_2,a\right)\leq V_a\left(s_1,a\right),$$

for all $s_2>s_1$.

Now, note that, by envelope theorem:

$$V_a\left(s,a\right)=u'\left(c\left(s,a\right)\right).$$

As s *increases,* $V_a\left(s,a\right)$ *decreases. This implies* $c\left(s,a\right)$ *must increase.*

Lemma 2 *There exists some* $\hat{a}\in\mathbb{R}_+$*, such that* $\forall a\in[0,\hat{a}]$*,* $a'\left(a,s_{\min}\right)=0$.

Proof 2 *It is easy to see that, for* $a=0$*,* $a'\left(a,s_{\min}\right)=0$*. First of all, note the first order condition:*

$$u_c\left(c\left(s,a\right)\right)\geq\frac{\beta}{q}\sum_{s'}\Gamma_{ss'}u_c\left(c\left(s',a'\left(s,a\right)\right)\right),$$

with equality when $a'(s,a) > 0$. Under the assumption that $\beta/q < 1$, we have:

$$\begin{aligned} u_c\left(c\left(s_{\mathsf{min}}, 0\right)\right) =& \frac{\beta}{q}\sum_{s'} \Gamma_{s_{\mathsf{min}}s'} u_c\left(c\left(s', a'\left(s_{\mathsf{min}}, 0\right)\right)\right) \\ &< \sum_{s'} \Gamma_{ss'} u_c\left(c\left(s', a'\left(s_{\mathsf{min}}, 0\right)\right)\right). \end{aligned}$$

By Lemma 1, if $a' = a'(0, s_{\mathsf{min}}) > a = 0$, then $c(s', a') > c(s_{\mathsf{min}}, 0)$ for all $s' \in S$, which leads to a contradiction.

Lemma 3 *$a'(s_{\mathsf{min}}, a) < a$, for all $a > 0$.*

Proof 3 *Suppose not; then $a'(s_{\mathsf{min}}, a) \geq a > 0$ and as we showed in Lemma 2:*

$$u_c\left(c\left(s_{\mathsf{min}}, a\right)\right) < \sum_{s'} \Gamma_{ss'} u_c\left(c\left(s', a'\left(s_{\mathsf{min}}, a\right)\right)\right).$$

Contradiction, since $a'(s_{\mathsf{min}}, a) \geq a$, and $s' \geq s_{\mathsf{min}}$, and the policy function in monotone.

Lemma 4 *There exits an upper bound for the agent's asset holding.*

Proof 4 *Suppose not; we have already shown that $a'(s_{\mathsf{min}}, a)$ lies below the 45 degree line. Suppose this is not true for $a'(s_{\mathsf{max}}, a)$; i.e. for all $a \geq 0$, $a'(s_{\mathsf{max}}, a) > a$. Consider two cases.*

In the first case, suppose the policy functions for $a'(s_{\mathsf{max}}, a)$ and $a'(s_{\mathsf{min}}, a)$ diverge as $a \to \infty$, so that, for all $A \in \mathbb{R}_+$, there exist some $a \in \mathbb{R}_+$, such that:

$$a'\left(s_{\mathsf{max}}, a\right) - a'\left(s_{\mathsf{min}}, a\right) \geq A.$$

Since S is finite, this implies, for all $C \in \mathbb{R}_+$, there exist some $a \in \mathbb{R}_+$, so that

$$c\left(s_{\mathsf{min}}, a\right) - c\left(s_{\mathsf{max}}, a\right) \geq C,$$

which is a contradiction, since c is monotone in s.

Next, assume $a'(s_{\text{max}}, a)$ and $a'(s_{\text{min}}, a)$ do not diverge as $a \to \infty$. We claim that, as $a \to \infty$, c must grow without bound. This is quite easy to see; note that, by envelope condition:

$$V_a(s,a) = u'(c(s,a)).$$

The fact that V is bounded, then, implies that V_a must converge to zero as $a \to \infty$, implying that $c(s,a)$ must diverge to infinity for all values of s, as $a \to \infty$. But, this implies, if $a'(s_{\text{max}}, a) > a$,

$$u_c(c(s_{\text{max}}, a'(s_{\text{max}}, a))) \to \sum_{s'} \Gamma_{s_{\text{max}} s'} u_c(c(s', a'(s_{\text{max}}, a))).$$

As a result, for large enough values of a, we may write:

$$\begin{aligned} u_c(c(s_{\text{max}}, a)) &= \frac{\beta}{q} \sum_{s'} \Gamma_{s_{\text{max}} s'} u_c(c(s', a'(s_{\text{max}}, a))) \\ &< \sum_{s'} \Gamma_{s_{\text{max}} s'} u_c(c(s', a'(s_{\text{max}}, a))) \\ &\approx u_c(c(s_{\text{max}}, a'(s_{\text{max}}, a))). \end{aligned}$$

But, this implies:

$$c(s_{\text{max}}, a) > c(s_{\text{max}}, a'(s_{\text{max}}, a)),$$

which, by monotonicity of policy function, means $a > a'(s_{\text{max}}, a)$, and this is a contradiction.