The Generalized Euler Equation and the Bankruptcy-Sovereign Default Problem

Based on Stuff by
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Motivation

- Models of debt with unilateral default – both household debt and sovereign debt – are workhorses in the quantitative literature.

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- These models are often solved numerically without characterizing the equilibrium.
- Precise characterization of trade-offs the agents face will help with intuition, and computation of these models.
- We want to open the "black box" and describe the tradeoffs in the model in terms of marginal costs and marginal benefits.

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- We give a formulation of optimality conditions in the long-term debt case that does not rely on prices.
- We have characterized the problem with commitment as well (won't talk about it today).

- Endowment $\epsilon \in [\underline{\epsilon}, \overline{\epsilon}]$ is iid with cdf F and density f.

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- Standard $u(c)$ and relative impatience, $\beta < R^{-1} = \bar{p}$.
- After default, agent reverts to financial autarky.

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The Problem With Commitment

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- Two alternative recursive timings
  1. Choose today when to default tomorrow
  2. Choose circumstances of when to default before realization of shock but commitment to expected value

$$\Omega(b,\epsilon) = \max_{c,b',\epsilon'^c} \; u(c) + \beta \int_{\underline{\epsilon}}^{\epsilon'^c} (u(\epsilon) + \beta \overline{v}) \; f(d\epsilon) + \beta \int_{\epsilon^c} \Omega(b',\epsilon') \; f(d\epsilon') \quad \text{s.t.}$$

$$c + b = b' \; \frac{[1 - F(\epsilon^c)]}{1+r} + \epsilon$$

Substituting in the constraints yields

$$\Omega(b,\epsilon) = \max_{b',\epsilon'^c} \; u\left(b' \; \frac{[1 - F(\epsilon'^c)]}{1+r} + \epsilon - b\right) +$$

$$\beta \int_{\underline{\epsilon}}^{\epsilon'^c} (u(\epsilon) + \beta \overline{v}) \; f(d\epsilon') + \beta \int_{\epsilon'^c} \Omega(b',\epsilon') \; f(d\epsilon')$$

$$
\begin{aligned}
\frac{[1-F(\epsilon'^c)]}{1+r}\; u_c\left(b'\;\frac{[1-F(\epsilon'^c)]}{1+r}+\epsilon-b\right) &= \beta\int_{\epsilon'^c}\Omega_b(b',\epsilon')\; f(d\epsilon') \\
\frac{-f(\epsilon^c)\; b'}{1+r}\; u_c\left(b'\;\frac{[1-F(\epsilon^c)]}{1+r}+\epsilon-b\right) &= \beta\; f(\epsilon^c)\;\left[u(\epsilon'^c)+\beta\overline{v}-\Omega(b',\epsilon'^c)\right] \\
\Omega_b(b,\epsilon) &= -u_c\left(b'\;\frac{[1-F(\epsilon'^c)]}{1+r}+\epsilon-b\right) \qquad \text{so} \\
\frac{[1-F(\epsilon'^c)]}{1+r}\; u_c\left(b'\;\frac{[1-F(\epsilon'^c)]}{1+r}+\epsilon-b\right) &= \beta\int_{\epsilon'^c}u_c\left(b''\;\frac{[1-F(\epsilon''^c)]}{1+r}+\epsilon'-b'\right)dF(\epsilon')
\end{aligned}
$$

or compactly

$$
\begin{aligned}
\frac{[1-F(\epsilon'^c)]}{1+r}\; u_c &= \beta\int_{\epsilon'^c}u_c'\; dF(\epsilon') \\
\frac{b'}{1+r}\; u_c &= \beta\left[\Omega(b',\epsilon'^c)-u(\epsilon'^c)-\beta\overline{v}\right]
\end{aligned}
$$

Timing 2: Using Long term debt $\lambda < 1$

$$v(b) = \max_{m, \epsilon^C, c(\epsilon), b'(\epsilon)} \left\{ \int_{\underline{\epsilon}}^{\epsilon^C} \left(u(\epsilon) + \beta \overline{v} \right) \; f(d\epsilon) + \quad \int_{\epsilon^C} u[c(\epsilon)] f(d\epsilon) + \beta \int_{\epsilon^C} v[b'(\epsilon)] f(d\epsilon) \right\} \qquad \textbf{s.t.}$$

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$$b + \frac{1-\delta}{r+\delta} b = [1 - F(\epsilon^c)] m$$

$$c(\epsilon) = \epsilon + \frac{b'(\epsilon)}{r+\delta} - m, \qquad \text{when } \epsilon > \epsilon^c$$

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Note that the price of debt is $\frac{1}{r+\delta}$. Substituting in the constraints yields

$$v(b) = \max_{\epsilon^c, b'(\epsilon)} \left\{ \int_{\mathbf{0}}^{\epsilon^c} \left(u(\epsilon) + \beta\overline{v} \right) f(d\epsilon) + \int_{\epsilon^c} u \left[\epsilon + \frac{b'(\epsilon)}{r+\delta} - \frac{b\frac{1+r}{r+\delta}}{1 - F(\epsilon^c)} \right] f(d\epsilon) + \beta \int_{\epsilon^c} v[b'(\epsilon)] f(d\epsilon) \right\}$$

- Repeating the problem

$$v(b) = \max_{\epsilon^c, b'(\epsilon)} \left\{ \int_0^{\epsilon^c} \left(u(\epsilon) + \beta \overline{v} \right) f(d\epsilon) + \right.$$

$$\left. \int_{\epsilon^c} u \left[\epsilon + \frac{b'(\epsilon)}{r+\delta} - \frac{b \frac{1+r}{r+\delta}}{1 - F(\epsilon^c)} \right] f(d\epsilon) + \beta \int_{\epsilon^c} v[b'(\epsilon)] f(d\epsilon) \right\}$$

Timing 2: Using Long term debt $\lambda < 1$

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- The first order condition with respect to $b'(\epsilon)$ and ϵ^c are

$$\begin{aligned} u_c(\epsilon) &= -\beta(r+\delta) \ \ v_b[b'(\epsilon)] \\ u(\epsilon^c) + \beta \overline{v} &= u[c(\epsilon^c)] + \beta v[b'(\epsilon^c)] + \int_{\epsilon^c} u_c[c(\epsilon)] \ \ \frac{b \frac{1+r}{r+\delta}}{[(1 - F(\epsilon^c)]^2} f(d\epsilon) \end{aligned}$$

Timing 2: More Algebra

The envelop condition with respect to b gives

$$v_b(b) = -\frac{1+r}{r+\delta} \frac{\int_{\epsilon^c} u_c[c(\epsilon)]f(d\epsilon)}{1 - F(\epsilon^c)}$$

Let $\epsilon^c = d^c(b)$, then forwarding the envelop condition yields

$$v_b[b'(\epsilon)] = -\frac{1+r}{r+\delta} \quad \frac{\int_{d[b'(\epsilon)]} u_c[c(\epsilon')]\ f(d\epsilon')}{1 - F(d^c[b'(\epsilon)])}$$

Combining the FOC wrt $b'(\epsilon)$ and the envelop condition yields

$$u_c[c(\epsilon)]\Big(1 - F(d^c[b'(\epsilon)])\Big) = \beta(1+r)\int_{d^c[b'(\epsilon)]} u_c[c(\epsilon')]\ f(d\epsilon')$$

Let $h^c(b, \epsilon)$ denote the choice of $b'(\epsilon)$, then the two policy functions are characterized by

$$u[d^c(b)] + \beta\overline{v} = u\left[d(b) + \frac{h(b,\epsilon)}{1+r} - \frac{b\frac{1+r}{r+\delta}}{1-F[d(b)]}\right] + \beta v[h(b,\epsilon)]$$

$$+ \int_{\epsilon^c} u_c\left[\epsilon + \frac{h(b,\epsilon)}{1+r} - \frac{b\frac{1+r}{r+\delta}}{1-F[d(b)]}\right] \frac{b\frac{1+r}{r+\delta}}{(1-F[d(b)])^2} f(d\epsilon)$$

$$u_c\left[\epsilon + \frac{h(b,\epsilon)}{1+r} - \frac{b\frac{1+r}{r+\delta}}{1-F[d(b)]}\right]\Big(1 - F(d[h(b,\epsilon)])\Big) =$$

$$\beta(1+r)\int_{d[h(b,\epsilon)]} u_c\left[\epsilon' + \frac{h[h(b,\epsilon),\epsilon']}{1+r} - \frac{h(b,\epsilon)\frac{1+r}{r+\delta}}{1-F(d[h(b,\epsilon)])}\right] f(d\epsilon')$$

Or compactly if $c^c(\epsilon, b) = \epsilon + \frac{h^c(b,\epsilon)}{1+r} - \frac{b\frac{1+r}{r+\delta}}{1-F[\epsilon^c]}$

$$u(\epsilon^c) + \beta v = u\left[c^c(\epsilon^c, b)\right] + \beta v(h^c) + \int_{\epsilon^c} u_c\left[c^c(\epsilon^c, b)\right] \frac{b\frac{1+r}{r+\delta}}{(1-F[\epsilon^c])^2}\ f(d\epsilon),$$

$$u_c\left[c^c(\epsilon, b)\right][1 - F(d'^c)] = \beta(1+r)\int_{d'^c} u_c\left[c^c(\epsilon', h)\right]\ f(d\epsilon').$$

The Problem Without Commitment

Value of honoring debt

$$V^R(\epsilon, b) = \max_{b'} \left\{ u[\epsilon - b + q(b')b'] + \beta \int_{\underline{\epsilon}}^{\bar{\epsilon}} \max \left\{ V^R(\epsilon', b'), V^A(\epsilon') \right\} dF \right\}$$

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Default threshold

$$d(b) = \min \{\{\epsilon : V^R(\epsilon, b) \geq V^A(\epsilon)\} \cup \{\bar{\epsilon}\}\}$$

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Value of honoring debt becomes

$$V^R(\epsilon, b) = \max_{b'} \left\{ u[\epsilon - b + q(b')b'] + \underbrace{\beta \int_{d(b')}^{\bar{\epsilon}} \left\{ V^R(\epsilon', b') - V^A(\epsilon') \right\} dF}_{\text{value of access to credit markets}} + \beta \bar{v} \right\}$$

Short-Term Debt: GEE

$$u_c(c)\underbrace{\left[q(b') + q_b(b')b'\right]}_{\text{marginal revenue}} = \beta \int_{d(b')}^{\bar{\epsilon}} u_c(c')dF$$

- Is this price differentiable? Almost, but not quite.

Default Threshold

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- For debt $b > b^*$ there is default risk.
- $d(b)$ not differentiable at b^*. $\partial^+ d(b) > 0$, but $\partial^- d(b) = 0$.
- No analytical solution for b^*, but we know it solves $V^R(\underline{\epsilon}, b^*) = V^A(\underline{\epsilon})$.

Bond Price

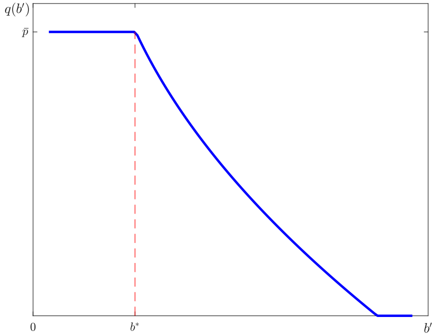
$$\frac{q(b')}{1+r} = \begin{cases} [1 - F(d(b))], & b^* < b', \\ 1, & b' \leq b^*. \end{cases}$$

Derivative is defined for $b' \neq b^*$ (inherited property of $d(b)$)

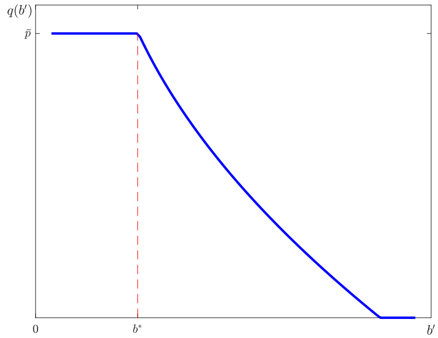
$$\frac{q_b(b')}{1+r} = -f[d(b')]\ d_b(b')$$

Marginal revenue of borrowing at b'

$$q(b') + q_b(b')b' = (1+r)\{[1 - F(d(b))] - f[d(b')]\ d_b(b')\ b'\}$$



- The kink in the price at the risk-free borrowing limit b^* makes b^* more attractive.



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- Agents will choose to state at b^* to avoid lowering the price of their debt.

Short-Term Debt: GEE

From Clausen and Strub (2020) we know either

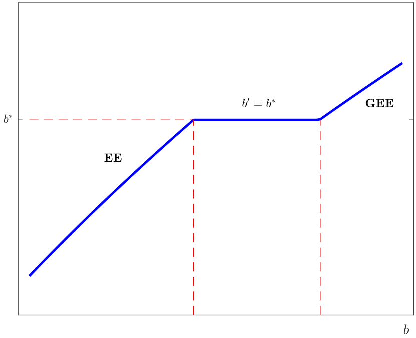
1. $b' = b^*$
2. or $b' > b^*$ and solves the GEE

$$u_c(c)[(1 - F(d(b))) - f(d(b'))d_b(b')b'] = \beta R \int_{d(b')}^{\bar{\epsilon}} u_c(c')dF$$

3. or $b' < b^*$ and solves EE

$$u_c(c) = \beta R \int u_c(c')dF$$

- No need to consider the price explicitly



- Agents stay at the risk-free limit b^* to avoid lowering price of debt

- Consumption with long maturity bonds

$$c = \epsilon - b + q(b') \; [b' - (1 - \lambda) b]$$

Long-Term Debt: What's Different? Dilution

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- Sovereign's choice of borrowing determines the value of outstanding debt $q(b')(1-\lambda)b$

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- Since debts can be diluted by sovereign, price today depends on future actions. Sovereign cannot commit not to borrow more in the future.

- This is a harder problem to characterize without the price.

The value of repaying debt

$$
\begin{aligned}
V^R(\epsilon, b) &= \max_{b'} \Big\{ u(\epsilon - b + q(b')\,[b' - (1-\lambda)b]\,) + \beta W(b') \Big\} \\
&= \max_{b'} \Big\{ u(\epsilon - b + q(b')\,[b' - (1-\lambda)b]\,) + \beta \int_{d(b')}^{\bar{\epsilon}} \Big\{ V^R(\epsilon', b') - V^A(\epsilon') \Big\}\, dF + \beta \bar{v} \Big\}
\end{aligned}
$$

What would a GEE look like (when it holds)?

$$
u_c(\cdot)[q(b') + q_b(b')[b' - (1-\lambda)b]] = -\beta W_b(b')
$$

- Depends on derivative of two objects $q_b(b')$ and $W_b(b')$

Lemma. $W(b')$ is differentiable everywhere in b'.

$$W(b') = \int_{d(b')}^{\bar{\epsilon}} \{V^R(\epsilon', b') - V^A(\epsilon')\} dF + \beta \bar{v}$$

$$W_b(b') = -\int_{d(b')}^{\bar{\epsilon}} u_c(c') \, [1 + (1-\lambda) q(b'')] dF$$

- The marginal cost of an additional unit of borrowing is the expected marginal utility loss of paying the coupon and rolling over unmatured debt at tomorrow's price in repayment states.

- The bond price equals discounted expected payoff of lending b'.

$$\frac{q(b')}{1+r} = \int_{\underline{\epsilon}}^{\bar{\epsilon}} 1_{\left\{V^R(\epsilon',b')\geq V^A(\epsilon')\right\}} \left[1 + (1-\lambda) q(h(\epsilon', b'))\right] dF$$

$$= \left[1 - F(d(b'))\right] + (1-\lambda) \int_{d(b')}^{\bar{\epsilon}} q(h(\epsilon', b')) dF$$

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- Changes in the price due to $d(b')$ reflect *default risk*, those due to $h(\epsilon', b')$ reflect *dilution risk*.

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- Price depends on both default $d(b')$ and future borrowing $h(b', \epsilon')$
- Changes in the price due to $d(b')$ reflect *default risk*, those due to $h(\epsilon', b')$ reflect *dilution risk*.
- Intuitively, more borrowing b' today increases borrowing tomorrow $h(\epsilon', b')$

What is known about the bond price?

Operator on prices

$$(Hq)(b') = \bar{p}[1 - F(d(b'; q))] + \bar{p}(1 - \lambda) \int_{d(b';q)}^{\bar{\epsilon}} q(h(\epsilon', b'; q))dF$$

- What do we know about this? Complicated by $d(\cdot; q)$ and $h(\cdot; q)$ being implicit functions of q.

Long-Term Debt: Bond Price

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- Chatterjee and Eyigungor (2012) show existence of a fixed point q^* that is decreasing in b'.
- We want to strengthen what we can say about $q(b')$, since the price derivative $q_b(b')$ effects the marginal incentive to borrow.

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- We impose a restriction on $q(b')$ that it be the limit of a finite horizon model as $T \to \infty$.
- Specifically, we consider the price of debt in the first period of a finite horizon model $q_1(b'; T)$ as T becomes large

Long-Term Debt: Bond Price

- We want to understand more about the properties of the bond price.

- We impose a restriction on $q(b')$ that it be the limit of a finite horizon model as $T \to \infty$.

- Specifically, we consider the price of debt in the first period of a finite horizon model $q_1(b'; T)$ as T becomes large

- We use backwards induction starting at $q_T(b'; T) = 0$ to get $q_{T-1}(b'; T) = \bar{p}1_{\{b'<0\}}, \; \ldots$, until $q_1(b'; T)$.

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- We use backwards induction starting at $q_T(b'; T) = 0$ to get $q_{T-1}(b'; T) = \bar{p}1_{\{b'<0\}}, \ \ldots$, until $q_1(b'; T)$.
- This is a restriction to say the $q(b)$ of interest is the limit of a *specific* sequence of functions

Bond Price

$$q(b') = \begin{cases} \bar{p}[1 - F(d(b'))] + \bar{p}(1-\lambda)\int_{d(b')}^{\bar{\epsilon}} q(h(\epsilon', b'))dF, & b^* < b' \\ \bar{p} + \bar{p}(1-\lambda)\int_{\underline{\epsilon}}^{\bar{\epsilon}} q(h(\epsilon', b'))dF, & 0 < b' \leq b^* \\ \frac{1}{r+\lambda}, & b' \leq 0 \end{cases}$$

- With short-term debt ($\lambda = 1$), $q(b') = \bar{p}$ when $b' < b^*$. No longer the case with long-term debt.

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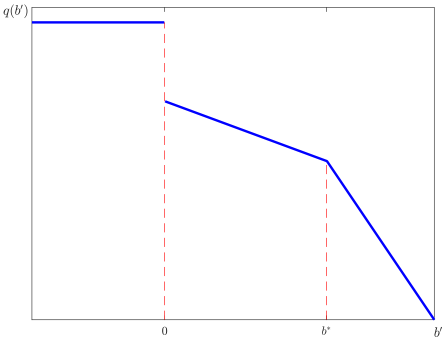
- Debt will be honored next period with certainty, but is discounted for dilution risk.
- Why? Intuitively, if there is probability of $b' > b^*$ at some point (after a sequence of bad shocks), the price today reflects this risk.

Long-Term Debt: Bond Price



- With long-term debt there is a discount for dilution risk at $b' = 0$.

Long-Term Debt: Bond Price

Derivative for $b' \notin \{0, b^*\}$

$$q_b(b') = \bar{p}(1-\lambda) \underbrace{\int_{d(b')}^{\bar{\epsilon}} q_b(h(\cdot))h_b(\cdot)dF}_{\textbf{Dilution},\ b'>0} - \bar{p} \overbrace{\overbrace{\left[1 + (1-\lambda)q(h(d(b'), b'))\right]}^{\textbf{Value of loss}} \overbrace{f(d(b'))d_b(b')}^{\textbf{Marginal P(default)}}}^{\textbf{Default},\ b'>b^*}$$

Leads to three cases for our GEE

1. Borrowing $b' > b^*$ has both *default* and *dilution* terms
2. Borrowing $0 < b' < b^*$ has *dilution* risk only
3. Saving $b < 0$ has neither

Is this dilution term well-defined? Yes

$$\int_{d(b')}^{\bar{\epsilon}} q_b(h(\cdot))h_b(\cdot)dF$$

There are three types of points $\epsilon \in [d(b'), \bar{\epsilon}]$.

1. Points s.t. $b' \notin \{0, b^*\}$, and h_b, $q_b(h)$ are defined.

2. Points s.t. $b' \in \{0, b^*\}$, and $h_b = 0$, $\Rightarrow q_b(h)h_b = 0$.

3. The remaining points where $b' \in \{0, b^*\}$, and h_b, hence the integrand $q_b(h)h_b$, is not well-defined.

The last set of points has zero measure.

Long-Term Debt: Eliminating $q_b(b')$

Use value of q_b implied by GEE, call it $B(h, d, q)$

$$q_b = B(h, d', q) = \frac{\int_{d'} u_c[1 + (1-\lambda)q']dF - u_c(c)q}{u_c[h - (1-\lambda)b]}$$

Substitute this into the expression for the bond price derivative

$$\frac{q_b}{1+r} = (1-\lambda)\int_{d(b')}^{\bar{\epsilon}} B(h', d'', q')h_b dF - [1 + (1-\lambda)\tilde{q}]\, f(d) d_b$$

Substitute back into GEE

$$u_c(c)\left[q(b') + \left\{\bar{p}(1-\lambda)\int_{d(b')}^{\bar{\epsilon}} B(h', d'', q')h_b dF - \bar{p}\,[1 + (1-\lambda)\tilde{q}]\, f(d) d_b\right\}[b' - (1-\lambda)b]\right]$$
$$= \beta \int_{d(b')}^{\bar{\epsilon}} u_c(c')[1 + (1-\lambda)q(b'')]dF$$

Long-Term Debt: GEE Effects

$$
u_c(c)\Bigg[\overbrace{q(b')}^{\text{consumption gain from marginal borrowing}} + \underbrace{\left\{\bar{p}(1-\lambda)\int_{d(b')}^{\bar{\epsilon}} B(h', d'', q')h_b dF\right\}}_{\text{dilution, } b'>0}[b'-(1-\lambda)b] - \underbrace{\left\{\bar{p}\left[1+(1-\lambda)\tilde{q}\right] f(d)d_b\right\}}_{\text{default, } b'>b*}[b'-(1-\lambda)b]\Bigg]
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Two borrowing regions that reflect different risks to creditors:

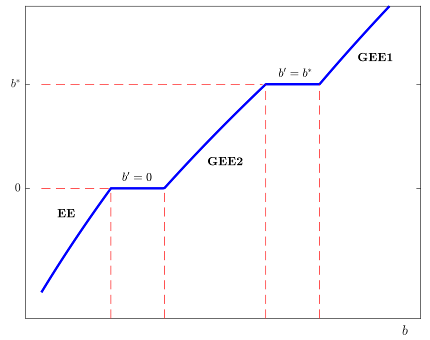
1. $b' > b^*$ the GEE reflects both default and dilution risk
2. $0 < b^{'} < b^*$ the GEE reflects only dilution risk

Long-Term Debt: GEE and Borrowing Policy

$$u_c(c)[q(b') + q_b(b')[b' - (1-\lambda)b]] = \beta \int_{d(b')}^{\bar{\epsilon}} u_c(c')[1 + (1-\lambda)q(b'')]dF$$

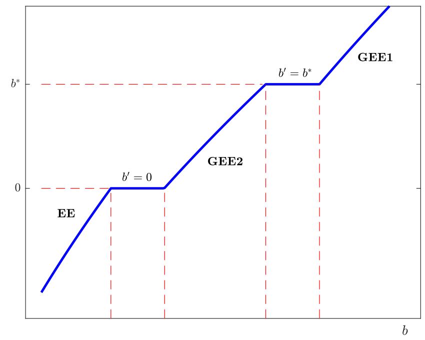
The borrowing policy $b' = h(\epsilon, b)$ satisfies:

1. $b' > b^*$ and solves the GEE1 (dilution and default risk)
2. $b' = b^*$
3. $0 < b' < b^*$ and solves the GEE2 (only dilution risk)
4. $b' = 0$
5. $b' < 0$ and solves the EE



- Agents wait to borrow, due to dilution lowering the price of borrowing.

Long-Term Debt: Borrowing Policy



- Agents wait to borrow, due to dilution lowering the price of borrowing.

- As with short-term debt, agents stay at risky borrowing limit b^*.

We can take a closer look at the derivative of the default threshold

$$d_b(b') = \frac{u_c(c(d(b'), b'))[1 + (1-\lambda)q(b'')]}{u_c(c(d(b'), b')) - u_c(d(b'))} > 1$$

- Numerator is marginal utility loss from additional debt after repayment.

Long-Term Debt: Default Threshold

We can take a closer look at the derivative of the default threshold

$$d_b(b') = \frac{u_c(c(d(b'), b'))[1 + (1 - \lambda)q(b'')]}{u_c(c(d(b'), b')) - u_c(d(b'))} > 1$$

- Numerator is marginal utility loss from additional debt after repayment.
- Denominator cost, in terms of marginal utility, to maintain access to financial markets.

We can describe equilibrium as set of functional equations in h and d

- Auxiliary Functions

$$q(h(\epsilon, b)) = \bar{p}\left\{[1 - F(d)] + (1-\lambda)\int_d q(h(h))dF\right\}$$

$$B(\epsilon, b; h, d, q) = \frac{\int_{d'} u_c[1 + (1-\lambda)q']dF - u_c q}{u_c[h - (1-\lambda)b]}$$

$$V^R(\epsilon, b) = u(\epsilon - bq[h - (1-\lambda)b) + \int_d V^R - V^A dF + \beta\bar{v}$$

$$u_c(c)\left[q(b') + \left\{\bar{p}(1-\lambda)\int_{d(b')}^{\bar{\epsilon}} B(h', d'', q')h_b dF - \bar{p}\left[1 + (1-\lambda)\tilde{q}\right] f(d)d_b\right\}[b' - (1-\lambda)b]\right]$$

$$= \beta\int_{d(b')}^{\bar{\epsilon}} u_c(c')[1 + (1-\lambda)q(b'')]dF$$

$$V^R(d, \epsilon) = V^A(d), \qquad V^R(\underline{\epsilon}, b^*) = V^A(\underline{\epsilon})$$

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- Equilibrium functional equations

$$u_c(c)\left[q(b') + \left\{\bar{p}(1-\lambda)\int_{d(b')}^{\bar{\epsilon}} B(h', d'', q')h_b dF - \bar{p}\,[1 + (1-\lambda)\tilde{q}]\,f(d)d_b\right\}[b' - (1-\lambda)b]\right]$$

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Computation

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- Hatchondo et al. (2010) compare various VFI algorithms to solve the short-term debt problem, but assess their accuracy using Euler residuals.
- Our characterization suggests using a numerical approach based on the GEE and auxiliary equations

Conclusion

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Conclusion

- We characterized the equilibrium of unilateral default problem without commitment.
- If marginal revenue is well-defined, the GEE describes the optimal borrowing policy.
- The GEE fails to capture tradeoffs at choices where the price is not differentiable, but we can still describe the optimal policy.
- Thank you!

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