Macro 7210 Lectures

Preliminary

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Introduction

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- It delivers some fundamental properties that are characteristics of industrialized economies. Kaldor (1957) summarizes six (plus one) stylized facts.

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    3. Recursive Competitive Equilibrium (RCE) directly.

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Neoclassical Growth Model satisfies those facts provided

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   - Log plus Constant Frisch: :

$$u(c, 1-\ell) = u(c, n) \log c + \chi \frac{n^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}}$$

Recursive Equilibria without Distortions

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  - Equilibrium Conditions/ Representative Agent Conditions

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- $c = c(K, a; G)$, $a' = g(K, a; G)$, $V(K, a; G)$ satisfy (use envelope)

$$u_c\left[c(K, a; G)\right] = \beta V_{a'}\left[G(K), g(K, a; G); G\right]$$

$$V_a\left(K, a; G\right) = R(K)\, u_c\left[c(K, a; G)\right]$$

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- In this case we can use the $G(K)$ that comes out of the social planner's dynamic programming problem as the candidate for RCE.

Economies with Distortions and Heterogeneity

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- If labor income tax, substitute $T(K)$ with $\tau(K)\ w(K)$.

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- Eq Cond: $P^*(K) = \tau(K) r^*(K) K$, and $R(K) = 1 + r(K)$ plus Rep Agent.
- The First Welfare Theorem fails and the RCE is not Pareto optimal. (if $\tau(K) > 0$ there will be a wedge, and the efficiency conditions will not be satisfied).

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Exercise

Derive the first order conditions in the above problem to see the wedge introduced by taxes.

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- So individual state is just a

- The household needs to know the evolution of capital and *debt*

$$\begin{aligned} V(K,B,a) &= \max_{c\geq 0, a'} \quad u(c, P(K,B)) + \beta V(K',B',a') \\ \text{s.t.} \quad c + a' &= w(K) + aR(K)\left(1-\tau(K,B)\right) \\ K' &= G(K,B) \\ B' &= H(K,B) \end{aligned}$$

with solution $g(K,B,a)$.

Definition

A *Rational Expectations Recursive Competitive Equilibrium given* $P(K, B)$ *and* $\tau(K, B)$*, are functions* V*,* g*,* G*,* H*,* w*,* q*, and* R*, s.t.*

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5. *Gov b constr:* $B + P(K, B) = \tau(K, B)R(K)K + q(K, B)H(K, B)$
6. *Government debt is bounded:* $\exists$ *some* $\bar{B}$*, such that for all* $K \in \left[0, \tilde{k}\right)$ *and* $B \leq \bar{B}$*,* $H\left(K, B\right) \leq \bar{B}$*.*

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Define it. Is the equilibrium optimum in this case?

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2. *Catching up with the Jones* $u\left(c, C^{-}\right)$*. Externality from aggregate consumption. Aggregate state* $\left\{K, C^{-}\right\}$*, while* c^{-} *is not a state.*

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How does the agent know C*? Is the equilibrium optimum?*

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- $P(K)$ is shares price; $D(K)$ dividends per share. Soltn, $a' = h(K, a)$

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$$\begin{aligned} V(K,a) = \max_{c,a'} \quad u(c) \quad &+ \quad \beta V[G(K), a'] \\ \text{s.t.} \quad c + P(K)a' \quad &= \quad a\,[D(K) + P(K)] \end{aligned}$$

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- Rep firms buy and install capital; own one unit of land used to produce $F(K, L)$.
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- d dividends (solution $d(K,k)$), $q\left[G(K)\right]$ is price of good tomorrow.

Definition

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1. Given prices, V and h solve the household's problem,

Exercise

Find missing condition. [Hint: it relates $q(G(K))$ with the price and dividends ($P(K), P(G(K))$, and $D(G(K))$).]

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Define the RCE if a were savings paying $R(K)$ instead of shares.

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5. Value of a representative firm equals price plus dividends

$$\Omega(K,K) = D(K) + P(K),$$

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Exercise

Define the RCE if a were savings paying $R(K)$ instead of shares.

- Two types of households differing only in wealth: R (rich) and P (poor) with measures μ and $1-\mu$. Otherwise identical.

$$
\begin{aligned}
V(K^R, K^P, a) = \max_{c,a'} \quad & u(c) \quad + \quad \beta V(K^{R'}, K^{P'}, a') \\
\text{s.t.} \qquad c + a' \quad & = \quad w\left[(\mu K^R + (1-\mu)K^P\right] + aR\left[\mu K^R + (1-\mu)K^P\right] \\
K^{i'} \quad & = \quad G^i(K^R, K^P) \qquad \text{for } i = R, P.
\end{aligned}
$$

Remark

Decision rules are not linear (even though they might be almost linear); therefore, we need two states, K^1 and K^2, not aggregate K.

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Remark

Note that $G^R\left(K^R, K^P\right) = G^P\left(K^P, K^R\right)$ (look at the arguments carefully). Why? (How are rich and poor different?)

- In steady state, the Euler equations for the two types simplify to

$$u'\left(c^{R^*}\right) = \beta R \; u'\left(c^{R^*}\right), \text{ and } u'\left(c^{P^*}\right) = \beta R \; u'\left(c^{P^*}\right).$$

$$\text{so } \beta R = 1, \text{ where } R = F_K\left(\mu K^{R^*} + (1-\mu) K^{P^*}, 1\right).$$

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- We have three equations (2 budget constraints and Euler equation) and four unknowns (a^{i^*} and c^{i^*} for $i = R, P$).
- The theory is silent about the steady state distribution of wealth!
- If savings are linear in a state (i.e. $g(K, a) = \mu^i(K) + \lambda(K)a$, and all have the same preferences, then aggregate capital can be expressed as the choice of a representative agent (with savings decision given by $g(K, K) = \bar{\mu}(K) + \lambda(K)K$).

- Type i has labor skill ϵ_i, $\mu^1 = \mu^2 = 1/2$. $\mu^1 \epsilon_1 + \mu^2 \epsilon_2 = 1$.

Exercise

Define the RCE.

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- The value functions are now indexed by type:

$$V^i(K^1, K^2, a) = \max_{c,a'} \quad u(c) \quad + \quad \beta V^i(K^{1'}, K^{2'}, a')$$

$$\text{s.t.} \quad c + a' \quad = \quad w\left(\frac{K^1 + K^2}{2}\right)\epsilon_i + aR\left(\frac{K^1 + K^2}{2}\right)$$

$$K^{i'} \quad = \quad G^i(K^1, K^2) \quad \text{for } i = 1, 2.$$

with solution $g^i(K^1, K^2, a)$.

Exercise

Define the RCE.

Remark

We can also rewrite this problem as

$$\begin{aligned} V^i(K,\lambda,a) = \max_{c,a'} \quad & \left\{ u(c) + \beta V^i(K',\lambda',a') \right\} \\ s.t. \quad & c + a' = R(K)\,a + W(K)\,\epsilon_i \\ & K = G(K,\lambda) \\ & \lambda' = H(K,\lambda), \end{aligned}$$

where K is aggregate capital, and λ is the share of type 1.

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where K is aggregate capital, and λ is the share of type 1.

Then the consistency conditions of the RCE must be:

$$G(K,\lambda) = \frac{1}{2}\left[g^1(K,\lambda,2\lambda K) + g^2(K,\lambda,2(1-\lambda)K)\right],$$
$$H(K,\lambda) = \frac{g^1(K,\lambda,2\lambda K)}{2G(K,\lambda)}.$$

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  - Need also a variable for wealth distribution, say, shares in country 1.

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s.t. \quad & c + Q(K^{1},K^{2},A)a' = w^{i}\left(K^{i}\right) + a\Phi(K^{1},K^{2},A) \\
& K^{i'} = G^{i}\left(K^{1},K^{2},,A\right), \quad \text{for } i = 1,2 \\
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- Mutual Funds' problem (note wages are country specific)

$$\Phi(K^1, K^2, A, k^1, k^2) \quad = \quad \max_{k^{1'}, k^{2'}, n^1, n^2} \sum_i \left[F^i(k^i, n^i) - n^i w^i(K_i) - k^{i'}\right] + \frac{1}{R(K^{1'}, K^{2'}, A)} \Phi(K^{1'}, K^{2'}, A', k^{1'}, k^{2'})$$

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Rec Comp Equil: $\left\{V^i, h^i, g^i, n^i, w^i, G^i\right\}_{i=1,2}$, Φ, H, Q, and R, S.t.:

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Exercise

Solve for the mutual fund's decision rules. Is next period capital in each country chosen by the mutual fund priced differently? What about labor?

Overlapping Generations

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What are they?

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    - May also happen in Aiyagari type economies Aguiar, Amador, and Arellano (2021)
- We may just want to be realistic about the finite nature of the length of life.

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- Many Bells and Whistles are possible.

What about Money?

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- Consider

$$m_t = \frac{\omega^y - c_t^y}{p_t}$$

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- Still, Why accept Money from older agents? Who needs them?

The Lucas Tree

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$$V(z,s) = \max_{c,s'} \quad u(c) + \beta \sum_{z'} \Gamma_{zz'} V(z', s')$$

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Definition

A Rational Expectations Recursive Competitive Equilibrium is a set of functions, V, g, d, and p, such that

1. V and g solves the household's problem given prices,
2. $d(z) = z$, and,
3. $g(z, 1) = 1$, for all z.

- Recall

$$u_c\left(c\left(z,s\right)\right)=\beta\sum_{z'}\Gamma_{zz'}\left[\frac{p\left(z'\right)+d\left(z'\right)}{p\left(z\right)}\right]\ u_c\left(c\left(z',s'\right)\right).$$

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$$p\left(z\right)u_c\left(z\right)=\beta\sum_{z'}\Gamma_{zz'}u_c\left(z'\right)\left[p\left(z'\right)+z'\right]\qquad\forall z.$$

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- A system of n_z equations. Denote $\mathrm{p}:=\left[p\left(z_1\right)\vdots p\left(z_n\right)\right]_{(n_z\times 1)}$ and

$$\mathrm{u}_c:=\begin{bmatrix}u_c\left(z_1\right) & & 0\\ & \ddots & \\ 0 & & u_c\left(z_n\right)\end{bmatrix}_{(n_z\times n_z)}.$$

- Then

$$\mathrm{u}_c.\mathrm{p} = \begin{bmatrix} p\left(z_1\right)u_c\left(z_1\right) \\ \vdots \\ p\left(z_n\right)u_c\left(z_n\right) \end{bmatrix}_{(n_z \times 1)},$$

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- where Γ is the transition matrix for z, as before. Hence, share prices are

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- This follows from a no-arbitrage argument.

$$p^m(z_0) = \sum_t \sum_{z^t \in H^t} q_t^0\left(z^t\right)\ a_t\left(z^t\right),$$

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- Given the $\left\{q_t^0(z^t)\right\}$, we can *replicate any possible asset by a set of state-contingent claims* and use this formula to price that asset.

- To find those q^0 consider a world where agents solve

$$\max_{c_t(z^t)} \quad \sum_{t=0}^{\infty} \beta^t \sum_{z^t} \pi_t\left(z^t\right) u\left(c_t\left(z^t\right)\right)$$

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- Note that this is the familiar Arrow-Debreu market structure, where the household owns a tree, and the tree yields $z \in Z$ amount of fruit in each period). The FOC for this problem imply:

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- This enables us to price the good in each history of the world and price any asset accordingly.

- Hhold Probl

$$V(z,s,b) = \max_{c,s',b'(z')} \quad u(c) + \beta \sum_{z'} \Gamma_{zz'} V\left(z', s', b'\left(z'\right)\right)$$

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- We can thus price *all types* of securities using p and q in this economy.

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- The European option to buy the day after tomorrow is

$$\bar{q}(z, P) = \sum_{z'} \sum_{z''} \max\{p(z'') - P, 0\} q(z', z'') q(z, z').$$

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- The unconditional gross risk free rate is

$$R^f = \sum_{z} \mu_z^* R(z)$$

where μ^* is the steady-state distribution of the shocks.

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- Use the expressions for p and q and the properties of the utility function to show that risk premium is positive.

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- Discussion of Demand vs Supply Shocks and what RBC vs Lucas trees are.

An Introduction to Search with a Particular Application: Endogenous Productivity in a Product Search Model

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- After meeting, trades may happen or not.

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$$\Psi^{f}(Q) := \frac{M(T, D)}{T} = Q^{-\varphi}$$

- There is matching function $M(T, D)$: Trees and Search Effort.
  - Constant Returns to Scale, e.g. $D^{\varphi}T^{1-\varphi}$. Let $\frac{1}{Q} := \frac{D}{T}$, i.e. the ratio of shoppers per trees, *the market tightness*.
  - Other more natural matching functions $D \leq M(T, D) \leq D$.
- The probability that a unit of shopping effort finds a tree is

$$= \Psi^h(Q) := \frac{M(T, D)}{D} = Q^{1-\varphi}$$

- The probability that a tree finds a shopper is

$$\Psi^f(Q) := \frac{M(T, D)}{T} = Q^{-\varphi}$$

- Here $T = 1$. The number of trees is constant.

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$$V(\theta, z, s) = \max_{c,d,s'} \quad u(c, d, \theta) + \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} \; V(\theta', z', s')$$

$$s.t. \quad c + P(\theta, z) s' = P(\theta, z) \left[s \left(1 + \widehat{R}(\theta, z) \right) \right]$$

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Exercise

Derive the Euler equation of the household from the problem defined above.

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- Consider a world consisting of a large number of islands. Each island has a sign that displays two numbers, $P(\theta, z)$ and $Q(\theta, z)$. (price and market tightness)
- Searchers and (trees and household effort) choose which island to go to. They have different trade-offs of price versus tightness.
- Equilibrium determines which island (Optimal so unique).

$$V(\theta, z, s) = \max_{c,d,s',P,Q} \quad u(\theta \; c, d) + \beta \sum_{\theta',z'} \Gamma_{\theta\theta'} \Gamma_{zz'} V\left(\theta', z', s'\right) \tag{1}$$

$$s.t. \quad c + Ps' = P\left[s\left(1 + \widehat{R}(\theta, z)\right)\right], \tag{2}$$

$$c = d \; \Psi^h(Q) \, z \tag{3}$$

$$\frac{z\Psi^f(Q)}{P} \geq \widehat{R}(\theta, z) \tag{4}$$

- The last constraint states that for a market to exist firms have to be guaranteed $\widehat{R}(\theta, z)$.

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Plug the first two constraints into the objective function (c and s' as functions of d) and (recall that $\Psi^h = Q^{1-\varphi}$) :

$$\theta Q^{1-\varphi} z u_c(\theta d Q^{1-\varphi} z, d) + u_d(\theta d Q^{1-\varphi} z, d) = \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} V_3\left(\theta', z', s(1+\widehat{R}(\theta, z)) - \frac{dQ^{1-\varphi} z}{P}\right) \frac{Q^{1-\varphi} z}{P} \quad (5)$$

Get rid of V_3 using original problem and use the envelope theorem

$$V_3(\theta, z, s) = \left[\theta u_c(\theta d Q^{1-\varphi} z, d) + \frac{u_d(\theta d Q^{1-\varphi} z, d)}{Q^{1-\varphi} z}\right] P(1+\widehat{R}(\theta, z))$$

Combining these two gives the Euler equation:

$$\theta u_c(\theta d Q^{1-\varphi} z, d) + \frac{u_d(\theta d Q^{1-\varphi} z, d)}{Q^{1-\varphi} z} = \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} \frac{P'(1+\widehat{R}(\theta', z'))}{P} \left[\theta' u_c(\theta' d' Q'^{1-\varphi} z', d') + \frac{u_d(\theta' d' Q'^{1-\varphi} z', d')}{Q'^{1-\varphi} z'}\right] \quad (6)$$

λ: Lagrange multiplier on the firm's participation constraint, then

$$\theta d(1-\varphi)Q^{-\varphi} z u_c(\theta d Q^{1-\varphi} z, d) = \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} V_3 \left(\theta', z', s(1+\widehat{R}(\theta,z)) - \frac{dQ^{1-\varphi} z}{P} \right) \frac{d(1-\varphi)Q^{-\varphi} z}{P} - \lambda \frac{\varphi Q^{-\varphi-1} z}{P} \quad (7)$$

and

$$\beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} V_3 \left(\theta', z', s(1+\widehat{R}(\theta,z)) - \frac{dQ^{1-\varphi} z}{P} \right) dQ = -\lambda \quad (8)$$

Combining these two equation gives us:

$$\theta u_c(\theta d Q^{1-\varphi} z, d) = \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'}$$
$$V_3 \left(\theta', z', s(1 + \widehat{R}(\theta, z)) - \frac{d Q^{1-\varphi} z}{P} \right) \left[\frac{1}{(1-\varphi)P} \right] \quad (9)$$

Recall $V_3(\cdot, \cdot, \cdot)$ so

$$(1-\varphi)\theta u_c(\theta d Q^{1-\varphi} z, d) = \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'}$$
$$\frac{P'(1 + \widehat{R}(\theta', z'))}{P} \left[\theta' u_c(\theta' d' {Q'}^{1-\varphi} z', d') + \frac{u_d(\theta' d' {Q'}^{1-\varphi} z', d')}{{Q'}^{1-\varphi} z'} \right] \quad (10)$$

Definition

An Eq with competitive search is functions $\{V, c, d, s', P, Q, \widehat{R}\}$ that:

1. Household's budget constraint, (condition 2)
2. Household's shopping constraint, (condition 3)
3. Household's Euler equation, (condition 6)
4. Market condition, (condition 10)
5. Firm's participation constraint, (condition 4), which gives us that the dividend payment is the profit of the firm, $\widehat{R}(\theta, z) = \frac{zQ^{-\varphi}}{P}$,
6. Market clearing, i.e. $s' = 1$ and $Q = 1/d$.

Firms maximize returns by choosing market, Q, P. It helps to use trees as numeraire, so $\widehat{P}(Q) = 1/P$ is the price of consumption. We want to characterize the set of available markets for firms, $\widehat{P}(Q)$ by looking at the implications for firms that face it:

$$\pi = \max_{Q} \ \widehat{P}(Q) \ \Psi^f(Q) \ z$$

with FOC

$$\widehat{P}'(Q)\Psi^f(Q) + \widehat{P}(Q)\Psi^{f'}(Q) = 0,$$

The set of pairs P a that satisfies FOC yields a relation of indifference between the firms the pairs $\{P, Q\}$ for the firms that implicitly determines $\widehat{P}(Q)$ as

$$\frac{\widehat{P}'(Q)}{\widehat{P}(Q)} = -\frac{\Psi^{f'}(Q)}{\Psi^f(Q)}.$$

Measure Theory

Measure theory is a tool that helps us aggregate.

Definition

For a set S, $\mathcal{S}$ is a family of subsets of S, if $B \in \mathcal{S}$ implies $B \subseteq S$ (but not the other way around).

Remark

Note that in this section we will assume the following convention

1. *small letters (e.g. s) are for elements,*
2. *capital letters (e.g. S) are for sets, and*
3. *fancy letters (e.g. $\mathcal{S}$) are for a set of subsets (or families of subsets).*

Definition

A family of subsets of S, $\mathcal{S}$, is called a σ-algebra in S if

1. $S, \emptyset \in \mathcal{S}$;
2. if $A \in \mathcal{S} \Rightarrow A^c \in \mathcal{S}$ (i.e. $\mathcal{S}$ is closed with respect to complements and $A^c = S \backslash A$); and,
3. for $\{B_i\}_{i \in \mathbb{N}}$, if $B_i \in \mathcal{S}$ for all $i \Rightarrow \bigcap_{i \in \mathbb{N}} B_i \in \mathcal{S}$ (i.e. $\mathcal{S}$ is closed with respect to countable intersections.

Example

1. The power set of S and $\{\emptyset, S\}$ are σ-algebras in S.
2. $\{\emptyset, S, S_{1/2}, S_{2/2}\}$, where $S_{1/2}$ means the lower half of S (imagine S as an closed interval in $\mathbb{R}$), is a σ-algebra in S.
3. If $S = [0, 1]$, then $\mathcal{S} = \left\{\emptyset, \left[0, \frac{1}{2}\right), \left\{\frac{1}{2}\right\}, \left[\frac{1}{2}, 1\right], S\right\}$ is *not* a σ-algebra in S. But $\mathcal{S} = \left\{\emptyset, \left\{\frac{1}{2}\right\}, \left\{\left[0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right]\right\}, S\right\}$ is.

It allows us to define sets where things happen and we can *weigh* those sets (avoiding math troubles)

Definition

Suppose $\mathcal{S}$ is a σ-algebra in S. A measure is a real-valued function $x : \mathcal{S} \to \mathbb{R}_+$, that satisfies

1. $x(\emptyset) = 0$;
2. if $B_1, B_2 \in \mathcal{S}$ and $B_1 \cap B_2 = \emptyset \Rightarrow x(B_1 \cup B_2) = x(B_1) + x(B_2)$ (additivity); and,
3. if $\{B_i\}_{i \in \mathbb{N}} \in \mathcal{S}$ and $B_i \cap B_j = \emptyset$ for all $i \neq j \Rightarrow x(\cup_i B_i) = \sum_i x(B_i)$ (countable additivity).

A set S, a σ-algebra in it ($\mathcal{S}$), and a measure on $\mathcal{S}$ x, define a measurable space, $(S, \mathcal{S}, x)$.

Definition

A Borel σ-algebra is a σ-algebra generated by the family of all open sets $\mathfrak{B}$ (generated by a topology). A Borel set is any set in $\mathfrak{B}$.

A Borel σ-algebra corresponds to complete information.

Definition

A probability measure is measure where $x(S) = 1$. $(S, \mathcal{S}, x)$ is a probab space. The probab of an event is then given by $x(A)$, where $A \in \mathcal{S}$.

Definition

Given a m'able space $(S, \mathcal{S}, x)$, a real-valued function $f : S \to \mathbb{R}$ is m'able (with respect to the m'able space) if, for all $a \in \mathbb{R}$, we have

$$\{b \in S \mid f(b) \leq a\} \in \mathcal{S}.$$

Interpret σ-algebras as describing available information.

Similarly, a function is m'able wrt a σ-algebra $\mathcal{S}$, if it can be evaluated

Example

Suppose $S = \{1, 2, 3, 4, 5, 6\}$. Consider a function f that maps the element 6 to the number 1 (i.e. $f(6) = 1$) and any other elements to -100. Then f is NOT measurable with respect to $\mathcal{S} = \{\emptyset, \{1, 2, 3\}, \{4, 5, 6\}, S\}$. Why? Consider $a = 0$, then $\{b \in S \mid f(b) \leq a\} = \{1, 2, 3, 4, 5\}$. But this set is not in $\mathcal{S}$.

Extend the notion of Markov stuff to any measurable space

Definition

Given a measurable space $(S, \mathcal{S}, x)$, a function $Q : S \times \mathcal{S} \to [0,1]$ is a transition probability if

1. $Q(s, \cdot)$ is a probability measure for all $s \in S$; and,
2. $Q(\cdot, B)$ is a measurable function for all $B \in \mathcal{S}$.

Intuitively, for $B \in \mathcal{S}$ and $s \in S$, $Q(s, B)$ gives the probability of being in set B tomorrow, given that the state is s today.

1. A *Markov chain* with transition matrix given by

$$\Gamma = \begin{bmatrix} 0.2 & 0.2 & 0.6 \\ 0.1 & 0.1 & 0.8 \\ 0.3 & 0.5 & 0.2 \end{bmatrix},$$

on $S = \{1, 2, 3\}$, with the the power set being the σ-algebra of S).

$$Q(3, \{1, 2\}) = \Gamma_{31} + \Gamma_{32} = 0.3 + 0.5 \; .$$

2. Consider a measure x on $\mathcal{S}$. x_i is the fraction of type i. Then

$$\begin{aligned} x_1' &= x_1\Gamma_{11} + x_2\Gamma_{21} + x_3\Gamma_{31}, \\ x_2' &= x_1\Gamma_{12} + x_2\Gamma_{22} + x_3\Gamma_{32}, \\ x_3' &= x_1\Gamma_{13} + x_2\Gamma_{23} + x_3\Gamma_{33}. \end{aligned}$$

In other words: $\mathrm{x}' = \Gamma^T \mathrm{x}$, where $\mathrm{x}^T = (x_1, x_2, x_3)$.

From a measure x today to one tomorrow x'

$$\begin{aligned} x'(B) &= T(x, Q)(B) \\ &= \int_S Q(s, B)\, x(ds), \quad \forall B \in \mathcal{S}, \end{aligned}$$

we integrated over all $s \in S$ to get the prob of B tomorrow.

A stationary distribution is a fixed point of T, that is x^* such that

$$x^*(B) = T(x^*, Q)(B), \quad \forall B \in \mathcal{S}.$$

Theorem

If Q has nice properties (American Dream and Nightmare) then $\exists$ a unique stationary distribution x^* and

$$x^* = \lim_{n \to \infty} T^n(x_0, Q), \qquad \textit{for any } x_0.$$

Exercise

Consider unemployment in a very simple economy (in which the transition matrix is exogenous). There are two states of the world: being employed and being unemployed. The transition matrix is given by

$$\Gamma = \begin{pmatrix} 0.95 & 0.05 \\ 0.50 & 0.50 \end{pmatrix}.$$

Compute the stationary distribution corresponding to this Markov transition matrix.

Industry Equilibrium

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- n^* is an increasing function of both arguments. Prove it.

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- Use x to define statistics of the industry: Since individual supply is $sf\left(n^*\left(s,p\right)\right)$, then the aggregate supply

$$Y^S\left(p\right) = \int_S sf\left(n^*\left(s,p\right)\right) x\left(ds\right). \tag{13}$$

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- Let Demand $Y^D\left(p\right)$. Then p^* clears the market:

$$Y^D\left(p^*\right) = Y^S\left(p^*\right). \tag{14}$$

Where does x come from?

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- The choice is static. The value of an s firm is

$$V(s;p) = \sum_{t=0}^{\infty} \left(\frac{\delta}{1+r} \right)^{t} \pi(s,p) = \left(\frac{1+r}{1+r-\delta} \right) \pi(s,p)$$

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- Every period a mass of firms die. To achieve a stationary equilibrium we need firms entry: assume that there is a constant flow of firms entering the economy in each as well, so that entry equals exit.

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- Entrants draw s from probability measure γ over $(S,\mathcal{S})$.

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- Assume a fixed entry cost, c^E before learning s. Value of an entrant

$$V^E(p) = \int_S V(s;p)\gamma(ds) - c^E. \quad (15)$$

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If $V^E > 0$ there will be entry.

- Equilibrium requires $V^E = 0$

- x_t : cross-sectional distribution of firms. For any $B \subset S$, fraction $1 - \delta$ of firms with $s \in B$ die and mass m of newcomers enter. Next period's measure of firms on set B is

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- Mass m of firms would enter $t+1$, with fraction $\gamma(B)$ having $s \in B$, $\forall B \in \mathcal{S}$.

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$$Tx(B) = \delta x(B) + m\gamma(B), \quad \forall B \in \mathcal{S}, \tag{17}$$

a stationary dbon is a fixed point, i.e. x^* such that $Tx^* = x^*$, so

$$x^*(B; m) = \frac{m}{1-\delta}\gamma(B), \quad \forall B \in \mathcal{S}. \tag{18}$$

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$$Y^D\left(p^*\left(m\right)\right) = \int_S s\ f\left[n^*\left(s;p\right)\right]\ dx^*\left(s;m\right), \tag{19}$$

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Definition

A stationary distribution for this environment consists of functions V, π^*, n^*, p^*, x^*, and m^*, that satisfy:

1. Given prices, V, π^*, and n^* solve the incumbent firm's problem;
2. $Y^D\left(p^*\left(m\right)\right) = \int_S s \;\, f\left[n^*\left(s; p\right)\right] \;\, dx^*\left(s; m\right)$;
3. $\int_s V\left(s; p\right)\gamma\left(ds\right) - c^E = 0$; and,
4. $x^*\left(B\right) = \delta x^*\left(B\right) + m^*\gamma\left(B\right), \quad \forall B \in \mathcal{S}$.

- Assume s follows a Markov process with transition Γ. This would change the mapping T in Equation (17) to

$$Tx(B) = \delta \int_S \Gamma(s, B)\, x(ds) + m\gamma(B), \quad \forall B \in \mathcal{S}. \tag{20}$$

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  - Then $\exists$ a threshold, $s^* \in S$, below which firms exit and above stay.

$$V(s;p) = \max \left\{ 0, \pi(s;p) + \frac{1}{(1+r)} \int_S V(s';p) \Gamma(s, ds') - c^v \right\}. \tag{21}$$

- Updating operator becomes

$$x'(B) = \int_{s^*}^{\bar{s}} \Gamma(s, B \cap [s^*, \bar{s}]) \, x(ds) + m\gamma(B \cap [s^*, \bar{s}]), \quad \forall B \in \mathcal{S}. \tag{22}$$

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  - Fraction of workers in largest top 10% of firms

$$\frac{\int_{\hat{s}}^{\bar{s}} n^*(s,p)\, x^*(ds)}{\int_{s^*}^{\bar{s}} n^*(s,p)\, x^*(ds)}.$$

Exercise

Compute the average growth rate of the smallest one third of the firms.

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5. Market clearing:

$$Y^d(p^\star) = \int_{s^\star}^{\bar{s}} s \; f(n^\star(s; p^\star)) x^\star(ds)$$

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- Gini coefficient.

Consider adjustment costs to labor $c\left(n^{-}, n\right)$ due to hiring n units of labor in t as

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$$x'\left(B^{S}, B^{N}\right) = m\gamma\left(B^{S} \cap [s^{*}, \bar{s}]\right) 1_{\{0 \in B^{N}\}} + \int_{s^{*}}^{\bar{s}} \int_{0}^{\bar{N}} 1_{\{g(s, n_{-}; p) \in B^{N}\}} \Gamma\left(s, B^{S} \cap [s^{*}, \bar{s}]\right) x\left(ds, dn_{-}\right),$$

$$\forall\, B^{S} \in \mathcal{S}, \forall\, B^{N} \in \mathcal{N}.$$

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- Consider demand shifters z_t so that $D(P, z_t)$ where $z_{t+1} = \phi(z_t)$ so we can choose to represent it as a sequence or recursively.

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3. **Law of motion**: $x_{t+1}(B) = m_{t+1}\gamma\left(\cap[s_{t+1}^*, \bar{s}]\right) + \int_{s_t^*}^{\bar{s}} \Gamma\left(s, B \cap [s_{t+1}^*, \bar{s}]\right)\ x_t(ds)$, $\forall B \in \mathcal{S}$.

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- Given a path $\{z_t\}_{t=0}^{\infty}$ and an initial x_0, an equilibrium defined in term of sequences is: Sequences $\{p_t, m_t, s_t^*\}$ of numbers, a sequence of measures x_t, and sequences $\{V_t(s), n_t(s)\}_{t=0}^{\infty}$ of functions:

1. **Optimality**: Given $\{p_t\}$, $\{V_t, s_t^*, n_t\}$ sole

$$V_t(s) = \max \left\{ 0, \max \; p_t \; s \; f(n) - wn - c^v + \frac{\int_S V_{t+1}(s') \Gamma(s, ds')}{1+r} \right\}$$

2. **Free-entry**: $\int V_t(s) \; \gamma(ds) \leq c^e$, with strict equality if $m_t > 0$.
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- Obviously You have to add the Expectations to the terms of one period later.

Numerical Approximations

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  3. Specify some tricks or procedures to effectively compute θ^* (say iterate backward from the future to the present using successive approximations).

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- Then do linear approximations in sequence space.

- Consider the social planner's problem (with full depreciation)

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\begin{aligned}
V(k_t) \quad = \quad & \max_{c_t, k_{t+1}} u(c_t) + \beta V(k_{t+1}) \\
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a second order difference equation with exactly two boundary conditions, k_0 and $k_\infty = k^*$.

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    1. guess k_1, use k_0 and $\psi(k_t, k_{t+1}, k_{t+2}) = 0$ to get $k_2, k_3, \ldots$ forward up until some T, and solve $k_T^\psi(k_1) = k^*$.

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- Either way you get a numerical solution starting from any k_0

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- This is in fact an impulse response function.

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Log-Linear Approximation in the Simplest Growth Model II

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&\vdots \\
\widetilde{k}_{t+1}(k_0, \epsilon^t) &= \sum_{\tau=0}^{t} \epsilon_t \ \widehat{k}_{t-\tau+1} \qquad \text{exact if } \epsilon_0 = 1, \ \epsilon_t = 0, \forall t \neq 0,
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4. *Describe a way to compute the evolution of the Gini Index or the Herfindahl Index of the industry over the first fifteen periods.*

Exercise

1. *What happens if demand suddenly doubles starting from a stationary equilibrium?*
2. *Define Formally the stochastic counterparts (sequentially and recursivrly) to the ones that we wrote above?*
3. *Sketch an algorithm to find the equilibrium prices.*
4. *Describe a way to compute the evolution of the Gini Index or the Herfindahl Index of the industry over the first fifteen periods.*
5. *Imagine now that the industry is subject to demand shocks that follow an AR(1). Describe an algorithm to approximate it.*

Incomplete Market Models

- Consider the problem of a farmer with storage possibilities

$$V(s,a) = \max_{c,a' \geq 0} \quad u(c) + \beta \sum_{s'} \Gamma_{ss'} \; V\left(s', a'\right) \qquad s.t.$$

$$c + qa' = a + s$$

a assets, c consumption, and $s \in \{s^1, \cdots, s^{N^s}\} = S$ has transition Γ. q units today yield 1 unit tomorrow. Only nonnegative storage.

- If s constant, then

$$V(a) = \max_{c,a' \geq 0} \left\{ u\left(a + s - qa'\right) + \beta V\left(a'\right) \right\}.$$

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- We can construct a prob distribution over states $S \times A$. Define $\mathcal{B}$ as all subsets of S times Borel-σ-algebra sets in A.
- For any such prob measure x its evolution is

$$x'\left(B\right) = \widetilde{T}(B,x;\Gamma,g) = \sum_{s}\int_0^{\bar{a}}\sum_{s'\in B_s}\Gamma_{ss'}\; 1_{\{g(s,a)\in B_a\}}\; x\left(s,da\right), \quad \forall B \in \mathcal{B}$$

where B_s and B_a are projections of B on S and A,

Theorem

With a well behaved Γ, there is a unique stationary probability x^*, so that:

$$\begin{aligned} x^*(B) &= \widetilde{T}(B, x^*; \Gamma, g)(B), \quad \forall B \in \mathcal{B}, \\ x^*(B) &= \lim_{n\to\infty} \widetilde{T}^n(B, x_0; \Gamma, g)(B), \quad \forall B \in \mathcal{B}, \end{aligned}$$

for all initial probability measures X_0 on $(E, \mathcal{B})$.

We use compactness of $[0, \overline{A}]$.

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$$a^n := -\frac{s_{\min}}{\left(\frac{1}{q} - 1\right)}. \tag{23}$$

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- Or it could be tighter which makes it likely to bind $0 > \underline{a} > a^n$.

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  3. $\lim_{q\to\infty} \int_{A\times S} a\, dX^*(q) < 0$. As $q \to \infty$, arbitrary large consumption is achievable by borrowing.

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- Now we need $\beta(1 + r) < 1$. We write

$$V(s, a) = \max_{c, a' \geq 0} \quad u(c) + \beta \int_{s'} V(s', a') \; \Gamma(s, d\, s') \qquad s.t.$$
$$c + a' = (1 + r)\, a + ws$$

where r is the return on savings and w is the wage rate.

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Rewrite the economy when households like leisure

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- Total transfer needed to compensate all agents to live in $\hat{\theta}$ is

$$\int_{A \times S} \eta(s, a)\, dX^*(\theta).$$

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- Otherwise the best tax policy in the Rep agent (which is Pareto Optimal) would be to subsidize capital to maximize steady state consumption.

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- The latter. Decision rules are not usually linear. But then $x' = G\left(z, x\right)$

$$\begin{aligned}
V\left(z, X, s, a\right) = \max_{c, a' \geq 0} \quad & u\left(c\right) + \beta \sum_{z', s'} \Pi_{zz'} \Gamma^{z'}_{ss'} V\left(z', X', s', a'\right) \\
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& K = \int a\ dX\left(s, a\right) \\
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(replaced factor prices with marginal productivities)

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(replaced factor prices with marginal productivities)

- Computationally, this problem is a beast! So, what then?

- They people believe tomorrow's capital depends only on K and not on x. This, obviously, is not an economy with rational expectations. The agent's problem in such a setting is

$$\begin{aligned} \widetilde{V}(z, K, s, a) = \max_{c,a'} \quad & u(c) + \beta \sum_{z',s'} \Pi_{zz'} \Gamma^{z'}_{ss'} \widetilde{V}\left(z', K', s', a'\right) \\ s.t. \quad & c + a' = a \; z \; f_k\left(K, \bar{N}\right) + szf_n\left(K, \bar{N}\right) \\ & K' = \widetilde{G}(z, K) \end{aligned}$$

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- We could approximate the equilibrium in the computer by posing a linear approximation to $\widetilde{G}$. A pain but doable. Krusell Smith (1997).

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$$\widetilde{V}(z, K, s, a) = \max_{c,a'} \quad u(c) + \beta \sum_{z',s'} \Pi_{zz'} \Gamma_{ss'}^{z'} \widetilde{V}\left(z', K', s', a'\right)$$

$$s.t. \quad c + a' = a\ z\ f_k\left(K, \bar{N}\right) + szf_n\left(K, \bar{N}\right)$$

$$K' = \widetilde{G}(z, K)$$

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- They found it works well in boring settings (things are pretty linear)

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- Valuable for SMALL shocks like all linear approximations.

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    - We look for a fixed point of this (not necessarily iterating mechanically but as solution of a system of equations)

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&\vdots \\
\widetilde{d}_{t+1}(x_0, \epsilon^t) &= \sum_{\tau=0}^{t} \frac{\epsilon_t}{\bar{\epsilon}_0}\ \widehat{d}_{t-\tau+1} \qquad \text{exact if } \epsilon_0 = \widetilde{\epsilon}_0,\ \epsilon_t = 0, \forall t \neq 0.
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- s is Markovian (Γ) labor labor productivity. Then the unemployed

$$V(s, 0, a) = \max_{c,h,a' \geq 0} u(c, h) + \beta \sum_{s'} \Gamma_{ss'} \left[\phi(h) V(s', 1, a') + (1 - \phi(h)) V(s', 0, a')\right]$$

$$s.t. \quad c + a' = h + (1 + r)a$$

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$$V(s, 1, a) = \max_{c,a' \geq 0} u(c) + \beta \sum_{s'} \Gamma_{ss'} \left[\delta V(s', 0, a') + (1 - \delta) V(s', 1, a')\right]$$

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- Define Stationary Equilibrium

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$$s.t. \quad c + a' = ws + (1+r)a$$

- Similarly, the entrepreneur's problem can be formulated as follows

$$V^e(s,\eta,a) = \max_{c,a'\geq 0, d\in\{0,1\}} u(c) + \beta \sum_{s',\eta'} \Gamma_{ss'}\Gamma_{\eta\eta'} \left[d \; V^w(s',\eta',a') + (1-d)V^e(s',\eta',a')\right]$$

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- Income is from profits $\pi(a,s,\eta)$ not wages. Assume entrepreneurs have a DRS technology f. Profits are

$$\pi(s,\eta,a) = \max_{k,n} \eta f(k,n) + (1-\delta)k - (1+r)(k-a) - w\max\{n-s,0\}$$

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- The constraint here reflects the fact that entrepreneurs can only make loans up to a fraction ϕ of his total wealth.

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- Who becomes an entrepreneur in this economy? Without financial constraints, wealth will play no role. $\exists \eta^*$ above which it becomes an entrepreneur.
- With financial constraints wealth matters. Wealthy agents with high h will while the poor with low η will not.
- For the rest, it depends. If η is persistent, poor individuals with high entrepreneurial ability will save to one day become entrepreneurs, while rich agents with low entrepreneurial ability will lend their assets and become workers.

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- What determines $q(a')$? A zero profit on lenders: Probability of default

Monopolistic Competition

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- Small in the Context of the Aggregate Economy. Hence Monopolistic Competition.

- Consumers have a taste for variety

$$u\left(\{c(i)\}_{i\in[0,n]}\right)=\left(\int_0^n c(i)^{\frac{\sigma-1}{\sigma}}di\right)^{\frac{\sigma}{\sigma-1}}$$

where σ is the elasticity of substitution, and $c(i)$ is the quantity consumed of variety i. For ease of notation, we rename $c(i)=c_i$.

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- They are endowed with one unit of time.

$$\max_{\{c_i\}_{i\in[0,n]}} \left(\int_0^n c_i^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}}$$

$$s.t. \quad \int_0^n p_i \; c_i \; di \leq I$$

- Deriving the FOC, and relating the demand for varieties i and j

$$c_j = c_i \left(\frac{p_j}{p_i} \right)^{-\sigma}$$

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- Multiplying both sides by p_j and integrating over j, yields

$$c_i^* = \frac{I}{\int_0^n p_j^{1-\sigma} dj} \, p_i^{-\sigma}$$

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- Here c_i^* depends on the price of i and an aggregate price

- Convenient to define the aggregate price index P as

$$P = \left(\int_0^n p_j^{1-\sigma} \, dj \right)^{\frac{1}{1-\sigma}}$$

Exercise

Show the following within this monopolistic competition framework

1. *σ is the elasticity of substitution between varieties.*
2. *Price index P is the expenditure to purchase a unit-level utility.*
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4. *Is there a missing n?*

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- and thus

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real income times a measure of the relative price of i.

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- $\frac{\sigma}{\sigma-1}$ is a constant mark-up over the marginal cost,
- When varieties are close substitutes ($\sigma \to \infty$), prices converge to W.

Set the wage as numeraire. An Eq is prices $\{p_i^*\}_{i\in[0,n]}$, the aggregate price index P, household's consumption, $\{c_i^*\}_{i\in[0,n]}$, income I, firm's labor demand $\{\ell_i^*\}_{i\in[0,n]}$ and profits $\{\pi_i^*\}_{i\in[0,n]}$, such that

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2. Given P and I, p_i^* and π_i^* solve the firm's problem $\forall i \in [0, n]$
3. Price Aggregation

$$P = \left(\int_0^n (p_j^*)^{1-\sigma} \, dj \right)^{\frac{1}{1-\sigma}}$$

Set the wage as numeraire. An Eq is prices $\{p_i^*\}_{i\in[0,n]}$, the aggregate price index P, household's consumption, $\{c_i^*\}_{i\in[0,n]}$, income I, firm's labor demand $\{\ell_i^*\}_{i\in[0,n]}$ and profits $\{\pi_i^*\}_{i\in[0,n]}$, such that

1. Given prices, $\{c_i^*\}_{i\in[0,n]}$ solves the household's problem
2. Given P and I, p_i^* and π_i^* solve the firm's problem $\forall i \in [0,n]$
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4. Markets clear

$$\begin{aligned} \int_0^n \ell_i^* \, di &= 1 \\ 1 + \int \pi_i^* \, di &= I \end{aligned}$$

A symmetric equilibria: $c_i^* = \bar{c}$, $p_i^* = \bar{p}$, $\ell_i^* = \bar{\ell}$, $\pi_i^* = \bar{\pi}$ for all i.

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    1. *Rotemberg pricing* (menu costs)
    2. *Calvo pricing* (some (randomly set) firms can change prices, others cannot).

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$$\Omega(S, p_j^-) = \max_{p_j} \; p_j c_j^* - W(S) c_j^* - \phi(p_j, p_j^-)$$

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- Without capital $S = P^-$ and Aggregate Shocks.

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$$\Omega^0(S, p_j^-) = [p_j^- - W(S)] c_j^* + \\ (1-\theta) E\{R^{-1}(S')\, \Omega^0(S', p_j^-)\} + \\ \theta\, E\{R^{-1}(S')\, \Omega^1(S', p_j^-)\}$$

Exercise

Derive the following for the dynamic model with Calvo pricing

1. *Solve the firm's problem in sequence space and write the firm's equilibrium pricing* $p_{j,t}$ *as a function of present and future aggregate prices, wages, and endowments:* $\{P_t, W_t, I_t\}_{t=0}^{\infty}$.
2. *Show that under flexible pricing* ($\theta = 1$)*, the firm's pricing strategy is identical to the static model.*
3. *Show that with price rigidity* ($\theta < 1$)*, the firm's pricing strategy is identical to the static model in a steady state with zero inflation.*

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  - Is this a nightmare? No. Log-linearization comes to help

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- We say Log Deviations

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- Smooth Functions $Z = f(X, Y) \Longrightarrow$

$$\overline{Z} \simeq \widehat{z} = f_x(\overline{X}, \overline{Y})\ \ \overline{X}\ \widehat{x} + \beta\ f_y(\overline{X}, \overline{Y})\ \overline{Y}\ \widehat{y}$$

- Recall the Law of motion for the price level

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$$\frac{1}{1-\sigma}\overline{P}\widehat{p} \simeq \theta\frac{1}{1-\sigma}\overline{P}\widehat{p}^{-} + (1-\theta)\frac{1}{1-\sigma}\overline{P}^{*}\widehat{p}^{*}$$

ignoring the consants which always cancels from both sides, noting that in St St $\overline{P} = \overline{P}^{*}$ we have $\widehat{p} = \theta\ \widehat{p}^{-} + (1-\theta)\ \widehat{p}^{*}$

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- Which implies for inflation that

$$\pi = p - p^{-} = (1-\theta)\ (\widehat{p}^{*} - \widehat{p^{-}})$$

- Price setting

$$P^* = \frac{\sigma}{\sigma - 1} \frac{E\left\{\sum_\tau (\theta\beta)^\tau u_c \ P_\tau^{\sigma-1} \ \varphi_\tau \ y_\tau\right\}}{E\left\{\sum_\tau (\theta\beta)^\tau u_c \ P_\tau^{\sigma-1} \ y_\tau\right\}}$$

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- Approximating the left hand side gives the terms

$$E\left\{\sum_\tau \; (\theta\beta)^\tau \; \overline{U}_c \; \overline{P}^{\sigma-1} \; \overline{Y} \; \overline{P}^* \left[\widehat{u}_{c,\tau} + (\sigma-1)\widehat{p}_\tau + \widehat{y}_\tau + \widehat{p}^*\right]\right\}$$

Steady state values $\overline{U}_c$, $\overline{P}$ etc are common to all terms in the sum

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- And Adding back in Steady State terms yield

$$\widehat{p}^{*} = \mu + (1-\theta\beta)\ E\left\{\sum_{\tau}(\theta\beta)^{\tau}\ [mc_{\tau} + p_{\tau}]\right\}$$

where log mark $\mu = \log\frac{\sigma}{\sigma-1}$ and where mv_{τ} is log real marginal cost

Extreme Value Shocks

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The General Problem 1: Estimation of Discrete Choice

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- Problem of correlated choices (blue/red bus). A Solution is to nest.

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$$V(s,a) = \max \ \{V^0(a), V^1(a)\} =$$

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- This gets rid of kinks and discontinuities as both choices are always possible for any a. But can cause problems.

- If ϵ follows i.i.d. $G(\mu, \alpha)$, where the mode μ is non-zero, we have

$$V^1 = E\{\epsilon\} = \mu + \alpha\ \gamma$$

$\gamma \simeq .57721$ is the Euler Mascheroni constant

$$\text{Mode}\ \ \{\epsilon\} = \mu$$

$$\text{Median}\{\epsilon\} = \mu - \alpha\ \ \ln(\ln 2)$$

$$\text{Var}\{\epsilon\} = \frac{\pi^2\ \alpha^2}{6}$$

$$\text{cdf}\{\epsilon\} = e^{\left\{-e^{\left[-\frac{(\epsilon-\mu)}{\alpha}\right]}\right\}}$$

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- We have

$$X^N \sim G\left(\mu + \alpha \; \ln N, \alpha\right)$$

$$\mathbb{E}\left[X^N\right] = \mu + \alpha \; \ln N + \alpha \; \gamma$$

- Expected maximum of N Gumbel random variables $G(\mu, \alpha)$. Let

$$X^N = \max\left\{\epsilon^1, \epsilon^2, \cdots, \epsilon^N\right\}$$

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$$\mathbb{E}\left[X^N\right] = \mu + \alpha \ \ln N + \alpha \ \gamma$$

- To make $\mathbb{E}\left[X^N\right]$ independent of the number of choices N, either

$$\mathbb{E}\left[X^N\right] = \bar{V} \Rightarrow \alpha(N) = \frac{\bar{V} - \mu}{\gamma + \ln N}$$

$$\mathbb{E}\left[X^N\right] = \bar{V} \Rightarrow \mu(N) = \bar{V} - \alpha \ \ln N - \alpha \ \gamma$$

better the latter so that they are all Gumbel

- η^i follows $G(\mu, \alpha)$, let $\epsilon^i = \eta^i + \delta^i$, $\epsilon^i \sim G(\mu + \delta^i, \alpha)$.

$$X^N \sim G\left(\alpha \ln \sum_i e^{\frac{\mu^i}{\alpha}}, \alpha\right) = G\left(\mu + \alpha \ln \sum_i e^{\frac{\delta^i}{\alpha}}, \alpha\right)$$

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- No closed-form solution for $\alpha(N)$

The continuum

- Consider an interval $C = [0, \overline{c}]$, and an $\epsilon(c), \forall c \in C$. We want

$$V^C = E\left\{\max_{c \in C} \{\epsilon(c)\}\right\}, \quad \epsilon(c) \sim G(0, \alpha(C)), \qquad \text{for some} \quad V^C > 0.$$

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- We choose $\alpha(V^C, N)$ so that $V^N = V^C$: $\qquad \alpha\left(V^C, N\right) = \frac{V^C}{\ln N + \gamma}$ for any N.

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- We have to think of V^C as a fundamental parameter that determines the size of the utility bonus for the richest agent (the one with the largest choice set).

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- Then we associate with choice set $C^{\widetilde{c}}$, a draw of $N^{\widetilde{c}}$ ϵ's with probability $\underline{p}(\widetilde{c}) = \frac{N^{\widetilde{c}}+1}{N^{\overline{c}}} - \frac{\widetilde{c}}{\overline{c}}$, and a draw of $N^{\widetilde{c}} + 1$ with probability $\overline{p}(\widetilde{c}) = \frac{\widetilde{c}}{\overline{c}} - \frac{N^{\widetilde{c}}}{N^{\overline{c}}}$.

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- Note that the utility bonus $V^{\overline{c}}$ is of the right size given V^C.

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- Now you can iterate on the value function that includes the utility bonus.

Agents in Aiyagari worlds with Extreme Value Shocks

- The fundamental problem

$$v(s,a) = \max_{a', c = sw + aR - a'} \left\{ \frac{c^{1-\sigma} - 1}{1 - \sigma} + \epsilon(c) + \sum_{s'} \Gamma_{s,s'} \; v(s', a') \right\}$$

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- Fix N, a large integer, we approximate the problem by

$$v(s,a) = \max_{a^{n\prime} = sw + aR - c^n, c^n} \left\{ \frac{c^{1-\sigma} - 1}{1-\sigma} + \epsilon^n + \sum_{s'} \Gamma_{s,s'} \; v(s', a^{n\prime}) \right\}$$

We have to impute the right probabilities

Endogenous Growth and R&D

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- So it has to be A: Exogenous
- Still, empirically, the problem is NOT accounting for **growth rate** differences but for output **LEVEL** differences

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- An explicit accumulation of technology

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  1. Final goods are competitive use labor and intermediate goods according to

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  2. Intermediate producers are monopolists. They have a differentiated technology of the form:

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  3. R&D sector. A new good is a new variety of the intermediate good produced using labor:

     $$\frac{A_{t+1}}{A_t} = 1 + \xi N_{2,t}.$$

     we can write $A_{t+1} - A_t = A_t \xi N_{2,t}$, so the flow of new intermediate goods is determined by the current stock of them in the economy (an externality).

     Right to produce new goods sold to new monopolists.

Remark

The reason we see A_t on the previous expression as an externality is that it is indeed used as an input in the process of R&D, while, it is not paid for. Thus, inventors, in a sense, do not pay the investors of the previous varieties, while using their inventions. They only pay for the labor they hire. Perhaps, the basic idea of this production function might be traced back to Isaac Newton's quote: "If I have seen further, it is only by standing on the shoulders of giants".

Exercise

If the price of all varieties are the same, what is the optimal choice of input vector for a producer?

Exercise

What if they do not have the same amount? Would a firm decide not to use a variety in the production?

HHOLDS

- Preferences

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Remark

In this economy, GDP is $Y_t = W_t + r_t K_t + \pi_t$, *where* π_t *are profits. In terms of expenditures, GDP is* $Y_t = C_t + K_{t+1} - (1-\delta) K_t + \pi_t$, *where* $K_{t+1} - (1-\delta) K_t$ *is the investment in physical capital. In terms of value added, it is* $Y_t = N_t^{\alpha} \int_0^{A_t} x_t(i)^{1-\alpha} di + p_t (A_{t+1} - A_t)$.

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- Not a model that maps well to the data, yet carefully crafted to convey ideas.

- Final good producer; it chooses $N_{1,t}$ and $x_t(i)$, $\forall i \in [0, A_t]$,

$$\max \; N_{1,t}^{\alpha} \int_0^{A_t} x_t(i)^{1-\alpha}\, di - w_t N_{1,t} - \int_0^{A_t} q_t(i)\, x_t(i)\, di,$$

where $q_t(i)$ is the price of variety i in period t. First order conditions are:

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$$x_t(i) = \left(\frac{(1-\alpha)}{q_t(i)} \right)^{\frac{1}{\alpha}} N_{1,t},$$

- which, given N_{1t}, is the *demand function* for variety i, by the final good producer.

•

$$\pi_t(i) = \max_{\{q_t(i)\}} \quad q_t(i)\, x_t(q_t(i)) - r_t \eta x_t(q_t(i))$$

$$s.t. \quad x_t(q_t(i)) = \left(\frac{(1-\alpha)}{q_t(i)}\right)^{\frac{1}{\alpha}} N_{1,t},$$

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- FOC wrt to $q_t(i)$, is $x_t(q_t(i)) + (q_t(i) - r_t\eta)\frac{\partial x_t(q_t(i))}{\partial q_t(i)} = 0$, which implies

  $$\frac{(1-\alpha)^{\frac{1}{\alpha}}}{q_t(i)^{\frac{1}{\alpha}}} N_{1,t} = \frac{(q_t(i) - r_t\eta)}{\alpha} \frac{(1-\alpha)^{\frac{1}{\alpha}}}{q_t(i)^{\frac{1+\alpha}{\alpha}}} N_{1,t}.$$

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- Rearranging yields $q_t(i) = \frac{1}{(1-\alpha)} r_t\eta$ and substituting
  $$x_t(i) = \left[\frac{(1-\alpha)^2}{r_t\eta}\right]^{\frac{1}{\alpha}} N_{1,t},$$
  and the demand for capital services is simply $\eta x_t(i)$.

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- With FOC $p_t = \frac{w_t}{A_t \xi}$.

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2. allocates labor demand for R&D, and that for final good production. For determining the labor choices $N_{1,t}$ and $N_{2,t}$. Note that as long as there are profits in the intermediate good sector, new monopolists will enter yielding a zero profit condition:

$$p_t = \sum_{s=t}^{\infty} \left(\prod_{\tau=t}^{s} \frac{1}{1+r_\tau - \delta} \right) \pi_s.$$

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Putting all Together yields two equations

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5. This model neatly delivers balanced growth, with just enough structure.

Growth Model with Many Firms Suitable for Pandemic Times

- This is a growth model suitable to study business cycles.
- Emphasis on small business creation not on inequality so rep hholds.
- Creation and destruction of small firms both for technological and for financial reasons.
- Household cannot help its small businesses in distress.
- We have in mind that even though Pandemic affects both Supply (want less work) and Demand (Less consumption) there is a reduction in output sold per unit of good produced of $\phi(S)$.

Environment: Technology

- Two sectors as in Quadrini (2000): Corporate and non corporate sector.
- Corporate sector uses capital and labor via aggr prod fn $F(K, N)$
- Non corporate sector: type/size firms $i \in \{1, \cdots, I\}$, $f^i(n)$, $f^i_n > 0$, (provided the firm has the required number of managers, λ^i).
- A firm requires creation: It costs ξ^i to open a new firm of size i.
- Some Firms are destroyed.
  - Firms invest m in maintenance.
  - Probability that a firm survives is $q^i(m)$, $q^i(0) = 0$, $q^i(\infty) < 1$, $q^i_m > 0$.
- Aggregate measure of type i firms is X_i
- The law of motion of new firms is:

$$X_i' = q^i(M_i)\ X_i + B_i$$

- The Aggregate Feasibility Constraint is

$$C + [K' - (1-\delta)K] + \sum_i X_i\ M_i + \sum_i B_i\ \xi_i = \sum_i X_i\ f_i(N_i) + F(K, N).$$

- Household owns measure x_i of firms of type $i \in \{1, \cdots, \mathcal{I}\}$
- The household may be rationed in its workforce: i.e. it may not be in its static Euler equation.
- Households create b^i new firms of type i at cost ξ^i each,
- Managers choose maintenance and profits.
- In addition to its firms, households own a units of corporate capital which they can increase by savings.
- Households allocate its members to managers, workers or enjoyers of leisure:

$$n + \sum_i \lambda^i x^i + \ell = 1.$$

(implicitly we are guessing (to be verified) that all business are operated).

- Households have preferences over consumption c and leisure ℓ, using utility function $u(c, \ell)$ and discounts the future at rate β.

- Small firms cannot access financing once they are born.
- They can only give benefits to the household:

$$\Omega^i(S) = \max_{n \geq 0, m \leq \psi(S) f^i(n) - w \; n} \psi(S) \; f^i(n) - w \; n - m + \frac{q^i(m)}{R(S')} \; \Omega^i(S')$$

Here, S is the aggregate state and s in the individual state, $\Psi(S) < 1$ is capacity used which is demand determined and $R(S')$ is the rate of return used by the firm.

- Implictly assuming that there is no need to index $\Omega^i(S)$ by s.

Exercise

Get the FOC assuming first that m is unrestricted and then that $m \leq \psi(S) f^i(n) - w \; n$.

$$V(S, a, x_1, \cdots, x_I) = \max_{c,n,b_1,\cdots,b_I,a'} \; u(c,\; 1 - n - \sum_i \lambda^i \; x^i) + \beta \; V(S', a', x_1', \cdots, x_I') \qquad s.t.$$

$$c + \sum_i \; b_i \; \xi_i + a' = n \; w(S) + a \; R(S) + \sum_i \; \pi_i(S) \; x_i$$

$$x_i' = q^i(M_i) \; x_i + b_i \qquad i \in \{1, \cdots, I\}.$$

Exercise

Get the FOCs for b^i a' *and* n *assuming first that* $\lambda^i = 0$ *and* $\pi^i > 0$ *and charaterize the solution (the relation between the FOC of* b^i, m^i *and* a'*). Then characterize the FOC when* $\lambda^i > 0$.

An Integraded Analysis Model of Climate Change

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- Goal: Derive the optimal policy —here a tax on carbon— so that the externality is internalized.

- Higher levels of carbon dioxide in the atmosphere contributes to global warming, which in turn causes damages like production shortfalls, poor health or deaths, capital destruction and much more.

- Map carbon concentration to climate, and then map climate to damages.

- **Expected sum of future damage elasticities:** the percentage change in output resulting from a percentage change in the amount of carbon in the atmosphere, caused by emitting a unit of carbon today.

- Discounted because of time preferences and because of carbon depreciating.

- Carbon circulation system: carbon is exchanged through various reservoirs such as the atmosphere, the terrestrial biosphere, and different layers of the ocean. A unit of Carbon will remain in the atmosphore s periods after emmited according to

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- We then have a non-linear function $T_{t+1} = \mathcal{T}(T_t, S_t)$ with a steady state like

$$T(f) = \frac{\eta}{(\kappa_{Planck} - \kappa_{other} - \kappa_{refl})} \frac{1}{\ln 2} \ln\left(\frac{S}{\bar{\bar{S}}}\right)$$

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- According to the latest (6th) IPCC report, σ_{CCR} is "likely" (2/3 confidence interval) between 1.0 and 2.3 degrees Celsius per 1000 GtC (corresponding to 0.27-0.63$^\circ / Tt$CO$_2$). This constant is called CCR (Carbon Climate Response, sometimes CRE or TCRE).

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- Nordhaus summarizes various studies of effects:
- Positive effects if initial temperature is below 11.5 degrees. Suggests quadratic damage $D(T) = \alpha_{ag}^1 \left(T + T_0^j\right) + \alpha_{ag}^2 \left(T + T_0^j\right)^2 + \alpha_{ag}^j$.

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Construct a General Equilibrium Model with various ingredients

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2. Production Process (GDP) affected by Climate Change
3. Households with preferences (needed to evaluate outcomes)
4. Explicit use of energy that both contributes to GDP and emits CO_2
5. Inclusion of Exhaustible Resources that induces savvy economic behavior.

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$$S_t = \tilde{S}_t \left(\sum_{j=1}^{J_g - 1} E_{j,-T}, \sum_{j=1}^{J_g - 1} E_{j,-T+1}, ..., \sum_{j=1}^{J_g - 1} E_{j,t} \right)$$

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- Here, $-T$ is defined as the start of industrialization.

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4. The function $\tilde{S}_t$ is linear and has the depreciation structure:

$$S_t - \bar{S} = \sum_{s=0}^{t+T} \sum_{j=1}^{J_g - 1} E_{j,t-s}$$

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- After that we worry about implementation

$$\max_{\substack{\{C_t, N_t, K_{t+1}, R_{j,t+1},\\ E_{j,t}, S_t\}_{t=0}^{\infty} \geq 0}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(C_t) \qquad \text{s.t.}$$

$$C_t + K_{t+1} = F_t(K_t, N_t, E_t, S_t) + (1-\delta)K_t \qquad \text{FB}$$

$$E_t = \sum_j E_{j,t}\ \alpha^j \qquad \text{AGE}$$

$$R_{j,t+1} = R_{j,t} - E_{j,t} \geq 0 \qquad \text{for all } j \qquad ExE$$

$$S_t = \tilde{S}_t \left(\sum_{j=1}^{J_g - 1} E_{j,-T}, \sum_{j=1}^{J_g - 1} E_{j,-T+1}, ..., \sum_{j=1}^{J_g - 1} E_{j,t} \right) \qquad CC$$

- $E_{j,t}$ is output of Energy of Sector (type) j measured in units of carbon emitted.

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- α^j Conversion of units of energy of type j from being in terms of carbon emissions to units of energy.

- The marginal externality damage is the same for all j:

$$\Lambda_t^s = \mathbb{E} \sum_{i=0}^{\infty} \beta^i \frac{U'(C_{t+i})}{U'(C_t)} \frac{\partial F_{t+i}}{\partial S_{t+i}} \frac{\partial S_{t+i}}{\partial E_{j,t}}$$

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- Under our specific assumptions, this expression simplifies to:

$$\Lambda_t^s = \mathbb{E} \sum_{i=0}^{\infty} \beta^i C_t \frac{Y_{t+i}}{C_{t+i}} \gamma_{t+i}(1 - d_i)$$

- The marginal externality damage is the same for all j:

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- Further, if the planner's problem implies a constant savings rate, then the expression can be written as:

$$\Lambda_t^s = Y_t\left[\mathbb{E}\sum_{i=0}^{\infty}\beta^i \gamma_{t+i}(1-d_i)\right]$$

- The FOC of the planner says

$$\alpha_j \frac{\partial F_t}{\partial E_t} - \xi_j - \Lambda_t^s = 0$$

$$\max_{\{C_t, N_t, K_{t+1}\}_{t=0}^{\infty}} \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(C_t)$$

$$\text{subject to} \qquad \mathbb{E}_0 \sum_{t=0}^{\infty} q_t(C_t + K_{t+1})$$

$$= \mathbb{E}_0 \sum_{t=0}^{\infty} q_t((1 + r_t - \delta)K_t + w_t N_t + T_t) + \Pi_t.$$

$$\Pi_0 = \max_{\{K_t, N_t, E_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} q_t \left[F_t(K_t, N_t, E_t, S_t) \right.$$

$$\left. - r_t K_t - w_t N_t - \sum_{j=1}^{J} p_{j,t} E_{j,t} \right]$$

- $\tau_{j,t} = \Lambda_t^s$ for the "dirty" energy firms, and $\tau_{j,t} = 0$ for the "clean" energy firms.

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- If there are multiple externalities (for instance an R&D component to the model) then a separate Pigouvian tax is required for each externality.

To understand the magnitude of the optimal tax rates given by this model, they can be compared with estimates from other models, and also with tax rates that are currently being used around the world.

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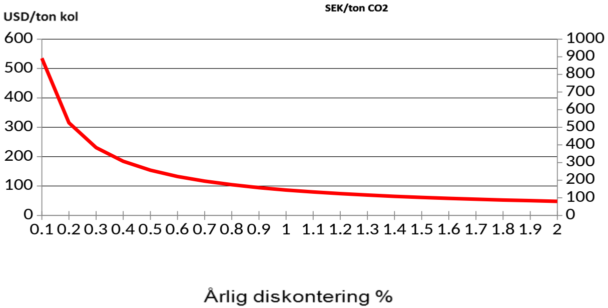
- In Sweden, the current tax on private consumption of carbon exceeds $600 per ton of carbon, which is larger than the estimates for the optimal tax in this paper. However, these taxes are significantly higher than many other countries, for instance the EU has a tax of around $77 per ton of carbon.

Sum damages over time => "optimal" tax!



Sweden has carbon tax ~ 600 USD/tC!

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- => Coal is the main threat!

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- So: bad for the coal industry (the world over), no big deal otherwise

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    - No: reduce them where they are least needed/least efficient (e.g., buy emission rights in EU trading system, pay to keep forests, ...)

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- a robust result (in Golosov, et al., 2013): optimal policy involves rather modest tax on CO 2 and would not pose threat to economic well-being
- some elements of analysis subject to substantial uncertainty

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- The quantitative magnitudes of feedback are disputed. The "average" view seems to be that feedbacks strengthen the direct warming effect considerably, but there is much uncertainty.

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- What is the appropriate level of the tax? For this, we use standard cost-benefit analysis.

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- For the second let $D(T)$ be Nordhaus's global damage function.
- Together, the two steps are $D(T(S))$ mapping additional atmospheric carbon to damages. Let's examine the mapping.

- It turns out that $1 - D(T(S))$, i.e., how much is left after damages as a function of S, is well approximated by the function $e^{-\gamma S}$: for $\gamma = 5.3 * 10^{-5}$ (black), it is quite close to $1 - D(T(S))$ (red dashed), as seen in the figure.

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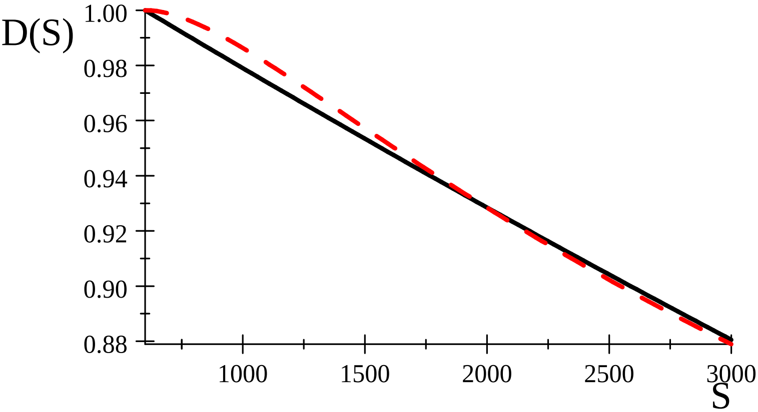

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- Robust?

Macro and COVID-19

- Short Horizons (No investment)
- Choose what Issues to Worry About
    1. *Mitigation Policy and Heterogeneity Age/Sector*
- Choose wich Allocation Mechanism to Model (large externality)
    1. All Econ choices are Government choices

- All variables are shares of a measure 1 population
- Three health states, $j \in \{s, i, r\}$ susceptible, infected, recovered or dead, with associated population shares S, I, R. Initial conditions $S(0), I(0), R(0)$.
- Two parameters: β governs rate of infection, κ the rate of recovery (or death)
- System of differential Equations

$$
\begin{aligned}
\dot{S}(t) &= -\beta S(t)I(t) \\
\dot{I}(t) &= \beta S(t)I(t) - \kappa I(t) \\
\dot{R}(t) &= \kappa I(t)
\end{aligned}
$$

- Basic Reproduction Number: define $R_0 = \frac{\beta}{\kappa}$

- Growth rate of infections given by $\frac{\dot{I}(t)}{I(t)} = \beta S(t) - \kappa$
- Let $I(0) = \epsilon, S(0) = 1 - I(0)$, when $\epsilon > 0$ is very small, $S(0) \approx 1$.
- Since $\dot{S}(t) = -\beta S(t) I(t)$ and for t close to zero,

  $I(t) \approx 0, S(t) \approx 1$, then $\dot{I}(t)/I(t)$ is roughly constant and equal to

  $$\dot{S}(t) = -\beta S(0) I(0) \qquad \text{So}$$

  $$I(t) = I(0) e^{\kappa\left(\frac{\beta}{\kappa} S(0) - 1\right)} \approx I(0) e^{\kappa\left(\frac{\beta}{\kappa} - 1\right)}$$
- If $R_0 = \frac{\beta}{\kappa} > 1$ exponential growth early (if $I(0) > 0$).
- If $R_0 = \frac{\beta}{\kappa} < 1$ then infections fall to zero and epidemic disappears immediately.

- The Ratio of differential equations: $\frac{\dot{I}(t)}{\dot{S}(t)} = -1 + \frac{1}{R_0 S(t)}$
- Integrating yields $I(t) = -S(t) + \frac{\ln(S(t))}{R_0} + q$

  where q is a constant of integration that does not depend on time.
- Evaluating at $t = 0$ yields (using $R(0) = 0$, thus $S(0) + I(0) = 1$

$$q = 1 - \frac{\ln(S(0))}{R_0}$$

- What is $S(\infty) = S^{\star}$? share of the population never to get infected
- Evaluating at $t = \infty$ and using the fact that $I(\infty) = 0$ yields

$$S^{\star} = 1 + \frac{\ln\left[S^{\star}/S(0)\right]}{R_0}$$

- Steady state satisfies the trascendental equation:

$$S^{\star} = 1 + \frac{\ln\left[S^{\star}/S(0)\right]}{R_0}$$

and $R^{\star} = 1 - S^{\star}, I^{\star} = 0$.

- If $R_0 > 1$ and $S(0) < 1$, $\exists$ a unique long-run S^*.

  Strictly decreasing in R_0 and strictly increasing in $S(0)$.

- For $R_0 \approx 1$ (but > 1), $S^{\star} = \frac{1}{R_0}$ and $R^{\star} = \frac{R_0 - 1}{R_0}$

  This approximation (a first good rule of thumb) uses $S(0) \approx 1$ and

$$\ln(1/R_0) = -\ln(R_0) = -\ln(1 + R_0 - 1) \approx 1 - R_0.$$

- With **costly** transfers across agents
- To Assess combination of two policies
    - Shutdown / mitigation (less output but also less contagion)
    - Redistribution toward those whose jobs are shuttered
- Characterize optimal policy
- Key interaction:
    - Mitigation creates the need for more redistribution
    - But if redistribution is costly, want less mitigation
    - Need heterogeneous-agent model to analyze this

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The SAFER SIR Model

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  - E to health-care workers

- Age $i \in \{y, o\}$
  - Only young work
  - Old have more adverse outcomes conditional on contagion
  - But young more prone to contagion (they work)
- Sector of production $\{b, \ell\}$
  - Basic (health care / food production etc.)
    - Will never want shut-downs in this sector
    - Workers in this sector care for the hospitalized
  - Luxury (restaurants, entertainment etc.)
    - Workers in this sector face shutdown unemployment risk
    - But they are less likely to get infected

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- What policy time paths do different types prefer? When (and how much) to shut down, when to open up? Size of Coronavirus check?
- How does the utilitarian optimal policy vary with the cost of redistribution?

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- Differences in expected longevity through $\rho_y \neq \rho_o$ (no aging)

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- Θ measures capacity of emergency health system, η its unit cost

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- Smart mitigation shutters most contact-intensive sub-sectors first

Flow into asymptomatic (out of susceptible)

$$
\begin{aligned}
\dot{x}^{ybs} &= -\beta_w(m)\left[x^{yba} + (1-m)x^{y\ell a}\right]x^{ybs} \\
&\quad - \left[\beta_c(m)x^a y(m) + \beta_h\left(x^a + x^f\right) + \beta_e x^e\right]x^{ybs} \\
\dot{x}^{y\ell s} &= -\left[\beta_w(m)\left[x^{yba} + (1-m)x^{y\ell a}\right](1-m)x^{y\ell s}\right] \\
&\quad - \left[\beta_c(m)x^a y(m) + \beta_h\left(x^a + x^f\right)\right]x^{y\ell s} \\
\dot{x}^{os} &= -\left[\beta_c(m)x^a y(m) + \beta_h\left(x^a + x^f\right)\right]x^{os}
\end{aligned}
$$

- For each type $j \in \{yb, y\ell, o\}$

$$\dot{x}^{ja} = -\dot{x}^{js} - \left(\sigma^{jaf} + \sigma^{jar}\right) x^{ja}$$
$$\dot{x}^{jf} = \sigma^{jaf} x^{ja} - \left(\sigma^{jfe} + \sigma^{jfr}\right) x^{jf}$$
$$\dot{x}^{je} = \sigma^{jfe} x^{jf} - \left(\sigma^{jed} + \sigma^{jer}\right) x^{je}$$
$$\dot{x}^{jr} = \sigma^{jar} x^{ja} + \sigma^{jfr} x^{jf} + (\sigma^{jer} - \varphi) x^{je}$$
$$\varphi = \lambda_o \max\{x^e - \Theta, 0\}.$$

- All flow rates σ vary by age
- $x^e - \Theta$ measures excess demand for emergency health care. Reduces flow of recovered (Increases flow into death)

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- Define measures of non-working and working as

$$\begin{aligned}
\mu^n &= x^{y\ell f} + x^{y\ell e} + x^{ybf} + x^{ybe} + m\left(x^{y\ell s} + x^{y\ell a} + x^{y\ell r}\right) + x^o \\
\mu^w &= x^{ybs} + x^{yba} + x^{ybr} + [1-m]\ \left(x^{y\ell s} + x^{y\ell a} + x^{y\ell r}\right) \\
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- Aggregate resource constraint

$$\mu^w c^w + \mu^n c^n + \mu^n T(c^n) = \mu^w - \eta\Theta$$

where $T(c^n)$ is per-capita cost of transferring c^n to non-workers

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$$\begin{aligned} W(x,m) &= \left[\mu^w + \mu^n\right] w(x,m) \\ w(x,m) &= \log(c^n) + \nu \log(1 + T'(c^n)) + \bar{u} + \sum_{i,j \in \{f,e\}} \frac{x^{ij}}{\mu^w + \mu^w} \widehat{u}^j \end{aligned}$$

- Assume $\mu^n T(c^n) = \mu^w \frac{\tau}{2} \left(\frac{\mu^n c^n}{\mu^w}\right)^2$
- Optimal allocation

$$
\begin{aligned}
c^n &= \frac{\sqrt{1 + 2\tau \frac{1-\nu^2}{\nu} \tilde{y}} - 1}{\tau \frac{1-\nu^2}{\nu}} \\
c^w &= c^n(1 + T'(c^n)) = c^n \left(1 + \tau \frac{1-\nu}{\nu} c^n\right)
\end{aligned}
$$

Where $\tilde{y} = \nu - \frac{\eta\Theta}{\mu^w + \mu^n}$.

- $\left(1 + \tau \frac{1-\nu}{\nu} c^n\right)$ is the effective marginal cost (MC) of transfers.
- It increases with c^n and τ, decreases with share of workers ν
- Higher mitigation m reduces ν, thus increases MC
- $\Rightarrow$ policy interaction between m, τ.

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