Lecture Notes in Macroeconomic Theory*
Econ 8108

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Contents

* These notes were started by graduate students at Penn and continued by the graduate students from Minnesota.

1 Introduction

A model is an artificial economy. Description of a model's environment may include specifying the agents' preferences and endowment, technology available, information structure as well as property rights. Neoclassical Growth Model becomes one of the workhorses of modern macroeconomics because it delivers some fundamental properties of modern economy, summarized by, among others, Kaldor:

1. Output per capita has grown at a roughly constant rate (2%).
2. The capital-output ratio (where capital is measured using the perpetual inventory method based on past consumption foregone) has remained roughly constant.
3. The capital-labor ratio has grown at a roughly constant rate equal to the growth rate of output.
4. The wage rate has grown at a roughly constant rate equal to the growth rate of output.
5. The real interest rate has been stationary and, during long periods, roughly constant.
6. Labor income as a share of output has remained roughly constant (0.66).
7. Hours worked per capita have been roughly constant.

Equilibrium can be defined as a prediction of what will happen and therefore it is a mapping from environments to outcomes (allocations, prices, etc.). One equilibrium concept that we will deal with is Competitive Equilibrium [1]. Characterizing the equilibrium, however, usually involves finding solutions to a system of infinite number of equations. There are generally two ways of getting around this. First, invoke the welfare theorem to solve for the allocation first and then find the equilibrium prices associated with it. The first way sometimes may not work due to, say, presence of externality. So the second way is to look at Recursive Competitive equilibrium, where equilibrium objects are functions instead of variables.

2 [Review]-Neoclassical Growth Model

We review briefly the basic neoclassical growth model.

[1] Arrow-Debreu or Valuation Equilibrium.

2.1 The Neoclassical Growth Model Without Uncertainty

The commodity space is

$$\mathcal{L} = \{(l_1, l_2, l_3) : l_i = (l_{it})_{t=0}^{\infty}\ l_{it} \in \mathbb{R},\ \sup_t |l_{it}| < \infty,\ i = 1, 2, 3\}$$

The consumption possibility set is

$$\begin{aligned} X(\overline{k}_0) = \{x \in \mathcal{L} : \exists & (c_t, k_{t+1})_{t=0}^{\infty} \\ & \text{such that } \forall t = 0, 1, \ldots \\ & c_t, k_{t+1} \geq 0 \\ & x_{1t} + (1-\delta)k_t = c_t + k_{t+1} \\ & - k_{t+1} \leq x_{2t} \leq 0 \\ & - 1 \leq x_{3t} \leq 0 \\ & k_0 = \overline{k}_0 \qquad \} \end{aligned}$$

The production possibility set is: $Y = \prod_t Y_t$ where

$$Y_t = \{(y_{1t}, y_{2t}, y_{3t}) \in \mathbb{R}^3 :\ 0 \leq y_{1t} \leq F(-y_{2t}, -y_{3t})\}$$

Definition 1. *An Arrow-Debreu equilibrium is $(x^*, y^*) \in X \times Y$, and a continuous linear functional ν^* such that*

i. $x^* \in \arg\max_{x \in X, \nu^*(x) \leq 0} \sum_{t=0}^{\infty} \beta^t u(c_t(x), -x_{3t})$.

ii. $y^* \in \arg\max_{y \in Y} \nu^*(y)$.

iii. $x^* = y^*$.

Now, let's look at the one-sector growth model's Social Planner's Problem:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, -x_{3t}) \qquad (SPP)$$

s.t.

$$c_t + k_{t+1} - (1-\delta)k_t = x_{1t}$$

$$0 \leq x_{2t} \leq k_t$$

$$0 \leq x_{3t} \leq 1$$

$$0 \leq y_{1t} \leq F(-y_{2t}, -y_{3t})$$

$$x = y$$

k_0 given.

Suppose we know that a solution in sequence form exists for (SPP) and is unique.

Homework: Clearly stating sufficient assumptions on utility and production function, show that (SPP) has a unique solution.

Two important theorems show the relationship between CE allocations and Pareto optimal allocations:

Theorem 2. *Suppose that for all $x \in X$ there exists a sequence $(x_k)_{k=0}^{\infty}$, such that for all $k \geq 0$, $x_k \in X$ and $U(x_k) > U(x)$. If (x^*, y^*, ν^*) is an Arrow-Debreu equilibrium then (x^*, y^*) is Pareto efficient allocation.*

Theorem 3. *If X is convex, preferences are convex, U is continuous, Y is convex and has an interior point, then for any Pareto efficient allocation (x^*, y^*) there exists a continuous linear functional ν such that (x^*, y^*, ν) is a quasiequilibrium, that is (a) for all $x \in X$ such that $U(x) \geq U(x^*)$ it implies $\nu(x) \geq \nu(x^*)$ and (b) for all $y \in Y$, $\nu(y) \leq \nu(y^*)$.*

Note that at the very basis of the CE definition and welfare theorems there is an implicit assumption of perfect commitment and perfect enforcement. Note also that the FWT implicitly assumes there is no externality or public goods (achieves this implicit assumption by defining a consumer's utility function only on his own consumption set but no other points in the commodity space).

From the First Welfare Theorem, we know that if a Competitive Equilibrium exits, it is Pareto Optimal. Moreover, if the assumptions of the Second Welfare Theorem are satisfied and if the SPP has unique a solution then the competitive equilibrium allocations is unique and they are the same as the PO allocations. Prices can be constructed using this allocations and first order conditions.

Homework: Show that

$$\frac{v_{2t}}{v_{1t}} = F_k(k_t, l_t) \qquad \frac{v_{3t}}{v_{1t}} = F_l(k_t, l_t)$$

One shortcoming of the AD equilibrium is that all trade occurs at the beginning of time. This assumption is unrealistic. Modern Economics is based on sequential markets. Therefore we define another equilibrium concept, Sequence of Markets Equilibrium (SME). We can easily show that SME is equivalent to ADE. Therefore all of our results still hold and SME is the right problem to solve.

Homework: Define a Sequential Markets Equilibrium (SME) for this economy. Prove that the objects we get from the AD equilibrium satisfy SME conditions and that the converse is also true. We should first show that a CE exists and therefore coincides with the unique solution of (SPP).

Note that the (SPP) problem is hard to solve, since we are dealing with infinite number of choice variables. We have already established the fact that this SPP problem is equivalent to the following dynamic problem:

$$\begin{aligned} v(k) = \max_{c,k'} \quad & u(c) + \beta v(k') \qquad (RSPP) \\ \text{s.t. } & c + k' = f(k). \end{aligned}$$

We have seen that this problem is easier to solve.

What happens when the welfare theorems fail? In this case the solutions to the social planners problem and the CE do not coincide and so we cannot use the theorems we have developed for dynamic programming to solve the problem. As we will see in this course, in this case we can work with Recursive Competitive Equilibria. In general, we can prove that the solution to the RCE coincides with a sequential markets problem but not the other way around (for example when we have multiple equilibria). However, in all the models we see this course, this equivalence will hold.

2.2 Review-Adding Uncertainty

2.2.1 Markov Process

In this part, we want to focus on stochastic economies where there is a productivity shock affecting the economy. The stochastic process for productivity that we are assuming is a first order Markov Process that takes on finite number of values in the set $Z = \{z^1 < \cdots < z^{n_z}\}$. A first order Markov process implies

$$\Pr(z_{t+1} = z^i | h_t) = \Gamma_{ij}, \qquad z_t(h_t) = z^j$$

where h_t is the history of previous shocks. Γ is a Markov matrix with the property that the elements of its columns sum to 1.

Let μ be a probability distribution over initial states, i.e.

$$\sum_i \mu_i = 1$$

and $\mu_i \geq 0 \ \forall i = 1, ..., n_z$.

Next periods the probability distribution can be found by the formula: $\mu' = \Gamma^T \mu$.

If Γ is "nice" then $\exists$ a unique μ^* s.t. $\mu^* = \Gamma^T \mu^*$ and $\mu^* = \lim_{m \to \infty} (\Gamma^T)^m \mu_0$, $\forall \mu_0 \in \Delta^i$.

Γ induces the following probability distribution conditional on z_0 on $h_t = \{z^0, z^1, ..., z^t\}$:

$\Pi(\{z^0, z_1\}) = \Gamma_i$ for $z^0 = z_i$.

$\Pi(\{z^0, z_1, z_2\}) = \Gamma^T \Gamma_i$ for $z^0 = z_i$.

Then, $\Pi(h_t)$ is the probability of history h_t conditional on z^0. The expected value of z' is $\sum_{z'} \Gamma_{zz'} z'$ and $\sum_{z'} \Gamma_{zz'} = 1$.

2.2.2 Problem of the Social Planner

Let productivity affects the production function in an arbitrary way, $F(z, K, N)$. Problem of the social planner problem (SPP) in sequence form is

$$\max_{\{c_t(h_t), k_{t+1}(h_t)\} \in X(h_t)} \sum_{t=0}^{\infty} \sum_{h_t} \beta^t \pi(h_t) u(c_t(h_t))$$
$$\text{s.t} \quad c_t(h_t) + k_{t+1}(h_t) = z^t F(k_t(h_{t-1}), 1).$$

Therefore, we can formulate the stochastic SPP in a recursive fashion:

$$V(z^i, K) = \max_{c, K'} \quad u(c) + \beta \sum_j \Gamma_{ji} V(z^j, K')$$
$$\text{s.t.} \quad c + K' = z^i F(K, 1).$$

This gives us a policy function $K' = G(z, K)$.

AD Equilibrium AD equilibrium can be defined by:

$$\max_{\{c_t(h_t), k'(h_t), x_{1t}(h_t), x_{2t}(h_t), x_{3t}(h_t)\} \in X(h_t)} \sum_{t=0}^{\infty} \sum_{h_t} \beta^t \pi(h_t) u(c_t(h_t))$$
$$\text{s.t} \quad \sum_{t=0}^{\infty} \sum_{h_t} p(h_t) x(h_t) \leq 0.$$

where $X(h_t)$ is the consumption feasibility set after history h_t occured. Note that we are assuming the markets are dynamically complete, i.e. there is complete set of securities for every possible history that can appear.

By the same procedure as before, SME can be written in following way:

$$\max_{c_t(h_t), b_{t+1}(h_t, z^j), k_{t+1}(h_t)} \sum_{t=0}^{\infty} \sum_{h_t} \beta^t \pi(h_t) u(c_t(h_t))$$
$$\text{s.t} \quad c_t(h_t) + k_{t+1}(h_t) + \sum_{z^j} b_{t+1}(h_t, z^j) q(h_t, z^j) = k_t(h_{t-1}) R(h_t) + w(h_t) + b_t(h_{t-1}).$$

Here we have introduced Arrow securities to allow agents to trade with each other against possible future shocks.

However, in equilibrium and when there is no heterogeneity, there will be no trade. Moreover, we have two ways of delivering the goods specified in an Arrow security contract: after production and before production. In an after production setting, the goods will be delivered after production takes place and can only be consumed or saved for the next period. This is the above setting. It is also possible to allow the consumer to rent the Arrow security income as capital to firms, which will be the before production setting.

An important condition which must hold true in the before production setting is the no-arbitrage condition, $\sum_{z_{t+1}} q(h_t, z_{t+1}) = 1$.

Homework: Describe the AD problem, in particular the consumption possibility set X and the production set Y.

Homework: Every equilibrium achieved in AD problem can be achieved by SME problem by the relation where $q(h_{t+1}) = p_1(h_{t+1})/p_1(h_t)$, $R(h_t) = -p_2(h_t)/p_1(h_t)$ and $w(h_t) = -p_3(h_t)/p_1(h_t)$. Check that from the FOCs the same allocations result.

Homework: The problem above state contingent goods are delivered in terms of consumption goods. Instead of this assume they are delivered in terms of capital goods. Show that the same allocation would be achieved in both settings.

2.2.3 A Comment on the Welfare Theorems

Situations in which the welfare theorems would not hold include externalities, public goods, situations in which agents are not price takers (e.g. monopolies), some legal systems or lacking of markets which rule out certain contracts which appears complete contract or search frictions. In all of these situation finding equilibrium through SPP is no longer valid. Therefore, in these situations, as mentioned before, it is better to define the problem in recursive way and find the allocation using the tools of Dynamic Programming.

3 Recursive Competitive Equilibrium

3.1 A Simple Example

What we have so far is that we have established the equivalence between allocation of the SPP problem which gives the unique Pareto optima (which is same as allocation of AD competitive equilibrium and allocation of SME). Therefore we can solve for the very complicated equilibrium allocation by solving the relatively easier Dynamic Programming problem of social planner. One handicap of this approach is that in a lot of environments, the equilibrium is not Pareto Optimal and hence, not a solution of a social planner's problem, e.g. when you have taxes or externalities. Therefore, the above recursive problem would not be the right problem to solve. In some of these situations we can still write the problem in sequence form. However, we would lose the powerful computational techniques of dynamic programming. in order to resolve this issue we will define Recursive Competitive Equilibrium equivalent to SME that we can always solve for.

In order to write the decentralized household problem recursively, we need to use some equilibrium conditions so that the household knows what prices are as a function of some economy-wide aggregate state variable. We know that if capital is K_t and there is 1 unit of labor, then $w\left(K\right) = F_n\left(K, 1\right), R\left(K\right) = F_k\left(K, 1\right)$. Therefore, for the households to know prices they need to know aggregate capital. Now, a household who is deciding about how much to consume and how much to work has to know the whole sequence of future prices, in order to make his decision. This means that he needs to know the path of aggregate capital. Therefore, if he believes that aggregate capital changes according to $K' = G(K)$, knowing aggregate capital today, he would be able to project aggregate capital path for the future and therefore the path for prices. So, we can write the household problem given function $G(\cdot)$ as follows:

$$\begin{aligned} \Omega(K, a; G) = \max_{c, a'} \quad & u(c) + \beta\Omega(K', a'; G) \qquad\qquad (RCE) \\ \text{s.t.} \quad & c + a' = w(K) + R(K)a \\ & K' = G(K), \\ & c \geq 0 \end{aligned}$$

The above problem, is the problem of a household that sees K in the economy, has a belief G, and carries a units of assets from past. The solution of this problem yields policy functions $c(K, a; G), a'(K, a; G)$ and a value function $\Omega(z, K, a; G)$. The functions $w(K), R(K)$ are obtained from the firm's FOCs (below).

$$u_c[c(K, a; G)] = \beta\Omega_a[G(K), a'(K, a; G); G]$$
$$\Omega_a[K, a; G] = (1 + r)u_c[c(K, a; G)]$$

Now we can define the Recursive Competitive Equilibrium.

Definition 4. *A Recursive Competitive Equilibrium with arbitrary expectations G is a set of functions*[2] $\Omega, g : \mathcal{A} \times \mathcal{K} \to \mathbb{R}, R, w, H : \mathcal{K} \to \mathbb{R}_+$ *such that:*

1. *given G; Ω, g solves the household problem in (RCE).*
2. *K'=H(K;G)=g(K,K;G) (representative agent condition).*
3. $w(K) = F_n(K, 1)$.
4. $R(K) = F_k(K, 1)$.

We define another notion of equilibrium where the expectations of the households are consistent with what happens in the economy:

Definition 5 (Rational Expectation Equilibrium)**.** *A Rational Expectations Equilibrium is a set of functions Ω, g, R, w, G^* such that*

1. $\Omega(K, a; G^*), g(K, a; G^*)$ *solves HH problem in (RCE).*
2. $G^*(K) = g(K, K; G^*) = K'$.
3. $w(K) = F_n(K, 1)$.
4. $R(K) = F_k(K, 1)$.

What this means is that in a REE, households optimize given what they believe is going to happen in the future and what happens in the aggregate is consistent with the household's decision. The proof that every REE can be used to construct a SME is left as an exercise. The reverse turns out not to be true. Notice that in REE, function G projects next period's capital. In fact, if we construct an equilibrium path based on REE, once a level of capital is reached in some period, next period capital is uniquely pinned down by the transition function. If we have multiplicity of SME, this would imply that we cannot construct the function G since one value of capital today could imply more than one value for capital tomorrow. We will focus on REE unless expressed otherwise.

[2] We could add the policy function for consumption $g_c(K, a; G)$.

3.2 Economy with Leisure

We may extend the previous framework to the elastic labor supply case. Note that aggregate employment level is not a state variable but is instead predicted by aggregate states. The households problem is as follows:

$$\begin{aligned} \Omega(K,a;G,H) = \max_{c,a',n} \quad & u(c,n) + \beta\Omega(K',a';G,H) \qquad\qquad (RCE) \\ \text{s.t.} \quad & c + a' = w(K,N)n + R(K,N)a \\ & K' = G(K), \\ & N = H(K) \end{aligned}$$

with solution $a^*(K,a;G,H), n^*(K,a;G,H)$.

We may thus define an RCE with rational expectation to be a collection of functions $(\Omega, a^*, n^*), (G,H), (R,w)$ such that

1. Given $(G,H),(R,w),(\Omega,a^*,n^*)$ solves HH problem
2. $G(K) = a^*(K,K;G,H)$.
3. $H(K) = n^*(K,K;G,H)$.
4. $w(K,N) = F_n(K,N)$.
5. $R(K,N) = F_k(K,N)$.

Note that condition 1 is the optimality condition. Condition 2 and 3 are imposed because of rational expectation. Condition 4 and 5 are marginal pricing equations.

3.3 Economy with Uncertainty

Go back to our simple framework with inelastic labor supply. But assume there is productivity shock z that can take on finite number of values, whose evolution follows a Markov process governed by $\Gamma_{zz'}$.Then Current period productivity shock, denoted by z,should be included into aggregate state variables that help predicting prices and future aggregate state variables. Moreover, assume that households can accumulate state contingent capital.

$$
\begin{aligned}
\Omega(K,z,a;G) = \max_{c,a'(z')} \quad & u(c) + \beta \sum_{z'} \Omega(K',z',a'(z');G)\Gamma_{zz'} \qquad (RCE) \\
\text{s.t.} \quad & c + \sum_{z'} q(K,z,z')\, a'(z') = w(K,z) + R(K,z)a \\
& K' = G(K,z), \\
& c \geq 0
\end{aligned}
$$

Solving this problem gives policy function $g(K,z,a,z';G)$

An RCE in this case is a collection of functions $(\Omega, g), (q, w, R, G)$ such that

1. Given $(q,w,R,G), (\Omega, a^*)$ solves HH problem
2. $g(K,z,K,z';G) = G(K,z), \forall K,z,z'$
3. $w(K,N) = F_n(K,N)$
4. $R(K,N) = F_k(K,N)$
5. $\sum_{z'} q(K,z,z') = 1$ (no arbitrage condition)

3.4 Economy with Government Expenditures

3.4.1 Lump Sum Tax

The government levies each period T units of goods in a lump sum way and spends it in a public good, say fireworks. Assume consumers do not care about medals. The household's problem becomes:

$$
\begin{aligned}
\Omega(K,a;G) = \max_{c,a'} \quad & u(c) + \beta\Omega(K',a';G) \\
\text{s.t.} \quad & c + a' + T = w(K) + R(K)a \\
& K' = G(K;M,T), \\
& c \geq 0
\end{aligned}
$$

A solution of this problem are functions $g_a^*(K,a;G,M,T)$ and $\Omega(K,a;G)$ and the equilibrium can be characterized by $G^*(K,M,T) = g_a^*(K,K;G^*,M,T)$ and $M^* = T$ (the government budget constraint is balanced period by period). We will write a complete definition of equilibrium for a version with government debt (below).

3.4.2 Income Tax

We have an economy in which the government levies taxes in order to purchase medals. Medals are goods which provide utility to the consumers (for this example).

$$\begin{aligned}
\Omega(K,a;G) = \max_{c,a'} \quad & u(c,M) + \beta\Omega(K',a';G) \\
\text{s.t.} \quad & c + a' = [w(K) + R(K)a](1-\tau) \\
& K' = G(K), \\
& M = \tau[w(K) + K\ R(K)] \\
& c \geq 0
\end{aligned}$$

Now the social planner function method cannot be used: the CE will not be Pareto optimal anymore (if $\tau > 0$ there will be a wedge, and the efficiency conditions will not be satisfied).

3.4.3 Taxes and Debt

Assume that government can issue debt. Note that in a sequence version of the household problem in SME, in order for households not to achieve infinite consumption, we need a no-Ponzi condition:

$$\lim_{t\to\infty} \frac{a_t}{\prod_{s=0}^{t} R_s} < \infty$$

This is the weakest condition that imposes no restrictions on the first order conditions of the household's problem. It is harder to come up with its analogue for the recursive case. One possibility is to assume that a' lies in a compact set $\mathcal{A}$ or a set that is bounded from below[3]. We will give a complete definition of RCE of this economy.

A government policy consists of taxes, spending (fireworks) as well as bond issuance:

$$\tau(K,B), F(K,B), B'(K,B)$$

In this environment, debt issued is relevant for the household because it permits him to correctly

[3] We must specify $\mathcal{A}$ such that the borrowing constraint implicit in $\mathcal{A}$ is never binding.

infer the amount of taxes. Therefore the household needs to form expectations about the future level of debt from the government. The government budget constraint now satisfies (with taxes on labor income):

$$F(K,B) + R(K) \cdot B = \tau(K,B)\, w(K) + B'(K,B)$$

Where also $B'(K,B) = G^B(\cdot)$. Notice that the household does not care about the composition of his portfolio as long as assets have the same rate of return which is true because of the no arbitrage condition. Therefore, the problem of a household with assets equal to a is given by:

$$\begin{aligned}\Omega(a,K,B) = \max_{c,a'} \quad & u(c,F) + \beta\Omega(a',K',B') \\ s.t. \quad & c + a' \leq w(K) + R(K)\, a\,(1-\tau(K,B)) \\ & K'^K(K,B) \\ & B'^B(K,B) \\ & \tau = \tau(K,B), \quad F = F(K,B)\end{aligned}$$

Definition 6 (Rational Expectation Equilibrium with Government Debt)**.** *A Rational Expectations Recursive Competitive Equilibrium given policies* $F(K,B)$,$\tau(K,B)$ *is a set of functions* $\Omega, g, R, w, G^K, G^B, B'$ *such that*

1. $\Omega(K,B,a;\cdot), g(K,B,a;G^K,G^B,F(\cdot),\tau(\cdot))$ *solves the HH problem.*
2. $w(K) = F_2(K,1), R(K) = F_1(K,1)$.
3. *Representative agent condition*
   $g(K,B,K+B;G^K,G^B,\cdot) = G^K(K,B;\cdot) + G^B(K,B;\cdot)$.
4. *Government Budget Constraint*
   $B'(K,B) = R(K)\,B + F(K,B) - \tau(K,B)\,R(K)\,(K+B)$
5. *Government debt is bounded*
   $\exists\ \underline{B}, \bar{B}$ *such that* $\forall B \in [\underline{B}, \bar{B}]$ *we have* $G^B(K,B) \in [\underline{B}, \bar{B}]$ *(and* $\forall K$*).*

Homework: Show that we do not need market clearing in the Recursive Competitive Equilibrium definition. (Hint: Walras Law)

3.5 Economy with Externalities

Let's consider an economy where the consumer cares about aggregate consumption in additions to his own, in particular a utility function taking the form $u(c, c/C)$.

$$\begin{aligned} \Omega(K,a) = \max_{c,a'} \quad & u(c,c/C) + \beta\Omega(K',a') \\ \text{s.t.} \quad & c + a' = w(K) + R(K)a \\ & K' = G(K), \\ & C = H(K) \end{aligned}$$

Homework: Show that the solution for the household's problem is not Pareto optimal (it is different from the social planner's solution).

In this case an RCE is Ω, g, H, G, W, R such that

1. $w(K) = F_2(K,1), R(K) = F_1(K,1)$.
2. Given other functions, Ω, g solves households problem
3. $g(K,K) = G(K)$
4. $H(K) = W(K) + R(K)K - G(K)$

Another possibility could be that the consumer cares about aggregate consumption from the previous period, $u(c, C_{-1})$, or "catching up with the Joneses". In that case we would have an additional state variable C_{-1}, since this information becomes relevant to the consumer when he is solving his problem. Other externalities could appear in how the consumer enjoys leisure, in the production function, etc. To write it out more explicitly:

$$\begin{aligned} \Omega(K,a,C^-) = \max_{c,a'} \quad & u(c,C^-) + \beta\Omega(K',a',C^{-\prime}) \\ \text{s.t.} \quad & c + a' = w(K) + R(K)a \\ & K' = G\left(K, C^-\right), \\ & C^{-\prime} = H\left(K, C^-\right) \end{aligned}$$

A RCE is Ω, g, H, G, W, R such that

1. $w(K) = F_2(K,1), R(K) = F_1(K,1)$.

2. Given other functions, Ω, g solves the household's problem.

3. $g(K, K, C^-) = G(K, C^-)$.

4. $H(K, C^-) = W(K) + R(K)K - G(K, C^-)$.

3.6 An Economy with Capital and Land

Consider an economy with with capital and land but without labor. The agent's problem is

$$\begin{aligned} V(K,a) = \max_{c,a'} \quad & u(c) + \beta V(K', a') \qquad \text{s.t.} \\ & c + a' = R(K)\, a, \\ & K' = G(K). \end{aligned}$$

The solution requires a function $a' = h(K,a)$. We could use yet another notation $c + q(K)\widehat{a}' = \hat{a}$, $\widehat{a}' = \hat{h}(K, \hat{a})$. The problem for the firm is the following:

$$\begin{aligned} \Omega(K,k) = \quad & \max_{k',d} d + q(K) \cdot \Omega(K', k') \\ & s.t. \\ & K' = G(K) \end{aligned}$$

Where $d = F(k,1) - k' + (1-\delta)k$ (full depreciation obviously means $\delta = 1$). An RCE consists of functions $\{V, \Omega, h, g, q, G, D, d\}$, and the conditions that have to hold are similar to the case with uncertainty. In particular:

- V, h and Ω, g solve the problems of the household and the firm, respectively.
- Firms are representative: $G(K) = g(K,K)$.
- Households are representative: $h[K, \Omega(K,K)/R(K)] = \Omega(G(K), G(K))/R(G(K))$.
- $q(K) = 1/R(K)$.

4 Adding Heterogeneity

In the previous section we looked at situations in which RCE were useful. In particular these were situations in which the welfare theorems failed and so we could not use the standard dynamic programming techniques learned earlier. In this section we look at another way in which RCE are helpful- in models with heterogeneous agents.

First, lets consider a model in which we have two types of households that differ only in the amount of wealth they own. Say there are two types of agents labelled type 1 and 2 of equal measure 1/2. Agents are identical other than their initial wealth position and there is no uncertainty in the model. The agent's problem is

$$\begin{aligned} V(K^1, K^2, a) = \max_{c,a'} \quad & u(c) + \beta V(K'^1, K'^2, a') \\ \text{s.t.} \quad & c + a' = R\left((K^1 + K^2)/2\right) a + W\left((K^1 + K^2)/2\right) \\ & K'^i = G^i(K^R, K^P) \;\; i \in 1, 2. \end{aligned}$$

Definition 7 (Rational Expectation Equilibrium with Agents that Differ in Wealth)**.** *A Rational Expectations Recursive Competitive Equilibrium is a set of functions V, g, R, w, G^1, G^2 such that*

1. *V, g solves the HH problem.*
2. *w, R are the marginal products of labor and capital respectively (watch out for arguements!).*
3. *Representative agent conditions*

$$\begin{aligned} g(K^1, K^2, K^1) &= G^1(K^1, K^2) \\ g(K^1, K^2, K^2) &= G^2(K^1, K^2). \end{aligned}$$

Now consider a slightly different economy where type i has labor skill ϵ_i. Measure of agents, μ^1, μ^2 satisfies $\mu^1\epsilon_1 + \mu^2\epsilon_2 = 1$ (below we will consider the case $\mu^1 = \mu^2 = 1/2$).

An important issue here is to determine what constitutes a sufficient statistic for describing household problem. In particular, we should determine whether K is enough for learning current prices and predict K'. It turns out that this is only true under particular assumptions that make the savings function of each individual linear in K.

The problem of the household $i \in \{1,2\}$ can be written as follows.

$$\begin{aligned} V^i(K^1,K^2,a) = \max_{c,a'} \quad & u(c) + \beta V^i(K'^1,K'^2,a') \\ \text{s.t.} \quad & c + a' = R\left((K^1+K^2)/2\right)a + W\left((K^1+K^2)/2\right)\epsilon_i \\ & K'^i = G^i(K^R,K^P) \;\; i \in 1,2. \end{aligned}$$

Notice that we have indexed the value function by the agent's type (if they were the same we would not need this index).

4.1 An International Economy Model

In an international economy model the specifications which determine the definition of country is an important one. We can introduce the idea of different locations, or geography, countries can be victims of different policies, additionally trade across countries maybe more difficult due to different restrictions.

Here we will see a model with two countries, A and B, such that labor is not mobile between the countries but with perfect capital markets. Two countries may have different technologies $F^A(K_A,1)$ and $F^B(K_B,1)$. We need one more variable, we can choose X, the share of total wealth for country A. So country i's problem becomes:

$$\begin{aligned} \Omega_i\left(K_A,K_B,X,a\right) = \max_{c,a'} \quad & u\left(c\right) + \beta\Omega_i\left(K'_A,K'_B,X',a'\right) \\ s.t. \quad & c + a' \le w_i(K_i) + aR\left(K_A,K_B,X\right) \\ & K'_j = G_j\left(K_A,K_B,X\right) \text{ for } j \in \{A,B\} \\ & X' = H\left(K_A,K_B,X\right). \end{aligned}$$

Definition 8. *A* ***RCE*** *for this economy is a set of functions Ω_i, g_i, $\{G_i,H,R,w_i\}$, such that the following conditions hold:*

1. Ω_i, $g_i(K_A,K_B,X,a)$ *solves the household's problem in each country.*
2. $H\left(K_A,K_B,X\right) = g_A\left(K_A,K_B,X,X\left(K_A+K_B\right)\right)/K\left(K_A,K_B,X\right)$ *where* $K\left(K_A,K_B,X\right) = g_A\left(K_A,K_B,X,X\left(K_A+K_B\right)\right) + g_B\left(K_A,K_B,X,(1-X)\left(K_A+K_B\right)\right).$
3. $G_A(\cdot) + G_B(\cdot) = g_A\left(K_A,K_B,X,X\left(K_A+K_B\right)\right) + g_B\left(K_A,K_B,X,(1-X)\left(K_A+K_B\right)\right).$

4. $w_i = F_n(K_i, 1), i = A, B.$

5. $R(K_A, K_B, X) = (1-\delta) + F_K^A(K_A, 1) = (1-\delta) + F_K^B(K_B, 1), w_i = F_N^i(K_i, 1).$

6. $F_K^A[G_A(\cdot), 1] = F_K^B[G_B(\cdot, 1)].$

5 Asset Pricing - Lucas Tree Model

We now turn to the simplest of all models in term of allocations as they are completely exogenous, the Lucas tree model. We want to characterize the properties of prices that are capable of inducing households to consume the endowment.

5.1 The Lucas Tree with Random Endowments (Productivity Shocks)

Consider an economy in which the only asset is a tree that gives fruit. The agents problem is

$$\begin{aligned} V(z,s) = \max_{c,s'} \quad & u(c) + \beta \sum_{z'} \Gamma_{zz'} V(z', s') \\ s.t. \quad & c + q \cdot s' = s(q+z) \\ & q = q(z). \end{aligned}$$

Definition 9. *A Rational Expectations Recursive Competitive Equilibrium is a set of functions V, g, q such that*

1. *V, g solves the HH problem.*
2. *$g(z, 1) = 1, \forall\, z.$*

The FOC for this problem using Envelope theorem is: $-q(z_i)u_c + \beta \sum_z' \Gamma_{zz'} u_c' = 0$. Arranging the terms this gives us:

$$q(z_i)\, u_c(z) = \sum_{z_j} \beta u_c(z_j)\, \Gamma_{ij}\, (q(z_j) + z_j).$$

Notice that this is just a system of n equations with unknowns $\{q(z_i)\}_{i=1}^n$. We can use the power of matrix algebra to solve it. Let

$$q = \begin{bmatrix} q(z_1) \\ \vdots \\ q(z_n) \end{bmatrix}; qu_c = \begin{bmatrix} q(z_1)\, u_c(z_1) \\ \vdots \\ q(z_n)\, u_c(z_n) \end{bmatrix}; zu_c = \begin{bmatrix} z_1 u_c(z_1) \\ \vdots \\ z_n u_c(z_n) \end{bmatrix}; \Gamma \text{ is the transition matrix for } z$$

and rewrite the system above as $qu_c = \beta\Gamma z u_c + \beta\Gamma q u_c$. Hence, the price for the shares is given by

$$\begin{bmatrix} u_c(z_1) & ... & 0 \\ 0... & ... & ... \\ 0 & ... & u_c(z_n) \end{bmatrix} q = qu_c = (I - \beta\Gamma)^{-1} \beta\Gamma z u_c.$$

Hence

$$q = \begin{bmatrix} u_c(z_1) & ... & 0 \\ ... & ... & ... \\ 0 & ... & u_c(z_n) \end{bmatrix}^{-1} (I - \beta\Gamma)^{-1} \beta\Gamma z u_c.$$

What happens if we add state contingent shares into the model? Then the agent's problem becomes

$$\begin{aligned} V(z, s, b) = \max_{c, s', b(z')} \quad & u(c) + \beta \sum_{z'} \Gamma_{zz'} V(z', s', b(z')) \\ s.t. \quad & c + q \cdot s' + \sum_{z'} p(z, z') b(z') = s(q + z) + b \\ & q = q(z) \end{aligned}$$

A characterization of p can be written as:

$$p(z, z') u_c(z) = \beta\Gamma_{zz'} u_c(z')$$

We can price ALL kinds of securities using p and q in this economy. For example, the option to sell tomorrow at price P if today's shock is z is priced as:

$$\hat{q}(z, P) = \sum_{z'} \max\{P - q(z'), 0\} p(z, z').$$

The option to sell at price P either tomorrow or the day after tomorrow is priced as:

$$\tilde{q}(z, P) = \sum_{z'} \max\{P - q(z'), \hat{q}(z', P)\} p(z, z').$$

Finally, note that $R(z) = \left(\sum_{z'} p(z, z')\right)^{-1}$ is the risk free rate given today's shock being z

5.2 Demand shock

Consider an economy in which the only asset is a tree that gives fruits. The fruit is constant over time (normalized to 1) but the agent is subject to preference shocks for the fruit each period, $\theta \in \Theta$. The agents problem is

$$\begin{aligned} V(\theta, s) = \max \quad & \theta u(c) + \beta \sum_{\theta'} \Gamma_{\theta\theta'} V(\theta', s') \\ s.t. \quad & c + q \cdot s' = s(q+1) \\ & q = q(\theta). \end{aligned}$$

All the analysis follow through once we write out the FOCs characterizing price $q(\theta)$ and state contingent prices $p(\theta, \theta')$.

6 Endogenous Productivity in a Product Search Version of the Lucas Tree Model

Let's model the situation where households need to find the fruit before consuming it.

Assume that households have to find the tree in order to consume the fruit. Finding trees is characterized by a constant return to scale matching function $M(T, D)$, where T is the number of trees and D is the *shopping effort* exerted by households when searching. Thus the probability that a tree finds a shopper is $M(T, D)/T$. And the probability that a unit of shopping effort finds a tree is $M(T, D)/D$. We further assume that M takes the form $D^{\varphi}T^{1-\varphi}$. And denote the probability of finding a tree by $\Psi_d(Q) = Q^{1-\varphi}$, where $Q = T/D$ is amount of tree per shopper, capturing market tightness. The households problem can be described as:

$$V(\theta, s) = \max_{c,d,s'} \quad u(c, d, \theta) + \beta \sum_{\theta'} \Gamma_{\theta\theta'} V(\theta', s') \tag{1}$$

$$\text{s.t.} \quad c = d\Psi_d(Q(\theta)) \tag{2}$$

$$c + P(\theta) \cdot s' = P(\theta) \cdot s(1 + R(\theta)). \tag{3}$$

where P is the price of tree relative to that of consumption and R is the dividend income (in units of tree). If we substitute the constraints in the objective, we get the simpler problem

$$V(\theta, s) = \max_d u\left[d\Psi_d(Q), d, \theta\right] + \beta \sum_{\theta'} \Gamma_{\theta\theta'}\, V\left(\theta', s[1 + R(\theta)] - \frac{1}{P(\theta)} d\Psi_d(Q)\right). \qquad (4)$$

with first order condition

$$u_c + \frac{u_d}{\Psi_d(Q(\theta))} = \beta \sum_{\theta'} \Gamma_{\theta\theta'} V_s\left(\theta', s(1 + R(\theta)) - \frac{1}{P(\theta)} d\Psi_d(Q)\right) \cdot \frac{1}{P(\theta)} \qquad (5)$$

To get rid of V_s, look at the initial household problem (1)-(3). Let the multiplier of the budget constraint be λ. Applying envelope theorem and taking FOCs, we get:

$$V_s(\theta, s) = \lambda P(\theta)(1 + R(\theta)),$$

$$u_c(\theta) + \frac{u_d(\theta)}{\Psi_d[Q(\theta)]} = \lambda,$$

and this implies

$$V_s(\theta, s) = P(\theta)(1 + R(\theta))\left(u_c(\theta) + \frac{u_d(\theta)}{\Psi_d[Q(\theta)]}\right).$$

Thus we get the Euler equation

$$u_c + \frac{u_d}{\Psi_d(Q)} = \beta \sum_{\theta'} \Gamma_{\theta\theta'} \frac{P(\theta')\left[1 + R(\theta')\right]}{P(\theta)} \left(u_c(\theta') + \frac{u_d(\theta')}{\Psi_d[Q(\theta')]}\right). \qquad (6)$$

We still need another functional equation to find an equilibrium. Note that P and Q have to be determined (other objects, C, R are known functions of P and Q). We now turn to various ways of doing so.

6.1 Competitive search

Competitive search is a particular search protocol of what is called a non-random search. Both firms and workers search for specific markets *indexed* by price P and market tightness. Agents can go to any such market provided that is operational. From the point of view of the firm, a pair would be operational if it guarantees enough utility to the household (an amount determined in equilibrium). First, solve the problem of a household given P and Q, and then let the firm choose which particular pair of P and Q gives earns the highest profit. Competitive search is magic.

It does not presuppose a particular pricing protocol (wage posting, bargaining) that other search protocols need.

We start by defining a useful object Ω, that tells us the value for a household of facing arbitrary tightness Q and price P today given V. In particular, Ω is defined as

$$\Omega\left(\theta, s, P, Q\right) = \max_{d} \quad u\left[d\Psi_d(Q), d, \theta\right] + \beta \sum_{\theta'} \Gamma_{\theta\theta'}\, V\left(\theta', s(1+R(\theta)) - \frac{1}{P} d\Psi_d(Q)\right). \tag{7}$$

Note that Ω is not an equilibrium object.

Let $\overline{V}$ denote the value for households shopping in the most attractive market, yet to be determined. To attract households, trees have to offer combinations of prices and market tightness that provide at least $\overline{V}$. The problem of a tree is to find a combination of price and tightness to maximize its profit while satisfying the participation constraint of households,

$$\max_{P,Q} \quad \Pi(P,Q) \quad = \quad \frac{1}{P}\Psi_T(Q) \tag{8}$$

$$\text{s.t.} \qquad \overline{V} \leq \Omega(\theta, s, P, Q) \tag{9}$$

where Ω is evaluated at households' optimal shopping effort d^* in response to (θ, s, P, Q),

$$\Omega(\theta, s, P, Q) = u(d^*\Psi_d(Q), d^*, \theta) + \beta \sum_{\theta'} \Gamma_{\theta\theta'} V\left(\theta', s(1+R(\theta)) - \frac{1}{P} d^*\Psi_d(Q)\right).$$

Let's characterize the tree's problem. Let the multiplier of the household's participation constraint (9) be γ. Using the definition $\Psi_T(Q) = Q^{-\varphi}$, $\Psi_d(Q) = Q^{1-\varphi}$, and $s = 1$ in equilibrium, we can write the FOC over Q as[4]

[4] Here is how we derive (10).

$$-\varphi \frac{1}{P} Q^{-\varphi-1} + \gamma \left[\frac{\partial \Omega}{\partial Q}\right] = 0$$

where

$$\begin{aligned}
\frac{\partial \Omega}{\partial Q} &= \left(u_c \Psi_d(Q) + u_d - \frac{1}{P}\Psi_d(Q)\beta \sum_{\theta'} \Gamma_{\theta\theta'} V_s(\theta')\right) \frac{\partial d^*}{\partial Q} + d^*(1-\varphi)Q^{-\varphi}\left[u_c - \frac{1}{P}\beta \sum_{\theta'} \Gamma_{\theta\theta'} V_s(\theta')\right] \\
&= \left(u_c + \frac{u_d}{\Psi_d(Q)} - \frac{1}{P}\beta \sum_{\theta'} \Gamma_{\theta\theta'} V_s(\theta')\right) \Psi_d(Q) \frac{\partial d^*}{\partial Q} + d^*(1-\varphi)Q^{-\varphi}\left[u_c - \frac{1}{P}\beta \sum_{\theta'} \Gamma_{\theta\theta'} V_s(\theta')\right]
\end{aligned}$$

$$\frac{1}{P} = \gamma \frac{1-\varphi}{\varphi} Q\ d^* \left[u_c - \frac{1}{P} \beta \sum_{\theta'} \Gamma_{\theta\theta'} V_s \left(\theta', 1 + R(\theta) - \frac{1}{P} d^* Q^{1-\varphi} \right) \right]. \tag{10}$$

The FOC with respect to P is given by,

$$1 = \gamma Q\ d^* \cdot \beta \sum_{\theta'} \Gamma_{\theta\theta'} V_s \left(\theta', 1 + R(\theta) - \frac{1}{P} d^* Q^{1-\varphi} \right) \tag{11}$$

Combining the two FOCs (10) and (11) to cancel γ, we have

$$\frac{1}{P} = \frac{(1-\varphi) u_c}{\beta \sum_{\theta'} \Gamma_{\theta\theta'} V_s \left(\theta', 1 + R(\theta) - \frac{1}{P} d^* Q^{1-\varphi} \right)}. \tag{12}$$

Definition 10. *An equilibrium with competitive search consists of* $(c, d, s, P, Q, R, \overline{V})$ *that satisfy households' shopping constraint (2), budget constraint (3), Euler equation (6), trees' FOC (12), households' participation constraint (9), market clearing conditions* $s = 1$ *and* $Q = d^{1-\varphi}$.

This definition is excessively cumbersome, and we can go to the core of the issue by writing the two functional equations that characterize the equilibrium. Note that In equilibrium, we have $c = Q^{-\varphi}$ and $c = PR$. Putting together the household's Euler (6) and the tree's FOC (12), we have that $\{Q, P\}$ have to solve

$$\varphi u_c(Q^{-\varphi}, Q^{-1}, \theta) = -\frac{u_d(Q^{-\varphi}, Q^{-1}, \theta)}{Q^{1-\varphi}}, \tag{13}$$

$$u_c(Q^{-\varphi}, Q^{-1}, \theta) = \beta \sum_{\theta'} \Gamma_{\theta\theta'} \frac{P(\theta') + Q\,(\theta')^{-\varphi}}{P} u_c(\theta'). \tag{14}$$

According to FOC with respect to d (5), the first term equals to zero. Thus,

$$\frac{\partial \Omega}{\partial Q} = d^*(1-\varphi) Q^{-\varphi} \left[u_c - \frac{1}{P} \beta \sum_{\theta'} \Gamma_{\theta\theta'} V_s(\theta') \right].$$

Plugging into the initial FOC, we have (10).

6.2 A note about dividend

In the previous section, we assume that dividends, $R(\theta)$, are paid out in units of the tree. So that equilibrium consumption $C(\theta) = P(\theta) R(\theta)$. Consider the following budget constraint:

$$c + P(\theta) \cdot s' = s(P(\theta) + H(\theta)) \tag{15}$$

where $H(\theta)$ is the dividend paid out in the form of fruit

Equation (7) becomes: :

$$\Omega(\theta, s, P, Q) = \max_{d} \quad u\left[d\Psi_d(Q), d, \theta\right] + \beta \sum_{\theta'} \Gamma_{\theta\theta'} V\left(\theta', s + \frac{1}{P}(s \cdot H(\theta) - d\Psi_d(Q))\right)$$

Notice that now:

$$\frac{\partial \Omega}{\partial P} = -\left(\beta \sum_{\theta'} \Gamma_{\theta\theta'} V\left(\theta', s + \frac{1}{P}(s \cdot H(\theta) - d\Psi_d(Q))\right)\right) \frac{sH(\theta) - d\Psi_d(Q)}{P^2}$$

Following the procedure in section 6.1, we can derive an analogous condition as (13).

6.3 Pareto Optimality

One of the fascinating properties of competitive search is that the equilibrium is optimal. To see this note that from the point of view of optimality there are no dynamic considerations, just static. So solve a social planner problem:

$$\max_{c,d,} \quad u(C, D, \theta) \tag{16}$$

$$\text{s.t.} \qquad C = D^{\Psi}. \tag{17}$$

It is trivial to see that the FOC condition of this problem is (13).

6.4 Random search and Nash bargaining

In the case of competitive search, many different markets can exist potentially, and consumers and firms (trees) choose to participate in the best one for them. Here, we consider another kind of market structure: only one market exists, and shoppers meet with trees randomly. After a shopper and a tree form a match, the price is determined via Nash bargaining.

With price P, the value for the firm (the tree) is simply $\frac{1}{P}$. Note that since the tree is already being found, $\Psi_T(Q(\theta))$ does not show up in the return of the tree. The value for the shopper is

$$u_c(c,d,\theta)-\beta\sum_{\theta'}\Gamma_{\theta\theta'}\frac{p(\theta')\left[1+R\left(\theta'\right)\right]}{P}\left(u_c(\theta')+\frac{u_d(\theta')}{\Psi_d[Q(\theta')]}\right).$$

The Nash bargaining problem is

$$\max_P\left[\frac{1}{P}\right]^{1-\mu}\left[u_c(c,d,\theta)-\beta\sum_{\theta'}\Gamma_{\theta\theta'}\frac{p(\theta')\left[1+R\left(\theta'\right)\right]}{P}\left(u_c(\theta')+\frac{u_d(\theta')}{\Psi_d[Q(\theta')]}\right)\right]^{\mu} \tag{18}$$

where μ is the bargaining power of the shopper. The first order condition is

$$(1-\mu)\left[u_c(c,d,\theta)-\beta\sum_{\theta'}\Gamma_{\theta\theta'}\frac{p(\theta')\left[1+R\left(\theta'\right)\right]}{P}\left(u_c(\theta')+\frac{u_d(\theta')}{\Psi_d[Q(\theta')]}\right)\right]$$
$$=\mu\beta\sum_{\theta'}\Gamma_{\theta\theta'}\frac{P(\theta')\left[1+R\left(\theta'\right)\right]}{P}\left(u_c(\theta')+\frac{u_d(\theta')}{\Psi_d[Q(\theta')]}\right)$$

Using the dynamic Euler Equation 6 and the equilibrium conditions, the equation above can be simplified to

$$\mu u_c(Q^{-\varphi},Q^{-1},\theta)=-\frac{u_d(Q^{-\varphi},Q^{-1}\theta)}{Q^{1-\varphi}} \tag{19}$$

Compared with Equation 13, if we set the Nash bargaining parameter μ equal to the goods matching elasticity φ, then the solution in the environment with random search and Nash bargaining coincides with the one under efficient competitive search.

If we set $\mu=0$, then $u_d=0$. This is because when the shopper has no bargaining power, the tree will obtain all the surplus and leave the shopper with

$$u_c(c,d,\theta)-\beta\sum_{\theta'}\Gamma_{\theta\theta'}\frac{p(\theta')\left[1+R\left(\theta'\right)\right]}{P}\left(u_c(\theta')+\frac{u_d(\theta')}{\Psi_d[Q(\theta')]}\right)=0$$

The shopping disutility u_d is not compensated, and consumers will not search at the first place. Unless consumers have non-zero bargaining power, a hold-up problem will show up that prevents the household from doing any investment in searching. Consequently, other issues have to be present in environments with price posting which is equivalent to $\mu = 0$.

6.5 Price posting by a monopoly that owns all trees

Here we study the case that all trees are run by a monopoly that posts the price but understands that Q is a function of P. If the household shows up, then she is charged P. The household sees P and chooses shopping effort d and consumption c. There is only one market. Clearly the equation that determines how market tightness depends on the price is the solution to the Euler equation of the household that defines an implicit equation:

$$u_d(Q^{-\varphi}, Q^{-1}, \theta) + Q^{1-\varphi}\left[u_c(Q^{-\varphi}, Q^{-1}, \theta) - \frac{1}{P}\beta \sum_{\theta'} \Gamma_{\theta\theta'} V_s\left(\theta', 1 + R(\theta') - \frac{1}{P}Q'^{-\varphi}\right)\right] = 0. \tag{20}$$

The monopoly chooses the optimal price subject to the household choosing Q according to its wishes.

$$\max_{P} \quad \Pi(P) \quad = \quad \frac{1}{P(\theta)}\Psi_T(Q(\theta, P)). \tag{21}$$

There are some issues of whether the monopoly understands what happens in the future that we will leave aside for now because nothing interesting happens dynamically here.

7 Measure Theory

This section will be a quick review of measure theory to be able to use in the subsequent sections.

Definition 11. *For a set S, $\mathcal{S}$ is a set (or family) of subsets of S. $B \in \mathcal{S}$ implies $B \subset S$, but not the other way around.*

Definition 12. *σ-algebra $\mathcal{S}$ is a set of subsets of S, with the following properties:*

1. $S, \emptyset \in \mathcal{S}$
2. $A \in \mathcal{S} \Rightarrow A^c \in \mathcal{S}$ *(closed in complementarity)*
3. *for* $\{B_i\}_{i=1,2...}$, $B_i \in \mathcal{S} \Rightarrow [\cap_i B_i] \in \mathcal{S}$ *(closed in countable intersections)*

A σ-algebra is a structure to organize information. Examples are the following:

1. Everything, aka the power set (all the possible subsets of a set S)
2. $\{\emptyset, S\}$
3. $\{\emptyset, S, S_{1/2}, S_{2/2}\}$ where $S_{1/2}$ means the lower half of S (imagine S as an closed interval on $\mathcal{R}$).

If $S = [0, 1]$ then the following is NOT a $\sigma-$ algebra

$$\mathcal{S} = \left\{\emptyset, [0, \frac{1}{2}), \left\{\frac{1}{2}\right\}, \left[\frac{1}{2}, 1\right], S\right\}$$

Remark 13. *A convention is (i) use small letters for elements, (ii) use capital letters for sets, (iii) use "fancy" letters for a set of subsets (or family of subsets).*

Definition 14. *A measure is a function $x : \mathcal{S} \to \mathcal{R}_+$ such that*

1. $x(\emptyset) = 0$
2. *if $B_1, B_2 \in \mathcal{S}$ and $B_1 \cap B_2 = \emptyset \Rightarrow x(B_1 \cup B_2) = x(B_1) + x(B_2)$*
3. *if $\{B_i\}_{i=1}^{\infty} \in \mathcal{S}$ and $B_i \cap B_j = \emptyset$ for all $i \neq j \Rightarrow x(\cup_i B_i) = \sum_i x(B_i)$ (countable additivity)*

Countable additivity means that measure of the union of countable disjoint sets is the sum of the measure of these sets.

Definition 15. *Borel-σ-algebra is a σ-algebra generated by the family of all open sets (generated by a topology).*

Since a Borel-σ-algebra contains all the subsets generated by intervals, you can recognize any subset of a set using Borel-σ-algebra. In other words, Borel-σ-algebra corresponds to complete information.

Definition 16. *Probability (measure) is a measure such that $x(S) = 1$.*

Definition 17. *Given a measure space $(S, \mathcal{S}, x)$, a function $f : S \to R$ is measurable if*

$$\forall a \quad \{b; f(b) \leq a\} \in \mathcal{S}$$

One way to interpret a σ-algebra is that it describes the information available based on observations. Suppose that S is comprised of possible outcomes of a dice throw. If you have no information regarding the outcome of the dice, the only possible sets in your σ-algebra can be $\emptyset$ and S. If you know that the number is even, then the smallest σ-algebra given that information is $\mathcal{S} = \{\emptyset, \{2,4,6\}, \{1,3,5\}, S\}$. Measurability has a similar interpretation. A function is measurable with respect to a σ-algebra $\mathcal{S}$, if it can be evaluated under the current measure space $(S, \mathcal{S}, x)$. We can also generalize Markov transition matrix to any measurable space.

Definition 18. *A function $Q : \mathcal{S} \times S \to [0,1]$ is a transition probability if*

- *$Q(\cdot, s)$ is a probability measure for all $s \in S$.*
- *$Q(B, \cdot)$ is a measurable function for all $B \in \mathcal{S}$.*

In fact $Q(B, s)$ is the probability of being in set B tomorrow, given that the state is s today. Consider the following example: a Markov chain with transition matrix given by

$$\Gamma = \begin{pmatrix} 0.2 & 0.2 & 0.6 \\ 0.1 & 0.1 & 0.8 \\ 0.3 & 0.5 & 0.2 \end{pmatrix}$$

Where Γ_{ij} is the probability of j given a present state i. Then

$$Q(\{1,2\}, 3) = \Gamma_{31} + \Gamma_{32} = 0.3 + 0.5 = 0.8$$

Suppose that x_1, x_2, x_3 is the fraction of types 1,2,3 today. We can calculate the fraction of types tomorrow using the following formulas

$$\begin{aligned} x_1' &= x_1\Gamma_{11} + x_2\Gamma_{21} + x_3\Gamma_{31} \\ x_2' &= x_1\Gamma_{12} + x_2\Gamma_{22} + x_3\Gamma_{32} \\ x_3' &= x_1\Gamma_{13} + x_2\Gamma_{23} + x_3\Gamma_{33} \end{aligned}$$

In other words

$$\mathbf{x}' = \Gamma^T \cdot \mathbf{x}$$

where $\mathbf{x}^T = (x_1, x_2, x_3)$. We can extend this idea to a general case with a general transition function. We define the *Updating Operator* as $T(x, Q)$ which is a measure on S with respect to the σ-algebra $\mathcal{S}$ such that

$$x'(B) = T(x, Q)(B) = \int_S Q(B, s)x(ds)$$

A stationary distribution is a fixed point of T, that is $x^* = T(x^*, Q)$. We know that if Q has nice properties[5] then a unique stationary distribution exists (for example, we discard "flipping" from one state to another) and $x^* = \lim_{n\to\infty} T^n(x_0, Q)\ \forall x_0$.

Example: Consider unemployment in a very simple economy (we have an exogenous transition matrix). There are two states: first one is employed(e) and second one is unemployed (ue). The transition matrix is

$$\Gamma = \begin{pmatrix} 0.95 & 0.05 \\ 0.50 & 0.50 \end{pmatrix}$$

As part of your homework you have to compute the stationary distribution.

8 Industry Equilibrium

8.1 Preliminaries

Now we are going to study a type of models initiated by Hopenhayn (1992). We will abandon the general equilibrium framework from the previous section to study the dynamics of distribution of firms in a partial equilibrium environment.

To motivate things let's start with the problem of a single firm that produces a good using labor input according to a technology described by the production function f. Let assume that this function is increasing, strictly concave and $f(0) = 0$. A firm that hires n units of labor is able to produce $sf(n)$. where s is a productivity parameter. Markets are competitive in the sense that a

[5]See SLP, Ch. 11.

firm takes prices as given and chooses n in order to solve

$$\pi(s,p) = \max_{n\geq 0} psf(n) - wn$$

A solution to this problem is a function $n^*(s,p)$[6]. Given the above assumptions, n^* is an increasing function of s (more productive firms have more workers) as well as p, the FOC is $psf'^*) = w$.

Suppose now there is a mass of firms in the industry, each associated with a productivity parameter $s \in S \subset \mathbb{R}_+$. Let x be a measure defined over the space $(S, \mathcal{B}_S)$ that describes the cross sectional distribution of productivity among firms. We will use this measure to define statistics of the industry. For example, at this point it is convenient to define the aggregate supply of the industry. Since individual supply is just $sf(n^*(s,p))$, the aggregate supply can be written as

$$y^s(p) = \int_S sf(n^*(s,p))\, x(ds)$$

Suppose now that the demand of the market is described by some function $y^d(p)$. Then the equilibrium price, p^* is determined by the market clearing condition $y^d(p^*) = y^s(p^*)$.

So far, everything is too simple to be interesting. The ultimate goal here is to understand how the object x is determined by the fundamentals of the industry. Hence we will be adding tweaks to this basic environment in order to obtain a theory of firms distribution in a competitive environment. Let's start by allowing firms to die.

8.2 A Simple Dynamic Environment

Consider now a dynamic environment. The situation above repeats every period. Firms discount profits at rate r which is exogenously given. In addition, assume that a single firm faces each period a probability δ of disappearing. We will focus on *stationary equilibria*, i.e. equilibria in which the price of the final output p stays constant through time.

Notice first that firm's decision problem is still a static problem. We can easily write the value of

[6] As we declared in advance, this is a partial equilibrium analysis. Hence, we ignore the dependence of the solution on w to focus on the determination of p.

an incumbent firm as follows

$$V(s,p) = \sum_{t=0}^{\infty} \left(\frac{1-\delta}{1+r}\right)^t \pi(s,p) = \left(\frac{1+r}{r+\delta}\right) \pi(s,p)$$

Note that we are considering that p is fixed (therefore we can omit it from the expressions above). Observe that every period there is positive mass of firms that die. As before, let x be the measure describing the distribution of firms within the industry. The mass of firms that die is given by $\delta x(S)$. We will allow these firms to be replaced by new entrants. These entrants draw a productivity parameter s from a probability measure γ.

One might ask what keeps these firms out of the market. The answer "nothing" gives a flaw to fix by assuming that there is a fixed entry cost that each firms must pay in order to operate in the market. Moreover, we will assume that the entrant has to pay this cost before learning s. Hence the value of a new entrant is given by the following function.

$$V^E(p) = \int_s V(s,p)\,\gamma(ds) - c_E$$

Entrants will continue to enter if this is bigger than 0 and continue to decide not to enter if this value is less than zero. So stationarity occurs where this is exactly equal to zero (this is the *free entry* assumption, and we are assuming that there is an infinite number (mass) of prospective firms).

Let's analyze how this environment shapes the distribution of firms in the market. Let x_t be the cross sectional distribution of firms in period t. For any $B \subset S$ the number of firms with $s \in B$ in period t is given by $x_t(B)$ Next period, some of them will die, and that will attract some newcomers. Hence next period measure of firms on set B will be given by:

$$x_{t+1}(B) = (1-\delta)\, x_t(B) + m\gamma(B)$$

That is, m firms enter and $\gamma(B)$ will belong in the set B. As you might suspect, this relationship must hold for every $B \in \mathcal{B}_S$. Moreover, since we are interested in stationary equilibria, the previous expression tells us that the cross sectional distribution of firms will be completely determined by γ. Stationarity on the system we implies:

$$x^*(B;m) = \frac{m}{\delta}\gamma(B)$$

Now note that demand supply relation has form

$$y^d\left(p^*(m)\right) = \int_S sf\left(n^*\left(s,p\right)\right)dx^*\left(s;m\right)$$

whose solution, $p^*(m)$, is continuous function under regularity conditions stated in SLP. We have two equations and two unknowns p and m.

Definition 19. *A stationary distribution for this environment consists of functions p^*, x^* and m^* such that*

1. $y^d\left(p^*(m)\right) = \int_S sf\left(n^*\left(s,p\right)\right)dx^*\left(s;m\right)$
2. $V^E\left(p\right) = \int_s V\left(s,p\right)\gamma(ds) - c_E$
3. $x^*(B) = (1-\delta)x^*(B) + m^*\gamma(B), \forall B \in \mathcal{B}_S$

8.3 Introducing Exit Decision

We want to introduce more (economic) content by making the exit of firms endogenous (a decision of the firm). Let's introduce now a cost of operation. Suppose firms have to pay c_v each period in order to stay in the market and assume $S = [\underline{s}, \overline{s}]$. By adding such a minor change, the solution still has reservation productivity property under some conditions (to be discussed below). In words, there will be a minimum s which will make profitable for the firm to stay in the market. To see that this will be the case you should prove that the profit before variable cost function $\pi\left(s,p\right)$ is increasing in s. Hence the productivity threshold is given by the s^* that satisfies the following condition:

$$\pi\left(s^*,p\right) = c_v$$

for an equilibrium price p. Now instead of considering γ as the probability measure describing the distribution of productivities among entrants, you must consider $\widehat{\gamma}$ defined as follows

$$\widehat{\gamma}\left(B\right) = \frac{\gamma\left(B \cap [s^*, \overline{s}]\right)}{\gamma\left([s^*, \overline{s}]\right)}$$

for any $B \in \mathcal{B}_S$.

One might suspect that this is an *ad hoc* way to introduce the exit decision. To make the things more concrete and easier to compute, we will assume that s is a Markov process. To facilitate the exposition, let's make S finite and assume s has transition matrix Γ. Assume further that Γ is regular enough so that it has a stationary distribution γ. For the moment we will not put any additional structure on Γ.

The operation cost c_v is such that the exit decision is meaningful. Let's analyze the problem from the perspective of the firm's manager. He has now two things to decide. First, he asks himself the question "Should I stay or should I go?". Second, conditional on staying, he has to decide how much labor to hire. Importantly, notice that this second decision is still a static decision. Later, we will introduce adjustment cost that will make this decision a dynamic one.

Let $\phi(s,p)$ be the value of the firm before having decided whether to stay or to go. Let $V(s,p)$ be the value of the firm that has already decided to stay. $V(s,p)$ satisfies

$$V(s,p) = \max_n spf(n) - n - c_v + \frac{1}{1+r}\sum_{s' \in S}\Gamma_{ss'}\phi(s',p)$$

Each morning the firm chooses d in order to solve

$$\phi(s,p) = \max_{d \in \{0,1\}} dV(s,p)$$

Let $d^*(s,p)$ be the optimal decision to this problem. Then $d^*(s,p) = 1$ means that the firm stays in the market. One can alternatively write:

$$\phi(s,p) = \max_{d \in \{0,1\}} d\left[\pi(s,p) - c_v + \frac{1}{1+r}\sum_{s' \in S}\Gamma_{ss'}\phi(s',p)\right]$$

or even

$$\phi(s,p) = \max\left[\pi(s,p) - c_v + \frac{1}{1+r}\sum_{s' \in S}\Gamma_{ss'}\phi(s',p), 0\right] \tag{22}$$

All these are valid. Additionally, one can easily add minor changes to make the exit decision more interesting. For example, things like scrap value or liquidation costs will affect the second argument of the $\max$ operator above, which so far is just zero.

What about $d^*(s,p)$? Given a price, this decision rule can take only finitely many values. Moreover, if we could ensure that this decision is of the form "*stay only if the productivity is high enough and go otherwise*" then the rule can be summarized by a unique number $s^* \in S$. Without doubt, that would be very convenient, but we don't have enough structure to ensure that such is the case. Because, although the ordering of s (lower s are ordered before higher s) gives us that the value of s today is bigger than value of smaller s', depending on the Markov chain, on the other hand, the value of productivity level s tomorrow may be lower than the value of s' (note $s' < s$) tomorrow. Therefore we need some additional regularity conditions.

In order to get a cutoff rule for the exit decision, we need to add an assumption about the transition matrix Γ. Let the notation $\Gamma(s)$ indicate the probability distribution over next period state conditional on being on state s today. You can think of it as being just a column of the transition matrix. Take s and $\widehat{s}$. We will say that the matrix Γ displays *first order stochastic dominance* $(FOSD)$ if $s > \widehat{s}$ implies $\sum_{s' \leq b} \Gamma(s' \mid s) \leq \sum_{s' \leq b} \Gamma(s' \mid \widehat{s})$ for any $b \in S$. It turns out that $FOSD$ is a sufficient condition for having a cutoff rule. You can prove that by using the same kind of dynamic programming tricks that we have used in the first semester for obtaining the reservation wage property in search problems. Try it as an exercise. Also note that this is just a sufficient condition.

Finally, we need to mention something about potential entrants. Since we will assume that they have to pay the cost c_E before learning their s, they can leave the industry even before producing anything. That requires us to be careful when we describe industry dynamics.
Now the law of motion becomes;

$$x'(B) = m\gamma(B \cap [s^*, \bar{s}]) + \int_S \sum_{s'} \mathbf{1}_{\{s' \in B \cap [s^*, \bar{s}]\}} \Gamma(s, s') x(ds)$$

8.4 Stationary Equilibrium

Now that we have all the ingredients in the table, let's define the equilibrium formally.

Definition 20. *A stationary equilibrium for this environment consists of a list of functions (ϕ, n^*, d^*), a price p^* and a measure x^* such that*

1. *Given p^*, the functions ϕ, n^*, d^* solve the problem of the incumbent firm*

2. $V^E(p^*) = 0$

3. *For any $B \in \mathcal{B}_S$ (assuming we have a cut-off rule with s^* is cut-off in stationary distribution)*[7]

$$x^*(B) = m\gamma(B \cap [s^*, \bar{s}]) + \int_S \sum_{s'} \mathbf{1}_{\{s' \in B \cap [s^*, \bar{s}]\}} \Gamma(s, s') x(ds) \tag{23}$$

You can think of condition (2) as a "*no money left over the table*" condition, which ensures additional entrants find unprofitable to participate in the industry.

We can use this model to compute interesting statistics. For example the average output of the firm is given by

$$\frac{Y}{N} = \frac{\sum s f(n^*(s)) x^*(ds)}{\sum x^*(ds)}$$

Next, suppose that we want to compute the share out output produced by the top 1% of firms. To do this we first need to compute $\tilde{s}$ such that

$$\frac{\sum_{\tilde{s}}^{\bar{s}} x^*(ds)}{N} = .01$$

where N is the total measure of firms. Then the share output produced by these firms is given by

$$\frac{\sum_{\tilde{s}}^{\bar{s}} s f(n^*(s)) x^*(ds)}{\sum_{\underline{s}}^{\bar{s}} s f(n^*(s)) x^*(ds)}$$

Suppose now that we want to compute the fraction of firms who are in the top 1% two periods in a row. This is given by

$$\sum_{s \geq \tilde{s}} \sum_{s' \geq \tilde{s}} \Gamma_{ss'} x^*(ds)$$

We can use this model to compute a variety of other statistics include the Gini coefficient.

[7] If we do not have such cut-off rule we have to define

$$x^*(B) = \int_S \sum_{s' \in S} \Gamma_{ss'} \mathbf{1}_{\{s' \in B\}} \mathbf{1}_{\{d(s', p^*) = 1\}} x^*(ds) + \mu^* \int_S \mathbf{1}_{\{s \in B\}} \mathbf{1}_{\{d(s, p^*) = 1\}} \gamma(ds)$$

where

$$\mu^* = \int_S \sum_{s' \in S} \Gamma_{ss'} \mathbf{1}_{\{d(s', p^*) = 0\}} x^*(ds)$$

8.5 Adjustment Costs

To end with this section it is useful to think about environments in which firm's productive decision is no longer static. A simple way of introducing dynamics is by adding adjustment costs.

We will consider labor adjustment costs. First let think of this sequentially, not recursively. These costs work pretty much like capital adjustment costs as you might suspect. Consider a firm that enters period t with n_{t-1} units of labor. We have then three alternatives (these are some particular specifications, there are endless others):

- *Convex Adjustment costs*: if the firm wants to vary the units of labor, it has to pay $\alpha\left(n_t - n_{t-1}\right)^2$ units of the numeraire good. The cost here depends on the size of the adjustment.
- *Training costs or hiring costs*: if the firm wants to increase labor, it has to pay $\alpha\left(n_t - (1-\delta)n_{t-1}\right)^2$ units of the numeraire good only if $n_t > n_{t-1}$, i.e. $\mathbf{1}_{\{n_t > n_{t-1}\}}\alpha\left(n_t - (1-\delta)n_{t-1}\right)^2$. δ measures attrition.
- *Firing costs*: Similarly we can also have firing costs.

The recursive formulation of the firm's problem conditional on staying would be

$$V\left(s, n_-, p\right) = \max_{n\geq 0} psf\left(n\right) - wn - \alpha\left(n - (1-\delta)n_-\right)^2 - c_v + \frac{1}{1+r}\sum_{s'\in S}\Gamma_{ss'}V\left(s', n, p\right)$$

for the case with pure adjustment costs. In class we also saw an example in which the firm pays a cost κ to post vacancies. In this case the firm's problem is

$$\Omega\left(s, n_-, p\right) = \max\{0, \max_{v\geq 0} psf\left(n\right) - wn - \kappa v + \frac{1}{1+r}\sum_{s'\in S}\Gamma_{ss'}V\left(s', n, p\right)\}$$

where

$$n = v\phi + (1-\delta)n_-.$$

9 Incomplete Market Models

9.1 A Farmer's problem

Consider the following problem of a farmer:

$$V(s,a) = \max_{\substack{c \geq 0 \\ 0 \leq a'}} u(c) + \beta \sum_{s'} \Gamma_{ss'} V(s', a') \tag{24}$$

$$c + qa' = a + s$$

Where a is his land holding which can only take positive values; c is his consumption and s is amount of 'fruits' he gets each period. s is a Markov process, with element drawn from set S. q is the inverse of the 'growth rate' of the land.

A crucial assumption for generating a bounded asset space is:

$$\beta/q < 1$$

which is saying that agents are sufficiently impatient so they tend to consume more and decumulate their asset when they get richer and far away from the non-negativity constraint $a' \geq 0$.

The first order condition is given by:

$$u_c(c(s,a)) \geq \frac{\beta}{q} \sum_{s'} \Gamma_{ss'} u_c(c(s', a'(s,a))), \text{ equality when } a'(s,a) > 0$$

where $c(.,.), a'(.,.)$ are policy functions from the farmer's problem.

Notice that $a'(s,a) = 0$ is possible for the farmer if we assume appropriate shock structure to the question. Specifically, it depends on the value of $s_{\min} \equiv \min S = \min(s_1, s_2, ...s_n)$

Suppose $s_{\min} = 0$, then the agent will optimally always choose $a' > 0$. Otherwise there is a strictly positive probability that the agent enters tomorrow into state $s_{\min}$ where he has no 'cash in hand' ($a' + s_{\min} = 0$) and is forced to consume 0, which is extremely painful to him. Hence he will raise his asset holding a' to insure himself against such risk.

If $s_{\min} > 0$, then the above argument no longer holds and it is indeed possible for agent to choose 0 asset holdings tomorrow.

Notice that the borrowing constraint $a' > 0$ is affecting agents asset accumulation decision even if he is away from the zero bound because he has an incentive to ensure against the risk of getting a series of back shock of s and is forced to 0 asset holdings. This is what we call 'precautionary savings motive'.

We make further assumptions on the transition matrix of s, Γ :

1. There is a unique stationary distribution of s^* such that $s^* = \Gamma' s^*$
2. (Monotonicity) $s_n > s_m$ implies that $E(s'|s_n) > E(s'|s_m)$

Under the two assumptions, one can show that the farmer's decision rule is monotonic. Namely: $a'(a_1, s) > a'(a_2, s), \forall a_1 > a_2, \forall s$

One can also show, under fairly general conditions, that $\exists\, \hat{a}$ such that $\forall a \in [0, \hat{a}]\,,\, a'(a, s_{\min}) = 0$.

Now we proceed to argue that under $\beta/q < 1$, there exits an upper bound for the agent's asset holding. Notice that when $a = 0$, FOC holds with equality, hence:

$$\begin{aligned} u_c(c(s,a)) &= \frac{\beta}{q}\sum_{s'}\Gamma_{ss'}u_c(c(s',a'(s,a))) \\ &< \sum_{s'}\Gamma_{ss'}u_c(c(s',a'(s,a))) \end{aligned} \tag{25}$$

As a grows larger, right hand side of (25) has a smaller variance because larger fraction of the farmer's 'cash in hand' comes from his wealth, rather than the shock s. Namely:

$$u_c(c(s',a'(s,a))) \to \sum_{s'}\Gamma_{ss'}u_c(c(s',a'(s,a))), \forall s', s \text{ ,as } a \ \to \infty \tag{26}$$

Combining (25) and (26), we can see that:

$$u_c(c(s,a)) < u_c(c(s,a'(s,a))), \text{for } a \text{ sufficiently large}$$

which implies

$$c(s,a) > c(s,a'(s,a))$$

which by monotonicity of decision rule implies

$$a > a'(s,a), \text{for } a \text{ sufficiently large}$$

Hence we get the theorem:

If $\beta/q < 1$ and u_c is convex, then there exist $\bar{a}$ such that $\forall s, a'(s,\bar{a}) \leq \bar{a}$

Note that convexity of u_c captures 'prudence' of preferences so that the agent will increase his saving if future uncertainty increases (in the mean-spread sense)

9.2 Huggett Economy

Now we modify the farmer's problem (24) a little bit:

$$V(s,a) = \max_{\substack{c \geq 0 \\ \underline{a} \leq a'}} u(c) + \beta \sum_{s'} \Gamma_{ss'} V(s', a')$$

$$c + qa' = a + s$$

Where $\underline{a}$<0, so now farmers can borrow and lend with each other, but with a borrowing limit. We make assumption $\beta/q < 1$. As shown in the last section, there is an upper bound of the asset space, call it $\bar{a}$. Solving this problem gives policy function $a'(s,a)$.

Now suppose there is a mass of farmers with distribution function $X(.,.)$, where $X(D,B)$ denotes fraction of people with shock $s \in D$ and $a \in B$, where B is a Borel set in [$\underline{a}$,$\bar{a}$]. Then distribution of farmers tomorrow is given by:

$$X'(S', B') = \int_{A \times S} 1_{\{a'(s,a) \in B'\}} \sum_{s' \in S'} \Gamma_{ss'} dX(s,a) \tag{27}$$

Implicitly this defines an operator T such that $T(X) = X'$. If T is sufficiently 'nice', then there exits a unique X^* such that $X^* = T(X^*)$ and $X^* = \lim_{n \to \infty} T^n(X_0), \forall X_0$. Note that the decision rule is obtained given q, hence the resulting stationary distribution X^* also depends on q. Denote it by $X^*(q)$.

We want to determine an endogenous q by looking at asset market clearing condition. We assume that there is no storage technology so that asset supply is 0. Hence price q should be such that asset demand equals asset supply:

$$\int_{A\times S} adX^*(q) = 0$$

We can show that a solution exists by invoking intermediate value theorem by showing that the following three conditions are satisfied, note that $q \in [\beta, \infty]$:

1. $\int_{A\times S} adX^*(q)$ is a continuous function of q.
2. $\lim_{q\to\beta} \int_{A\times S} adX^*(q) \to \infty$. This is intuitive because as $q \to \beta$, interest rate increases, hence agents would like to save more. Together with precautionary savings motive, they accumulate unbounded asset in the stationary equilibrium
3. $\lim_{q\to\infty} \int_{A\times S} adX^*(q) < 0$. This is also intuitive because as $q \to \infty$, interest rates converges to 0. Hence everyone would rather borrow.

9.3 Aiyagari Economy

In an Aiyagari Economy there is physical capital. The shock is to effective labor supply. Specifically, consider the following problem for the households:

$$V(s,a) = \max_{\substack{c\geq 0 \\ \underline{a}\leq a'}} u(c) + \beta \sum_{s'} \Gamma_{ss'} V(s', a')$$

$$c + a' = (1+r)a + ws$$

Where r is return to asset and w is the wage rate.

We assume standard production technology for the firm:

$$Y = AK^{1-\alpha}L^{\alpha}$$

with depreciation rate δ. Hence

$$\begin{aligned} r &= Y_k - \delta = (1-\alpha)A\left(\frac{K}{L}\right)^{-\alpha} - \delta \equiv r\left(\frac{K}{L}\right) \\ w &= Y_L = \alpha A\left(\frac{K}{L}\right)^{1-\alpha} \equiv w\left(\frac{K}{L}\right) \end{aligned}$$

Prices faced by the agent are all functions of the capital-labor ratio, so is the stationary distribution, denoted by $X^*\left(\frac{K}{L}\right)$. Equilibrium condition is thus given by:

$$\frac{K}{L}=\frac{\int_{A\times S} a dX^*\left(\frac{K}{L}\right)}{\int_{A\times S} s dX^*\left(\frac{K}{L}\right)}$$

Using this one can solve for capital-labor ratio. Note that the agents always supply 1 unit of labor into the market, hence $\int_{A\times S} s dX^*\left(\frac{K}{L}\right)$ is the steady state effective labor supply.

9.4 Policy

In Aiyagary(or Huggett) economy, model parameters can be summarized by $\theta=\{u,\beta,s,\Gamma\}$. In stationary equilibrium, value function $v(s,a;\theta)$ as well as $X^*(\theta)$ can be obtained, where $X^*(\theta)$ is a mapping from model parameters to stationary distribution of agent's asset holding and shocks. Suppose now there is a policy change that shifts θ to $\hat{\theta}=\{u,\beta,s,\hat{\Gamma}\}$. Associated with this new environment there is a new value function $v\left(s,a;\hat{\theta}\right)$ and $X^*\left(\hat{\theta}\right)$. Define $\eta(s,a)$ to be the solution of:

$$v\left(s,a+\eta(s,a),\hat{\theta}\right)=v(s,a,\theta)$$

which is the transfer payment necessary to the households so that they are indifferent between living in the old environment and in the new one. Hence total payment needed to compensate households for this policy change is given by:

$$\int_{A\times S}\eta(s,a)\,dX^*(\theta)$$

Notice that the changes do not take place when the government is trying to compensate the households. Hence we use the original stationary distribution associated with θ to aggregate the households.

If $\int_{A\times S} v(s,a)\,dX^*\left(\hat{\theta}\right) > \int_{A\times S} v(s,a)\,dX^*(\theta)$, does this necessarily mean that households are willing to accept this policy change? The answer is not necessarily because the economy may well spend a long time in the transition path to the new steady state, during which there may be severe welfare loss.

10 Models with Growth

Previously we have seen the Neo-classical Growth Model as our benchmark model and built on it for the analysis of more interesting economic questions. One peculiar characteristic of our benchmark model, unlike its name suggested, was lack of growth (after reaching steady state). Many interesting questions in economics are related to the cross-country differences of growth rates and we will cover some models that will allow for growth so that we will be able to attempt to answer such questions.

10.1 Exogenous Growth

We know that in our standard NGM there are basically two ways of growth, one in which everything grows, which is not necessarily a per-capita growth, and the other is per-capita growth. We will be focusing on per-capita growth. The title 'exogenous' growth refers to the structure of models in which the growth rate is determined exogenously, and is not an outcome of the model. The simplest one of these is one in which the determinant of the growth rate is population growth.

10.1.1 Population Growth

Suppose the population of our economy grows at rate γ and we have the classical CRS technology in capital and labor inputs.

$$Y_t = AF(K_t, N_t)$$

$$N_t = N_0\gamma^t$$

Note that our economy is no longer stationary but as we will see, within the exogenous growth framework, we can make these economies look like stationary ones by re-normalizing the variables. Thus, at the end of the day it will only be a mathematical twist on our standard growth model. Once we do that, we will be looking for the counterpart of a steady state that we have in our stationary economies, the Balanced Growth Path, in which all the variables will be growing at constant but not necessarily equal rates. Back to our population growth model, we know that

$$AF(K, N) = A[KF_K(K, N) + NF_N(K, N)]$$

If N is growing at rate γ, can this economy have a balanced growth path? Can we construct one? We know that by the CRS property F_K and F_N are homogeneous of degree zero. If we assume that capital stock grows at rate γ as well, then prices stay constant, and per-capita variables are constant, and output grows at the same rate. So we get a growth on a balanced growth path without per-capita growth. One question is how to model population growth in our representative agent model. One way is to assume that there is a constant proportion of immigration to our economy from outside but this has to assume the immigrants are identical to our existing agents, which is a bit problematic. The other way could be to assume growing dynasties which preserves the representative agent nature of our economy. If we do so, the problem of the social planner becomes:

$$\begin{aligned}\max \sum_{t=0}^{\infty} & \beta^t N_t U(\frac{C_t}{N_t}) \\ & s.t. \\ & C_t + K_{t+1} = AF(K_t, N_t) + (1-\delta)K_t \\ & nonegativity\end{aligned}$$

To transform the feasibility set to per capita terms, divide all terms by N_t and to make the environment stationary by dividing all the variables by γ^t and assume $N_0 = 1$ we get,

$$\begin{aligned}\max \sum_{t=0}^{\infty} & (\beta\gamma)^t N_0 U(\hat{c}_t) \\ & s.t. \\ & \hat{c}_t + \gamma\hat{k}_{t+1} = AF(\hat{k}_t, 1) + (1-\delta)\hat{k}_t \\ & nonegativity\end{aligned}$$

So how is this transformed model any different than our NGM? By the discount factor, the agents in this economy with growth discount future less but everything else is identical to NGM, of course with the exception of this economy growing at a constant rate.

10.1.2 Labor Augmenting Productivity Growth

Now suppose we have a *labor augmenting* productivity growth with constant population normalized to one, i.e. have the following CRS technology:

$$Y_t = AF(K_t, \gamma^t N_t)$$

$$AF(K_t, \gamma^t N_0) = A[K_t F_K(K_t, \gamma^t N_0) + \gamma^t N_0 F_N(K_t, \gamma^t N_0)]$$

$$\max \sum_{t=0}^{\infty} \beta^t U(\frac{C_t}{N_t})$$
$$s.t.$$
$$C_t + K_{t+1} = AF(K_t, \gamma^t N_t) + (1-\delta)K_t$$
$$nonegativity$$

and since we have a population of one, these variables are already per-capita terms. For stationarity, we have to normalize the variables to *per productivity* units, by dividing all by γ^t. Then the problem becomes:

$$\max \sum_{t=0}^{\infty} \beta^t U(\gamma^t \hat{c}_t)$$
$$s.t.$$
$$\hat{c}_t + \gamma \hat{k}_{t+1} = AF(\hat{k}_t, 1) + (1-\delta)\hat{k}_t$$
$$nonegativity$$

Suppose we have a CRRA preferences, then the question is how can we represent the preferences as a function of $\hat{c}_t$ only. Using CRRA:

$$\sum_{t=0}^{\infty} \beta^t \frac{(\gamma^t \hat{c}_t)^{1-\sigma} - 1}{1-\sigma} \approx \sum_{t=0}^{\infty} (\beta\gamma^{1-\sigma})^t \frac{\hat{c}_t^{1-\sigma} - 1}{1-\sigma}$$

Note that we have added some constant to right hand side, but we know that adding and subtracting constants to objective function does not change the solution of the problem. Once again it is the exact same problem as the NGM with a different discount factor. Note that the existence of a solution to this problem depends on $(\beta\gamma^{1-\sigma})$. In this set-up we get per-capita growth at rate γ. Also note that CRRA is the only functional form for preferences that is compatible with BGP. This is because as per-capita output grows, for consumption to grow at a constant rate, our agent has to face the same trade-off at each period.

10.2 Endogenous Growth

So far in the models we covered growth rate has been determined exogenously. Next we will look to models in which the growth rate is chosen by the model itself. We do know that for a fixed amount of labor, the curvature of our technology limits the growth due to diminishing marginal return on capital and with depreciation there is an upper limit on capital accumulation. So if our economy is to experience sustainable growth for a long period of time, we either give up the curvature assumption on our technology or we have to be able to shift our production function up. Given a fixed amount of labor, this shift is possible either by an increasing TFP parameter or increasing labor productivity. The simplest of such models where we can see this is the AK model, where the technology is linear in capital stock so that diminishing marginal return on capital does not set in.

10.2.1 AK Model

Recall the simple one-sector growth model from the first mini. When there is curvature in the production function as a function of capital, there is no long-run growth. When the production function is linear in capital there is a balanced growth path but there is no transitional dynamics. From, this simple observation, we can see that in order to get long-run growth we need a model that behaves similar to the AK growth model. One way to achieve this is to assume there is human capital and production function is CRS w.r.t physical capital and human capital. When investment in human capital is done using consumption goods, this economy behaves as an AK economy. This is not true when the cost of investment in human capital is time. In this section, we build another model that behaves similar to an AK model.

$$\begin{aligned}
\max \sum_{t=0}^{\infty} & \beta^t U(c_t) \\
& s.t. \\
& y_t = c_t + k_{t+1} = Ak_t \\
& c_t, k_{t+1} \geq 0
\end{aligned}$$

This problem will give following Euler equation:

$$U_c(c_t) = \beta U_c(c_{t+1})A$$

When preferences are CRRA, i.e.

$$U(c) = \frac{c^{1-\sigma} - 1}{1-\sigma}$$

and when we are on a balanced growth path (consumption is growing at a rate γ_c) then Euler equation gives the following growth rate: $\gamma_c = (\beta A)^{\frac{1}{\sigma}}$.

10.2.2 Human Capital Models

Now assume production technology is such that we have another production factor, namely human capital. Then followings are the feasibility condition and law of motion for the human capital

$$\begin{aligned}
&C_t + I_t^K + [I_t^H] = AK_t^\theta (H_t n_t^1)^{(1-\theta)} \\
&Kt + 1 = (1-\delta)K_t + I_t^K \\
&[n_t^1 + n_t^2 = 1] \\
&C_t, K_{t+1} \geq 0
\end{aligned}$$

Equations and variables in brackets appear when they are needed in the following specifications of law of motion for capital. Law of motion for the capital can be different for different interpretation of human capital formation. If we think human capital is built "with bricks":

$$H_{t+1} = (1-\delta_h)H_t + I_t^H$$

If we think it is formed by learning by doing, then we can define individual human capital formation in two ways:

$$h_{t+1} = g(n_t^2, h_t)$$

or

$$h_{t+1} = g(n_t^2, H_t)$$

Lucas defined a human capital model with the specification of schooling and inelastic labor supply. Now that there is no limit to the accumulation of human capital and sustainable growth on a BGP is feasible. Furthermore, an analysis of the characterization of the balanced growth path will indicate that this model indeed has transitional dynamics. So, unlike the AK model, if we starts out of this optimal growth path, economy can adjust and will converge to it by responding to prices in a de-centralized setting. If one defines a learning by doing model, s/he can see there is a natural limit to the growth of human capital and such an economy might not have a BGP. The key ingredient of endogenous growth with labor is then the reproducibility of the human capital without such a limit.

10.2.3 Growth Through Externalities by Romer

Then let's write the growth model with externality (Romer, 1986). We have seen in the AK model that the growth rate is determined solely by model primitives and endogenized but still it is not directly or indirectly determined by the agents' choices. In Lucas' human capital model, the growth rate is determined by the choice of agents, specifically by the optimal ratio of human and physical capital.The source of growth in Lucas' model is reproducibility of human capital. In the next model, Romer introduces the notion of externality generated by the aggregate capital stock to go through the problem of diminishing marginal returns to aggregate capital. In this model, the source of growth will be the aggregate capital accumulation, which is possible with a linear aggregate technology in capital as we saw in the AK model.The firms in our model will not be aware of this externality and will have the usual CRS technology and observe the source of growth coming from the TFP parameter. As usual with externalities, the equilibrium outcome will not be optimal. Each firm has the following technology:

$$y_t = A_t k_t^\theta (K_t n_t)^{1-\theta}$$

We can write this technology as follows

$$y_t = \bar{A}_t k_t^\theta n_t^{1-\theta}$$

where $\bar{A}_t = AK_t^{1-\theta}$. With this technology and assuming CRRA utility, Euler equation for household in balanced growth path reduces to

$$1 = \beta\gamma^{-\sigma}(1 + r)$$

where $r = MP_K$.

10.2.4 Monopolistic Competition, Endogenous Growth and R&D (Romer)

There are three sectors in the economy: a final good sector, an intermediate goods sector and a R&D sector. Final goods are produced using labor (as we will see there is only one wage since there is only one type of labor) and intermediate goods according to the production function

$$N_{1t}^{\alpha} \int_0^{A_t} x_t(i)^{1-\alpha} \, di$$

where $x(i)$ denotes the consumption of intermediate good of variety $i \in [0, A_t]$[8]. The intermediate goods are produced using capital according to a linear technology

$$x(i) = \frac{k(i)}{\eta}$$

The aggregate demand of capital from this sector is $\int_0^{A_t} \eta x(i) \, di$.

A new good is a new variety of the intermediate good. In every period, a flow of new intermediate goods is created by using labor according to the following production technology

$$\frac{A_{t+1}}{A_t} = 1 + \xi N_{2t}$$

Notice that with after some manipulation, one can express growth in the stock of intermediate goods as follows

$$A_{t+1} - A_t = A_t \xi N_{2t}$$

Hence, the flow of new intermediate goods is determined by the stock of them in the economy. This type of externality is the key feature of the model[9]. This assumption provides us with a constant returns to scale technology in the R&D sector.

Certainly, this is not a model that one can map to the data. Instead it has been carefully crafted to deliver what is desired and it provides an interesting insight in thinking about endogenous growth.

Now let's introduce a form of competition between sectors. We will assume that final good

[8] The function that aggregates consumption of intermediate goods is often referred as Dixit-Stiglitz aggregator.

[9] Perhaps, the basic idea of this production function might be traced back to Isaac Newton's quote "If I have seen further, it is only by standing on the shoulders of giants".

producers act as price takers. Intermediate good firms act as monopolistic competitors, which means that they set the price of their variety although they take the rental price of capital as given. Finally, the R&D producers act as price takers.

Solving the model

Let's consider first the problem of a final good producer. In every period they choose N_{1t} and $x_t(i)$ for every i in order to solve

$$\max N_{1t}^{\alpha} \int_0^{A_t} x_t(i)^{1-\alpha}\, di - w_t N_{1t} - \int_0^{A_t} q_t(i)\, x_t(i)\, di$$

where $q_t(i)$ is the price of variety i in period t. First order conditions for this problem are

$$\begin{aligned} N_{1t} &: \alpha N_{1t}^{\alpha-1} \int_0^{A_t} x_t(i)^{1-\alpha}\, di = w_t \\ x_t(i) &: (1-\alpha) N_{1t}^{\alpha} x_t(i)^{-\alpha} = q_t(i) \qquad \forall i \end{aligned}$$

From the second condition one obtains

$$x_t(i) = \left(\frac{(1-\alpha)}{q_t(i)} \right)^{\frac{1}{\alpha}} N_{1t} \tag{1}$$

which is the demand for variety i of the final good producer.

Let's consider now the problem of an intermediate firm. These firms acts as price setters. They choose $q_t(i)$ in order to solve

$$\pi_t(i) = \max q_t(i)\, x_t(q_t(i)) - r_t \eta x_t(q_t(i))$$

where $x_t(q_t(i))$ is the demand function in (1). Notice that we have plugged the functional form for the technology, i.e. $x_t(q_t(i)) = \frac{1}{\eta} k_t(i)$. First order conditions for this problem are

$$q_t(i) : x_t(q_t(i))\, N_{1t} + (q_t(i) - r_t \eta) \frac{\partial x_t(q_t(i))}{\partial q_t(i)} = 0$$

which can be written as

$$\frac{(1-\alpha)^{\frac{1}{\alpha}}}{q_t(i)^{\frac{1}{\alpha}}}N_{1t}=\frac{(q_t(i)-r_t\eta)}{\alpha}\frac{(1-\alpha)^{\frac{1}{\alpha}}}{q_t(i)^{\frac{1+\alpha}{\alpha}}}N_{1t}$$

Rearranging yields

$$q_t(i)=\frac{r_t\eta}{(1-\alpha)} \tag{2}$$

Note what happens when $\alpha=0$. Plugging this into the demand function yields

$$x_t(i)=\left(\frac{(1-\alpha)^2}{r_t\eta}\right)^{\frac{1}{\alpha}}N_{1t} \tag{3}$$

and the demand for capital services is just $\eta x_t(i)$.

Let Y_t be the production of the final good and plug (2) and (3) in its corresponding production function to get

$$Y_t=N_{1t}A_t\left(\frac{(1-\alpha)^2}{r_t\eta}\right)^{\frac{1-\alpha}{\alpha}} \tag{4}$$

Hence the model displays constant returns to scale in A_t.

Let's study now the problem of the R&D firm. This firm chooses N_{2t} in order to solve the following problem

$$\max p_tA_{t-1}\xi N_{2t}-w_tN_{2t}$$

An equilibrium with positive production of R&D goods require $p_t=w_t/A_{t-1}\xi$, which pins down the price of them. It remains to find an equation to pin down the actual production of these goods. Well, in equilibrium it must be that no more of the R&D goods are produced because nobody finds it profitable. That is, the present discounted value of a firm producing it must be equal to the price of introducing an additional variety

$$p_t = \sum_{s=t}^{\infty} \frac{\pi_s(i)}{\prod_{\tau=t}^{s} (1+r_\tau)^{\tau-t}} \tag{5}$$

Notice that the stock of new goods will be growing at a rate ξN_{2t}. In fact, in a balanced growth path, aggregate variables will be growing at this rate whereas labor, of course, will remain constant.

So far we have been silent about the consumer side of the model. Let just add it for the sake of completeness. We will assume that the consumer chooses positive streams of consumption, labor services and capital services in order to solve

$$\begin{aligned} \max \quad & \sum_{t=0}^{\infty} \beta^t u(c_t) \\ s.t. \quad & c_t + k_{t+1} \leq r_t k_t + w_t \quad \text{for all } t \\ & k_0 \text{ given} \end{aligned}$$

Finally, market clearing condition are the usual ones

$$\begin{aligned} N_{1t} + N_{2t} &= 1 \\ C_t + K_{t+1} &= Y_t \end{aligned}$$

for every t. One final comment, in this economy we have $GDP = Y_t + p_t \xi A_t N_{2t} = C_t + [K_{t+1} - (1-\delta)K_t] + p_t \xi A_t N_{2t}$.