Lecture Notes in Macroeconomic Theory

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1 Introduction

- What is equilibrium? There are two ways to define it:

  1. Techincal: An equilibrium is where agents maximize and what they do is compatible with what others do.
  2. Equilibrium is a prediction of what will happen and therefore it is a mapping from environments to allocations.

- One equilibrium concept that we will deal with is Competitive Equilibrium[1]. What we know about Compettive Equilibrium is:

  1. The allocation is Pareto Optimal.
  2. Under mild conditions, they exist and there is a finite number of them; see Debreu (1970).
  3. Agents do not have to consider other agents behavior. They only care about prices.
  4. It can be applied to environments with many agents where no one has the ability to manipulate prices.

- Now, let's look at the one-sector growth model's Social Planner's Problem:

$$\begin{aligned} \max_{\{c_t,k_{t+1}\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t u(c_t) \qquad\qquad (SPP) \\ \text{s.t.} \quad & c_t + k_{t+1} = f(k_t) \\ & k_0 : \text{ given.} \end{aligned}$$

  Suppose we know that a solution in sequence form exists for (SPP) and is unique. From the First Welfare Theorem, we know that if a Competitive Equilibrium exits, it is Pareto Optimal. So we should first show that a CE exists and therefore coincides with the unique solution of (SPP).

  So we use the Second Welfare Theorem. Second welfare theorem states that if there is a PO allocation, there is a price that makes the allocation a CE.

  Notice that we have not set up the household problem yet. We have to write it in a way so that we can apply Hahn-Banach Separating Hyperplane Theorem and therefore the Second Welfare theorem. The AD version of household problem is the following:

$$\begin{aligned} \max_{x \in X} \quad & u(x) \\ \text{s.t.} \quad & p(x) \leq 0 \end{aligned}$$

[1] Arrow-Debreu or Valuation Equilibrium

where $p(\cdot)$ is a continuous linear function. The issue that should be dealt with here is that in the (SPP) we are looking for infinite sequences of numbers. Therefore the right commodity space is

$$\mathcal{L} = \{\{\ell_t\}^{t=0,\infty} = \{\ell_{it}\}_{i=1,2,3}^{t=0,\infty}, \quad \ell_{it} \in \mathbb{R} : \sup_t |\ell_t| < \infty\}$$

In this environment, we should show that the pricing function is linear. We can use a theorem from Prescott and Lucas (1972) that shows that under discounting and bounded utility function, prices exist. Hence there exists prices

$$\{p_t\}_{t=0}^{\infty} = \{(p_{t1}, p_{t2}, p_{t3})\}_{t=0}^{\infty}$$

such that

$$p(x) = \sum_{t=0}^{\infty} p_t \cdot x_t$$

- One shortcoming of the AD equilibrium is that all trade occurs at the beginning of time. this assumption is unrealistic. Modern economy is based on sequence of markets. Therefore we define another equilibrium concept, Sequence of Markets Equilibrium (SME). We can easily show that SME is equivalent to ADE. Therefore all of our results still hold and SME is the right problem to solve. In a SME, the household problem given prices $\{w_t, R_t\}_{t=0}^{\infty}$ is the following:

$$\begin{aligned} \max_{\{c_t, k_{t+1}\}} \quad & \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{s.t.} \quad & c_t + k_{t+1} = w_t + R_t k_{t+1} \\ & k_0 : \text{ given.} \end{aligned}$$

  The correspondence from SME to ADE is clear once we define prices as $w_t = \frac{p_{2t}}{p_{1t}}, R_t = \frac{p_{1t}}{p_{1,t+1}}$.

- Solving (SPP) in the form given above is a hard problem since we have to solve for an infinite dimensional object. To resolve this issue, we write (SPP) in a recursive from that is solvable using computers:

$$\begin{aligned} v(k) = \max_{c,k'} \quad & u(c) + \beta v(k') \qquad (RSPP) \\ \text{s.t.} \qquad & c + k' = f(k). \end{aligned}$$

- So far we've taken a complicated equilibrium definition and rewritten it in a way that is solvable. One shortcoming of this approach is that in a lot of environments, the equilibrium is not Pareto Optimal and hence, not a solution of a social planner's problem, e.g. when you have taxes or externalities. Therefore, the above recursive problem would not be the right problem to solve. to resolve this issue we will define Recursive Competitive Equilibrium equivalent to SME that we can always solve for.

March 27th, 2008

2 Recursive Competitive Equilibrium

In order to write decentralized household problem recursively, we need to use some equilibrium conditions so that the household know what prices are as a function of some economy-wide aggregate state variable. We know that if capital is K_t and there is one unit of labor, then $w_t = f(K_t) - f'(K_t)K_t, R_t = f'(K_t)$. Therefore, for the households to know prices they need to know aggregate capital. Now, a household who is deciding about how much to consume and how much to work. He has to know the whole sequence of future prices, in order to make his decision. This means that he needs to know the path of aggregate capital. Therefore, if he believes that aggregate capital changes according to $K' = G(K)$, knowing aggregate capital today, she would be able to project aggregate capital path for the future and therefore path for prices. So, we can write household problem given function $G(\cdot)$ as follows:

$$\begin{aligned} V(K, a; G) = \max_{c,a'} \quad & u(c) + \beta V(K', a; G) \qquad\qquad (RCE) \\ \text{s.t.} \quad & c + a' = w(K) + R(K)a \\ & K' = G(K), \\ & c \geq 0 \end{aligned}$$

The above problem, is the problem of a household that sees K in the economy, has a belief G, and carries a units of assets from past.

In a sequence version of the household problem in SME, in order for households not to achieve infinite consumption, we need a no-Ponzi condition:

$$\lim_{t\to\infty} \frac{a_t}{\prod_{s=0}^{t} R_s} < \infty$$

This is the weakest condition that imposes no restriction on the first order conditions of the household problem. It is harder to come up with its analog for the recursive case. One possibility is to assume that a' lies in a compact set $\mathcal{A}$ or a set that is bounded from below[2].

Definition 1 *The solution to (RCE) given G is a pair of functions, (V, g) such that V satisfies (RCE) and we have*

$$V(K, a; G) = u(w(K) + R(K)a - g(K, a; G)) + \beta V(G(K), g(K, a; G); G)$$

Now we can define the Recursive Competitive Equilibrium

Definition 2 *A Recursive Competitive Equilibrium with arbitrary expectation G is a set of functions $V, g : \mathcal{A} \times \mathcal{K} \to \mathbb{R}, R, w, H : \mathcal{K} \to \mathbb{R}_+$ such that:*

[2]We must specify $\mathcal{A}$ such that the borrowing constraint implicit in $\mathcal{A}$ is never binding.

1. *Given G, V, g solves the household problem in (RCE).*
2. *K'=H(K;G)=g(K,K;G)*

The above definition lacks finding the evolution of aggregate capital over time which is endogenous in SME, while it is given here. Therefore, we define another notion of equilibrium

Definition 3 (Rational Expectation Equilibrium) *A Rational Expectations Equilibrium is a set of function V, g, R, w, G^* such that*

1. *$V(K, a; G^*), g(K, a; G^*)$ solves HH problem in (RCE).*
2. *$G^*(K) = g(K, K; G^*)$.*

What this means is that in a REE, household optimize given what they believe is going to happen in the future and what happens in aggregate is consistent with household's decision. The proof that every REE can be used to construct a SME is left as an exercise. The reverse turns out not to be true. Notice that in REE, function G projects next period capital. In fact, if we construct an equilibrium path based on REE, once a level of capital is reached in some period, next period capital is uniquely pinned down by the transition function. If we have multiplicity of SME, this would imply that we cannot construct the function G since one value of capital today could imply more than value for capital tomorrow.

April 1st, 2008

Now, we look at different examples where we write problems in RCE form. The baseline model will be the growth model and we will try different twists.

2.1 Public Goods with Lump-Sum Taxes

Suppose that a public good exists, P, and it enters period utility of consumer in a separable way, so the utility function would be $u(c) + \nu(P)$. Public good is supplied by government and is financed using lump-sum taxes. We assume that it is one-to-one exchangeable with consumption good. Therefore, the feasibility at each date is $C+I+P = F(K, N)$. Therefore, the recursive formulation is:

$$
\begin{aligned}
V(K, a; G, \bar{P}, \bar{T}) = \max_{c,a'} \quad & u(c) + \nu(P) + \beta V(K', a'; G, \bar{P}, \bar{T}) \\
\text{s.t.} \quad & c + a' = R(K)a + w(K) - T \\
& K' = G(K) \\
& P = \bar{P} \\
& T = \bar{T}.
\end{aligned}
$$

Therefore we can define our equilibrium concept as the following:

Definition 4 *A RERCE given policy $\bar{P}$ is a set of function $V^*, G^*, g^*, R^*, w^*, \bar{T}$, s.t.*

(i) *V^*, g^* solves HH problem given $G^*, \bar{T}, \bar{P}$.*

(ii) *$G^*(K) = g^*(K, K; G^*, \bar{T}, \bar{P})$.*

(iii) *Governments budget constraint is satisfied: $\bar{T} = \bar{P}$.*

(iv) *$R^*(K), w^*(K)$ are given by firm's maximization problem.*

Feasibility constraint, here, is worth discussing. Notice that we have 5 commodities: consumption good, public good, investment good, labor services, and capital services. By assumption, 3 of these are one-to-one interchangeable: consumption, public and investment good. We know from Walras law that if $n-1$ markets clear and budget constraints are satisfied, the other market clears. The fact that $w^*(K), R^*(K)$ are given by firm's problem, means that market for labor and capital services clear. Since, HH budget constraint is satisfied from the definition of the value function, feasibility will be implied. Therefore, we do not have to include feasibility in the definition of RERCE.

2.2 Public Goods with Capital Income Tax

Because of LS taxes in section 2.1, the allocation is the solution of a recursive SPP. In this section we introduce distortionary taxes to capital income. The HH recursive problem is:

$$\begin{aligned} V(K, a; G, \bar{P}, \tau) = \max_{c, a'} \quad & u(c) + \nu(P) + \beta V(K', a'; G, \bar{P}, \tau) \\ \text{s.t.} \quad & c + a' = a + (R(K) - 1)(1 - \tau)a + w(K) \\ & K' = G(K) \\ & P = \bar{P} \\ & \tau = \tau(K). \end{aligned}$$

Equivalently, equilibrium is defined as

Definition 5 *A RERCE given policy $\bar{P}$ is a set of function $V^*, G^*, g^*, R^*, w^*, \tau^*$, s.t.*

(i) *V^*, g^* solves HH problem given $G^*, \bar{P}, \tau^*$.*

(ii) *$G^*(K) = g^*(K, K; G^*, \bar{P}, \tau^*)$.*

(iii) *Period-by-period government budget constraint is satisfied: $\tau^*(K)(R^*(K) - 1)K = \bar{P}$.*

(iv) *$R^*(K), w^*(K)$ are given by firm's maximization problem.*

2.3 An Economy with Two Types of Agents

Here, we introduce an economy with two types of agents, that are identical in all aspects except their initial level of capital. Half of the agents are poor and half of them are rich. We know from last mini that heterogeneity implies that state variable is not aggregate capital anymore. We need the wealth distribution to be a state variable unless homotheticity allows us to neglect wealth distribution is determinant of aggregate movements[3]. Therefore, here we set aggregate state variable to be (K, s) where K is capital per capita and s is the ratio of rich agent's capital to aggregate capital per capita. Hence, the ratio of poor's capital to capital per capita is $2 - s$. The recursive formulation of household problem is the following:

$$\begin{aligned} V(K, s, a; G) = \max_{c, a'} \quad & u(c) + \beta V(K', s', a'; G) \\ \text{s.t.} \quad & c + a' = R(K)a + w(K) \\ & K' = G_K(K, s) \\ & s' = G_s(K, s). \end{aligned}$$

Equilibrium definition:

Definition 6 *A RERCE is a set of function V^*, G^*, g^*, R^*, w^*, s.t.*

(i) V^*, g^* *solves HH problem given G^*.*

(ii) $K' = G_K^*(K, s) = \frac{1}{2} g^*(K, s, sK; G^*) + \frac{1}{2} g^*(K, s, (2 - s)K; G^*)$.

(iii) $s' = \frac{g^*(K, s, sK; G^*)}{G_K^*(K, s)} = G_s^*(K, s)$.

(iv) $R^*(K), w^*(K)$ *are given by firm's maximization problem.*

April 10th, 2008

2.4 An Economy with Debt

Consider the economy in section 2.1 where government has to finance a constant stream of expenditure using debt and distortionary labor income taxes. The aggregate state variables are (K, B), capital and debt. Then individual problem becomes the following:

$$\begin{aligned} V(K, B, a) = \max_{c, a'} \quad & u(c) + \beta V(K', B', a') \\ \text{s.t.} \quad & c + a' = aR(K) + w(K)(1 - \tau) \\ & K' = G(K, B) \\ & B' = H(K, B) \\ & \tau = \tau(K, B). \end{aligned}$$

[3] For a general discussion, see Duffie et al. (1994).

Notice that in the above problem, debt and capital holding have the same rate of return, by no-arbitrage. The equilibrium is defined as follows:

Definition 7 *A RERCE is a set of function $V^*, G^*, H^*, \tau^*, g^*, R^*, w^*$, s.t.*

(i) *V^*, g^* solves HH problem given G^*, H^*, τ^*.*

(ii) *$K' = G^*(K, B) + H^*(K, B) = g(K, B, K + B)$.*

(iii) *$H(K, B) = BR(K) + P - \tau(K, B)w(K)$.*

(iv) *$\exists \bar{B}, \bar{K} s.t, \forall K, B, K \in [0, \bar{K}], B \in [-\bar{B}, \bar{B}]; H(K, B) \in [-\bar{B}, \bar{B}]$.*

(v) *$R^*(K), w^*(K)$ are given by firm's maximization problem.*

where condition (iv) is a no-Ponzi condition for government so that it cannot issue unlimited debt.

April 3rd, 2008

3 Stochastic Growth Model

In this part, we want to focus on stochastic economies where there is a productivity shock affecting the economy. The stochastic process for productivity that we are assuming is a first order Markov Process that takes on finite number of values in the set $Z = \{z^1 < \cdots < z^{n_z}\}$. Being first order Markov process implies

$$\Pr(z_{t+1} = z^i | h_t) = \Gamma_{ij}, \qquad z_t(h_t) = z^j$$

where h_t is the history of previous shocks. Γ is a Markov matrix with the property that the elements of its columns sum to 1.

Productivity affects the production function in an arbitrary way, $F(z, K, N)$. Therefore, we can formulate the stochastic SPP in a recursive fashion:

$$\begin{aligned} V(z^i, K) = \max_{c, a'} \qquad & u(c) + \beta \sum_j \Gamma_{ji} V(z^j, K') \\ \text{s.t.} \quad & c + a' = F(z^i, K). \end{aligned}$$

This gives us a policy function $K' = G(z, K)$. As we discussed before, this is not an equilibrium. We can use AD equilibrium concept to decentralize the above allocation. To do so, we need to define commodities in the following way: for every history h_t, there is a good and has price $p(h_t)$ provided that the pricing function has the linear form. Hence, the consumer's problem in AD equilibrium is:

$$\begin{aligned}
\max_{c_t(h_t),i_t(h_t),a_t(h_{t-1})} \quad & \sum_{t=0}^{\infty}\sum_{h_t}\beta^t\pi(h_t)u(c_t(h_t)) \\
\text{s.t} \quad & \sum_{t=0}^{\infty}\sum_{h_t}\left(i_t(h_t)+c_t(h_t)\right)p_1(h_t)-p_2(h_t)-p_3(h_t)a_t(h_{t-1})\leq 0 \\
& a_{t+1}(h_t)=a_t(h_{t-1})(1-\delta)+i_t(h_t).
\end{aligned}$$

By the same procedure as before, we can construct the SME problem for the consumer:

$$\begin{aligned}
\max_{c_t(h_t),i_t(h_t),a_t(h_{t-1})} \quad & \sum_{t=0}^{\infty}\sum_{h_t}\beta^t\pi(h_t)u(c_t(h_t)) \\
\text{s.t} \quad & c_t(h_t)+a_{t+1}(h_t)+\sum_{z^j}b(h_t,z^j)q(h_t,z^j)=a_t(h_{t-1})R(h_t)+w(h_t)+b(h_{t-1},z_t(h_t)).
\end{aligned}$$

Here we have introduced Arrow securities to allow agents to trade with each other against possible future shocks. However, in equilibrium and when there is no heterogeneity, there will be no trade. Moreover, we have two ways of delivering the goods specified in an Arrow security contract: after production and before production. In an after production setting, the goods will be delivered after production takes place and can only be consumed or saved for the next period. This is the above setting. It is also possible to allow the consumer to rent the arrow security income as capital to firms, which will be the before production setting.

Now, we need to show that the AD equilibrium is equivalent to SME. To this end, we need to construct prices and show that AD solves the SME problem. For prices, we have:

$$\begin{aligned}
w(h_t) &= \frac{p_2(h_t)}{p_1(h_t)} \\
q(h_t,z^j) &= \frac{p_1(h_t,z^j)}{p_1(h_t)} \\
R(h_t) &= \frac{p_3(h_t)}{p_1(h_t)}+1-\delta
\end{aligned}$$

April 8th, 2008

In this part, we want to write the stochastic growth model in a decentralized and recursive way; RCE. We will formulate so as the Arrow securities are paid in the beginning of period and hence can be used in production. This is in contrast to the last formulation of the sequence problem, where Arrow securities are paid after the production decision and can only be consumed or saved. The Recursive problem of the agent is the following[4]:

$$\begin{aligned} V(z,K,a) = \max_{c,y,\mathbf{b}} \quad & u(c) + \beta \sum_{z'} \Gamma_{z,z'} V(z', \mathbf{K}'(z'), y + \mathbf{b}(z')) \qquad (P1) \\ \text{s.t.} \quad & c + y + \mathbf{q}(z,K) \cdot \mathbf{b} = R(K,z)a + w(z,K) \\ & \mathbf{K}' = \mathbf{G}(K,z). \end{aligned}$$

Comment on notation - Bold letters represent vectors. So, $\mathbf{b} \in \mathbb{R}^{n_z}$ is a vector of numbers each of which is associated with a state tomorrow; $\mathbf{b}(z')$. The same is true for $\mathbf{q}(z,K)$ which is a vector in $\mathbb{R}^{n_z}_+$.

Notice that aggregate capital cannot depend on z' since it is decided about today. Hence, the following theorem:

Theorem 1 *There exists a function $G : \mathbb{R}^2_+ \to \mathbb{R}$, s.t. $\forall z', \mathbf{G}(z,K)(z') = G(z,K)$.*

Assume that the solution to the above problem is $y(z,K,a), \mathbf{b}(z,K,a)$. Similar to the previous cases, we cna define the RERCE as follows:

Definition 8 *A RERCE is a set of function $V^*, G^*, y^*, \mathbf{b}^*, R^*, w^*$, s.t.*

(i) *$V^*, y^*, \mathbf{b}^*$ solves HH problem given G^*.*

(ii) *$K' = G^*(z,K) = y^*(z,K,K)$.*

(iii) *$\forall z', \mathbf{b}^*(z,K,K)(z') = 0$.*

(iv) *$\mathbf{q}(z,K) \cdot \mathbf{1} = 1$.*

(v) *$R^*(K), w^*(K)$ are given by firm's maximization problem.*

where $\mathbf{1} = \overbrace{(1, \cdots, 1)}$.

Notice that condition (iv) comes from a no-arbitrage condition. To give an intuition, suppose $\mathbf{q}(z,K) \cdot \mathbf{1} > 1$, then the agent can issue one unit of each arrow securities, sell them for $\mathbf{q}(z,K) \cdot \mathbf{1}$. She can save one unit of the proceeds and consume the rest which is a positive amount. Tomorrow, she has to pay back the issued securities entitlements. Given that she has saved one unit, she can pay it back using the extra unit of saving at the beginning of period. Therefore, she is able to make a trade and increase her consumption. She can do this trade for an infinite amount and make infinite profit which cannot be true. Therefore, $\mathbf{q}(z,K) \cdot \mathbf{1} > 1$ cannot hold. A similar argument goes through if the reverse of the inequality holds. So no-arbitrage implies condition (iv).

You can see how the modification we made about the timing of trade, helps us a lot in recursive formulation and the no-arbitrage condition. In the other formulation, the no-arbitrage is not as easy as this case.

[4] The exogenously given laws of motion are suppressed for ease of notation.

Condition (iii) in the definition of equilibrium, implies that there is no borrowing and lending, since there is only a representative agent. This means that in the original problem, we are allowing for some trades that will not happen. That is, we are allowing agents to insure themselves against a recession in the future but equilibrium forces them not to do so, since there is no one who provides that insurance[5].

The same is true for capital. There are some states that capital is more desirable than others. We make the price of capital in those states so high such that every state has the same capital.

Indeterminacy - One thing that should be noticed is that the solution of agent's maximization problem is not unique. That is, there are a lot of saving plans and asset holdings that maximize agent's utility. However, we only focus on those with no borrowing and lending.

We can define the following recursive problem using equilibrium condition (iii):

$$\begin{aligned} V(z,K,a) = \max_{c,y} \quad & u(c) + \beta \sum_{z'} \Gamma_{z,z'} V(z',K',y) \qquad (P2) \\ \text{s.t.} \quad & c + y = R(K,z)a + w(z,K) \\ & K' = G(K,z). \end{aligned}$$

You are asked in a homework to show that (P1) and (P2) are equivalent given equilibrium conditions.

3.1 A Model of International Economics

Here, to extend our stochastic growth model to heterogenous environments, we consider a model of international economics in the spirit of Baxter and Crucini (1995). Baxter and Crucini (1995) solve the planning problem with two countries. A country is identified by a pair (z_i, K_i). Then the SPP in recursive form is the following:

$$\begin{aligned} V(z_1,z_2,K_1,K_2) = \max_{c_1,c_2,{K_1}',{K_2}'} \quad & u(c_1) + \lambda u(c_2) + \beta \sum_{\mathbf{z}'} \Gamma_{\mathbf{z},\mathbf{z}'} V(\mathbf{z}', {K_1}', {K_2}') \\ \text{s.t.} \quad & c_1 + c_2 + {K_1}' + {K_2}' = F(z_1,K_1) + F(z_2,K_2) \\ & \mathbf{K}' = G(\mathbf{K}, z). \end{aligned}$$

Assume that $\lambda = 1$. Therefore, in any solution, the two countries have equal consumption. This is only possible if the two countries are completely symmetric, regarding their initial level of capital and TFP processes. So the question is how we can decentralize this allocation.

One way to decentralize it is to assume that we are dealing with two neighboring farmers where each has access to a backyard technology and there is no capital market. Then the recursive problem of each farmer becomes the following with the relevant equilibrium

[5] You can see from here that having heterogenous agents - different beliefs, endowments, etc. - changes this implication.

conditions[6]:

$$
\begin{aligned}
V(z_1, z_2, K_1, K_2, b) = \max_{c_1, K_1', \mathbf{b}'} \quad & u(c_1) + \beta \sum_{\mathbf{z}'} \Gamma_{\mathbf{z},\mathbf{z}'} V(\mathbf{z}', K_1{}', K_2{}', \mathbf{b}'(z')) \\
\text{s.t.} \quad & c_1 + \mathbf{q}(\mathbf{s}) \cdot \mathbf{b}' = F(z_1, K_1) + b \\
& K_2' = G_2(\mathbf{s}) \\
& \text{given } \mathbf{q}(\mathbf{s}).
\end{aligned}
$$

If we want to associate above problem with the SPP, we have to determine what initial level of borrowing and lending we should consider. In fact, we need to find b such that the above problem lead to equal consumption. Therefore, the equilibrium concept in Baxter and Crucini (1995) is an equilibrium with transfers. Furthermore, the backyard technology together with non-existence of capital markets is an unrealistic assumption. Therefore, we will try to write their problem according to the methods that we have produced so far. We know from before that we need to keep track of distribution of asset holdings. So the aggregate state here is $(z_1, z_2, K_1, K_2, A_1)$ where A_1 is the financial wealth of country 1. So the agent's decision problem is

$$
\begin{aligned}
V(s, A_1, a) = \max_{c, \mathbf{a}'} \quad & u(c) + \beta \sum_{\mathbf{z}'} \Gamma_{\mathbf{z},\mathbf{z}'} V(s'(\mathbf{z}'), \mathbf{A_1'}(\mathbf{z}'), \mathbf{a}'(\mathbf{z}')) \\
\text{s.t.} \quad & c_1 + K_1' + \mathbf{q}(\mathbf{s}, \mathbf{A_1}) \cdot \mathbf{a}' = w_1(s) + aR(s, A_1) \\
& s'(\mathbf{z}') = (\mathbf{z}', G_1(s, A_1), G_2(s, A_1)) \\
& \text{given } \mathbf{q}(\mathbf{s}, \mathbf{A_1}).
\end{aligned}
$$

Here, a is the individual's wealth in the international stock market and $R(s, A_1)$ is the rate of return in the stock market. Notice that, wages depend on the specific country[7], while they do not depend on distribution of assets since there labor supply is inelastic.

In this model, in contrast to the models so far, the household is not deciding about how to invest the capital stock. They are shareholders of the single international firm and care about their return in the stock market. There is a single international firm, who owns plants in each country and decides where to invest the capital efficiently. Therefore, here we need to model the firm explicitly. We will do that in the context of a deterministic one-sector growth model and modeling firm in this model is left as an exercise.

3.2 A Model of Firm Decision in the Growth Model

In this part, we look at a model with land, where firm is the owner of capital and households own firms and trade their share in the stock market. Moreover, production requires two factors, capital and land which are both owned by the firm and not traded. The household

[6] $\mathbf{s} = (z_1, z_2, K_1, K_2)$.

[7] No international labor market.

problem is simple and the same as before:

$$\begin{aligned} V(K,a) = \max_{c,a'} \quad & u(c) + \beta V(K', a') \\ \text{s.t.} \quad & c_1 + a' = aR(K) \\ & K' = G(K). \end{aligned}$$

Households are not endowed with anything. a is financial wealth before interest.

We model firm's problem as a dynamic programming problem:

$$\Omega(K,k) = \max_{k'} \quad F(k,1) - k' + \frac{1}{R(K')}\Omega(K',k')$$

Here k is individual firm's capital and K is aggregate capital. Also, total land endowment is normalized to 1. We are ready to define the equilibrium:

Definition 9 *A RERCE is a set of function $V^*, \Omega^*, G^*, h^*, g^*, R^*$, s.t.*

(i) $V^*(K,a), h^*(K,a)$ *solves HH problem given G^*.*

(ii) $\Omega^*(K,k), g^*(K,k)$ *solves firm's problem given G^*.*

(iii) $K' = G^*(K) = g^*(K,K)$.

(iv) $h\left(K, \frac{\Omega(K,K)}{R(K)}\right) = \frac{\Omega(G(K),G(K))}{R(G(K))}$.

(v) $R(K) = F_K(K,1)$.

$\Omega(K,K)$ is the value of the firm and contains the annualized value of land. Notice that we use land in stead of labor not to deal with slavery!

April 10th, 2008

3.3 Stochastic Growth with Two Agents

Here, we do the same as in section 2.3 for the stochastic growth model. Assume that $\mathcal{B}$ is the set of markets available to each trader. It contains a partition of Z[8], the set of possible

[8]A partition of Z is

$$\mathcal{B} = \{\Phi_1, \cdots, \Phi_n\}$$

where each Φ_i is a subset of Z and $\Phi_i \cap \Phi_j = \emptyset$ and $\bigcup_i \Phi_i = Z$. For each $z \in Z$, $\Phi(z)$ is the member of $\mathcal{B}$ containing z. Each $\Phi \in \mathcal{B}$ represents a security that pays off if any of the states in Φ happens. A partition consisted of all the singleton subsets is associated with availability of all Arrow securities.

states, reflecting the fact that risk free bond is always available. Therefore, each household problem is the following:

$$
\begin{aligned}
V(z,K,A,a) = \max_{c,y,\mathbf{b}} \quad & u(c) + \beta \sum_{z' \in Z} \Gamma_{z,z'} V(z', K', \mathbf{A}'(z'), \mathbf{b}(\Phi(z')) + y) \qquad (P3) \\
\text{s.t.} \quad & c + y + \sum_{\Phi \in \mathcal{B}} q(z,K,A)(\Phi)\mathbf{b}(\Phi) = w(z,K) + aR(z,K,A) \\
& K' = G(z,K,A) \\
& \mathbf{A}'(z') = \mathbf{H}(z,K,A)(z').
\end{aligned}
$$

Equilibrium is defined as follows:

Definition 10 *A RERCE is a set of function $V^*, y^*, \mathbf{b}^*, G^*, H^*, R^*, w^*, \mathbf{q}^*$, s.t.*

(i) V^*, y^*, b^* *solves HH problem given* $G^*, H^*, \mathbf{q}^*$.

(ii) $K' = G^*(z,K,A) = \frac{1}{2}y(z,K,A,A) + \frac{1}{2}y(z,K,A,2K-A)$.

(iii) $\mathbf{H}(z,K,A)(z') = y(z,K,A,A) + \mathbf{b}^*(z,K,A,A)(\Phi(z')), \forall z'$

(iv) $\frac{1}{2}\mathbf{b}^*(z,K,A,A)(\Phi) + \frac{1}{2}\mathbf{b}^*(z,K,A,2K-A) = \mathbf{0}$.

(v) *No-arbitrage:* $\sum_{\Phi \in \mathcal{B}} \mathbf{q}^*(z,K,A)(\Phi) = 1$.

One can show that when $\mathcal{B}$, contains all the singletons, equilibrium is Pareto Optimal. However, for smaller $\mathcal{B}$, this result does not necessarily hold. You are asked to show this in an exercise.

3.3.1 Characterization of Equilibrium

We have the following equations

$$
-u'(c(z,K,A,a))q(z,K,A)(\Phi) + \beta \sum_{z' \in \Phi} \Gamma_{z,z'} V_4(z', K', A'(z'), \mathbf{b}(\Phi) + y) = 0 \tag{1}
$$

$$
V_4(z,K,A,a) = R(z,K,A)u'(c(z,K,A,a)) \tag{2}
$$

where (1) is the first order condition in the maximization problem in (P3) and (2) is implied by the Envelope Theorem. Combining the two, we get

$$
q(z,K,A)(\Phi) = \beta \sum_{z' \in \Phi} \Gamma_{z,z'} \frac{R(z',K',A'(z'))u'(c(z',K',A'(z'),a'))}{u'(c(z,K,A,a))} \tag{3}
$$

When $\mathcal{B}$ contains all of the singletons, (3) shrinks to

$$
q(z,K,A)(z') = \beta \Gamma_{z,z'} \frac{R(z',K',A'(z'))u'(c(z',K',A'(z'),a'))}{u'(c(z,K,A,a))} \tag{4}
$$

Equation (4) implies that ratios of marginal utilities are equalized among the agents which is true in social planner's problem.

April 15th, 2008

4 Lucas Tree Model

In this section, we will focus on Lucas tree model based on Lucas (1978). Using that we would be able to talk about finance and asset prices.

4.1 Model

A Lucas tree is a tree that pays dividend. Dividend process, d_t, follows a Markov process with transition matrix $\Gamma_{d,d'}$ and takes on values in the set $\mathcal{D} = \left\{d^1, \cdots, d^I\right\}$ with d_0 being given. The environment is an exchange economy with no storage. There is a representative agent in the economy or equivalently a measure one of identical agents who choose a symmetric allocation. We want to find prices such that the representative agent does not want to trade so that markets clear. Notice that if you focus on symmetric allocations, the only Pareto Optimal allocation is to have $c_t = d_t$ at every period. As a first step we can formulate the Arrow-Debreu economy as follows:

$$\begin{aligned} \max_{c(h_t)} \quad & \sum_{t=0}^{\infty} \sum_{h_t} \beta^t \pi(h_t) u(c_t(h_t)) \\ \text{s.t.} \quad & \sum_{t=0}^{\infty} \sum_{h_t} p(h_t) c(h_t) \leq \sum_{t=0}^{\infty} \sum_{h_t} p(h_t) d_t(h_t) \end{aligned}$$

Market clearing implies that $c(h_t) = d_t(h_t)$.
First order conditions imply:

$$\beta^t \pi(h_t) u'(d_t(h_t)) = u'(d_0) p(h_t) \Rightarrow p(h_t) = \beta^t \pi(h_t) \frac{u'(d_t(h_t))}{u'(d_0)}$$

Now that we derive Arrow-Debreu prices, there are several ways to decentralize this economy. One way is to allow for the usual Arrow securities in SME. Then household problem becomes the following:

$$\begin{aligned} \max_{c_t(h_t), b(h_t)} \quad & \max \sum_{t=0}^{\infty} \beta^t \pi(h_t) u(c_t(h_t)) \\ \text{s.t.} \quad & c_t(h_t) + \sum_{d_i \in \mathcal{D}} \hat{p}(h_t, d^i) b(h_t, d^i) = b(h_t) + d_t(h_t). \end{aligned}$$

Again in equilibrium consumption should be equal to dividend and there should not be any trade. As before we can derive from the first order conditions that

$$\hat{p}(h_t, d^i) = \beta \frac{\pi(h_t, d^i)}{\pi(h_t)} \frac{u'(d_{t+1}(h_t, d^i))}{u'(d_t(h_t))}$$

Due to the Markov property of the dividend process, we can say that there is I^2 number of these Arrow security prices[9]:

$$\hat{p}(d^j, d^i) = \hat{p}_{ji} = \beta \Gamma_{ji} \frac{u'(d^i)}{u'(d^j)} \tag{5}$$

where d^j is today's shock and d^i is tomorrow's shock. Notice that higher order markov processes, imply higher degree of history dependence.
Now, we decentralize the economy using a stock. The sequence problem is the following:

$$\begin{aligned} \max_{c_t(h_t), s_{t+1}(h_t)} \quad & \max \sum_{t=0}^{\infty} \beta^t \pi(h_t) u(c_t(h_t)) \\ \text{s.t.} \quad & c_t(h_t) + s_{t+1}(h_t) q_t(h_t) = s_t(h_{t-1})(q_t(h_t) + d_t(h_t)). \end{aligned}$$

where q's are share prices. Equilibrium condition would be $s_t(h_{t-1}) = 1$ for all histories. The recursive equivalent of the above problem is the following:

$$\begin{aligned} V(d^i, s) = \quad \max_{c, s'} \quad & \max u(c) + \beta \sum_i \Gamma_{ji} V(d^i, s') \\ \text{s.t.} \quad & c + q_j \cdot s' = s \cdot (q_j + d^j). \end{aligned}$$

First order condition and Envelope theorem imply the following equations - dividend is substituted for consumption:

$$\begin{aligned} u'(d^j) &= \beta \sum_i \Gamma_{ji} V_2(d^i, s') \qquad\qquad (FOC) \\ V_2(d^j, s) &= (q_j + d^j) \cdot u'(d^j) \end{aligned}$$

substitution implies

$$u'(d^j) q_j = \beta \sum_i \Gamma_{ji} (q_j + d^i) u'(d^i), \qquad \forall j = 1, \cdots, I \tag{6}$$

The above equations are a system of linear equations in q's. We can rewrite it as

$$\underbrace{\begin{pmatrix} u_{c1} & 0 & \cdots & 0 \\ 0 & u_{c2} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & u_{cI} \end{pmatrix}}_{\mathbf{U_c}} \cdot \underbrace{\begin{pmatrix} q_1 \\ \vdots \\ q_I \end{pmatrix}}_{\mathbf{q}} = \beta \Gamma \cdot \mathbf{U_c} \cdot \mathbf{q} + \beta \Gamma \cdot \mathbf{U_c} \cdot \underbrace{\begin{pmatrix} d^1 \\ \vdots \\ d^I \end{pmatrix}}_{\mathbf{d}} \tag{7}$$

which implies that

$$\mathbf{q} = \left(\mathbf{U_c} - \beta \Gamma \mathbf{U_c}\right)^{-1} \beta \Gamma \mathbf{U_c} \mathbf{d}$$

[9]These are sometimes called Pricing Kernel or Stochastic Discount Factor

4.2 Asset Pricing

The decentralization above shows that we can price stocks.[10] Basically, when we have complete markets and therefore a pricing kernel, we can price any asset given that we know how it pays. Here we go over examples of pricing different types of securities.

- **1-Period Bond**
  A 1-period bond is a risk free security that pays 1 unit of consumption good in every state tomorrow. Therefore, its price is equal to

  $$q_j^b = \sum_i 1 \cdot \hat{p}_{ji}$$

  where j is the state today.

- **2-Period Bond** A 2-period bond is similar to a 1-period bond except that it pays for 2 subsequent periods regardless of the state of economy. So

  $$q_j^{2b} = \sum_i \hat{p}_{ji} \cdot (1 + q_i^b)$$

- **1-Period Call Option**
  A Call Option at strike price $\bar{q}$ is a security that gives its holder, the right to sell one unit of an underlying stock at price $\bar{q}$. Notice that the holder of an option, only exercise it if the strike price is higher than market price. Upon exercising, the holder buys a stock at market price and sells it at strike price. Therefore, we can write the following pricing equation:

  $$q_j^{CO1} = \sum_i \hat{p}_{ji} \cdot \max\{0, \bar{q} - q_i\}$$

- **European 2-Period Call Option**
  A European Option is an option that can only be exercised at the expiration date. Hence, we have

  $$q_j^{COE2} = \sum_i \hat{p}_{ji} q_i^{CO1}$$

- **American 2-Period Call Option**
  An American option can be exercised any time before the expiration date. So,

  $$q_j^{COA2} = \sum_i \hat{p}_{ji} \max\{\bar{q} - q_i, q_i^{CO1}\}$$

[10] Notice that equation (6) is in the same spirit. A stock is a security that pays dividend and its price next period.

April 17th, 2008

4.3 Equity Premium Puzzle

In this section, we briefly discuss the equity premium puzzle as introduced in Mehra and Prescott (1985). Consider equation (5) and assume that the representative agent represents an average consumer for the US economy. Then d^j would be per capital consumption for the US economy. Using time series data on consumption, we can construct a Markov transition matrix with an approximate set of possible consumptions. Moreover, we can assume a functional form $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$, where a reasonable value of σ lies between 1 and 3.[11] The same as before, we can price assets. If the price of risk free bond at state i is q_i^f, then we can define its rate of return as $1 + r_i^f = \frac{1}{q_i^f}$ and expected return $R^f = E[1 + r^f]$. We can also calculate the same rate of return for equity using our pricing kernel. The difference between the two numbers would be the premium that a consumer is willing to pay for the risky equity. Using the pricing kernel, the maximum equity premium that they calculated was 0.35 percent.[12] The average equity premium over the period 1889-1978 calculated from stock market data is 6.18% which is far larger than 0.35%. This observation is called the Equity Premium Puzzle. One way to resolve this is to increase σ to a number around 50. When people are very risk averse, they need to be paid a lot to hold a risky asset. However, this value of σ is not consistent with micro observations.

5 Measure Theory

In this section, we do a quick review of measure theory to be able to use in the subsequent sections.

Definition 11 *For a set S, $\mathcal{S}$ is a set of subsets of S.*

Definition 12 *σ-algebra $\mathcal{S}$ is a set of subsets of S, with the following properties:*

1. S, $\emptyset \in \mathcal{S}$
2. $A \in \mathcal{S} \Rightarrow A^c \in \mathcal{S}$ *(closed in complementarity)*
3. *for* $\{B_i\}_{i=1,2...}$, $B_i \in \mathcal{S} \Rightarrow [\cap_i B_i] \in \mathcal{S}$ *(closed in countable intersections)*

Examples of σ-algebra are the follows:

1. Everything, aka the power set (all the possible subsets of a set S)
2. $\{\emptyset, S\}$

[11] Based on micro observations.

[12] By changing β between 0 and 1 and σ from 0 to 10.

3. $\{\emptyset, S, S_{1/2}, S_{2/2}\}$ where $S_{1/2}$ means the lower half of S (imagine S as an closed interval on $\mathcal{R}$).

If $S = [0, 1]$ then the following is NOT a $\sigma-$ algebra

$$\mathcal{S} = \left\{\emptyset, [0, \frac{1}{2}), \left\{\frac{1}{2}\right\}, \left[\frac{1}{2}, 1\right], S\right\}$$

Remark 1 *A convention is (i) use small letters for elements, (ii) use capital letters for sets, (iii) use "fancy" letters for set of subsets.*

Definition 13 *A measure is a function $x : \mathcal{S} \to \mathcal{R}_+$ such that*

1. $x(\emptyset) = 0$
2. *if $B_1, B_2 \in \mathcal{S}$ and $B_1 \cap B_2 = \emptyset \Rightarrow x(B_1 \cup B_2) = x(B_1) + x(B_2)$*
3. *if $\{B_i\}_{i=1}^{\infty} \in \mathcal{S}$ and $B_i \cap B_j = \emptyset$ for all $i \neq j \Rightarrow x(\cup_i B_i) = \sum_i x(B_i)$ (countable additivity)*

Countable additivity means that measure of the union of countable disjoint sets is the sum of the measure of these sets.

Definition 14 *Borel-σ-algebra is a σ-algebra generated by the family of all open sets (generated by a topology).*

Since a Borel-σ-algebra contains all the subsets generated by intervals, you can recognize any subset of a set using Borel-σ-algebra. In other words, Borel-σ-algebra corresponds to complete information.

Definition 15 *Probability (measure) is a measure such that $x(A) = 1$*

Definition 16 *Given a measure space $(S, \mathcal{S}, x)$, a function $f : S \to \mathbb{R}$ is measurable if*

$$\forall a \quad \{b; f(b) \leq a\} \in \mathcal{S}$$

One way to interpret a σ-algebra is that it describes the information available based on observations. Suppose that S is comprised of possible outcomes of a dice throw. If you have no information regarding the outcome of the dice, the only possible sets in your σ-algebra can be $\emptyset$ and S. If you know that the number is even, then the smallest σ-algebra given that information is $\mathcal{S} = \{\emptyset, \{2, 4, 6\}, \{1, 3, 5\}, S\}$. Measurability has a similar interpretation. A function is measurable with respect to a σ-algebra, if it cannot give us more information about possible outcomes than the σ-algebra. We can also generalize Markov transition matrix to any measurable space.

Definition 17 *A function $Q : \mathcal{S} \times S \to [0, 1]$ is a transition probability if*

- *$Q(\cdot, s)$ is a probability measure for all $s \in S$.*
- *$Q(B, \cdot)$ is a measurable function for all $B \in \mathcal{S}$.*

In fact $Q(B, s)$ is the probability of being in set B tomorrow, given that the state is s today. Consider the following example. Consider a Markov chain with transition matrix given by

$$\Gamma = \begin{pmatrix} 0.2 & 0.2 & 0.6 \\ 0.1 & 0.1 & 0.8 \\ 0.3 & 0.5 & 0.2 \end{pmatrix}$$

Then

$$Q(\{1, 2\}, 3) = \Gamma_{31} + \Gamma_{32} = 0.3 + 0.5 = 0.8$$

Suppose that x_1, x_2, x_3 is the fraction of types 1,2,3 today. We can calculate the fraction of types tomorrow using the following formulas

$$\begin{aligned} x_1' &= x_1\Gamma_{11} + x_2\Gamma_{21} + x_3\Gamma_{31} \\ x_2' &= x_1\Gamma_{12} + x_2\Gamma_{22} + x_3\Gamma_{32} \\ x_3' &= x_1\Gamma_{13} + x_2\Gamma_{23} + x_3\Gamma_{33} \end{aligned}$$

In otherwords

$$\mathbf{x}' = \mathbf{x} \cdot \Gamma$$

where $\mathbf{x} = (x_1, x_2, x_3)$. We can extend this idea to a general case with a general transition function. We define the *Updating Operator* as $T(x, Q)$ which is a measure on S with respect to the σ-algebra $\mathcal{S}$ such that

$$x'(B) = T(x, Q)(B) = \int_S Q(B, s)x(ds)$$

A stationary distribution is a fixed points of T, that is $x^* = T(x^*, Q)$. We know that if Q has nice properties then a unique stationary distribution exists.[13]

In the next section, we use the concepts defined here to look at an economic problem.

6 Industry Equilibrium

This section is based on Hopenhayn (1992). Consider a set of firms in an industry. There is a competitive market for the single good produced in this industry. Given price p, demand for the good is given by $y^d(p)$. Every firm has a productivity level s and uses a single factor n, say labor, to produce according to the production function, $sf(n)$ where s can also be think of as quality. Firms maximize profit, taking price of output and labor as given,

$$\max_n psf(n) - wn$$

[13] See Stokey et al. (1989).

FOC implies that $psf'(n) = w$ and this gives us an optimal choice $n^*(s,p)$.[14] We define, $\pi(s,p) = psf(n^*(s,p)) - wn^*(s,p)$.
Suppose that $s \in S = [\underline{s}, \bar{s}]$ and that $\mathcal{S}$ is a σ-algebra over S together with a measure $x(\cdot)$ of firms. As an example, when there is only one firm with $s = 10$, x can be defined as

$$x(B) = \begin{cases} \text{\# of firms} & \text{if } 10 \in B \\ 0 & o/w \end{cases}, \qquad \forall B \in \mathcal{S}$$

Given the optimal decision of firms, the total output produced in the economy as a function of p, is

$$\int_S sf(n^*(s,P))x(ds)$$

Therefore, in equilibrium

$$y^d(p) = \int_S sf(n^*(s,P))x(ds)$$

The model discussed so far is not very interesting if we want a theory about size of firms. That is, there is no theory of x. To make the problem more interesting, we introduce dynamics into the model.

Now, suppose firms live forever and they die with probability δ. Then the value of the firm would be:

$$\Pi(s,p) = \sum_{t=0}^{\infty} \frac{(1-\delta)^t}{(1+r)^t}\pi(s,p) = \frac{1+r}{r+\delta}\pi(s,p)$$

Now suppose that firms can enter, there is a single level of productivity, $\bar{s}$, and there is an entry cost c_f. In this new environment, we still have market clearing which determines the price. What we need to calculate is the number of entering firms. What we need is a free entry condition that says there should not be any left-over profits. This implies that in equilibrium, a potential entrant should not gain more than what it costs to enter, otherwise there would be a mass of entry and that cannot be an equilibrium. Hence, equilibrium is characterized be the following equations:

$$y^d(p^*) = \bar{s}f(n^*(s,p^*))x(\bar{s}) \qquad \text{– market clearing} \tag{8}$$
$$\text{\# of entrants} = \delta x(\bar{s}) \qquad \text{– stationarity} \tag{9}$$
$$c_f = \Pi(\bar{s}, p^*) \qquad \text{– Free Entry} \tag{10}$$

So, the equilibrium solves for $p^*, x^*(\bar{s})$, equilibrium price and number of firms. What we eventually want to do is to make productivity stochastic over time.

[14] Notice that w is exogenous to the model.

April 22nd, 2008

Now, suppose that upon entry, firms draw productivity from the probability measure $\gamma(\cdot)$ and assume that economy is static. Then equilibrium is comprised of e^*, p^* where e^* is the number of firms. The measure of firms is $x^*(s) = e^*(s)\gamma(s)$. Therefore, the equilibrium conditions are the following:

$$\begin{aligned} y^d(p^*) &= \int_S sf(n^*(s,p^*))x^*(ds) \\ c_f &= \int_S \left(psf(n^*(s,p^*)) - wn^*(s,p^*)\right)\gamma(ds) \end{aligned}$$

If we make this economy dynamic where entry only happens at date zero, then free entry can be rewritten as

$$c_f = \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} \int_S \left(psf(n^*(s,p^*)) - wn^*(s,p^*)\right)\gamma(ds)$$

If we assume firms exogenously exit with probability $\delta(s)$, then the value of a firm of type s is

$$V(s,p) = \pi(s,p) + \frac{1-\delta(s)}{1+r}V(s,p) \Rightarrow V(s,p) = \frac{1+r}{1-\delta(s)}\pi(s,p)$$

Therefore, free entry is the following:

$$c_f = \int_S V(s,p^*)\gamma(ds) = \int_S \frac{1+r}{1-\delta(s)}\pi(s,p^*)\gamma(ds)$$

To find the stationary distribution, we need to find the updating operator. Suppose the measure of firms is x today and x' is the measure of firms tomorrow. Then,

$$x'(B) = \int_B (1-\delta(s))x(ds) + e\gamma(B), \qquad \forall B \in \mathcal{S}$$

where e is the number of entrants. Now assume that productivity follows a Markov process with transition $\Gamma(s', s)$. Then, updating operator would be the following:

$$x'(B) = \int_S \int_S \mathbf{1}_{s' \in B}(1-\delta(s))\Gamma(ds', s)x(ds) + e\gamma(B), \qquad \forall B \in \mathcal{S}$$

Now, let's make a theory of $\delta(\cdot)$. Suppose there is a fixed cost of $\bar{c}$ of staying open every period. A firm can decide to stay or leave every period. Therefore, the decision problem of the firm can be written as a dynamic programming

$$V(s,p) = \max\{0, -\bar{c} + \pi(s,p) + \frac{1}{1+r}\int_S V(s')\Gamma(ds', s)\}$$

If we assume that Γ satisfies first order stochastic dominance, that is

$$\forall s_1 > s_2 \Rightarrow \forall f(s) : nondecreasing; \quad \int_S f(s')\Gamma(ds', s_1) \geq \int_S f(s')\Gamma(ds', s_2)$$

then one can show that firm's decision satisfies a cutoff rule. That is above a certain productivity level, a firm stays and otherwise it exits.

April 24th, 2008

Given the previous analysis, we see that there is a cutoff rule, that is

$$\begin{aligned}
&\forall p, \exists s > s^*(p) \quad s.t. && (11)\\
&\forall s > s^*, \quad psf(n^*(s,p)) - wn^*(s,p) - \bar{c} + \frac{1}{1+r}\int v(s',p)\Gamma(ds',s) \geq 0\\
&\forall s \leq s^*, \quad psf(n^*(s,p)) - wn^*(s,p) - \bar{c} + \frac{1}{1+r}\int v(s',p)\Gamma(ds',s) < 0
\end{aligned}$$

Therefore, we have a theory of entry and exit and firm size distribution. Upon entry γ affects the distribution. Once firms enter, $\Gamma(\cdot,\cdot)$ shapes the next period productivity and firms in the lower tail of the distribution exit.

Now, let's assume that there is a labor adjustment cost. This could come from a specific policy or frictions in the labor market. Then then recursive formulation would be the following:

$$V(s,n^-;p) = \max\{0, \max_n [psf(n) - wn - \bar{c} - g(n^-,n) + \frac{1}{1+r}\int V(s,n;p)\Gamma(ds',s)]\}$$

where $g(n^-,n)$ is the adjustment cost of labor. Here are some examples of what determines g:

1. **Adjustment Cost**: $\alpha(n-n^-)^2$. This means that any direction of changing labor demand is costly for the firms.
2. **Firing Cost**: $\alpha(n-n^-)^2\mathbf{1}[n<n^-]$, which means that only decreasing labor is costly.
3. **Training Cost**: $\alpha(n-n^-)^2\mathbf{1}[n>n^-]$, which means that only increasing labor is costly.

Therefore, the firm's policy function is the pair $n(s,n^-;p), d(s,n^-;p) \in \{0,1\}$. Then the equilibrium conditions will be as follows - assume a new entrant starts with 0 employees:

$$\int_S V(s,0)\gamma(ds) = c_f \tag{12}$$

$$y^d(p^*) = \int_S sf(n(s,n^-;p^*))x^*(ds,dn^-;p^*) \tag{13}$$

$$x^* = Tx^* \tag{14}$$

where x^* is a measure over $S \times \mathbb{R}_+$ with respect to product σ-algebra and T is the updating operator defined as follows:

$$\begin{aligned}
&\forall B \in \mathcal{S} \otimes \mathcal{B}_{\mathbb{R}_+}\\
&Tx(B) = \int_{S\times\mathbb{R}_+}\int_B \mathbf{1}[(s',n(s,n^-)) \in B, d(s',n(s,n^-)) = 1]\Gamma(ds',s)x(ds,dn^-)\\
&\qquad\qquad + M\int_S \mathbf{1}[(s,0) \in B]\gamma(ds)
\end{aligned}$$

where M is the number of entrants. Stationarity implies

$$M = \int_{S \times \mathbb{R}_+} \mathbf{1}[d(s', n(s, n^-)) = 0] \Gamma(ds', s) x^*(ds, dn^-)$$

7 Growth

Recall the simple one-sector growth model from the first mini. In this, when there is curvature in the production function as a function of capital, there is no long-run growth. When the production function is linear in capital there is a balanced growth path but there is no transitional dynamics. From, this simple observation, we can see that in order to get long-run growth we need a model that behaves similar to the AK growth model. One way to achieve this is to assume there is human capital and production function is CRS w.r.t physical capital and human capital. When investment in human capital is done using consumption goods, you have seen that this economy behaves as an AK economy. This is not true when the cost of investment in human capital is time. In this section, we build another model that behaves similar to an AK model based on Romer (1986).

7.1 Long-Run Growth Through Externality

Assume that production function for each firm is of the form:

$$y_t = AK_t^{1-\theta} k_t^{\theta} n_t^{1-\theta}$$

where K is aggregate level of capital and k_t is individual level of capital. Firms take aggregate level of capital as given and there exists only one firm. In this environment, optimal decision of firm is to have:

$$r_t = A\theta K_t^{1-\theta} k_t^{\theta-1} n_t^{1-\theta}, w_t = A(1-\theta) K_t^{1-\theta} k_t^{\theta} n_t^{-\theta}$$

Therefore, for Euler equation we have - assume log preferences:

$$\frac{1}{c_t} = \beta \frac{1}{c_{t+1}} [1 - \delta + A\theta K_{t+1}^{1-\theta} k_{t+1}^{\theta-1} n_{t+1}^{1-\theta}] = \beta \frac{1}{c_{t+1}} [1 - \delta + A\theta n_{t+1}^{1-\theta}]$$

where the last equality is implied by the fact that $k_t = K_t$. Therefore, on a BGP where labor supply is constant, this model behaves the same as a an AK model. Notice that the return to capital for a social planner in this economy is $An_t^{1-\theta}$ while for the individual firm, it is $A\theta n_t^{1-\theta}$. So we can see that equilibrium is inefficient which is due to the externality. One downside of this model is that it implies big externality from investment which is not present in the data.

April 29th, 2008

8 Models of Search in Labor Market

In this section, we discuss models of search in labor market. Most of the material is based on Rogerson et al. (2005).

8.1 Basic Search Model

The basic search model is the problem of a worker who receive wage offers over time and can decide to accept them or not. The upside of accepting a job is the income that the worker receives. The downside of accepting is that waiting might result in a higher offer next period. A worker that receives an offer an accepts it is has the same job forever. Assume that wage offer is drawn from c.d.f. $F(w)$. The value of having a job with wage w is $\Omega(w) = \frac{w}{1-\beta}$. The value of being unemployed is

$$U = b + \beta \int_w \max\{U, \Omega(w)\} F(dw) \tag{15}$$

We can see that U is a constant number and $\Omega(w)$ is a linear function. Therefore, there exists a w^* that is $\Omega(w) \geq U$ for $w \geq w^*$ and the reverse holds for lower values of w. w^* is called *reservation wage*. We can also define the ex-ante value of having offer w at hand as follows:

$$V(w) = \max\{U, \Omega(w)\} = \begin{cases} \frac{w}{1-\beta} & w \geq w^* \\ U = \frac{w^*}{1-\beta} & w < w^* \end{cases} \tag{16}$$

Notice that this reservation wage is definitely not strictly lower than b. Because the strategy of always rejecting an offer if $w < b$ is feasible to the worker and gives a higher income since she gets $b > w$. Now suppose that $w^* = b$. The value of being unemployed is equal to the value of being employed with wage w^* according (16). Since an unemployed person receives b tomorrow and with positive probability a wage higher than $w^* = b$, the value of unemployment would be higher than value of receiving b every period which is a contradiction. Therefore, we can say that reservation wage is higher than unemployment benefit.

When time is finite, we can solve a similar problem. Consider the very last period, the household decision can be formulated as follows:

$$V_0(w) = \max\{w, b\} \Rightarrow w_0^* = b$$

we can iterate as follows:

$$V_1(w) = \max\{(1+\beta)w, b + \beta \int V_0(w')F(dw')\} \tag{17}$$

$$\vdots$$

$$V_{t+1}(w) = \max\{(1+\beta+\cdots+\beta^{t+1})w, b + \beta \int V_t(w')F(dw')\}$$

It can be shown that V_t converges with the sup-norm to the value function for the infinite horizon case. One reformulation of reservation wage can be obtained by manipulating (15):

$$\frac{w^*}{1-\beta} = b + \frac{\beta}{1-\beta}\int \max\{w^*, w'\}F(dw') = b + \frac{\beta}{1-\beta}\int \max\{0, w' - w^*\}F(dw') + \frac{\beta}{1-\beta}w^*$$
$$\Rightarrow w^* = b + \frac{\beta}{1-\beta}\int_{w^*}^{\infty}(w' - w^*)F(dw') \tag{18}$$
$$\Rightarrow w^* = b + \frac{\beta}{1-\beta}\int_{w^*}^{\infty}[1 - F(w)]dw$$

Equation (18) can be interpreted as benefit per period of accepting the reservation wage and benefit of not accepting it. A worker upon accepting the reservation wage receives w^* per period. If she rejects the offer she receives, she collects b and annuity value of future expected jump in her wage.[15]

We can also write modifications of above problem. For example, if there is an exogenous probability of being fired, we will have the following decision problem:

$$\Omega(w) = w + \beta\pi\Omega(w) + \beta(1-\pi)U$$

where a fired worker has to stay unemployed for a period. U can be characterized as before.

We can also allow for stochastic raise over time according to the Markov Process $F(w'|w)$:

$$\Omega(w) = w + \beta\int \max\{\Omega(w'), U\}F(dw'|w)$$

8.2 Search Model in Continuous Time

One convenient way to formulate search problem is in continuous time. Here we formulate the same problem as before in continuous time. Suppose household preferences are given by

$$\int_0^{\infty} e^{-\rho t} c_t dt$$

Then the value of accepting a job is

$$\Omega(w) = \int_0^{\infty} e^{-\rho t} w dt$$

To deal with continuous time issues, we assume that arrival of offers is a Poisson stochastic process with arrival rate α.[16] If there were offers arrive at every moment independently, a

[15] RHS can be interpreted as a payment today and option that can be exercised every day in future.

[16] A Poisson process with arrival rate α is a stochastic process $X(t)$ such that

$$\Pr[X(t+\tau) - X(t) = k] = \frac{e^{-\alpha\tau}(\alpha\tau)^k}{k!} \tag{19}$$

worker could wait for a positive amount of time to get the highest offer for sure since law of large number holds. The offers conditional on arrival are independent of each other over time. If we consider an interval of time of length Δ, according to (19), the probability of arrival of an offer for the unemployed at the end of interval is $e^{-\alpha\Delta}(\alpha\Delta)$. Now, we try to formulate decision problem of workers:

$$\Omega(w) = w\int_0^{\Delta} e^{-\rho t}dt + e^{-\rho\Delta}\Omega(w) \tag{20}$$

$$U = \int_0^{\Delta} e^{-\rho t}bdt + e^{-\rho\Delta}[e^{-\alpha\Delta}(\alpha\Delta)\int \max\{U,\Omega(w)\}F(dw) + (1-e^{-\alpha\Delta}(\alpha\Delta))U] \tag{21}$$

where the last equation is written assuming that the agent is unemployed for a period Δ and the probability of receiving multiple offers by the end of period is small since Δ is small. Notice that $\int e^{-\rho t}dt = -1/\rho e^{-\rho t}$. Therefore

$$\int_0^{\Delta} e^{-\rho t}dt = \frac{1}{\rho}\left(1-e^{-\rho\Delta}\right) \simeq \frac{1-(1-\rho\Delta)}{\rho} = \Delta$$

We also have

$$e^{-\rho\Delta} \simeq \frac{1}{1+\rho\Delta}$$
$$e^{-\alpha\Delta}(\alpha\Delta) \simeq (1-\alpha\Delta)\alpha\Delta \simeq \alpha\Delta$$

Replacing in (20) and (21), implies the following equations:

$$\Omega(w) = \Delta w + \frac{1}{1+\rho\Delta}\Omega(w)$$
$$U = \Delta b + \frac{\alpha\Delta}{1+\rho\Delta}\int \max\{U,\Omega(w)\}F(dw) + \frac{1-\alpha\Delta}{1+\rho\Delta}U$$

Dividing both sides of above equations by Δ and taking Δ to zero implies

$$\rho\Omega(w) = w \tag{22}$$

$$\rho U = b + \alpha\int \max\{0,\Omega(w)-U\}F(dw) \Rightarrow w^* = b + \frac{\alpha}{\rho}\int_{w^*}^{\infty}(1-F(w))dw \tag{23}$$

We can interpret these equations again as flow values. ρU is the annuity value of receiving reservation wage and RHS is the current benefit plus the future expected jump in value.

Now suppose at a monetary cost household can increase arrival rate endogenously as function of cost paid, $\alpha(e)$. As before

$$w^*(e) = \rho U(e) = b - e + \frac{\alpha(e)}{r}\int_{w^*(e)}^{\infty}[1-F(w)]dw$$

An unemployed person picks e to maximize $w^*(e)$. You are ask to characterize it in a homework.

May 1st, 2008

8.2.1 Job-to-Job Transition

In this section, we use model developed in (8.2) to model job-to-job transition. Suppose a worker receives new offers at arrival rate α_1 and an unemployed person receives new offer at arrival rate α_1 both drawn from $F(w)$. There is also exogenous lay-off from work at rate λ. We can write the Bellman equations the same way as in (8.2):

$$\begin{aligned}\rho U &= b + \alpha_0 \int_{w^*}^{\infty} (\Omega(w) - U)\, F(dw) \\ \rho\Omega(w) &= w + \alpha_1 \int_0^{\infty} \max\{\Omega(w') - \Omega(w), 0\} F(dw') + \lambda[U - \Omega(w)]\end{aligned}$$

where the second term in second equation is the expected future jump in value due to arrival of a new offer and third term is the jump from exogenous lay-off. If $\Omega(\cdot)$ is increasing - this can be proved - then a new offer will be accepted if it is above the current wage. Therefore

$$\rho\Omega(w) = w + \alpha_1 \int_w^{\infty} (\Omega(w') - \Omega(w))\, F(dw') + \lambda[U - \Omega(w)]$$

The reservation wage w^* is characterized by $U = \Omega(w^*)$. Using the above equations, we have

$$\begin{aligned}&w^* + \alpha_1 \int_{w^*}^{\infty} (\Omega(w') - \Omega(w^*))\, F(dw') = b + \alpha_0 \int_{w^*}^{\infty} (\Omega(w') - U)\, F(dw') \\ &w^* = b + (\alpha_0 - \alpha_1) \int_{w^*}^{\infty} (\Omega(w') - \Omega(w^*))\, F(dw')\end{aligned}$$

Therefore, for reservation wage to be higher than unemployment benefit, we need $\alpha_0 > \alpha_1$ that is an unemployed person gets offers more often than a worker. One can then show that the we have the following:

$$w^* = b + (\alpha_0 - \alpha_1) \int_{w^*}^{\infty} \frac{1 - F(w)}{r + \lambda + \alpha_1(1 - F(w))} dw$$

Once we know the decision rule of workers and unemployed, we can put these into a model of labor market. Assume we have a continuum of agents with measure 1. Suppose the fraction unemployed is $u(t)$. At every day, a fraction of unemployed that receive wages higher than w^* exit the unemployed status. Moreover, a fraction λ of workers are laid-off and become unemployed. Therefore, we have

$$\dot{u}(t) = \lambda(1 - u(t)) - \alpha_0(1 - F(w^*))u(t)$$

In a stationary equilibrium for labor market, we will have an unemployment rate:

$$u = \frac{\lambda}{\lambda + \alpha_0(1 - F(w^*))}$$

8.3 Matching and Bargaining

We now use the methods developed so far to think about vacancies and unemployment and how they can exists in an economy at the same time. To do so, we need to put forward a General equilibrium model with firms and households. Assume that unemployment is represented by u and number of vacancies are v. There exists an exogenous matching function $m(u, v)$ that is the rate of matching between firms and workers. If we assume all unemployed and vacancies are homogenous, then the arrival rate of a job for a worker, α_w and arrival rate of a worker for a firm, α_e are given by

$$\alpha_w = \frac{m(u, v)}{u}, \qquad \alpha_e = \frac{m(u, v)}{v}$$

$m(\cdot, \cdot)$ is a CRS, concave and increasing function. Firms and workers match given the described matching technology. We need a theory of how wages get determines upon matching. One way to model this is to use Nash bargaining. Assume that the value of an unemployed worker is is U and the value of a vacancy for a firm is V. Moreover, suppose that $\Omega(w)$ is the value of a worker with wage w and $J(y - w)$ is the value of a firm paying w to a worker. y is output produced by a worker. Then, Nash bargaining implies that w is chosen to solve the following problem:

$$\max_w (\Omega(w) - U)^\mu (J(y - w) - V)^{1-\mu}$$

where μ is the bargaining power for the worker. Given the bellman equations we developed before, we have

$$\begin{aligned}
r\Omega(w) &= w + \lambda(U - \Omega(w)) \\
rJ(\pi) &= \pi + \lambda(V - J(\pi)) \\
rU &= b + \alpha_w(\Omega(w) - U) \\
rV &= -c + \alpha_e(J(\pi) - V)
\end{aligned}$$

where c is the operating cost of a vacancy.

The FOC of the Nash Bargaining problem is given by

$$\mu[J(y - w) - V]\Omega'(w) = (1 - \mu)[\Omega(w) - U]J'(y - w)$$

Given the bellman equations,

$$J'(y - w) = \Omega'(w) = \frac{1}{r + \lambda}$$

which implies

$$\mu[J(y - w) - V] = (1 - \mu)[\Omega(w) - U]$$

The surplus from the match is $J(y - w) - V + \Omega(w) - U$ and the above equation says that workers receive fraction μ of the total surplus and firms receive the rest.

In order to characterize equilibrium, we either need a free entry condition for firms or a fixed number of vacancies. One free-entry condition would be $V = 0$ which say firms can freely create vacancies. This section closely follows Chapter 2 of Pissarides (2000).

References

BAXTER, M. AND M. J. CRUCINI (1995): "Business Cycles and the Asset Structure of Foreign Trade," *International Economic Review*, 36, 821–854.

DEBREU, G. (1970): "Economies with a Finite Set of Equilibria," *Econometrica*, 38, 387–392.

DUFFIE, D., J. GEANAKOPLOS, A. MAS-COLELL, AND A. MCLENNAN (1994): "Stationary Markov Equilibria," *Econometrica*, 62, 745–781.

HOPENHAYN, H. (1992): "Entry, Exit, and Firm Dynamics in Long Run Equilibrium," *Econometrica*, 60, 1127–1150.

LUCAS, ROBERT E., J. (1978): "Asset Prices in an Exchange Economy," *Econometrica*, 46, 1429–1445.

MEHRA, R. AND E. C. PRESCOTT (1985): "THE EQUITY PREMIUM A Puzzle," *Journal of Monetary Economics*, 15, 145–161.

PISSARIDES, C. (2000): *Equilibrium Unemployment Theory*, MIT Press.

PRESCOTT, E. AND R. LUCAS (1972): "A Note on Price Systems in Infinite Dimensional Space," *International Economic Review*, 13, 416–422.

ROGERSON, R., R. SHIMER, AND R. WRIGHT (2005): "Search-Theoretic Models of the Labor Market: A Survey," *Journal of Economic Literature*, 43, 959–988.

ROMER, P. M. (1986): "Increasing Returns and Long-Run Growth," *The Journal of Political Economy*, 94, 1002.

STOKEY, N., R. LUCAS, AND E. PRESCOTT (1989): *Recursive Methods in Economic Dynamics*, Harvard University Press.