Econ 8108, Macroeconomic Theory
Problem Set 1
Suggested solutions by Ali Shourideh

Note: These are only outline of solutions and you should complete some details regarding each problem.

Problem 1

Define the following space of sequences as

$$\ell^\infty = \{x = (x_0, x_1, \cdots); x_t = (x_{1t}, x_{2t}) \in \mathbb{R}_+^2\}$$

Then using the techniques introduced in the first mini we know that $(\ell^\infty, ||\cdot||_\infty)$ is a Banach Space where $||x||_\infty = \sup_t\{|x_t|_2\}$ where $|\cdot|_2$ is the Euclidean norm in $\mathbb{R}^2$. Now define:

$$U(x) = \sum_{t=0}^{\infty} \beta^t u(x_{2t}), \qquad \forall x \in \ell^\infty$$
$$\Gamma(k_0) = \{x; x \in \ell^\infty, x_{10} = k_0, x_{2t} + x_{1t+1} = f(x_{1t}), \quad \forall t \geq 0\}$$

For U to be well defined, we need a bounded-ness condition on u similar to those in SLP chapter 4. Now the sequence problem becomes the following problem:

$$\max_{x \in \Gamma(k_0)} U(x)$$

Notice that $U : \ell^\infty \to \mathbb{R}$ is continuous. So to proof existence of solution, we can use Extreme Value Theorem. There is a version of extreme value theorem for general metric spaces and it states that the image of any **sequentially compact** set under a continuous function, is also compact. Therefore, we should give conditions on f so that $\Gamma(k)$ is a sequentially compact subset of ℓ^∞. We know that in metric spaces, sequential compactness is equivalent to **total bounded-ness** and completeness. The definitions of the terms defined are the following:

- Sequentially compact: A set X is said to be sequentially compact if every sequence in X has a convergent subsequence.

- Totally bounded: A subset A of a metric space X is said to be totally bounded if

$$\forall \epsilon > 0, \quad \exists\{x_1, \cdots, x_n\} \subset A; s.t. \quad A \subset \cup_{k=1}^n B(x_k, \epsilon)$$

We know from first mini that $(\ell^\infty, ||\cdot||_\infty)$ is complete, problem set 1. Therefore, we need a condition on f to for $\Gamma(k)$ to be closed and totally bounded. Closed-ness can be implied by continuity of f(why?). If we impose a condition similar to the condition in exercise 5.1 in SLP, we can get total bounded-ness of $\Gamma(k)$ which is the following:

$$\exists k^*; s.t. \quad k < k^* \Rightarrow k < f(k) < k^* = f(k^*); k > k^* \Rightarrow k^* < f(k) < k$$

Under this condition it can be easily shown that we get our desired property and therefore, our sequence problem has a solution.

For uniqueness, we should impose that u, f are strictly concave. Then, if there exists $x_1 \neq x_2 \in \Gamma(k)$ such that $U(x_1) = U(x_2)$, by concavity of f, $x_\lambda = \lambda x_1 + (1-\lambda)(x_2) \in \Gamma(k)$ -why?. Moreover, by strict concavity of u, $U(x_\lambda) > \lambda U(x_1) + (1-\lambda)U(x_2)$. Therefore, two maximums cannot exist.[1]

Q.E.D.

Problem 3
The only condition that we need is continuity of $G(K)$, so that we can apply the theorem of maximum. Define the contraction as the following:

$$T_G v(K, a) = \max_{a' \in \mathcal{A}} u(aR(K) + w(K) - a') + \beta v(G(K), a')$$

where T is defined over $C(\mathcal{K} \times \mathcal{A})$. By theorem of maximum, Tv is continuous in a, K. Therefore, if we prove that T satisfies Blackwell properties, we can show that T is a contraction. The Blackwell properties are obviously satisfied using the same reasoning as in chapter 4 of SLP. Therefore, T is a contraction.

Q.E.D.

Problem 4
Here, we will show that V, the solution to the dynamic programming problem is weakly concave. We cannot use the methods developed in SLP to show that V is strictly concave, since the technology in terms of asset holding is linear. I suspect that there is still a way to prove st. concavity, but I have not written it down! To show weak concavity, we use the corollary of Contraction Mapping Theorem in chapter 3 of SLP. That is we show that T_G takes concave functions to concave functions. Now define the following set:

$$S = \{v \in C(\mathcal{K} \times \mathcal{A}); \forall K \in \mathcal{K}, v(\cdot, K) : \mathcal{A} \to \mathbb{R} \text{ is concave.}\}$$

Consider a $v \in S$. consider a_1, a_2 and suppose that the optimal choice of a' under these asset holdings are a'_1, a'_2. Therefore, we have

$$\begin{aligned}
&T_G v(K, a_i) = u(R(K)a_i + w(K) - a'_i) + \beta v(G(K), a'_i) \\
&\Rightarrow u\left(R(K)a_\lambda + w(K) - a'_\lambda\right) + \beta v(G(K), a'_\lambda) \geq \\
&\lambda[u(R(K)a_1 + w(K) - a'_1) + \beta v(G(K), a'_1)]] + (1-\lambda)[u(R(K)a_2 + w(K) - a'_2) + \beta v(G(K), a'_2)]
\end{aligned}$$

where we have used concavity of v, u. Notice that by definition of the max operator

$$T_G v(K, a_\lambda) \geq u\left(R(K)a_\lambda + w(K) - a'_\lambda\right) + \beta v(G(K), a'_\lambda) \geq \lambda T_G v(K, a_1) + (1-\lambda) T_G v(K, a_2)$$

so $T_G v \in S$.

Q.E.D.

Problem 5
To show the equivalence between the two problems, we need an extra assumption:

[1] There is an obvious fixable mistake in the solution to the last part, what is it??!!

Assumption 0.1

$$\forall K_0, K_t = G(K_{t-1}), a_{t+1} \leq R(K_t)a_t + w(K_t); \lim_{t\to\infty} \beta^t V(K_t, a_t) = 0$$

One sufficient condition for this is u bounded.
Now, suppose we have a RERCE. We can construct capital stocks and prices as follows:

$$a_t = K_t = G(K_{t-1}), t \geq 1; w_t = w(K_t), R_t = R(K_t), c_t = R(K_t)K_t + w(K_t) - K_{t+1}$$

Now consider the sequence problem of the household:

$$\begin{aligned} \max_{a_t, c_t} \quad & \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{s.t.} \quad & c_t + a_{t+1} = R_t a_t + w_t \\ & \text{given } a_0 = K_0 \end{aligned}$$

We want to show that the allocation constructed above is the solution to this sequence problem. Consider any other allocation $\{\hat{c}_t, \hat{a}_t\}$ that satisfies the budget constraint. We will have the following:

$$\begin{aligned} \sum_{t=0}^{\infty} \beta^t u(c_t) &= V(K_0, a_0) = u(c_0) + \beta V(K_1, a_1) \geq u(\hat{c}_0) + \beta V(K_1, \hat{a}_1) \\ &\geq u(\hat{c}_0) + \beta[u(\hat{c}_1) + \beta V(K_2, \hat{a}_2)] \geq u(\hat{c}_0) + \beta[u(\hat{c}_1) + \beta[u(\hat{c}_2) + \beta V(K_3, \hat{a}_3)]] \\ &\geq \cdots \geq u(\hat{c}_1) + \beta u(\hat{c}_2) + \cdots + \beta^t u(\hat{c}_t) + \beta^{t+1} V(K_{t+1}, \hat{a}_{t+1}) \quad \text{- by induction.} \\ &\geq \sum_{t=0}^{\infty} \beta^t u(\hat{c}_t) + \lim_{t\to\infty} \beta^{t+1} V(K_{t+1}, \hat{a}_{t+1}) = \sum_{t=0}^{\infty} \beta^t u(\hat{c}_t) \quad \text{- by assumption 0.1} \end{aligned}$$

where we have used the definition of max operator in the above derivations. So we have proved that $\{c_t, a_t\}$ is a solution to household's sequence problem.