Jan 10, 2006

- What is a model?

A model is a specification of the world, endowed with (i) an environment, (ii) agents and (iii) characteristics of the agents. Once a model is defined, we need to know what happens, i.e., an equilibrium.

- What is an equilibrium?
  An equilibrium is a statement about what the outcome of an economy is. Tells us **what happens in an economy,** and by an ecomomy we mean a well defined environment in terms of primitives such as preferences and technology.
  Then an equilibrium is a particular mapping from the environment (preference, technology, information, market structure) to allocations where,

  1. Agents maximize
  2. Agents' actions are compatible with each other.

- One of the important questions is, given the environment what type of equilibria we should look at. The economist doesn't have the right to choose what happens, but is free to define the environment.

- For the theory to be able to predict precisely what is going to happen in a well defined environment, the outcome we define as the equilibrium needs to **exist** and must be **unique**. For this reason uniqueness is property that we want the equilibrium to have. We also know with certain assumptions that will be covered we can ensure the existence and uniqueness of an equilibrium outcome.

1 Arrow-Debreu Equilibrium

- In macroeconomics, we are interested in infinite- dimensional commodity spaces. We want to look at the relationship between competitive equilibrium and Pareto optimality in models with infinite-dimensional spaces. You looked at competitive equilibrium and Pareto optimality in 701, but the proofs of the FBWT and SBWT were done in the context of finite-dimensional commodity spaces. Here we want to show that the welfare theorems hold for economies with infinite dimensional spaces. To do this, we introduce the equilibrium concept of '*valuation (or AD) equilibrium*'.

- Before defining valuation equilibrium, we first need to define the environment, unlike the social planner problem, which is a problem of allocation, in a AD world we will have exchange among agents. This requires definition of markets in which the relevant commodities to be defined are traded.

1. $\mathcal{L}$, Commodity space:
   $\mathcal{L}$ is a topological vector space.

   Definition 1 (Vector Space)**.** *A vector space is a space where the operations addition and scalar multiplication are defined, and where the space is closed under these two operations. i.e. If we take two sequences $a = \{a_i\} \in \mathcal{L}$ and $b = \{b_i\} \in \mathcal{L}$, it must be that $a + b \in \mathcal{L}$. And if we take $k \in \mathcal{R}^+, k > 0$, it must be that $a \in \mathcal{L} \Rightarrow d = ka \in \mathcal{L} \ \forall \ k > 0$.*

   Definition 2 (Topological Vector Space)**.** *A topological vector space is a vector space which is endowed with a topology such that the maps $(x, y) \to x+y$ and $(\lambda, x) \to \lambda x$ are continuous. So we have to show the continuity of the vector operations addition and scalar multiplication.*

2. $X \subset \mathcal{L}$, Consumption Possibility Set:
   Specification of the 'things' that people could do (that are feasible to them). X contains every (individually) technologically feasible consumption point.
   Characteristics of X: non-empty, closed and convex. Also, note that we will use the convention that output is positive while inputs are negative.

3. $U : X \to \mathcal{R}$, Specifies the preference ordering (utility function)

4. Y, Production possibility set, which must be non-empty, closed, convex and must have an interior point.

A simplifying assumption (which will be relaxed in a couple of weeks) we'll impose, is that there are many identical firms and agents. With this, we guarantee that they act competitively (take prices as given) and we only have to consider a representative agent who chooses what everyone else chooses (although everyone could do differently).

1.1 Prices

Prices (p) are continuous linear functions that are defined on our commodity space. More specifically, $p \in \mathcal{L}^*$, where $\mathcal{L}^*$ is the 'dual' of $\mathcal{L}$ (the set of all linear functions over $\mathcal{L}$); it may not always be possible to find a sequence of real numbers to represent this function as a dot product formulation (as we think of prices in finite dimensions)

Remark 1. *if $\mathcal{L} = \mathcal{R}^n \Rightarrow \mathcal{L}^* = \mathcal{R}^n$. In other words $p(\ell) = \sum_{n=1}^{N} p_n \ell_n$ defines the value of a bundle ℓ, composed of $\{\ell_1, \ell_2, ..., \ell_N\}$*

if $\mathcal{L} = \mathcal{R}^n \Rightarrow \mathcal{L}^* = \mathcal{R}^n$. In particular

$$p(\ell) = \sum_{n=1}^{N} p_n \ell_n$$

1.1.1 Definition of the Arrow-Debreu equilibrium

Definition 3 (Arrow-Debreu/Valuation Equilibrium)**.** *ADE equilibrium is a feasible allocation (x^*, y^*) and a continuous linear function p^* such that,*

1. *x^* solves the consumer's problem:*

$$x^* \in Arg \max_{x \in X} u(x)$$

st

$$p^*(x) \leq 0$$

2. *y^* solves the firm's problem:*

$$y^* \in Arg \max_{y \in Y} p^*(y)$$

3. *markets clear (compatibility of actions)*

$$x^* = y^*$$

Jan 12, 2006

1.2 Welfare Theorems

Theorem 1 (First Basic Welfare Theorem)**.** *Suppose that for all $x \in X$ there exists a sequence $\{x_n\}_{n=0}^{\infty}$ in$X$ converging to x with $u(x_n) \geq u(x)$ for all n (local nonsatiation). If an allocation (x^*, y^*) and a continuous linear functional p constitute a competitive equilibrium, then the allocation (x^*, y^*) is Pareto optimal.*

The FBWT tells us that the market is a good mechanism to allocate resources, which imply that 'you cannot do better'.

Definition 4 (Feasible allocation)**.** *For all $x = y$ st $x \in X$ and $y \in Y$*

$$F(\varepsilon) = X \cap Y$$

where ε describes, through primitives, a particular economy

Theorem 2. *If u is strictly concave and $X \cap Y$ is convex and compact, then the solution to*

$$\max_{x \in X \cap Y} u(x) \tag{1}$$

exists and its unique.

This last result plus the FBWT imply that if an Arrow Debreu equilibrium exists, then it's a Pareto Optimal allocation.

From (1) we can calculate the allocation (x^*, y^*). Nevertheless, to construct an AD equilibrium, we still need a price function. The Second Basic Welfare Theorem will provide us with one.

Theorem 3 (Second Basic Welfare Theorem)**.** *If (i) X is convex, (ii) preference is convex (for $\forall x, x' \in X$, if $x' < x$, then $x' < (1-\theta)x' + \theta x$ for any $\theta \in (0,1)$), (iii) $U(x)$ is continuous, (iv) Y is convex, (v) Y has an interior point, then with any PO allocation (x^*, y^*) such that x^* is not a saturation point, there exists a continuous linear functional p^* such that (x^*, y^*, p^*) is a Quasi-Equilibrium with transfers((a) for $x \in X$ which $U(x) \geq U(x^*)$ implies $p^*(x) \geq p^*(\nu^*)$ and (b) $y \in Y$ implies $p^*(y) \leq p^*(y^*)$)*

Note that an additional assumption we are making for SBWT to go through in infinitely dimensional spaces is that Y has an interior point i.e.

$$\exists \bar{y} \in Y,\ B \subset Y,\ B \text{ open and } \bar{y} \in B$$

Also that the SBWT states that under certain conditions listed above, we can find prices to support any Pareto optimal allocation as a quasi equilibrium with transfers. Transfers are not relevant in our case since we are working in an representative agent environment with identical households. Taking care of the transfers still leaves us with Quasi-Equilibrium so SBWT by itself it does not say anything about the existence of Arrow-Debreu equilibrium. The following lemma takes care of this.

Lemma 1. *If, for (x^*, y^*, ν^*) in the theorem above, the budget set has cheaper point than x^* ($\exists x \in X$ such that $\nu(x) < \nu(x^*)$), then (x^*, y^*, ν^*) is a ADE.*

With the SBWT, we established that there exists a p that will support our PO allocation as a competitive equilibrium. What's the problem with this approach? SBWT only tells us that such a p exists, it doesn't tell us what it is. Also, we are not sure that p has a dot product representation. The next theorem deals with this nuance

Theorem 4. *(based on Prescott and Lucas 1972) If, in addition to the conditions of the SBWT, agents discount remote and/or unlikely states and u is bounded, then $\exists \{q_t\}$ such that*

$$p(x) = \sum_{t=0}^{\infty} q_t x_t \tag{2}$$

i.e. the price system has an inner product representation.

Remember, our main purpose is to be able to apply the welfare theorems to the most commonly used models in macroeconomics where we have an infinite-dimensional commodity space. Until now, we set up an environment (Arrow-Debreu economy) which consisted of the commodity space, consumption possibility set, production possibility set, and preferences) with infinite-dimensional commodity space and we stated that under certain conditions the Welfare Theorems hold in this environment. Now we will map the growth model into the environment that we talked about until here, and show that in the context of the growth model the assumptions we need for the Welfare Theorems are satisfied. Then we can conclude that any competitive equilibrium allocation is Pareto optimal and moreover we can support a PO allocation with some prices as a competitive equilibrium. This result is very important in macroeconomics. It helps us in solving for the equilibria. With the FBWT and SBWT, we can just solve for the PO allocations and then get the prices. This makes life much easier.

2 Growth Model

2.1 Technology

- Agents have 1 unit of labor and own capital which can be transformed in output.

- Production function:

$$f : R_+^2 \to R_+ \quad \text{such that } c_t + k_{t+1} = f(k_t, n_t) \tag{3}$$

- We assume (i) Constant Returns to Scale (CRS, or homogeneous of degree one, meaning $f(\lambda k, \lambda n) = \lambda f(k, n)$), (ii) strictly increasing in both arguments, and ((iii) INADA condition, if necessary)

2.2 Preferences

- We assume infinitely-lived representative agent (RA).[1]
- We assume that preference of RA is (i) time-separable (with constant discount factor $\beta < 1$), (ii) strictly increasing in consumption (iii) strictly concave
- Our assumptions let us use the utility function of the following form:

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \tag{4}$$

- Initial capital stock k_0 is given.

  With these in hand the problem is,

$$\max_{\{c_t, n_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{5}$$

subject to[2]

$$k_{t+1} + c_t = f(k_t, n_t) \tag{6}$$

$$c_t, k_{t+1} \geq 0 \tag{7}$$

$$k_0 \text{ is given} \tag{8}$$

A solution to this problem is a sequence of consumption and capital accumulation decisions $\left\{c_t^*, k_{t+1}^*\right\}_{t=0}^{\infty}$. We would like this solution to exists,

[1]For now, let's treat the economy as if there were only one agent in the economy. We might interpret it as the result of normalization (so the number of population is 1) of the economy with FINITE number of identical (sharing the same technology, preference, and allocation) agents. If we proceed to the economy with mass of zero measure agents, things will be not so trivial because changing allocation of one agent does not change the aggregate amount of resources in the economy (since, by assumption, measure of an agent is zero), but let's forget it for now.

[2]We can also define f (the production function) as including depreciation of capital. In the 1st class, Victor actually took this approach, but I modified the notation to make notation consistent across classes.

be unique and be Pareto Optimal. How do we do that? We can show those properties by rewriting the model in Arrow-Debreu language and using the theorems we learnt so far.

What is an allocation in this environment? An allocation is a pair (x, y). On the other hand, a feasible allocation is (x, y) such that x=y (agents' actions need to be compatible). What are the commodities we need to make tradable in this environment? Output, labor services, capital services. So lets define the commodity space.

$$\mathcal{L} = \{\{\ell_t\}^{t=0,\infty} = \{\ell_{it}\}_{i=1,2,3}^{t=0,\infty},\ s_{it} \in R : \sup_t |s_t| < \infty\ \}$$

so our commodity space will be the set of bounded sequences in sup norm. Subindexes 1,2 and 3 represent output, capital services and labor services respectively. The interested reader can refer to Stokey and Lucas (1989) for the reasons behind the choice of this particular space. Next is the definition of consumption possibility set X

$$X(k_0) = \{x \in \mathcal{L} = l_\infty^3 : \exists \{c_t, k_{t+1}\}_{t=0}^{\infty} \geq 0 \text{ such that}$$

$$\begin{aligned} k_{t+1} + c_t &= x_{1t} \qquad \forall t && (9) \\ x_{2t} &\in [-k_t, 0] \qquad \forall t && (10) \\ x_{3t} &\in [-1, 0] \qquad \forall t\} \end{aligned}$$

Now, let's define the production possibility set Y. The firm's problem is relatively simple as firms do not have intertemporal decisions. Firms just rent production factors and produce period by period.

$$Y = \Pi_{t=0}^{\infty} \widehat{Y}_t : \widehat{Y}_t = \{y_{1t} \geq 0, y_{2t}, y_{3t} \leq 0 : y_{1t} \leq f(-y_{2t}, -y_{3t})\} \tag{11}$$

Finally, preferences over this space $U : X \to R$

$$U(x) = \sum_{t=0}^{\infty} \beta^t u(c_t(x)) \tag{12}$$

c_t is unique given x, because each x implies a sequence $\{c_t, k_{t+1}\}_{t=0}^{\infty}$. If $x_{2t} = k_t$, $c_t = x_{1t} - x_{2t+1}$.

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To support a Pareto Optimal allocation as a solution to the growth model presented before, we have to take care of certain issues that arise when we apply the SBWT to get our equilibrium. Those issues/solutions are listed below:

- What are the 'transfers' of the conclusion of the SBWT in terms of the growth model? / we don't need transfers; agents are homogeneous, so even if they can act differently, they choose to do the same as everyone else.

- Do we have to worry about the 'Quasi' part of the equilibrium? / If we can find a cheaper point in the feasible set, then the Quasi equilibrium is equivalent to the AD equilibrium

- representation of prices/ if we can check the conditions of the Prescott & Lucas Theorem, then we have a dot product representation of prices.

2.3 Characterization of the solution to the growth model

The solution to the growth model is triplet of sequences $\{c_t^*, k_{t+1}^*, q_t^*\}_{t=0}^{\infty}$. As you proved in the homeworks, you can use the Arrow-Debreu apparatus in order to argue that such an equilibrium exists. To characterize more carefully the equilibrium, we have to impose additional restrictions:

- u, f are C^2 (twice continuously differentiable)

- Inada conditions (see the Stockey and Lucas textbook for specifics)

With these conditions, we can restrict our attention to interior solutions, which means that first order conditions are sufficient to characterize equilibria.

Rewriting the growth model (replacing consumption in the utility function using the budget constraint):

$$\max_{\{k_{t+1}\}_{t=0}^{\infty}} u[f(k_t) - k_{t+1}]$$

Taking the FOC with respect to k_{t+1} and replacing for c_t to ease notation, we get (note that we are using variables with * to denote that the following are *equilibrium* conditions)

$$-\beta^t u'[c_t^*] + \beta^{t+1} u'[c_{t+1}^*] f'(k_{t+1}^*) = 0$$

rearranging terms

$$\frac{u'[c_t^*]}{\beta u'[c_{t+1}^*]} = f'(k_{t+1}^*) \tag{13}$$

Therefore, the solution to the growth model has to satisfy the condition in (13).

Now, for prices, we can rewrite the budget equation from the AD setting (if the conditions of Prescott and Lucas are satisfied so that prices have a dot product representation) as follows

$$p(x) \equiv \sum_{t=0}^{\infty}(q_{1t}x_{1t} + q_{2t}x_{2t} + q_{3t}x_{3t}) \leq 0 \tag{14}$$

Since $c_t + k_{t+1} = x_{1t}$, $k_t \geq -x_{2t} \geq 0$ and $1 \geq -x_{3t} \geq 0$, (14) becomes

$$\sum_{t=0}^{\infty}(q_{1t}^*(c_t + k_{t+1}) - q_{2t}^*k_t - q_{3t}^*) \leq 0 \tag{15}$$

Note that in (15), we have used the fact that there is no waste (agents rent their full capital and labor services) and that agents take the equilibrium prices as given. The maximization problem now can be set as a Lagrangian:

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \pounds = \sum_{t=0}^{\infty} \beta^t u[c_t] - \lambda\{\sum_{t=0}^{\infty} q_{1t}^*(c_t + k_{t+1}) + q_{2t}^*k_t + q_{3t}^*\} \tag{16}$$

The first order conditions of this problem with respect to c_t and k_{t+1} are respectively

$$\frac{\beta^t u'[c_t^*]}{q_{1t}^*} = \lambda \tag{17}$$

$$\lambda q_{1t}^* - \lambda q_{2,t+1}^* = 0 \tag{18}$$

Note that (18) implies that $q_{1t}^* = q_{2,t+1}^*$, which pins down one sequence of prices (specifically, the price of capital services). From (17), we get

$$\lambda = \frac{\beta^t u'[c_t^*]}{q_{1t}^*} = \frac{\beta^{t+1} u'[c_{t+1}^*]}{q_{1,t+1}^*}$$
$$\Rightarrow \frac{u'[c_t^*]}{\beta u'[c_{t+1}^*]} = \frac{q_{1t}^*}{q_{1,t+1}^*}$$

From before, we know that the left hand side of the last equation equals $f'(k_{t+1}^*)$. Hence

$$\frac{q_{1t}^*}{q_{1,t+1}^*} = f'(k_{t+1}^*) \tag{19}$$

Since f' represents the (technical) rate of exchange between goods today and goods tomorrow, (19) tells us exactly what the sequence of output prices should be. Finally, to obtain q_{3t}^*, we turn to problem of the producer

$$\max_{y \in Y_t} q^*(y) = q_{1t}^* y_{1t} + q_{2t}^* y_{2t} + q_{3t}^* y_{3t}$$

$$st$$

$$y_{1t} = f(-y_{2t}, -y_{3t})$$

Again, we know that $\{k^*_{t+1}, 1\}^\infty_{t=0}$ solve this problem. Then, the problem of the firm is equivalent to

$$\max_{y \in Y_t} f(k_t, y_{3t}) - q^*_{2t} k_t - q^*_{3t} y_{3t}$$

Taking FOCs with respect to k^*_t and y_{3t} respectively

$$q^*_{1t} f_k(k^*_t, 1) = q^*_{2t}$$

$$q^*_{1t} f_n(k^*_t, 1) = q^*_{3t}$$

Hence, the price of labor services must satisfy $f_n(k^*_t, 1) = q^*_{3t}/q^*_{1t}\ \forall t$.

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We know now how to characterize the sequence of prices at equilibrium in the growth model. The problem with the AD framework however, is that we have a triple infinite (!) number of prices. Together with the assumption that all trade takes place at $t = 0$, this implies that agents must know a triple infinite number of prices in order to solve their problem.

We want to depart from this assumption of all trading happening at the beginning of time, so we will define *sequential markets* and a corresponding *sequential markets equilibrium* (SME). Note that we would like to maintain existence, uniqueness and optimality of the equilibrium, so we would like $ADE \Leftrightarrow SME$.

2.4 Sequential Markets Equilibrium

- We need a spot market at every period of time where agents would be able to trade output, capital and labor services and a new good (which we will specify below) which are 'loans'.
- Agents must be able to move resources across time.

Clearly, the budget constraint will change from the previous setup. In ADE

$$\sum_{i=1}^{3} \sum_{t=0}^{\infty} q_{it} x_{it} \leq 0$$

In SME, we introduce the concept of 'loans' (l), to enable agents to move resources across time. Loans are rights to a R units of output/consumption tomorrow, in exchange of 1 unit of output/consumption today. So, the budget constraint becomes

$$-l_t R_t + l_{t+1} + \sum_{i=1}^{3} q_{it} x_{it} \leq 0 \quad \forall t$$

Definition 5. *A Sequence of Markets Equilibrium is* $\{x_{it}^*, q_t^*, y_{it}^*, l_{t+1}^*, R_t^*\}_{t=0}^{\infty}$ *such that*

- *Agents maximize, i.e.*

$$\{x_{it}^*, l_{t+1}^*\} \in \arg\max_{x \in X} \sum_{t=0}^{\infty} \beta u[c_t(x)]$$

$$st$$

$$c_t + k_{t+1} + l_{t+1}^* = R_t^* l_t + q_{2t}^* k_t + q_{3t}^*$$

$$k_0, l_0 \quad given$$

- *Firms maximize*
- $x^* = y^*$ *(market clears)*
- $l_{t+1}^* = 0 \quad \forall t$ *(loan market clears)*

To show that $ADE \Leftrightarrow SME$, we need to check that allocations and choices of the agents in both worlds are the same. In the $SME \Rightarrow ADE$ direction, it's easy to see that if we have a SME, we can construct an ADE just by ignoring $\{l_{t+1}\}$ (it's zero at equilibrium anyway).

Conversely ($ADE \Rightarrow SME$), if we have an ADE, we need l_{t+1}^* and R_t^* to construct a SME. Again, given the condition for the clearing of the loans market, l_{t+1}^* comes trivially. For R_t^*, we use an arbitrage condition: since loans and capital perform the same function (move resources from one period of time to another), then their price should be the same. Specifically

$$R_t = \frac{q_{1t}^*}{q_{1,t+1}^*}$$

Finally, we have a close relationship between prices between both equilibriums. If $\{x^*, y^*, q^*\}$ is an ADE and $\{x^*, y^*, \widehat{q}^*, R^*, l^*\}$ is a SME, the following is true since at a SME, the budget constraint is priced with respect to output/consumption at each point of time

$$\widehat{q}_{it}^* = \frac{q_{it}^*}{q_{1t}^*} \quad \forall t$$

2.5 SME 'easy'

Now we will define a simpler version of the SME. Basically, we will simplify the definition of equilibrium by ignoring loans and using the properties of the production function

Definition 6. *A SMEE is* $\{c_t^*, k_{t+1}^*, w_t^*, R_t^*\}$ *such that*

- *Agents maximize:*

$$\{c_t^*, k_{t+1}^*\} \in \arg \max_{\{c_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u[c_t]$$

$$st$$

$$c_t + k_{t+1} = R_t^* k_t + w_t^*$$

$$k_0 \quad given$$

- *Firms maximize:*

$$\{k_{t+1}^*, 1\} \in \arg \max_{k_t, n_t} f(k_t, n_t) - R_t^* k_t - w_t^* n_t$$

- *Market clearing:*

$$c_t^* + k_{t+1}^* = f(k_t^*, 1)$$

Note that the last condition is redundant, because the production function is homogenous of degree one, i.e., production is exhausted in the payment to production factors.

After all this work, we still have the problem of how to calculate the equilibrium. From the FOCs we know that to get a solution, we have to solve a second order difference equation, with an initial conditions plus a transversality condition. Nevertheless, the Growth model has infinite dimensions, which complicate things a bit.

The next step is to reformulate the problem in a recursive form. This is much better, since we will be able to solve the problem recursively, that is, every new period, the agent faces the same problem.

January 24, 2006

3 Recursive Competitive Equilibrium

As you have seen in 704, the beauty of the recursive representation lies in the fact that, in a stationary environment, the nature of the problem do not change with passage of time. Unlike the sequential formulation, in which the solution to the problem depends on at what point in time you solve it, the solution to the recursive problem do not depend on time and we do not have to keep track of time.

So what do we keep track of? Everything that matters to the structure of our problem. These are the variables that our agents respond to either directly or indirectly and we call them the STATE VARIABLES. State variables need to satisfy the following criteria:

1. PREDETERMINED: when decisions are made, the state variables are taken as given and cannot be effected by the agent.
2. It must MATTER for decisions of agents: there is no sense of adding irrelevant variables as state variable.
3. It must VARY across time and state: otherwise, we can just take it as a parameter.

One important thing is to be able to distinguish the aggregate and individual state variables. Aggregate state is not affected by individual choice. But aggregate state should be consistent with the individual choice (we will consider the meaning of "consistency" more formally later), because aggregate state represents the aggregated state of individuals. In particular, in our RA-NGM aggregate state turns out to be the same as individual state in equilibrium, but this does not mean that the agent decide the aggregate state or the agent is forced to follow the average behavior, but rather the behavior of the agent turns out to be the aggregate behavior, in equilibrium.

Also note that prices (wages, and rental rates of capital) is determined by aggregate capital, rather than individual capital, and since individual takes aggregate state as given, she also takes prices as given (because they are determined by aggregate state). Again, the aggregate capital turns out to coincide with the individual choice, but it is not because of the agent's choice, rather it is the result of consistency.

One notational note. Victor is going to use a for individual capital and K for aggregate capital, in order to avoid the confusion between K and k. But the problem with aggregate and individual capital is often called as "big-K, small-k" problem, because the difference of aggregate capital and individual capital is crucial. So for our case, the counterpart is "big-K, small-a" problem.

What does the agent has to know in order to solve her problem? Our agent has to know how wealthy she is. She also needs to know the prices but we do not need $\{R, w\}$ directly. Why? Because they are redundant: K is a sufficient statistic to calculate $\{R, w\}$ as they must be equal to marginal products in the firm problem at equilibrium.If we put K as a state variable instead of these prices, we do not need $\{R, w\}$. So are we done? Not yet. As the problem of the HH is formulated below, our agent not only needs to know $\{R, w\}$ but $\{R', w'\}$ as well thus K'. But this is a variable that our agent has no control over and the best she can do is to have a 'belief' about it. These beliefs in our model are parameterized by the G^F function which maps today's state to a unique belief about next period's value. As it is formulated, it is an exogenous parameter, i.e. we can solve this problem for any sort of beliefs under which the problem is well defined. But from the beginning of this course we want to be able to 'predict' the outcome, once we setup our environment as precise as possible. To continue to be able to do so, as we will see later, we will impose an additional constraint on the beliefs and make them an equilibrium object as well, i.e. endogenize them.

The recursive form of the problem of the agent (that does not value leisure) is:

$$V(K, a; G^F) = \max_{c,a'}[u(c) + \beta V(K', a'; G^F)]$$

$$\begin{gathered} s.t: \quad c + a' = Ra + w \\ R = R(K) = F_K(K, 1) \\ w = w(K) = F_N(K, 1) \\ K' = G^F(K) \end{gathered}$$

Note that F is indexing the beliefs. This means that for different beliefs F, different equilibria will arise. Of course, in this course we will only focus on rational expectations equilibria (which we will define below).

How do we deal with the problem written above? The first step is to assume that the solution (for the decision rule) is of the form, $a' = g\left(a, K; G^F\right)$. So use this fact to write:

$$\begin{gathered} V(K, a; G^F) = \\ u(R(K)\,a - g\left(a, K; G^F\right)) + \beta V(G^F(K), g\left(a, K; G^F\right); G^F)] \qquad (1) \end{gathered}$$

Note that the FOC implies that:

$$\text{-}u_c(c) + \beta \frac{\partial V}{\partial a'} = 0 \qquad (2)$$

and in order to obtain a more useful expression for the derivative we use the Envelope condition. This is where equation (1) is useful. From (1), take the derivative with respect to a :

$$\frac{\partial V}{\partial a} = u_c(c)\left[R(K) - \frac{\partial g}{\partial a}\right] + \beta \frac{\partial V}{\partial a'}\frac{\partial g}{\partial a} = u_c(c)\,R(K) + \frac{\partial g}{\partial a}\left[-u_c(c) + \beta \frac{\partial V}{\partial a'}\right],$$

so by (2): $\frac{\partial V}{\partial a} = u_c(c)\,R(K)$.

January 26, 2006

Recursive Competitive Equilibrium (continued)

Before we proceed to the definition of RCE, recall the issue of compactness that we discussed in class. We will assume that there exist a $\bar{K}$, such that $\bar{K} \geq F\left(\bar{K}, 1\right) + (1-\delta)\bar{K}$. In words $\bar{K}$ is the amount of capital stock which is impossible to reproduce. The assumption is that such a value of K exists.

Definition 1:

A Recursive Competitive Equilibrium with arbitrary beliefs G^F is a list of functions $\{V^*(.), g^*(.), G^*(.), R(.), w(.)\}$ such that:

1) Given $\{G^F(.), R(.), w(.)\}$, $\{V^*(.), g^*(.)\}$ solves the household problem above,

2) $\{R(.), w(.)\}$ are characterized by the optimal decisions of firms, and

3) $G^*(K; G^F) = g^*(K, K; G^F)$ (agent is representative).

Some comments on the third condition. This condition is called 'Representative agent condition' and is a specific case of the compatibility condition that any equilibrium must satisfy. It basically means that if a consumer turns out to be average this period (her individual capital stock is K, which is aggregate capital stock), the consumer will choose to be average in the next period (she chooses $G^*(K)$, which is a belief on the aggregate capital stock in the next period if today's aggregate capital stock is K). This condition guarantees that in an equilibrium, individual choice turns out to be consistent with the aggregate law of motion. This is true not because our agent is constrained to do so but because the prices are such that she choses to do so.

Definition 2:

A RCE with rational expectations is a list of functions $\{V^*(.), g^*(.), G^*(.), R(.), w(.)\}$ such that conditions 1) to 3) from the above definition are satisfied, plus:

4) $G^*(K) = G^F(K)$, in other words:

A RCE with rational expectations is a list of functions $\{V^*(.), g^*(.), G^*(.), R(.), w(.)\}$ such that:

1) Given $\{G^*(.), R(.), w(.)\}$, $\{V^*(.), g^*(.)\}$ solves the household problem above,

2) $\{R(.), w(.)\}$ are characterized by the optimal decisions of firms, and

3) $G^*(K; G^*) = g^*(K, K; G^*)$.

Comment: Why do we need the RCE? Isn't it more easy to work with the Social Planner's Problem? The answer is yes. However, there are lots of cases where the solution to SPP does not coincide with that of the Competitive Economy. The most characteristic example is the presence of an externality or a distortionary tax. Moreover, we will also see models where we don't have a representative agent (heterogeneous agents models), and so we don't really know what the SP Problem looks like. In all the above cases the SPP is not helpful anymore, and we have to "attach" the Competitive Equilibrium directly.

The most common way to find and characterize A RCE (given its complexity) is to use computational methods. But we will not cover any of these techniques in 702.

January 31, 2006

Suppose now that leisure appears in the utility function of the agent. How does the analysis of the previous lecture change? The firm's problem (and thus its optimal choices) surely don't change. However, l (or equivalently n, hours worked) will now be a choice variable for the agent. The problem of the agent now is the following:

$$V(K, a; G, H) = \max_{c,l,a'}[u(c, l) + \beta V(K', a'; G, H)]$$

$$\begin{aligned} s.t: \quad c + a' &= Ra + (1 - l)\, w \\ R = R(K) &= F_K(K, N) \\ w = w\,(K) &= F_N(K, N) \\ K' &= G(K) \\ N = N\,(K) &= H\,(K) \end{aligned}$$

Note that introducing leisure in the utility function created a potential problem: Now N, the aggregate labor supply is present in the production function. This means that our problem is not well defined (remember: a recursive problem is well defined if every variable is either a state, or a control, or an explicit function of the above). We overcome this problem by adding the last condition which gives N as a function of the aggregate state.

Definition

A RCE in this context is a list $\{V^*(.), g^*(.), h^*\,(.)\,, G^*(.), H^*\,(.)\,, R(.), w(.)\}$ such that:

1) Given $\{G^*(.), H^*\,(.)\,, R(.), w(.)\}$, $\{V^*(.), g^*(.), h^*\,(.)\}$ solves the household problem above,

2) $\{R(.), w(.)\}$ are characterized by the optimal decisions of firms, and

3) Agent is representative, which here means:
$G^*(K) = g^*(K, K; G^*, H^*)$ and $H^*\,(K) = h^*(K, K; G^*, H^*)$.

Introducing Government

The simplest model with government is the one where a constant income tax is imposed. Typically the government returns the earnings in the form of a lump sum tax. But it wouldn't make any difference if the tax earnings where thrown into the ocean.

The recursive formulation of this simple problem is as follows (from now on we will not repeat the G function as an argument (more precisely an index) of the value function):

$$V(K,a) = \max_{c,l,a'}[u(c,l) + \beta V(K',a')]$$

$$s.t: \quad c + a' = (1-\tau)\left[w(K)(1-l) + a(R(K)-1)\right] + a + T$$

$$\begin{gathered} K' = G(K) \\ N = H(K) \\ T = T(K) \end{gathered}$$

Here a RCE is a list $\{V^*(.), g^*(.), h^*(.), G^*(.), H^*(.), T^*(.), R(.), w(.)\}$ such that:

1) Given $\{G^*(.), H^*(.), T^*(.), R(.), w(.)\}$, $\{V^*(.), g^*(.), h^*(.)\}$ solves the household problem above,

2) $\{R(.), w(.)\}$ are characterized by the optimal decisions of firms,

3) Agent is representative: $G^*(K) = g^*(K, K; G^*, H^*, T^*)$ and $H^*(K) = h^*(K, K; G^*, H^*, T^*)$, and

4) Government budget constraint is satisfied:

$$T^*(K) = \tau\left[w(K) H^*(K) + (R(K)-1)K\right]$$

A Model with two social classes

Assume that in the economy under consideration there are two social classes, A and B. Let the measures of these classes be μ_A and μ_B, with $\mu_A + \mu_B = 1$. Assume that people in the two groups have same preferences. Note that in this model, unless the decision rules are linear in the (aggregate) state, owenership matters. So the states are K^A and K^B (per capita capital in each group). The recursive problem is:

$$V(a, K^A, K^B) = \max_{c,a'}[u(c) + \beta V(a', K'^A, K'^B)]$$

$$\begin{gathered} s.t: \quad c + a' = R(K)\, a + w(K)\,, \text{ where } K = \left(K^A, K^B\right) \\ R(K) = F_K(K, 1) \\ w(K) = F_N(K, 1) \\ K'^A = G^A(K) \\ K'^B = G^B(K) \end{gathered}$$

Note that we don't write a'^A and a'^B because from the point of view of the agent it doesn't matter in which group he belongs. Of course K^A, K^B are mentioned explicitly because their values will play a critical role for the determination of the prices.

Here a RCE is a list of functions $\{V^*(.), g^*(.), G^{A*}(.), G^{B*}(.)R(.), w(.)\}$ such that:

1) Given $\{G^{A*}(.), G^{B*}(.)R(.), w(.)\}$, $\{V^*(.), g^*(.)\}$ solves the household problem above,

2) $w(K) = F_2\left(\mu_A K^A + \mu_A K^A, 1\right)$ and $R(K) = F_1\left(\mu_A K^A + \mu_A K^A, 1\right)$, and

3) $G^{A*}(K^A; K^B) = g^*(K^A, K^A, K^B)$, $G^{B*}(K^A; K^B) = g^*(K^B, K^A, K^B)$

An economy with two countries that have different production functions

We will assume that there is perfect capital integration, but no labor mobility. Also assume equal size of the two countries. The main difference between this model and the previous one is that now it matters in which country you are. The reason is that there is no labor mobility and hence the wage will be different in these countries. For $i = A, B$ the recursive specification of the problem is:

$$V_i(a, K^A, K^B, x) = \max_{c,a'}[u(c) + \beta V_i(a', K'^A, K'^B, x')]$$

$$\begin{gathered} s.t: \quad c + a' = Ra + w^i, \\ R = F_K^A(K^A, 1) = F_K^A(K^A, 1) \text{ (from perfect capital integration)} \\ w^i = F_N^i(K^i, 1) \\ K'^A = G^A(K^A, K^B, x) \\ K'^B = G^B(K^A, K^B, x) \\ x' = \chi\left(K^A, K^B, x\right) \end{gathered}$$

In the problem above we know how much capital there is in each country, but not who owns it (whether it is owned by a country A or B citizen). This is why we introduce the new variable x which denotes the percentage of capital that belongs to the citizens of country A.

Here a RCE will be as usual a set of functions such that 1) given prices and the las of motion of the aggregate states agents maximize, 2) firms maximize, plus the following equilibrium conditions:

$$g^{A*}\left(x\left(K^A+p^*K^B\right), K^A, K^B, x\right) = \\ \chi^*\left(K^A, K^B, x\right)\left[G^{*A}(K^A, K^B, x) + p'^*G^{*B}(K^A, K^B, x)\right] \quad (1)$$

$$g^{B*}\left((1-x)\left(K^A+p^*K^B\right), K^A, K^B, x\right) = \\ \left(1-\chi^*\left(K^A, K^B, x\right)\right)\left[G^{*A}(K^A, K^B, x) + p'^*G^{*B}(K^A, K^B, x)\right] \quad (2)$$

$$R^A(G^{*A}(K^A, K^B, x)) = R^B(G^{*B}(K^A, K^B, x)) \quad (3)$$

$$p^*\left((K^A, K^B, x)\right) = 1 \quad (4)$$

Note that p is the relative price of capital between the countries. We use it to make sure that we don't add "apples" with "oranges". However, by the assumption of perfect capital integration, in equilibrium $p^* = 1$.

February 2, 2006

We will continue in the same spirit as in the previous lecture to define RCE for different environments.

Assume that in the economy of this example the firms own the capital. They also own land and there is no market for this commodity. Moreover, set $L = 1$.Of course, the firms are owned by the households. In this setting the firms will have a dynamic problem. This is given by:

$$\Omega(K, k) = \max_{k'}\left\{F(k, 1) - k' + q(K')\Omega(K', k')\right\}$$

$$\text{s.t: } K' = G\left(K\right)$$

The households problem is:

$$V(K, a) = \max_{a', c}\left\{u(c) + \beta V[K', a']\right\}$$

$$\text{s.t: } c + a' = q(K') = a$$

Assume that the solutions are of the form $k' = g(k, K)$, $a' = y(a, K)$. The equilibrium conditions here are:

$$q^{-1}(K') = F_1(K, 1) \quad (1)$$

$$g(K, K) = G(K) \quad (2) \text{ (firm is representative)}$$

$$y(\Omega(K, k), K) = \Omega(G(K), G(K)) \quad (3) \text{ (household owns the firm and is representative)}$$

$$\Omega(K, K) = F(k, 1) - G(K) + q(G(K))\Omega(G(K), G(K)) \quad (4) \text{ (Ω satisfies the value function).}$$

Government Debt

In this last example we consider a model with government that issues debt and imposes a consumption tax. What are the state variables for this economy?

Aggregate states: B, K,individual states: a
The problem of the representative agent is:

$$V(k, B, a) = \max_{c,a'} [u(c) + \beta V(k', B', a')]$$

$$\begin{aligned} s.t: \quad c(1-\tau) + a' &= w(k) + aR(k) \\ k' &= G(k, B) \\ B' &= H(k, B) \\ \\ w(k) &= F_2(k, 1) \\ R(k) &= F_1(k, 1) \\ \tau &= \tau(k, B) \end{aligned}$$

Let the solution have the form $a' = g(a, k, B)$.

DEFINITION: A Recursive Competitive Equilibrium is a list of functions $\{V^*(k, B, a),\ g^*(k, B, a),\ G^*(k, B),\ H^*(k, B), \tau^*(k, B)\}$ such that:

1) Given $G^*(k, B),\ H^*(k, B), \tau^*(k, B)$ the functions $V^*(k, B, a),\ g^*(k, B, a)$

solve the agent's maximization problem.
2) The Government Budget Constraint is balanced

$$R(k)\,B + \bar{G} = H^*(k,B) + \tau^*(k,B)\left[F(k,1) - G^*(k)\right]$$

3) (Consistency Equilibrium Condition)

$$g^*(k+B,k,B) = G^*(k,B) + H^*(k,B)$$

Note that the unknown functions here are three (G^*, H^*, τ^*), but the equilibrium conditions are only two (namely (2) and (3) above). What's missing? The problem here is that we don't have a theory for government, i.e, we don't have a theory of how τ is specified. In other words, the above equilibrium is indexed by τ. Only after specifying the government's objective can we more precise about how the tax equation looks like. However, we can redefine the above set of RCE by restricting attention only to feasible policies. This can be archived if we assume that there exist $\underline{B}, \bar{B}$ and $\bar{k}$ such that for every $(k,B) \in \left[0,\bar{k}\right] \times \left[\underline{B},\bar{B}\right]$, $G(k,B) \in \left[0,\bar{k}\right]$ and $H(k,B) \in \left[\underline{B},\bar{B}\right]$.

February 7th.

4 Stochastic Processes

4.1 Markov Process

In this course, we will concentrate on Markov productivity shocks. Considering shocks is really a pain, so we want to use less painful ones. A Markov shock is a stochastic process with the following properties:

1. there are FINITE number of possible states for each time. More intuitively, no matter what happened before, tomorrow will be represented by one finite set.
2. the only thing that matters for the realization of tomorrow's shock is today's state. More intuitively, no matter what kind of history we have, the only thing you need to predict the realization of the shock tomorrow is TODAY's realization.

More formally, for each period, suppose either z^1 or z^2 happens [3]. Denote z_t is the state today and Z_t is the set of possible states today, i.e. $z_t \in$

[3] Here we restrict our attention to the 2-state Markov process, but increasing the number of states to any finite number does not change anything fundamentally.

$Z_t = \{z^1, z^2\}$ for all t. Since the shock follows a Markov process, the state tomorrow will only depend on today's state. So let's write the probability that z^j will happen tomorrow, conditional on today's state being z^i as $\Gamma_{ij} = prob[z_{t+1} = z^j | z_t = z^i]$. Since Γ_{ij} is a probability, we know that

$$\sum_j \Gamma_{ij} = 1 \qquad for \ \forall i \tag{20}$$

Notice that a 2-state Markov process is summarized by 6 numbers: z^1, z^2, Γ_{11}, Γ_{12}, Γ_{21}, Γ_{22}.

The great beauty of using a Markov process is that we can use the explicit expression of probability for future events, instead of using the ambiguous operator called expectation, which very often people don't know what it means when they use it.

4.2 Representation of History

- Let's concentrate on a 2-state Markov process. In each period the state of the economy is $z_t \in Z_t = \{z^1, z^2\}$.
- Denote the history of events up to t (which one of $\{z^1, z^2\}$ happened from period 0 to t, respectively) by $h_t = \{z_1,\ z_2, ...,\ z_t\} \in H_t = Z_0 \times Z_1 \times ... \times Z_t$.
- In particular, $H_0 = \emptyset$, $H_1 = \{z^1,\ z^2\}$, $H_2 = \{(z^1, z^1),\ (z^1, z^2),\ (z^2, z^1),\ (z^2, z^2)\}$.
- Note that even if the state today is the same, past history might be different. By recording history of events, we can distinguish the two histories with the same realization today but different realizations in the past (think that the current situation might be "you do not have a girl friend", but we will distinguish the history where "you had a girl friend 10 years ago" and the one where you didn't
- Let $\Pi(h_t)$ be the unconditional probability that the particular history h_t does occur. By using the Markov transition probability defined in the previous subsection, it's easy to show that (i) $\Pi(h_0) = 1$, (ii) for $h_t = (z^1,\ z^1)$, $\Pi(h_t) = \Gamma_{11}$ (iii) for $h_t = (z^1,\ z^2,\ z^1,\ z^2)$, $\Pi(h_t) = \Gamma_{12}\Gamma_{21}\Gamma_{12}$.

With this, we can rewrite the growth model when these shocks affect the production function (usual convention in Macro). Preferences are given by

$$u(x) = \sum_t \beta^t \sum_{h_t \in H_t} \pi(h_t)\, u\left[c_t(h_t)\right].$$

In an Arrow -Debreu world the constraint is

$$\sum_t \sum_{h_t \in H_t} \sum_j p_t^j(h_t)\ x_t^j(h_t) \leq 0, \quad \text{where} \quad j = 1, 2, 3.$$

In a SM setting we need to give to the agent enough tools, so that she can consume different quantities in different states of the world. In other words, we have to make sure that whatever she was able to do in an AD setting, she will also be able to do it in the SM setting. To that end, we will introduce the notion of a state contingent claim. For example, $b_t\left(h_{t-1}, z^i\right)$ is a claim that the agent bought in period $t-1$, and will pay 1 unit of consumption for sure if state i occurs. In the SM world, the budget constraint will be

$$\begin{gathered} c_t(h_t) + k_{t+1}(h_t) + \sum_{z_{t+1}} q_t(h_t, z_{t+1}) b_{t+1}(h_t, z_{t+1}) = \\ k_t(h_{t-1})\ R_t(h_t) + w_t(h_t) + b_t(h_{t-1}, z_t), \end{gathered}$$

where $q_t(h_t, z_{t+1})$ is the price of the state contingent claim that pays 1 in period $t+1$ if state z_{t+1} occurs.

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One way of representing the recursive stochastic growth model is

$$V(z, K, a, b_z) = \max_{c, a', b_{z'}} [u(c) + \beta \sum_{z'} \Gamma_{zz'} V(z', K', a', b_{z'})]$$

st

$$\begin{aligned} c + a' + \sum_{z'} q^{z'}(z, K)\, b_{z'} &= w(z, K) + R(z, K)\, a + b_z \\ K' &= G(z, K) \end{aligned}$$

But we can reduce the number of individual state variables (a, b_z) into a only, since the household only cares about wealth. Using the arbitrage condition stated above, we have that the recursive problem is

$$V(z,K,a)=\max_{c,a'}[u(c)+\beta\sum_{z'}\Gamma_{zz'}V(z',K',a')]$$

st

$$\begin{aligned} c+a' &= w\,(z,K)+R\,(z,K)\,a \\ K' &= G(z,K) \end{aligned}$$

This is because the household can secure herself a unit of consumption for sure next period either by saving or having a portfolio that pays 1 unit for sure next period at each possible state. Since the last option is an overkill, we drop it in order to work with our usual formulation.

Again, a solution to this problem is an optimal policy for asset accumulation $a'=g\,(z,K,a)$

Definition 7. *A RCE with stochastic shocks is a list $\{V,G,g,w,R,q\}$ such that*

1. *Given $\{G,R,w,q\}$, V and g solve the consumer problem*
2. *R and w solve the firm's problem*
3. *Representative agent condition is satisfied, i.e.*

$$g\,(z,K,K)=G\,(z,K)$$

5 Lucas Tree Model (Lucas 1978)

5.1 The Model

Suppose there is a tree which produces random amount of fruits every period. We can think of these fruits as dividends and use d_t to denote the stochastic process of fruits production. Further, assume d_t follows Markov process. Formally:

$$d_t\sim\Gamma(d_{t+1}=d_i\mid d_t=d_j)=\Gamma_{ji} \tag{21}$$

Let h_t be the history of realization of shocks, i.e., $h_t=(d_0,\ d_1,\ ...,\ d_t)$. Probability that certain history h_t occurs is $\pi(h_t)$.

Household in the economy consumes the only good, which is fruit. With usual assumption on preference retained, consumers maximize:

$$\sum_t \beta^t \sum_{h_t \in H_t} \pi(h_t) u(c_t) \tag{22}$$

Since we assume representative agent in the economy, and there is no storage technology, in an equilibrium, the representative household eats all the dividends every period. So the lifetime utility of the household will be:

$$\sum_t \beta^t \sum_{h_t \in H_t} \pi(h_t) u(d_t) \tag{23}$$

with

$$p_0 = 1 \tag{24}$$

Note that we are considering the Arrow-Debreu market arrangement, with consumption goods in period 0 as a numeraire.

5.2 First Order Condition

Take first order condition of the above maximization problem:

$$FOC \quad c(h_t) \qquad \frac{p(h_t)}{p_0} = p_t(h_t) = \frac{\beta^t \pi(h_t) u'(c(h_t))}{u'(c(h_0))} \tag{25}$$

By combining this FOC with the following equilibrium condition:

$$c(h_t) = d_t \ \forall t, h_t \tag{26}$$

We get the expression for the price of the state contingent claim in the Arrow-Debreu market arrangement.

$$p_t(h_t) = \frac{\beta^t \pi(h_t) u'(d(h_t))}{u'(d(h_0))} \tag{27}$$

5.3 LT in Sequential Markets

In sequential markets, the household can buy and sell fruits in every period, and the tree (the asset). To consider the trade of the asset, let s_t be share of asset and q_t be the asset price at period t. The budget constraint at every time-event is then:

$$q_t s_{t+1} + c_t = s_t(q_t + d_t) \tag{28}$$

Thus, the consumer's optimization problem turns out to be:

$$\max_{\{c_t(h_t), s_{t+1}(h_t)\}_{t=0}^{\infty}} \sum_t \beta^t \sum_{h_t \in H_t} \pi(h_t) u(c_t(h_t)) \tag{29}$$

subject to

$$q_t(h_t) s_{t+1}(h_t) + c_t(h_t) = s_t(h_{t-1})[q_t(h_t) + d_t] \tag{30}$$

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5.4 LT recursively

Looking at the recursive version of the same problem with denoting discrete state variable as subscripts (a note on notation: this is the same as having $V(d, s)$, but since the amount of fruit is linked one to one to the shock, we can drop d and use the state as a subscript)

$$V_i(s) = \max_{s',c} u(c) + \beta \sum_{d'} \Gamma_{ij} V_j(s')$$

$$s.t. \quad c + s\prime q_i = s[q_i + d_i]$$

In equilibrium, the solution has to be such that $c = d$ and $s^{'} = 1$. Impose these on the FOC and get the prices that induce the agent to choose that particular allocation. Then the FOC for a particular state i would imply,

$$q_i = \beta \sum_j \Gamma_{ij} \frac{u'(d_j)}{u'(d_i)} [q_j + d_j] \tag{31}$$

where

$$q_i = \frac{p(h_{t-1}, d_i)}{p(h_t)}$$

and $p(.)$ are the prices we derived from the AD setting.

A closer look tells us that we can calculate all prices in just one system of equations. Taking FOCs, at an equilibrium we have

$$p_i u_c(d_i) + \beta \sum_j \Gamma_{ij} \frac{\partial V^j(s')}{\partial s'} = 0$$

using the envelope condition

$$\frac{\partial V^j(s')}{\partial s'} = [p_i + d_i] u_c(d_i)$$

hence,

$$p_i u_c(d_i) = \beta \sum_j \Gamma_{ij} [p_j + d_j] u_c(d_j) \qquad \forall i$$

Stacking each equation and forming matrices

$$\begin{bmatrix} u_c(d_1) & 0 & \cdots & 0 \\ 0 & u_c(d_2) & \ddots & \vdots \\ 0 & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & u_c(d_{n_d}) \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_{n_d} \end{bmatrix} = \beta \begin{bmatrix} u_c(d_1) & 0 & \cdots & 0 \\ 0 & u_c(d_2) & \ddots & \vdots \\ 0 & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & u_c(d_{n_d}) \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_{n_d} \end{bmatrix} + \beta\Gamma \begin{bmatrix} u_c(d_1) & 0 & \cdots & 0 \\ 0 & u_c(d_2) & \ddots & \vdots \\ 0 & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & u_c(d_{n_d}) \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{n_d} \end{bmatrix}$$

In matrix notation

$$\begin{aligned} p &= \beta u_c^{-1}\Gamma u_c p + \beta u_c^{-1}\Gamma u_c d \\ \left[I - \beta u_c^{-1}\Gamma u_c\right] p &= \beta u_c^{-1}\Gamma u_c d \\ p &= \left[I - \beta u_c^{-1}\Gamma u_c\right]^{-1} \left[\beta u_c^{-1}\Gamma u_c d\right] \end{aligned}$$

5.5 Pricing an Arbitrary Asset

Because in a complete market any asset can be reproduced by buying and selling contingent claims at every node, we can use this model as a powerful asset pricing formula. For example, take the option of selling shares at price $\bar{p}$ tomorrow. Since tomorrow we'll have the option to sell, we exercise only if

$$\bar{p} - p_i > 0 \qquad \forall i$$

Then, the value of this option (if we are in state i), is

$$\varphi_i(\bar{p}) = \sum_j q_{ij} \max\{\bar{p} - p_j, 0\}$$

where $q_{ij} = \beta\Gamma_{ij} u_c(d_j)\left[u_c(d_i)\right]^{-1}$.

Our next example is the option that can only be executed 2 periods from now. In that case, we have

$$\varphi_i^2(\bar{p}) = \sum_j q_{ij} \sum_l q_{jl} \max\{\bar{p} - p_l, 0\}$$

Finally, take the option that can be exercised tomorrow or the day after tomorrow. The day after tomorrow, we exercise the option iff

$$\bar{p} - p_l > 0$$

where l is the state the day after tomorrow. At the previous node (if we haven't exercised the option yet and the state is j), the value of the option is

$$\sum_l q_{jl} \max\{\bar{p} - p_l, 0\}$$

Hence, if the state today is i, the value of the option is

$$\hat{\varphi}_i^2(\bar{p}) = \sum_j q_{ij} \max\left\{\bar{p} - p_j, \sum_l q_{jl} \max\{\bar{p} - p_l, 0\}\right\}$$

February 23rd

6 Economy with Heterogeneous Agents

6.1 Introduction

So far, in environments we analyzed, the type of agents do not change over time. If the number of type of agents is small (as the example we did in class with only two different types) it's easy to keep track of all the types, and so is to define an equilibrium. From now on, we will consider economies with (i) many agents who are very different among themselves at a given time (crosssection), and (ii) change their types over time.

An immediate question is: what is a considered MANY agents?

- N large, countable
- a continuum, uncountable and infinite

We will use models with continuum of agents, since we can use math tools (such as measure theory) and other useful concepts (like continuity) to describe agents.

6.2 Introduction to Measure Theory

6.2.1 Intuition

Measure theory can be understood nicely by comparing to the notion of weight. Measure is about "measuring" a mass in a mathematically consistent way, which is similar to weighting a mass. Therefore, intuitively the following properties are expected to be satisfied by measures:

1. $measure(nothing) = 0$

2. if $A \cap B = \emptyset \Rightarrow measure(A + B) = measure(A) + measure(B)$

These properties are intuitive with weight. The weight of nothing is zero. If a body is 200 pounds, and you chop off a hand from the body and put the hand and the rest of the body together on the scales, they must weight 200 pounds. Now consider an economy with many agents. The measure of nobody in the economy is zero. If a measure of the total population is normalized to one, and you take away the rich people from the population and measure the sum of rich people and the rest of the population, they must have measure one.

In macro models with heterogenous agents, we are interested in how to measure agents with different characteristics (wealth, earnings, etc.).

6.2.2 Definitions

Definition 8. *For a set S, $\mathcal{S}$ is a set of subsets of S.*

Definition 9. *σ-algebra $\mathcal{S}$ is a set of subsets of S, with the following properties:*

1. $S, \emptyset \in \mathcal{S}$

2. $A \in \mathcal{S} \Rightarrow A^c \in \mathcal{S}$ *(closed in complementarity)*

3. *for* $\{B_i\}_{i=1,2...}$, $B_i \in \mathcal{S} \Rightarrow [\cap_i B_i] \in \mathcal{S}$ *(closed in countable intersections)*

The intuition of the property 2 of σ-algebra is as follows. If we chop off a hand from a body, and if the hand is an element of $\mathcal{S}$, the rest of the body is also an element of $\mathcal{S}$. Soon we will define measure as a function from σ-algebra to a real number Then the property of σ-algebra implies that if we can measure the chopped hand, we can measure also the rest of the body.

Examples of σ-algebra are the follows:

1. Everything, aka the power set (all the possible subsets of a set S)

2. $\{\emptyset, S\}$

3. $\{\emptyset, S, S_{1/2}, S_{2/2}\}$ where $S_{1/2}$ means the lower half of S (imagine S as an closed interval on $\mathcal{R}$).

If $S = [0, 1]$ then the following is NOT a $\sigma-$ algebra

$$\mathcal{S} = \left\{ \emptyset, [0, \frac{1}{2}), \left\{\frac{1}{2}\right\}, \left[\frac{1}{2}, 1\right], S \right\}$$

Remark 2. *A convention is (i) use small letters for elements, (ii) use capital letters for sets, (iii) use "fancy" letters for set of subsets.*

Definition 10. *A measure is a function $x : \mathcal{S} \to \mathcal{R}_+$ such that*

1. $x(\emptyset) = 0$

2. *if* $B_1, B_2 \in \mathcal{S}$ *and* $B_1 \cap B_2 = \emptyset \Rightarrow x(B_1 \cup B_2) = x(B_1) + x(B_2)$

3. *if* $\{B_i\}_{i=1}^{\infty} \in \mathcal{S}$ *and* $B_i \cap B_j = \emptyset$ *for all* $i \neq j \Rightarrow x(\cup_i B_i) = \sum_i x(B_i)$ *(countable additivity)*

In English, countable additivity means that measure of the union of countable disjoint sets is the sum of the measure of these sets.

Definition 11. *Borel-σ-algebra is (roughly) a σ-algebra which is generated by a family of open sets (generated by a topology).*

Since a Borel-σ-algebra contains all the subsets generated by intervals, you can recognize any subset of a set using Borel-σ-algebra. In other words, Borel-σ-algebra corresponds to complete information.

You might find that a σ-algebra is similar to a topology. Topology is also a set of subsets, but its elements are open intervals and it does not satisfy closedness in complementarity (complement of an element is not an element of a topology). Very roughly, the difference implies that topologies are useful in dealing with continuity and σ-algebra is useful in dealing with measure.

Definition 12. *Probability (measure) is a measure such that $x(A) = 1$*

Lets apply these basic notions of measure theory to a simple set of problems: industry equilibria with many firms

7 Partial Equilibrium Industry Theory

We will consider models were prices are given exogenously, hence our analysis is a partial equilibrium one. For that reason, think that the environment is a small industry producing "flip flops" (things nobody cares about).

There is an inverse demand function $y^d(p)$, where p is the price of the good. A firm in this industry is indexed by it's productivity $s \in S = [\underline{s}, \bar{s}]$ and produces according to $sf(n)$ (the production function depends on labor only). Firms are competitive in the output market as well as in the labor market.

Problem of the firm is

$$\max_n psf(n) - wn$$

from the FOC, we get $psf'(n) = w$, which implicitly defines the solution of the firm $n^* = n(s,p)$. The profits are defined

$$\Pi(s,p) = psf(n^*[s,p]) - wn[s,p]$$

Given a price p, to calculate the output of the industry we need a measure X of firms (the distribution of firms according to their s).

Let

- $\mathcal{S}$ be the borel σ-algebra of $[\underline{s}, \bar{s}]$
- $X : \mathcal{S} \to \mathbb{R}$ be a measure

Then, the supply of the industry is defined as

$$y^s(p) = \int_{\underline{s}}^{\bar{s}} sf(n[s,p])\, X(ds)$$

Definition 13. *An Industry equilibrium is a measure function $X^* : \mathcal{S} \to \mathbb{R}$ and a price p^* such that*

$$y^d(p^*) = \int_{\underline{s}}^{\bar{s}} sf(n[s,p^*])\, X^*(ds)$$

8 February 28, 2006

Now suppose that the firm will only operate next period with probability $(1-\delta)$. With probability δ it will die. In that case, the two period profit of the firm is,

$$\pi_2 = [p^* sf(n^*) - wn^*]\left[1 + \frac{1-\delta}{1+r}\right]$$

Now consider the infinite periods profit of the firm,

$$\begin{aligned}
\pi_\infty &= [p^* sf(n^*) - wn^*] \sum_{t=0}^{\infty} \left(\frac{1-\delta}{1+r}\right)^t \\
&= [sf(n^*) - wn^*] \left(\frac{1+r}{r+\delta}\right)
\end{aligned}$$

The zero profit condition is that the profit from entry is equal to the cost of entry, denoted by c_e. This condition says that there are no further incentives to enter the industry:

$$c_e = \pi_\infty$$

Define $x : \mathcal{S} \to R$ as the measure of firms, where $\mathcal{S}$ is the $\sigma - algebra$ defined on the set S

An industry equilibrium is a set $\{p^*, y^*, n^*, x^*(s)\}$, such that:

1) $p^* = p(y^*)$ (demand is satisfied)

2) $y^* = x^*(s, p^*)\ \ s\ f(n^*)$ (feasibility)
3) Firms optimize: $n^* \in \arg\max_n \ \ p^* s f(n) - w^* n$

4) Zero profit condition: $c_e = \pi_\infty$

March 2, 2006

Industry equilibrium (continued)

TWEAK #1
Suppose that each firm has to pay a cost of entry c_∞, and the productivity shock is drawn from the distribution $\gamma(s)$. Once the firm draws s it keeps it forever.

An industry equilibrium is a set $\{p^*, n^*(s, p^*), N_e^*, x^*\}$, such that:

1) $n^*(s, p^*) \in \arg\max\ [p^* s f(n) - wn]$
2) $y^D(p^*) = \int\limits_S\ s\ f(n^*(s, p^*))\ dx^*$
3) $c_\infty = \int\limits_S \hat{\Pi}_\infty(s, p^*) d\gamma(s)$, where $\hat{\Pi}_\infty = \sum_{t=0}^{\infty} \left(\frac{1-\delta}{1+r}\right)^t \Pi(s, p^*)$.
4) $x^*(B) = (1-\delta)\, x^*(B) + N_e^* \gamma(B)$, for all $B \in \mathcal{S}$, and
5) $N_e^* = \delta x^*(S)$.

Note that here, the distribution of firms completely reflects the distribution from which they draw their productivity shocks, $\gamma(s)$. This is because what types of firms remain or what types of firms exit is not an issue since there is exogenous entry and exit. For example, if exit was endogenous we would expect the 'bad' firms to exit and the better ones to stay, and therefore the type distribution of incumbent firms would be different than the initial distribution $\gamma(s)$. But in our case, the distribution of incumbents and the initial type distribution are identical.

So this model is not interesting because it has no economics.

TWEAK #2 (Changing productivity)

Here s is drawn from $\gamma(s)$ as before, but after the initial shock is obtained, $s' \sim \Gamma_{ss'}$. We will assume that Γ satisfies First Order Stochastic Dominance. This means that

$$For\ s_1, s_2 \in S,\ \ s_1 < s_2 \Rightarrow \int_{\tilde{s}}^{\bar{s}} \Gamma(s_1, s)\, ds \leq \int_{\tilde{s}}^{\bar{s}} \Gamma(s_2, s)\, ds$$

Moreover, we introduce a fixed cost that the firm has to pay in every period, c_f .

The recursive formulation of the firm's problem is:

$$\Omega(s) = \max\left[0, -c_f + \max_n\{psf(n) - wn\} + \frac{1}{1+r}\int \Omega(s')\,\Gamma(ds' \mid s)\right]$$

March 7, 2006

In the last lecture we introduced a second Tweak of the Industry equilibrium model, where the firm draws a productivity s from $\gamma(s)$ as before, but after the initial shock is obtained, $s' \sim \Gamma_{ss'}$. We assumed that Γ satisfies First Order Stochastic Dominance.

In order to analyze this version of the model we will need a few more mathematical tools. Hence we start with a small introduction about measurability and transition functions.

Definition 14. *A function $f : S \rightarrow R$ is measurable with respect to the $\sigma - A\lg ebra$ $\mathcal{S}$, if for every $a \in R$, $B \equiv \{s \in S : \ f(s) \leq a\} \in \mathcal{S}$.*

Definition 15. *A transition function is a mapping $Q : S \times \mathcal{S} \rightarrow [0, 1]$, such that:*

1) $Q(s, .)$ is a probability measure for every $s \in S$.
2) $Q(., B)$ is a measurable function with respect to $\mathcal{S}$, for every $B \in \mathcal{S}$.

Based on the transition function Q we can define an updating operator T. This new object satisfies $x'(B) = T(x(B), Q)$. In words, T gives us the measure of (here) firms in the subset B in the next period $(x(B))$, based on the transition function and the measure of firms in this subset today.

In one of this week's homework you had to show that there exists an s^* such that if the firm's shock is $s \geq s^*$ it stays in the market. If, on the other hand, $s < s^*$ the firm leaves the market. Based on this result we can now define stationary equilibrium for the "Tweak 2" model as follows.

Definition 16. *A stationary equilibrium for the economy described by Tweak 2 is a list* $\{p^*, N_e^*, s^*, n^*(s, p^*), x^*, Q^*\}$ *such that:*

1) $n^*(s, p^*)$ maximizes profits and $s^* = s^*(p^*)$ (threshold is optimal).
2) Free entry is satisfied: $c_\infty = \Pi_\infty$.
3) Market clearing: $y^D(p^*) = \int_S s\ f(n^*(s, p^*))\, dx^*$.
4) The measure of firms is stationary: $\mathrm{x}^*(B) = \int_S Q(s, B)\ \ dx^* + N_e^* \gamma(B \cap [s^*, \bar{s}])$
5) $Q(s, B) = \Gamma\left[s, B \mid [\underline{s}, s^*)\right]$, (where Γ is implied by the Markovian transition matrix).

March 9, 2006

TWEAK 3

Employment protection with hiring/firing Costs

Note that in this problem the labor force of the last period is a state variable for the firm. This means that the state space will also be different. The new state space is given by $X = S \times N$. N is the set of the possible values of labor force. For convenience assume that it is bounded, i.e, $N = \left[0, \bar{N}\right]$, where $\bar{N} < \infty$.

Assuming that there is a cost of firing equal to a per worker, the profit function is given by

$$\Pi\left(s, n^{-1}\right) =$$
$$\max_n \left[p\ s\ f(n) - w\ n - c_f - \ \phi[n - (1-\delta)\, n^{-1}] \ \ + \tfrac{1}{1+\ r}\ \int_S \Pi(s', n)\, \Gamma(ds', s)\right],$$

where δ is the quitting rate of workers, $\phi(0) = 0,\ \phi' > 0,\ \phi'' > 0$, and for every x, $\phi(x) = 0$ (so here there is no firing cost).

We obtain the following First-order condition:

$$p\ s\ f'(n) - w\ \ - \phi'[n - (1-\delta)\, n^{-1}]\ \ + \tfrac{1}{1+\ r}\ \int_S \Pi_n\left(s', n\right) \Gamma\left(ds', s\right) = 0.$$

Moreover, the envelope condition is:

$$\Pi_{n^{-1}}\left(s, n^{-1}\right) = (1-\delta)\, \phi'[n - (1-\delta)\, n^{-1}].$$

Combining FOC and EC we get:

$$p\ s\ f'(n) - w\ - \phi'[n - (1-\delta)\, n^{-1}]\ \ - \tfrac{1-\delta}{1+\ r} \int_S' \phi'[n^{+1} - (1-\delta)\, n] \Gamma\left(ds', s\right).\ = 0,$$

which is a second order difference equation in n. The solution yields a cutoff point, $s^*\left(n^{-1}\right)$ as well as the optimal hiring solution, $n\left(s, n^{-1}\right)$. IN order to be sure that our state space is compact we assume that for every period $n \leq\ \bar{n}$. We can assume that $\bar{n}$ is very large, and come back at the end to verify that indeed $n \leq\ \bar{n}$ holds at every period.

The construction of the updating operator here is a bit trickier. Let $\mathcal{S}_N$ be the Borel set for our state space. Also let $Q\left(s, n^{-1}, B\right)$ be the probability that a firm with current state s, n^{-1} is in B tomorrow. Then

$Q\left(s, n^{-1}, B\right) = \Gamma\left(s, B_S \cap [s^*\left(n^{-1}\right), \bar{s}]\right) \{n\left(s, n^{-1}\right) \in B_N\}$, where $\{.\}$ is the indicator function.

Now $\mathrm{x}'(B) = \displaystyle\int_{S \times N} Q\left(s, n^{-1}, B\right)\ dx^* + N_e^* \gamma\left(B \cap [s^*(0), \bar{s}]\right)$.

A complete definition of equilibrium was left for you to show as a homework.

From firms we go back to consumers. Now the agents will be heterogeneous. Our goal is to define stationary equilibrium in an environment where each agent gets an idiosyncratic shock, and there is a certain storage technology.

March 14, 2006

9 HUGGETT Economy

Imagine a Archipelago that has a continuum of islands (instead of the pig farmers we used in class). There is a fisherman on each island. The fishermen get an endowment e each period which follows a Markov process with transition $\Gamma_{ee'}$ and,

s$\in \{e^1, ..., e^{n_e}\}$

The fishermen cannot swim. There is a storage technology such that, if the fishermen store q units of fish today, they get 1 unit of fish tomorrow.

(e,a) is the type of a fisherman and the set consisting of all possible such pairs is,

$$E \times A = \{e^1, e^2, ..., e^n\} \times [0, \overline{a}]$$

Let $\mathcal{A}$ be the set of Borel sets on SxA. And define a probability measure x on $\mathcal{A}$,

$$x : \mathcal{A} \to [0, 1]$$

The fisherman's problem is:

$$V(e, a) = \max_{c, a' \geq 0} u(c) + \beta \sum_{e'} \Gamma_{ee'} V(e', a')$$

subject to

$$c + qa\prime = e + a$$

$$c \succeq 0 \text{ and a' } \in [0, \overline{a}]$$

With the decision rule $a' = g(e, a)$ and the transition matrix for the endowment process $\Gamma_{ee'}$

The First Order Conditions are,

$$u_c(e + a - qa\prime) = \frac{\beta}{q} \sum_{s'} \Gamma_{ee'} u_c(e\prime + a\prime - q'a'')$$

You'll notice that a$'$ $\in [0, \overline{a}]$ is already one of the constraints of the above maximization problem. But now rather than just imposing such a constraint, we will find a natural reason that savings should have a lower bound and we will consider a condition that ensures an upper bound for savings.

For the lower bound, we assume that there is no technology which allows negative amount of saving and this sounds natural since storing a negative amount of fish does not make much sense. So savings has a lower bound because Mother Nature says so.

Here, the fisherman has the risk of getting a very bad shock tomorrow. So the fisherman would save just in case he has this bad shock; he would want to store some fish today in order to insure himself against getting very small number of fish tomorrow so he is not hungry in case that happens. In this case we need to think more about how to put an upper bound on savings, because with uncertainty even if $\beta < q$, the fisherman is willing to save due to gains from insurance. The kind of savings to protect oneself from risk in the future in the absence of state contingent commodity markets which can be used to insure against any contingency to make sure consumption is constant across states, which is usually called precautionary savings. In order to ensure an upper bound for savings, we need to bound the gains from insurance somehow. The way to do this is to impose the condition on the utility function that its negative curvature (keeping in mind that the utility function is concave) is diminishing as wealth increases. This means that wealthier agents are less risk-averse. Formally, that u' is convex. The wealthier the agent is, the smaller the variance of his endowment next period proportional to his wealth so he doesn't want to save if he is very wealthy. This is simply because of the fact that the wealth is not subject to any uncertainty but income is thus as the income wealth ratio rise, the overall uncertainty the agent faces diminishes.

So in the economy with uncertainty, in order to have an upper bound on savings, we need the first derivative of the utility function to be convex so that the following Jensen's Inequality holds:

$$\frac{\beta}{q}\int \Gamma_{ss'}u_c(c') > \frac{\beta}{q}u_c(\int \Gamma_{ss'}c')$$

Theorem 5. *If $\beta < q$ and u′is convex then $\exists \overline{a}$ such that $a_0 < \overline{a}$, g(s,a)< $\overline{a}$ $\forall s$.*

Now consider the case of lower bound. Suppose we let the fisherman borrow and lend to each other but not store any fish, how can we make sure that our agents always has the capability to pay back what they owe. What would be the endogenous lower bound to ensure this? Such a condition would make sure that in the worst case scenario our agent should be able pay the interest rate on its debt and roll over the same amount (the lowest possible amount). Thus, letting the lower bound be $\underline{a}$ and the lowest possible shock be $\underline{e}$ then,

$$\begin{aligned} 0 + q\underline{a} &= \underline{a} + \underline{e} \\ \underline{a} &= \frac{\underline{e}}{q-1} \end{aligned}$$

This is called the solvency constraint rather than a borrowing constraint. Note that when we let the fishermen to get into lending contracts with each other, we need a consistency condition to make sure agents actions are compatible with each other. Here the price will be endogenously determined will ensure the agents hold just the right amount of assets.

The stationary equilibrium of such an economy is defined as,

Definition 17. *A stationary equilibrium for an Huggett(1993) economy is a set $\{q^*,\ x^*(q^*),\ Q(e,a,B;q^*),\ g(e,a;q^*)\}$ such that*

1. *(Agent Optimization) Given q^*, $g(e,a;q^*)$ solves the agent's problem.*
2. *(Consistency) $Q(e,a,B;q^*)$ is a transition matrix associated with $\Gamma_{ee'}$ and $g(e,a;q^*)$.*

3. *(Stationarity) x^* is the unique stationary distribution associated with $Q(e, a, B; q^*)$, that is $x^* = T(x^*, Q)$.*

4. *(Market clear)*

$$\int a dx^*(q^*) = 0$$

March 16, 2006

In the beginning of this lecture we studied a model with heterogeneous agents that are either employed or unemployed. This problem is given by

$$V(s, a) = \max_{c \geq 0, a'} \left[u(c) + \beta \sum_{s'} \Gamma_{s\ s'} V(a', s') \right]$$
$$\text{s.t: } c + qa' = s + a$$

In PS 8 you had to show some important properties of the decision rule. In particular we saw that $g(e, \bar{a}) = \bar{a}$, and that $g(e, a)$ never crosses the 45 degree line again (here $s = e$ stands for the state of employment, i.e. the good state).

Next we finished the description of a Huggett economy (In these notes this has already been done in Lecture 16).

Finally we incorporated growth in the Huggett economy setting. Here we still have heterogeneity and each agent obtains an idiosyncratic shock, s. We can think of this as a shock on the efficiency of labor provided by that particular agent. The difference is that now the agent can work and also rent his capital (a) to firms at the market price R. The new definition is as follows.

Definition 18. *A stationary equilibrium for the Huggett economy with growth is a list $\{x^*, g^*(s, a), R^*, w^*\}$ such that*

1) x^* is the unique stationary distribution associated with $Q(e, a, B)$, that is $x^* = T(x^*, Q)$ (nothing new),
2) $g^*(s, a)$ solves the agent's problem,
3) $F_1\left[\int a dx^*, \int s dx^*\right] = R^*$, where $\int a dx^*$ is total capital and $\int s dx^*$ is total efficient units of labor, and
4) $F_2\left[\int a dx^*, \int s dx^*\right] = w^*$.

March 23, 2006

9.1 Technology Improvements in the Fishermen Economy (Transitions-simple)

In the model we were studying, we get a stationary distribution X^* for the assets of all fishermen. This equilibrium object clearly depends on the 'fundamentals' of the economy, namely $\{\beta, \Gamma, q, u(c)\}$. In general, the problem is defined as

$$\begin{aligned} V(s,a;q) &= \max_{a'} \left\{ u[c] + \beta \sum_{s'} \Gamma_{ss'} V(s', a'; q) \right\} \\ & \text{subject to} \\ c + qa' &= s + a \end{aligned}$$

with solution $y(s,a;q)$

Note that we are showing explicitly the importance of q in the value functions. Now imagine that the economy is in a steady state with some q^0. We would like to know what happens if unexpectedly the cost of storage decreases permanently from q^0 to q^1. In a first step, we can solve the above problem with this new $q^1 < q^0$ and get a new policy function $y(s,a;q^1)$. Applying the updating operator for this new economy

$$T\left[X^*\left(q^0\right), y\left(s,a;q^1\right)\right] \neq X^*\left(q^0\right)$$

Clearly, since a fundamental of the economy has changed, the distribution of the fishermen will move away from $X^*(q^o)$ into a new stationary distribution. This will be true if Γ satisfies the 'AD-AN' conditions. Then, we know for sure that

$$\lim_{n\to\infty} T^n\left[X\left(q^0\right), y(.)\right] = X^*\left(q^1\right)$$

How do we understand transitions for this economy? Some economist forget about transitions and just compare steady states, which is not correct if we are interested in welfare. The correct way to proceed is to guess some N (number of periods necessary to achieve the new stationary equilibrium) and then compare the sequence of distributions $\left\{T^n\left[X\left(q^0\right), y(.)\right]\right\}_{n=1}^N$ to $X^*\left(q^0\right)$. Note that we can do this because we can calculate the decision rules independently from $X^*(.)$.

9.2 Welfare Questions

How much would this society be willing to pay for a reduction of q^0 to q^1? We can answer this for a particular household and for the whole society. For a particular household, we have three different values (call them $\mathcal{V}$)

- Utility gain from the switch

$$\mathcal{V}_1 = V\left(s, a; q^1\right) - V\left(s, a; q^0\right)$$

  Note that this is a calculation that we can perform for particular households, defined by particular (s, a).

- Consumption/assets willing to be sacrificed (for each household) in order to have q^1

$$\mathcal{V}_2 = V\left(s, a - z; q^1\right) - V\left(s, a; q^0\right)$$

  where z represents the amount of consumption/assets that the household would sacrifice in order to get the better technology. Note that we don't make the distinction between consumption and assets, since in this economy we have only one good. Also, $\mathcal{V}_1$ is not defined for households with $a = 0$ i.e., no assets. One way of avoiding this problem is to use the next calculation

- Consumption/assets needed to compensate the household if it stays with the old technology

$$\mathcal{V}_3 = V\left(s, a; q^1\right) - V\left(s, a + z; q^0\right)$$

Now, for the whole society, the gains from technical improvements are given by the sum of the individual gains. The formula is given by

$$\mathcal{V}_{soc} = \int z\left(s, a\right) dX^*\left(q^0\right)$$

where $z\left(s, a\right)$ is the one calculated from $\mathcal{V}_3$, for each household (pairs of s, a) and the integration is with respect to $X^*\left(q^0\right)$, the distribution of fishermen at the beginning.

9.3 Technology Improvements in the Fishermen Economy (Transitions-difficult)

Suppose now that instead of storage technology, we have an aggregate technology of the form

$$Y = A_0 K^{\theta} N^{1-\theta}$$

where $K = \int a dX^*$ and $L = \int s dX^*$. In this case, if the economy receives an improvement in total factor productivity (i.e., $A_1 > A_0$), answering the welfare questions becomes much harder. Why? simply, because now the measure matters. Recall that in the previous example, the measure of fishermen was exogenous to the calculation of the policy functions. Now, given the new value A_1, we will be moving away from a steady state, which means that the interest rate ($F_K = \partial Y/\partial K$) and the wage ($F_N = \partial Y/\partial N$) are not constant. Hence, we need the entire sequences of prices to get the policy function. The problem is that to get prices, we need the whole distribution (measure) of agents in the economy, which makes this a non-trivial problem. The problem is still solvable with computer-intensive methods and comparing steady states is still wrong.

9.4 Growth

In our analysis so far, we have used Neo-classical Growth Model as our benchmark model and built on it for the analysis of more interesting economic questions. One peculiar characteristic of our benchmark model, unlike its name suggested, was lack of growth. Many interesting questions in economics are related to the cross-country differences of growth rates. We will cover some models that will allow for growth so that we will be able to attempt to answer such questions.

9.4.1 Exogenous growth

What does it take for an economy to grow? Before answering that question, we know in our standard NGM there is basically two ways of growth, one in which everything grows, which is not necessarily a per-capita growth, and the other is per-capita growth. We will be focusing on per-capita growth, hence, the next definition is useful

Definition 19 (Balanced Growth Path)**.** *is a situation where all variables of a model grow at a constant rate (not necessarily at the same rate)*

The title exogenous growth refers to the structure of models in which growth rate is determined exogenously, and is not an outcome of the model. First and the simplest one of these is one in which the determinant of the growth rate is population growth.

March 24, 2006

9.5 Growth with population

Suppose the population of our economy grows at rate γ and we have the classical CRTS technology in capital and labor inputs.

$$\begin{aligned} Y_t &= AF(K_t, N_t) \\ N_t &= N_0 * \gamma^t \end{aligned} \tag{32}$$

Note that our economy is no longer stationary but as we will see, within the exogenous growth framework we can make these economies look like stationary ones by re-normalizing the variables. Thus, at the end of the day it will only be a mathematical twist on our standard growth model. Once we do that, we will be looking for the counterpart of a steady state that we have in our stationary economies, the Balanced Growth Path, in which all the variables will be growing at constant rates but not necessarily equal. Back to our population growth model, we know

$$AF(K, N) = A[KF_k(K, N) + NF_N(K, N)] \tag{33}$$

Question is, if N is growing at rate γ, can this economy have a balanced growth path. Can we construct one? We know by CRTS property F_K and F_N are homogenous of degree zero. If we assume capital stock grows at rate γ as well, then prices stay constant and per-capita variables are constant and output grow at the same rate. So we get growth on a balanced growth path without per-capita growth. One question is how do we model population growth in our representative agent model. One way is to assume there is a constant proportion of immigration to our economy from outside but this has to assume the immigrants are identical to our existing agents in our economy, which is a bid problematic. The other way could be to assume growing dynasties which preserves the representative agent nature of our economy. If we do so, the problem of the social planner becomes,

$$\begin{aligned} & \max \sum_{t=0}^{\infty} \beta^t N_t U(\frac{C_t}{N_t}) \\ st \quad C_t + K_{t+1} &= AF(K_t, N_t) + (1-\delta)K_t \end{aligned} \tag{34}$$

To transform the budget set to per capita terms, divide all terms by N_t and to make the environment stationary by dividing all the variables by γ^t and assume $N_0 = 1$, we get,

$$\begin{aligned} & \max \sum_{t=0}^{\infty} (\beta\gamma)^t U(\widehat{c}_t) && (35) \\ st \quad \widehat{c}_t + \gamma \widehat{k}_{t+1} &= AF(\widehat{k}_t, 1) + (1-\delta)\widehat{k}_t \end{aligned}$$

So how is this transformed model any different than our NGM? By the discount factor, the agents in this economy with growth discounts the future less but everything else is identical to NGM of course with the exception of this economy growing at a constant rate.

Now suppose we have a 'labor augmenting' productivity growth with constant population normalized to one, i.e. have the following CRTS technology,

$$Y_t = AF(K_t, \gamma^t N_t) \tag{36}$$

$$AF(K_t, \gamma^t N_t) = A[K_t F_k(K_t, \gamma^t N_0) + \gamma^t N_0 F_N(K_t, \gamma^t N_0)] \tag{37}$$

Can we have an BGP? The problem is,

$$\begin{aligned} & \max \sum_{t=0}^{\infty} \beta^t U(C_t) && (38) \\ st \quad C_t + K_{t+1} &= AF(K_t, \gamma^t N_0) + (1-\delta)K_t \end{aligned}$$

and since we have a population of one, these variables are already per-capita terms. For stationarity, we have to normalize the variables to 'per productivity' units, by dividing all by γ^t. Then the problem becomes,

$$\begin{aligned} & \max \sum_{t=0}^{\infty} \beta^t U(\gamma^t \widehat{c}_t) && (39) \\ st \quad \widehat{c}_t + \gamma \widehat{k}_{t+1} &= AF(\widehat{k}_t, 1) + (1-\delta)\widehat{k}_t \end{aligned}$$

Suppose we have a CRRA preferences, then the question is how can we represent the preferences as a function of $\widehat{c}_t$ only. Writing the CRRA,

$$\sum_{t=0}^{\infty} \beta^t \frac{(\gamma^t \widehat{c}_t)^{1-\sigma} - 1}{1-\sigma} = \sum_{t=0}^{\infty} (\beta(\gamma^{1-\sigma}))^t \frac{\widehat{c}_t^{\,1-\sigma} - 1}{1-\sigma} \tag{40}$$

and the problem becomes,

$$\begin{aligned} & \max \sum_{t=0}^{\infty} (\beta(\gamma^{1-\sigma}))^t \frac{\widehat{c}_t^{\,1-\sigma} - 1}{1-\sigma} && (41) \\ st \quad \widehat{c}_t + \gamma \widehat{k}_{t+1} &= AF(\widehat{k}_t, 1) + (1-\delta)\widehat{k}_t \end{aligned}$$

and once again it is exact same problem as the NGM with a different discount factor. Note that the existence of a solution to this problem depends on $\beta(\gamma^{1-\sigma})$.In this set-up we get per-capita growth at rate γ. Also note that CRRA is the only functional form for preferences that is compatible with BGP. This is because as per-capita output grows, for consumption to grow at a constant rate, our agent has to face the same trade-off at each period.

Now suppose we have the TFP growing at rate γ with a CRTS Cobb-Douglas technology

$$\begin{aligned} Y_t &= A_t F(k_t, 1) \\ \frac{A_{t+1}}{A_t} &= \gamma \end{aligned}$$

What would be the growth rate of this economy? We can show that like the previous cases the growth rate of the economy is the growth rate for the productivity of labor, which is $\gamma^{\frac{1}{1-\alpha}}$ in this case.

9.6 Endogenous Growth

So far in the models we covered growth rate has been determined exogenously. Next we will look to models in which the growth rate is chosen by the model itself. We do know for a fixed amount of labor, the curvature of our technology limits the growth due to diminishing marginal return on capital and with depreciation there is an upper limit on (physical) capital accumulation. So if our economy is to experience sustainable growth for a long period of time, we either give up the curvature assumption on our technology or we have to be able to shift our production function up. Given a fixed amount of labor, this shift is possible either by an increasing TFP parameter or increasing labor productivity, . The simplest of such models where we can see that is the AK model, where the technology is linear in capital stock so that diminishing marginal return on capital does not set in.

9.7 AK Model

We have the usual social planner's problem with linear technology and full depreciation,

$$\max \sum_{t=0}^{\infty} \beta^t U(C_t)$$

such that

$$C_t + K_{t+1} = AK_t$$

and the FOCs

$$(c_t) \quad : \quad \beta^t U_c(.) = \lambda_t \tag{42}$$
$$(k_{t+1}) \quad : \quad \lambda_t = \lambda_{t+1} \tag{43}$$

together implies the Euler equation,

$$U_c(c_t) = A\beta U_c(c_{t+1}) \tag{44}$$

and on the BGP with consumption growing at rate γ with CRRA utility we get,

$$\gamma = (A\beta)^{1/\sigma} \tag{45}$$

and the growth rate is determined by the model parameters endogenously. Note that capital also grows at rate γ and the fate of the economy is determined by the fundamentals of the model. The capital stock will diverge to infinity if $(A\beta)^{1/\sigma} > 1$ or the economy is destined to vanish if $(A\beta)^{1/\sigma} < 1$. Also note that there is no transitional dynamics in this model (we loose the state variables in the euler equation after substituting for the balanced growth rate relation) and conditional on γ,asymptotically all economies are same regardless of the initial capital level. If we de-centralize this economy we know wages will be zero since labor has no use and gross rental rate of capital will be fixed at A. This is at odds with what we observe in the real world. We would rather like to have a model that allows for both transitional dynamics, labor and growth at the same time. Allowing for labor implies that we need a variable that proxies the increasing productivity of labor endogenously and be reproducible in terms of output, such that we are able to shift our production function continually in the output-capital space without hitting a natural bound.

9.8 Human Capital and Growth

Another way of getting our models to grow "endogenously", is by introducing the variable 'Human Capital' as an input of production. This will proxy continuous and endogenous increasing labor efficiency. We have two ways of modeling human capital:

- one way is to see it very much like physical capital, in the sense output has to be invested to increase the existing stock of human capital. That is the Lucas' approach, in which you can think of investing in education by building more schools as a way to increase the existing human capital stock.
- The alternative way would be to reserve a part of the leisure time for increasing the human capital stock, which can be thought of studying

harder to get better in a fraction of the leisure time. Unfortunately, the second approach puts limit on the rate human capital can grow and might fail to generate sustainable endogenous growth. Next, we look at the Lucas' human capital model.

Lucas' Human Capital Model We have an Cobb-Douglas technology with CRTS and human capital (H) as an input of production instead of labor and the laws of motion for the inputs,

$$\begin{aligned} F(H,K) &= AK^{\alpha}H^{1-\alpha} && (46) \\ K' &= i_k + (1-\delta_k)K && (47) \\ H' &= i_h + (1-\delta_h)H && (48) \end{aligned}$$

Now that there is no limit to the accumulation of human capital and sustainable growth on a BGP is feasible. Furthermore, an analysis of the characterization of the balanced growth path will indicate that this model indeed has transitional dynamics, so unlike the AK model if economy starts out of this optimal growth path economy can adjust and converge to it by responding to prices in a de-centralized setting. If we model the law of motion for human capital as,

$$H' = (1-N) + (1-\delta_h)H \tag{49}$$

where $(1-N)$ is the time devoted to accumulating human capital, say by studying harder, we see there is a natural limit to the growth of human capital and such an economy might not have a BGP. The key ingredient of endogenous growth with labor is then the reproducibility of the human capital without such a limit.

Growth through Externalities (Romer)

We have seen in the AK model that the growth rate is endogenous and determined solely by model primitives. Still, it is not directly or indirectly determined by the agents' choices in our model. In Lucas' human capital model, the growth rate is determined by the choice of agents, specifically by the optimal ratio of human and physical capital. The source of growth in Lucas' model is reproducibility of human capital. In this next model, Romer introduces the notion of externality generated by the aggregate capital stock to go through the problem of diminishing marginal returns to aggregate capital. In this model, the source of growth will be the aggregate capital accumulation, which is possible with a linear aggregate technology in capital as we saw in the AK model. The firms in our model will not be aware of this externality and will have the usual CRTS technology and observe the source of growth coming from the TFP parameter. As usual with externalities, the equilibrium outcome will not be optimal. Each firm has the following technology,

$$y_t = AK_t^{1-\alpha}k_t^{\alpha}n_t^{1-\alpha} \tag{50}$$

but since the firms are not aware of the positive externality they are facing they are solving the problem with the following technology.

$$\begin{aligned} y_t &= \overline{A}_t k_t^{\alpha} n_t^{1-\alpha} && (51) \\ & && (52) \end{aligned}$$

where

$$\overline{A}_t = A_t K_t^{1-\alpha}$$

We can see that the social planner in fact is solving an AK model in per-capita terms. So does the de-centralized version of this economy have a BGP and if it does, how would it look like? Assuming CRRA preferences without leisure we can derive the BGP condition and pin down the growth rate from the euler equation of a typical household,

$$1 = \beta\gamma^{-\sigma}(1+r) \tag{53}$$

where $\gamma = \dfrac{c_{t+1}}{c_t}$ is the growth rate at the balanced path as usual and r=MP$_k$. So to find out the marginal product of capital for the firm we differentiate the technology w.r.t. k_t,

$$1 + r_t = \alpha A K_t^{1-\alpha}k_t^{\alpha-1}n_t^{1-\alpha} + (1-\delta) \tag{54}$$

and since the prices are determined by aggregate state variables $K_t = k_t$ gives,

$$A\alpha - \delta = r \tag{55}$$

and substituting this into the euler equation we get the growth rate of consumption.

$$[(A\alpha - \delta + 1)\beta]^{\frac{1}{\sigma}} = \gamma \tag{56}$$

Solving the AK problem the SP faces we can verify the optimal growth rate for consumption is,

$$[(A - \delta + 1)\beta]^{\frac{1}{\sigma}} = \gamma^{sp}. \tag{57}$$

The important properties of the decentralized model are,

1. It is sub-optimal due to firms' unawareness of the externality they are facing and thus have lower growth rate.

2. Once again, the rental rate does not depend on the capital stock (due to the aggregate linear technology, the states variables drop out from the euler equation) and there is no transitional dynamics generated by the model.

To sum up what we have done so far, we have started with models that had exogenous growth and saw that we can make these models look and behave like our NGM after appropriate transformation. Then we went on to look at models that generate growth endogenously and saw that a prerequisite for growth in these models is linearity of the technology in reproducible factors. We looked at the simple AK model, where the technology is linear in capital stock and analyzed the BGP of such an economy. Then we looked at Lucas' human capital model, in which we had two forms of capital, human and physical, both of which are reproducible in terms of output. Then we analyzed the model by Romer, which again has linearity in the reproducible factor at the aggregate level (capital stock), but firms were facing the CRTS technology with diminishing marginal return on capital and not aware of the positive externality they face. Next we will see another model by Romer with monopolistic competition and a R&D sector which can generate endogenous growth.

March 28 & March 30, 2006

9.9 Monopolistic Competition, Endogenous Growth and R&D

Romer's monopolistic competition model has three production sectors, the final goods production, intermediate goods production and R&D i.e. variety production. Our usual TFP parameter in production function will represent the 'variety' in production inputs and as we will see, the growth of varieties through research and development firms will make sure a balanced growth path is sustainable. The production function in this economy is,

$$Y_t = L_{1t}^{\alpha} \int_0^{A_t} x_t(i)^{1-\alpha} di \tag{58}$$

where $x_t(i)$ is the type i intermediate good and there is a measure A_t of different intermediate goods and L_{1t} is the amount of labor allocated to the final good production. The production function exhibits CRTS. The intermediate goods are produced with the following linear technology,

$$\int_0^{A_t} \eta x_t(i) di = K_t \tag{59}$$

Now suppose the variety of goods grows at rate γ,that is $A_{t+1} = \gamma A_t$. Is long run sustainable growth possible? The answer to this question will depend whether our final goods production technology is linear in growing terms. We do know that by the curvature of the technology, optimality implies equal amount of each variety will be used in production, $x_t(i) = x_t$,then we have,

$$A_t \eta x_t = K_t \tag{60}$$

and our output at this equal variety becomes,

$$Y_t = L_{1t}^{\alpha} A_t {x_t}^{1-\alpha} \tag{61}$$

then substituting for x_t we have,

$$Y_t = \frac{L_{1t}^{\alpha}}{\eta^{1-\alpha}} A_t^{\alpha} K_t^{1-\alpha} \tag{62}$$

thus if both A_t and K_t are growing at rate γ, then production function is linear in growing terms and long run balanced growth is feasible. Note that this model becomes very similar to our previous exogenous labor productivity growth under these assumptions. The purpose of this model is to determine γ endogenously. *What will be the source of growth, where does γ come from?* As we will see, there will be incentives for R&D firms to produce new 'varieties' because there will be a demand for them. These new varieties will be patented to intermediate good production firms, where a patent will mean exclusive rights to produce that intermediate good. So we will have monopolistic competition in the intermediate goods production. Now suppose the law of motion for 'varieties', which is the technology in R&D sector, is given by

$$A_{t+1} = (1 + L_{2t}\zeta) A_t \tag{63}$$

where L_{2t} is the labor employed in R&D sector. Note that this is not a regular law of motion in the sense that every new variety produced helps the production of further new varieties. Hence, there is a positive externality to variety production. Also, assume leisure is not valued and we have an aggregate feasibility condition for labor

$$L_{2t} + L_{1t} = 1 \tag{64}$$

As a homework, we have calculated the BG rate of SP version of this economy, now we will de-centralize this economy and characterize the equilibrium growth rate (it is sub-optimal). The period t problem of a firm in the competitive final good production sector is

$$\max_{x_t(i), L_{1t}} \{ L_{1t}^{\alpha} \int_0^{A_t} x_t(i)^{1-\alpha} di - w_t L_{1t} - \int_0^{A_t} q_t(i) x_t(i) di \} \tag{65}$$

and since we have CRTS with perfect competition we have zero profit with following FOCs,

$$\begin{aligned} w_t &= \alpha L_{1t}^{\alpha-1} \int_0^{A_t} x_t(i)^{1-\alpha} di && (66) \\ q_t(i) &= (1-\alpha) L_{1t}^{\alpha} x_t(i)^{-\alpha} && (67) \end{aligned}$$

notice that the inverse demand function for good of variety i is,

$$\left(\frac{q_t(i)}{(1-\alpha)L_{1t}^{\alpha}} \right)^{\frac{-1}{\alpha}} = x_t(i) \tag{68}$$

The intermediate goods industry will show monopolistic competition, in which there is only one firm, that is one patent holder, producing each variety. Each firm takes the demand of its variety and prices as given, and solves the following problem each period

$$\begin{aligned} \Pi_t(i) &= \max_{x_t(i), K_t(i)} \{q_t(i)x_t(i) - R_t K_t(i)\} && (69) \\ s.t. \quad x_t(i) &= \frac{K_t(i)}{\eta} \end{aligned}$$

plugging in the inverse demand function and the technology constraint, the FOC is,

$$(1-\alpha)^2 x_t(i)^{-\alpha} L_{1t} = R_t \eta \tag{70}$$

and because of the symmetry we mentioned ($x_t(i) = x_t = \frac{K_t}{\eta A_t}$) we can write this FOC as,

$$(1-\alpha)^2 (\frac{K_t}{\eta A_t})^{-\alpha} L_{1t} = R_t \eta \tag{71}$$

i.e. the rental price of capital is not equal to it's marginal product and there is opportunities for positive profit. But also remember there is a fixed cost of entering this industry, namely the price paid for the patent. Then as we will see, the relation between the two will be one of our equilibrium conditions. Now lets look at the problem of R&D firms,

$$\max_{A_{t+1}, L_{2t}} \{p_t^P (A_{t+1} - A_t) - w_t L_{2t}\} \tag{72}$$

$$s.t. A_{t+1} = (1 + L_{2t}\zeta) A_t$$

where p_t^P is the patent of the price. Free entry is assumed, thus there will be zero profits in equilibrium. Notice also the R&D firm is solving a static problem without realizing the positive externality this period's decision creates on next periods production. As we will see, this and the monopoly power of the patent owners will be the sources of sub-optimality in the decentralized solution.The FOC is,

$$p_t^P = \frac{w_t}{\zeta A_t} \tag{73}$$

where the wage (w_t) is determined in the final goods market and given this price, equilibrium quantity will come from the demand function. As we mentioned before, one equilibrium condition will be that at any point in time, total profit a patent generates will be equal to price of it such that there will also be zero profit in the intermediate goods market.

$$p_t^P = \sum_{\tau=t}^{\infty} \frac{\Pi_t(i)}{(1+r)^{\tau-t}} \tag{74}$$

These conditions with constant growth equations for the growing variables is sufficient to characterize the equilibrium growth rate of this economy.

April 4, 2006

10 Optimal unemployment insurance

10.1 (with observable effort)

Consider an economy where the probability of finding a job $p(a)$ is a function of effort $a \in [0,1]$. And we assume that once the agent gets a job, she will have wage w for ever. Thus, the individual problem is

$$\max_{a_t} E \sum_t \beta^t \left[u(c_t) - a_t\right]$$

There are two cases: when the agent has got a job, she will pay no effort and enjoy w for ever. The life long utility is

$$V^E = \sum_t \beta^t u(w) = \frac{u(w)}{1-\beta} \tag{75}$$

When the agent is still unemployed, she will have nothing to consumer. Her problem is

$$V^u = \max_a \left\{u(0) - a + \beta\left[p(a) V^E + \left(1 - p(a) V^{u\prime}\right)\right]\right\} \tag{76}$$

If the optimal solution of a is interior, $a \in (0,1)$, then the first order condition gives

$$-1 + \beta p'(a)\left(V^E - V^u\right) = 0 \tag{77}$$

And since the V^u is stationary,

$$V^u = \max_a \left\{u(0) - a + \beta\left[p(a) V^E + (1 - p(a) V^u)\right]\right\} \tag{78}$$

Solving (77)(78) gives the optimal a and V^u. Another way is to successively substitute a and obtain solution because (**??**) defines a contraction mapping operator. We can fix V_0^u, then solve (78) to get $a\left(V_0^u\right)$ and obtain V_1^u. Keeping going until $V_n^u = V_{n+1}^u$. In a word, optimal effort level a^* solves (78) with $V^u = V^{u\prime}$.

The probability of finding a job $p(a)$ is called hazard rate. If agents did not find a job with effort level a^*, next period, she will still execute the same effort level a^*. Why? Because the duration of unemployment is not state variable in agent's problem. (If agents do not have enough realization about the difficulty of getting a job. With learning, their effort a will increase as they revise their assessment of the difficulty. But such revision of belief is not in this model.)

Now suppose resource is given to people who is unemployed to relive her suffering by a benevolent planner. This planner has to decide the minimal cost of warranting agent a utility level V, $\Psi(V)$. To warrant utility level V, the planner tells the agent how much to consume, how much effort to exert and how much utility she will get if she stay unemployed next period. Obviously, the cost function $\Psi(V)$ is increasing in V.

The cost minimization problem of the planner can be written in the following recursive problem:

$$\Psi(V) = \min_{c,a,V^u} c + [1 - p(a)] \frac{1}{1+r} \Psi(V^u) \tag{79}$$

subject to

$$V = u(c) - a + \beta \left[p(a) V^E + (1 - p(a)) V^u\right] \tag{80}$$

To solve the problem, construct Lagrangian function

$$\mathcal{L} = c + [1 - p(a)] \beta \Psi(V^u) + \theta \left[V - u(c) + a - \beta \left[p(a) V^E + (1 - p(a)) V^u\right]\right]$$

FOC: (c)

$$\theta = \frac{1}{u_c} \tag{81}$$

(a)

$$\Psi(V^u) = \theta \left[\frac{1}{\beta p'(a)} - \left(V^E - V^u\right)\right] \tag{82}$$

(V^u)

$$\Psi'(V^u) = \theta \tag{83}$$

Envelope condition

$$\Psi'(V) = \theta \tag{84}$$

We will work on some implication of these conditions:

1. Compare (82) and (77), we can see that the substitution between consumption and effort is different from the one in agent's problem without unemployment insurance. This is because the cost of effort is higher for work that it is from the viewpoint of planner.

2. (83) tells us that the marginal cost of warranting an extra unit of utility tomorrow is θ.,provided that tomorrow V^u is optimally chosen when today's promise is V. And (84) tells us that the marginal cost of warranting an extra unit of V today is θ.

3. Given that Ψ is strictly concave, $V = V^u$.

4. Regardless of unemployment duration, $V = V^u$. So, effort required the planner is the same over time. Hazard rate is still constant.

Next , we will study the case when effort is not observable. Planner can only choose consumption and V^u. Effort level is chosen optimally by worker and it is unobservable.

April 6, 2006

10.2 (with UN-observable effort)

When a is not observable, planner can only choose c and V^u. And households choose a optimally. Now it becomes a principle-agent problem. We will solve the problem backward.

If given c and V^u, the agent will solve

$$\max_a u(c) - a + \beta \left[p(a) V^E + (1 - p(a)) V^u\right] \tag{85}$$

FOC is

$$\left[p'(a)\beta\right]^{-1} = V^E - V^u \tag{86}$$

This FOC gives an implicit function of a as a function of V^u: $a = g(V^u)$. (Because c and a are separate in the utility function, a is not a function of c).

Then, the planner solve her cost minimization problem, in which the optimality condition is also one constraint.

$$\Psi(V) = \min_{c,a,V^u} c + [1 - p(a)]\beta\Psi(V^u)$$

subject to

$$\begin{aligned} V &= u(c) - a + \beta\left[p(a) V^E + (1 - p(a)) V^u\right] && (87) \\ 1 &= \left[p'(a)\beta\right]\left[V^E - V^u\right] && (88) \end{aligned}$$

Lagrangian is

$$\begin{aligned} &c + [1 - p(a)]\beta\Psi(V^u) + \theta\left[V - u(c) + a - \beta\left[p(a) V^E + (1 - p(a)) V^u\right]\right] \\ &+\eta\left[1 - \left[p'(a)\beta\right]\left[V^E - V^u\right]\right] \end{aligned}$$

FOC: (c)

$$\theta^{-1} = u_c$$

(a)

$$\Psi\left(V^u\right) = \theta\left[\frac{1}{\beta p'\left(a\right)} - \left(V^E - V^u\right)\right] - \eta\frac{p''\left(a\right)}{p'\left(a\right)}\left(V^E - V^u\right) \tag{89}$$

(V^u)

$$\Psi'\left(V^u\right) = \theta - \eta\frac{p'\left(a\right)}{1 - p\left(a\right)} \tag{90}$$

Envelope condition

$$\Psi'\left(V\right) = \theta \tag{91}$$

Again, (90) tells the marginal cost to warrant additional amount of delayed promise. (91) gives the marginal cost to increase today's utility. The Lagrangian multiplier associated with constraint (88) is positive, $\eta > 0$, which means that the constraint is binding. So,

$$\eta\frac{p'\left(a\right)}{1 - p\left(a\right)} > 0$$

Therefore, we have

$$\Psi'\left(V^u\right) < \Psi'\left(V\right) \Rightarrow V^u < V$$

from the strict concavity of $\Psi\left(.\right)$. The delayed promised utility decreases over time.

Let $\theta^u = \theta - \eta\frac{p'(a)}{1-p(a)}$, then $\theta^u < \theta$,which tells us about the consumption path. Consumption decreases over time because $\theta^{-1} = u_c$.

Overall, we get the following model implications: optimal unemployment insurance says that longer unemployment period the agent stays, the less insurance she will be insured for. In this way, the planner induces the higher effort level. Although you cannot let people do what is optimal, such behavior can be achieved by giving out less consumption and promised utility over time. This model implies that time-varying unemployment insurance plan is optimal, under which the replacement rate θ goes down over time.

April 11, 2006

11 Models with one sided lack of commitment

We now study an economy where the agent cannot commit to the contract that the Social Planner offers her. This means that as long as what the Planner offers is better than what the agent can do alone she stays around.

If in some period the shock that the agent receives is so good that her value under autarky is higher than what the planner is offering she will be willing to walk away and be on her own (at no cost). On the other hand, if the planner wants to keep her around, he now has to offer even more. In this model there is no storage or financial markets, and so the agent consumes the fruit (or fish) which is just the shock that she receives in each period.

Consider the village of fisherladies, where young granddaughters receive $s \in \{s_1, s_2, ..., \bar{s}\}$ in every period. We assume that s is iid. The probability that a certain s is realized is Π_s. h_t is a history of shocks up to period t.

First, if the granddaughter stays autarky, she will enjoy total utility,

$$V_{AUT} = \sum_{t=0}^{\infty} \beta^t \sum_s \Pi_s u(s) = \frac{\sum_s \Pi_s u(s)}{1-\beta}$$

Note that here V^A is the utility of the young lady before endowment shock is realized.

Now we assume that the grandmother offers a contract to the granddaughter, which transfer resources and provide insurance to her. Grandmother can commit. But the young granddaughter may leave grandmother and break her word. Thus, this model is one-sided commitment model: an agent can walk away from a contract but the other cannot. Therefore, the contract should be always in the interest of granddaughter for her to stay.

We define a contract $f_t : H_t \to c \in [0, \tau]$. We will see next class that incentives compatibility constraint requires that at each node of history H_t, the contract should guarantee a utility which is higher than that in autarky.

Notice that the problem is different from Lucas tree model because of the shock realization timing. In Lucas tree model, shock is state variable because action takes place after shock is realized. Thus, action is indexed by shock. Here action is chosen before shock realization. Therefore, shock is not a state variable and action is state contingent.

In Lucas tree model, $V(s) = \max_c u(c) + \beta \sum_{s'} \Pi_{ss'} V(s')$. Here, if we write the problem recursively, it is $V = \max_{c_s} \sum_s \Pi_s u(c_s) + \beta V$.

Remember, the grandmother will make a deal with her granddaughter. They sign a contract to specify what to do in each state. $h_t \in H_t$. Contract is thus a mapping $f_t(h_t) \to c(h_t)$. With this contract, granddaughter gives y_t to the grandmother and receives $c_t = f_t(h_{t-1}, y_t)$. But if the granddaughter decided not to observe the contract, she consumes y_t this period and cannot enter a contract in the future, i.e. she has to live in autarky in the future.

For grandmother to keep granddaughter around her, the contract has to be of interest to granddaughter because although grandmother keeps her promise, granddaughter does not. There are two possible outcome if this contract is broken. One is that granddaughter goes away with current and future endowment. The other is that they renegotiate. We ignore the second possibility as no renegotiation is allowed. But we need deal with the possibility that the granddaughter says no to the contract and steps away.

The first best outcome is to warrant a constant consumption c_t to granddaughter who is risk averse. But because of the one-side lack of commitment, the first best is not achievable. The contract should always be attractive to granddaughter, otherwise, when she gets lucky with high endowment y_s, she will feel like to leave. So, this is a dynamic contract problem which the grandmother will solve in order to induce good behavior from granddaughter. The contract is dynamic because the nature keeps moving.

We say the contract $f_t(h_t)$ is incentive compatible or satisfies participation constraint if for all h_t,

$$u(f_t(h_t)) + \sum_{\tau=1}^{\infty} \beta^{\tau} \sum_s \Pi_s u(f_{t+\tau}(h_{t+\tau})) \geq u(y_s(h_t)) + \beta V^A \tag{92}$$

The left hand side is utility guaranteed in the contract. And the right hand side is the utility that granddaughter can get by herself. The participation constraint is not binding if y_s is low. And when y_s is high, PC is binding.

11.1 Problem of the grandmother

In this model, problem of the grandmother is to find an optimal contract that maximizes the value of such a contract of warranting V to her. We define the problem using recursive formula. Firstly, let's define the value of contract to grandmother if she promised V to her granddaughter by $\Phi(V)$. $\Phi(V)$ can be defined recursively as the following:

$$\Phi(V) = \max_{\{c_s, \omega_s\}_{s=1}^{S}} \sum_s \Pi_s[(s - c_s) + \beta \Phi(\omega_s)] \tag{93}$$

subject to

$$u(c_s) + \beta \omega_s \geq u(s) + \beta V^A \quad \forall s \tag{94}$$

$$\sum_s \Pi_s[u(c_s) + \beta \omega_s] \geq V \tag{95}$$

Notice that there are $1+S$ constraints. The choice variables c_s, ω_s are state-contingent where ω_s is the promised utility committed to granddaughter in each state. In the objective function, $\sum_s \Pi_s(s - c_s)$ is the expected value of net transfer.

There are two sets of constraints. (94) is IC and (95) is promise keeping constraint.

The First Order Conditions to the grandmother's problem are:

(c_s)

$$\Pi_s = (\theta_s + \lambda\Pi_s)u'(c_s) \tag{96}$$

(ω_s)

$$-\Pi_s\Phi'(\omega_s) = \lambda\Pi_s + \theta_s \tag{97}$$

(μ)

$$\sum_s \Pi_s[u(c_s) + \beta\omega_s] = V \tag{98}$$

(λ)

$$u(c_s) + \beta\omega_s \geq u(s) + \beta V^A \tag{99}$$

In addition, Envelope Theorem tells that:

$$\Phi'(v) = -\lambda \tag{100}$$

Interpreting the first order conditions:

1. (96) tells that in an optimal choice of c_s, the benefit of increasing one unit of c equals the cost of doing so. The benefit comes from two parts: first is $\lambda\Pi_s u'(c_s)$ as increasing consumption helps grandmother to fulfill her promise and the second part is $\theta_s u'(c_s)$ since increase in consumption helps alleviated the participation constraint. And the cost is the probability of state s occurs.

2. (97) equates the cost of increasing one unit of promised utility and the benefit. The cost to grandmother is $-\Pi_s\Phi'(\omega_s)$ and the benefit is $\mu\Pi_s + \lambda_s$ which helps grandmother deliver promise and alleviate participation constraint.

How about the contract value $\Phi(V)$. First, $\Phi(V)$ can be positive or negative.

Claim: There exits V such that $\Phi(V) > 0$.

What's the largest V we will be concerned with? When PC will be binding for sure. If PC binds for the best endowment shock s, then PC holds for all the shock s. When granddaughter gets the best shock, the best autarky value is then

$$V_{AM} = u(\bar{s}) + \beta V_A$$

And the cheapest way to guarantee V_{AM} is to give constant consumption $\overline{c_S}$, such that

$$V_{AM} = \frac{u(c_{\bar{s}})}{1-\beta}$$

From this case, we can see that because of lack of commitment, the grandmother will have to give more consumption in some states. While when there is no lack of commitment, strict concavity of $u(.)$ implies that constant stream of consumption beats any $\{c_t\}$ that have the same present value, as there is no PC.

11.2 Characterizing the Optimal Contract

We will characterize the optimal contract by considering the two cases: (i) $\theta_s > 0$ and (ii) $\theta_s = 0$.

Firstly, if $\theta_s = 0$, we have the following equations from FOC and EC:

$$\Phi'(\omega_s) = -\mu \tag{101}$$

$$\Phi'(V) = -\mu \tag{102}$$

Therefore, for s where PC is not binding,

$$V = \omega_s$$

c_s is the same for all s. For all s such that the Participation Constraint is not binding, the grandmother offers the same consumption and promised future value.

Let's consider the second case, where $\theta_s > 0$. In this case, the equations that characterize the optimal contract are:

$$u'(c_s) = \frac{-1}{\Phi'(\omega_s)} \tag{103}$$

$$u(c_s) + \beta\omega_s = u(s) + \beta V^A \tag{104}$$

Note that this is a system of two equations with two unknowns (c_s and ω_s). So these two equations characterize the optimal contract in case $\theta_s > 0$. In addition, we can find the following properties by carefully observing the equations:

1. The equations don't depend on V. Therefore, if a Participation Constraint is binding, promised value does not matter for the optimal contract.

2. From the first order condition with respect to ω_s, $\Phi'(\omega_s) = \Phi'(V) - \frac{\theta_s}{\Pi_s}$, where $\frac{\theta_s}{\Pi_s}$ is positive. Besides, we know that Φ is concave. This means that $V < \omega_s$. In words, if a Participation Constraint is binding, the moneylender promises more than before for future.

Combining all the results we have got, we can characterize the optimal contract as follows:

1. Let's fix V_0. We can find a $s^*(V_0)$, such that $\forall s < s^*(V_0)$, the participation constraint is not binding and $\forall s \geq s^*(V_0)$, the constraint is binding, i.e. $\theta_s > 0$. (For a formal proof of this fact see PS12 / exercise 1).

2. The optimal contract that the moneylender offers to an agent is the following:

   If $s_t \leq s^*(V_0)$, the moneylender gives $(V_0, c(V_0))$. Both of them are the same as in the previous period. In other words, the moneylender offers the agent the same insurance scheme as before.

If at some point in time $s_t > s^*(V_0)$, the moneylender gives $(V_1, c(V_1))$, where $V_1 > V_0$ and c doesn't depend on V_0. In other words, the moneylender promises larger value to the agent to keep her around.

So the path of consumption and promised value for an agent is increasing with steps.

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12 Models with two-sided lack of commitment

12.1 The Model

- Two brothers, A and B, and neither of them has access to a commitment technology. In other words, the two can sign a contract, but either of them can walk away if he does not feel like observing it.
- This is an endowment economy (no production) and there is no storage technology. Endowment is represented by $(y_s^A, y_s^B) \in Y \times Y$, where y_s^i is the endowment of brother i. s=(y_s^A, y_s^B) follows a Markov process with transition matrix $\Gamma_{ss'}$.

12.2 First Best Allocation

We will derive the first best allocation by solving the social planner's problem:

$$\max_{\{c_i(h_t)\}_{\forall h_t, \forall i}} \lambda^A \sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t) u(c^A(h_t)) + \lambda^B \sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t) u(c^B(h_t))$$

subject to the resource constraint:

$$\sum_i c^i(h_t) - y^i(h_t) = 0 \qquad \forall h_t \quad \text{w/ multiplier } \gamma(h_t)$$

The First Order Conditions are:

$$\begin{aligned} FOC(c^A(h_t)) &: \lambda^A \beta^t \Pi(h_t) u'(c^A(h_t)) - \gamma(h_t) = 0 \\ FOC(c^B(h_t)) &: \lambda^B \beta^t \Pi(h_t) u'(c^B(h_t)) - \gamma(h_t) = 0 \end{aligned}$$

Combining these two yields:

$$\frac{\lambda^A}{\lambda^B} = \frac{u'(c^A(h_t))}{u'(c^B(h_t))}$$

The first best allocation will not be achieved if there is no access to a commitment technology. Therefore, the next thing we should do is look at the problem the planner is faced with in the case of lack of commitment. Due to lack of commitment, the planner needs to make sure that at each point in time and in every state of the world, h_t, both brothers prefer what they receive to autarky. Now we will construct the problem of the planner adding these participation constraints to his problem.

12.3 Constrained Optimal Allocation

The planner's problem is:

$$\max_{c^A(h_t),c^B(h_t)} \lambda^A \sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t) u(c^A(h_t)) + \lambda^B \sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t) u(c^B(h_t))$$

$$\sum_i c^i(h_t) - y^i(h_t) = 0 \qquad \forall h_t \quad \text{w/ multiplier } \gamma(h_t)$$

$$\sum_{r=t}^{\infty} \beta^{r-t} \sum_{h_r} \Pi(h_r|h_t) u(c^i(h_r)) \geq \Omega_i(h_t) \qquad \forall h_t, \forall i \quad \text{w/ multiplier } \mu_i(h_t)$$

where $\Omega_i(h_t) = \sum_{r=0}^{\infty} \beta^{r-t} \sum_{h_r} \Pi(h_r|h_t) u(y_i(h_t))$ (the autarky value)

- How many times does $c^A(h_{17})$ appear in this problem? Once in the objective function, once in the feasibility constraint, and it appears in the participation constraint from period 0 to period 16.
- We know that the feasibility constraint is always binding so that $\gamma(h_t) > 0 \ \forall h_t$. On the other hand the same is not true for the participation constraint.
- Both participations cannot be binding but both can be nonbinding.
- Define $M_i(h_{-1}) = \lambda^i$

  and $M_i(h_t) = \mu_i(h_t) + M_i(h_{t-1})$

  (We will use these definitions for the recursive representation of the problem in the next class)

12.4 Recursive Representation of the Constrained SPP

We want to transform this problem into the recursive, because it would be easier to solve the optimal allocation with a computer. Now we will show how to transform the sequential problem with the participation constraints into its recursive representation.

Before we do this transformation, first recall the Lagrangian associated with the sequential representation of the social planner's problem:

$$\lambda^A \sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t) u(c^A(h_t)) + \lambda^B \sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t) u(c^B(h_t))$$
$$+ \sum_{t=0}^{\infty} \beta^t \sum_{h_t \in H_t} \Pi(h_t) \sum_{i=1}^{2} \mu_i(h_t) \left[\sum_{r=t}^{\infty} \beta^{r-t} \sum_{h_r} \Pi(h_r|h_t) u(c^i(h_r)) - \Omega_i(h_t) \right]$$
$$+ \sum_{t=0}^{\infty} \sum_{h_t \in H_t} \gamma(h_t) \left[\sum_{i=1}^{2} c_i(h_t) - \sum_{i=1}^{2} y_i(h_t) \right]$$

Note that here the Lagrangian multiplier associated witih the participation constraint for brother i after history h_t is $\beta^t \Pi(h_t) \mu_i(h_t)$.

Now we will use the definitions from the previous class (for $M_i(h_t)$) to rewrite the above Lagrangian in a more simple form,

Collect terms and rewrite,

$$\sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t) \sum_{i} \left\{ \lambda^i u(c^i(h_t) + \mu_i(h_t) \left[\sum_{r=t}^{\infty} \beta^{r-t} \sum_{h_r} \Pi(h_r|h_t) u(c^i(h_r)) - \Omega_i(h_t) \right] \right\}$$
$$+ \sum_{t=0}^{\infty} \sum_{h_t \in H_t} \gamma(h_t) \left[\sum_{i=1}^{2} c_i(h_t) - \sum_{i=1}^{2} y_i(h_t) \right]$$

Note that, $\sum_{r=t}^{\infty} \beta^{r-t} \sum_{h_r} \Pi(h_r|h_t) u(c^i(h_r)) - \Omega_i(h_t) = u(c^i(h_t)) + \sum_{r=t+1}^{\infty} \beta^{r-t} \sum_{h_r} \Pi(h_r|h_t) u(c^i(h$
$\Omega_i(h_t)$,

and that $\Pi(h_r|h_t)\Pi(h_t) = \Pi(h_r)$ so using these, rewrite as,

$$\sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t) \sum_{i} \left\{ \lambda^i u(c^i(h_t) + \mu_i(h_t) u(c^i(h_t)) \right\}$$
$$+ \sum_{t=0}^{\infty} \sum_{h_r} \sum_{i} \mu_i(h_t) \left[\sum_{r=t+1}^{\infty} \beta^{r} \sum_{h_r} \Pi(h_r) u(c^i(h_r)) - \Omega_i(h_t) \right]$$
$$+ \sum_{t=0}^{\infty} \sum_{h_t \in H_t} \gamma(h_t) \left[\sum_{i=1}^{2} c_i(h_t) - \sum_{i=1}^{2} y_i(h_t) \right]$$

Collect the terms of $u(c^i(h_r)$,

$$\sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t) \sum_{i} \left\{ \left[\lambda^i + \sum_{r=0}^{t-1} \mu_i(h_r) \right] u(c^i(h_t) + \mu_i(h_t) \left[u(c^i(h_t) - \Omega_i(h_t) \right] \right\}$$
$$+ \sum_{t=0}^{\infty} \sum_{h_t \in H_t} \gamma(h_t) \left[\sum_{i=1}^{2} c_i(h_t) - \sum_{i=1}^{2} y_i(h_t) \right]$$

Introduce the variable $\mathrm{M}_i(h_t)$ and define it recursively as,

$$\begin{aligned} M_i(h_t) &= M_i(h_{t-1}) + \mu_i(h_t) \\ M_i(h_{-1}) &= \lambda^i \end{aligned}$$

where $\mathrm{M}_i(h_t)$ denotes the Pareto weight plus the cumulative sum of the Lagrange multipliers on the participation constraints at all periods from 1 to t.

So rewrite the Lagrangian once again as,

$$\sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t) \sum_i \left\{ M_i(h_{t-1}) u(c^i(h_t) + \mu_i(h_t) \left[u(c^i(h_t) - \Omega_i(h_t) \right] \right\}$$
$$+ \sum_{t=0}^{\infty} \sum_{h_t \in H_t} \gamma(h_t) \left[\sum_{i=1}^{2} c_i(h_t) - \sum_{i=1}^{2} y_i(h_t) \right]$$

Now we are ready to take the First Order Conditions:

$$\frac{u'(c^A(h_t))}{u'(c^B(h_t))} = \frac{M_A(h_{t-1}) + \mu_A(h_t)}{M_B(h_{t-1}) + \mu_B(h_t)}$$

$$\left[\sum_{r=t}^{\infty} \beta^{r-t} \sum_{h_r} \frac{\Pi(h_r)}{\Pi(h_t)} u(c^i(h_r)) - \Omega_i(h_t) \right] \mu_i(h_t) = 0$$

$$\sum_{i=1}^{2} c_i(h_t) - \sum_{i=1}^{2} y_i(h_t) = 0$$

12.5 Recursive Formulation

Our goal is make the problem recursive, which is very nice when we work with computer. To do this, we need to find a set of state variables which is sufficient to describe the state of the world. We are going to use x as a state variable.So the state variables are the endowment: $y = (y^A, y^B)$ and weight to brother 2: x. Define the value function as follows:

$V = \{(V_0, V_A, V_B)$ such that $V_i : X \times Y \to \mathcal{R}$, $i = 1, 2, V_0(x, y) = V_A(x, y) + xV_B(x, y)\}$

What we are going find is the fixed point of the following operator (operation is defined later):

$$T(V) = \{T_0(V), T_1(V), T_2(V)\}$$

Firstly, we will ignore the participation constraints and solve the problem:

$$\max_{c_A, c_B} u(c^A(y, x)) + xu(c^B(y, x)) + \beta \sum_{y'} \Gamma_{yy'} V_0(y', x)$$

subject to

$$c^A + c^B = y^A + y^B$$

First Order Conditions yield:

$$\frac{u'(c_A)}{u'(c_B)} = x$$

Second, we will check the participation constraints. There are two possibilities here:

1. Participation constraint is not binding for either 1 or 2. Then set $x(h_t) = x(h_{t-1})$. In addition,

$$\begin{aligned} V_0^N(y,x) &= V_0(y,x) \\ V_i^N(y,x) &= u(c^i(y,x)) + \beta \sum_{y'} \Gamma_{yy'} V_i(y',x) \end{aligned}$$

2. Participation constraint is not satisfied for one of the brothers (say A).

This means that agent A is getting too little. Therefore, in order for the planner to match the outside opportunity that A has, he needs to change x so that he guarantees person A the utility from going away. We need to solve the following system of equations in this case:

$$\begin{aligned} c^A + c^B &= y^A + y^B \\ u(c^A) + \beta \sum_{y'} \Gamma_{yy'} V_A(y',x) &= u(y_A) + \beta \sum_{y'} \Gamma_{yy'} \Omega_A(y') \\ x' &= \frac{u'(c_A)}{u'(c_B)} \end{aligned}$$

This is a system of three equations and three unknowns. Denote the solution to this problem by,

$$\begin{aligned} & c^A(y,x) \\ & c^B(y,x) \\ & x'(y,x) \end{aligned}$$

So that,

$$\begin{aligned} V_0^N(y,x) &= V_A^N(y,x) + x V_B^N(y,x) \\ V_i^N(y,x) &= u(c^i(y,x)) + \beta \sum_{y'} \Gamma_{yy'} V_i(y',x'(y,x)) \end{aligned}$$

Thus we have obtained T(V)=V^N. And the next thing we need to do is find V* such that T(V*) $= V^*$.

Final question with this model is "how to implement this allocation?" or "Is there any equilibrium that supports this allocation?". The answer is yes. How? Think of this model as a repeated game. And define the strategy as follows: keep accepting the contract characterized here until the other guy walks away. If the other guy walks away, go to autarky forever. We can construct a Nash equilibrium by assigning this strategy to both of the brothers.