Jan 30th, 2007

1 Existence and Pareto Optimality in the Growth Model

To support a Pareto Optimal allocation as a solution to the growth model presented before, we have to take care of certain issues that arise when we apply the SBWT to get our equilibrium. Those issues/solutions are listed below:

- What are the 'transfers' of the conclusion of the SBWT in terms of the growth model? / we don't need transfers; agents are homogeneous, so even if they can act differently, they choose to do the same as everyone else.
- Do we have to worry about the 'Quasi' part of the equilibrium? / If we can find a cheaper point in the feasible set, then the Quasi equilibrium is equivalent to the AD equilibrium
- representation of prices/ if we can check the conditions of the Prescott & Lucas Theorem, then we have a dot product representation of prices.

1.1 Characterization of the solution to the growth model

The solution to the growth model is triplet of sequences $\{c_t^*, k_{t+1}^*, q_t^*\}_{t=0}^{\infty}$. As you proved in the homeworks, you can use the Arrow-Debreu apparatus in order to argue that such an equilibrium exists. To characterize more carefully the equilibrium, we have to impose additional restrictions:

- u, f are C^2 (twice continuously differentiable)
- Inada conditions (see the Stockey and Lucas textbook for specifics)

With these conditions, we can restrict our attention to interior solutions, which means that first order conditions are sufficient to characterize equilibria.

Rewriting the growth model (replacing consumption in the utility function using the budget constraint):

$$\max_{\{k_{t+1}\}_{t=0}^{\infty}} u[f(k_t) - k_{t+1}]$$

Taking the FOC with respect to k_{t+1} and replacing for c_t to ease notation, we get (note that we are using variables with * to denote that the following are *equilibrium* conditions)

$$-\beta^t u'[c_t^*] + \beta^{t+1} u'[c_{t+1}^*] f'(k_{t+1}^*) = 0$$

rearranging terms

$$\frac{u'[c_t^*]}{\beta u'[c_{t+1}^*]} = f'(k_{t+1}^*) \tag{1}$$

Therefore, the solution to the growth model has to satisfy the condition in (1).

Now, for prices, we can rewrite the budget equation from the AD setting (if the conditions of Prescott and Lucas are satisfied so that prices have a dot product representation) as follows

$$p(x) \equiv \sum_{t=0}^{\infty} (q_{1t}x_{1t} + q_{2t}x_{2t} + q_{3t}x_{3t}) \leq 0 \tag{2}$$

Since $c_t + k_{t+1} = x_{1t}$, $k_t \geq -x_{2t} \geq 0$ and $1 \geq -x_{3t} \geq 0$, (2) becomes

$$\sum_{t=0}^{\infty} (q_{1t}^*(c_t + k_{t+1}) - q_{2t}^* k_t - q_{3t}^*) \leq 0 \tag{3}$$

Note that in (3), we have used the fact that there is no waste (agents rent their full capital and labor services) and that agents take the equilibrium prices as given. The maximization problem now can be set as a Lagrangian:

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \pounds = \sum_{t=0}^{\infty} \beta^t u[c_t] - \lambda \{ \sum_{t=0}^{\infty} q_{1t}^*(c_t + k_{t+1}) + q_{2t}^* k_t + q_{3t}^* \} \tag{4}$$

The first order conditions of this problem with respect to c_t and k_{t+1} are respectively

$$\frac{\beta^t u'[c_t^*]}{q_{1t}^*} = \lambda \tag{5}$$

$$\lambda q_{1t}^* - \lambda q_{2,t+1}^* = 0 \tag{6}$$

Note that (6) implies that $q^*_{1t} = q^*_{2,t+1}$[1], which pins down one sequence of prices (specifically, the price of capital services). From (5), we get

$$\lambda = \frac{\beta^t u'[c^*_t]}{q^*_{1t}} = \frac{\beta^{t+1} u'[c^*_{t+1}]}{q^*_{1,t+1}}$$

$$\Rightarrow \frac{u'[c^*_t]}{\beta u'[c^*_{t+1}]} = \frac{q^*_{1t}}{q^*_{1,t+1}}$$

From before, we know that the left hand side of the last equation equals $f'(k^*_{t+1})$. Hence

$$\frac{q^*_{1t}}{q^*_{1,t+1}} = f'(k^*_{t+1}) \tag{7}$$

Since f' represents the (technical) rate of exchange between goods today and goods tomorrow, (7) tells us exactly what the sequence of output prices should be. Finally, to obtain q^*_{3t}, we turn to problem of the producer

$$\max_{y \in Y_t} q^*(y) = q^*_{1t} y_{1t} + q^*_{2t} y_{2t} + q^*_{3t} y_{3t}$$

$$st$$

$$y_{1t} = f(-y_{2t}, -y_{3t})$$

Again, we know that $\{k^*_{t+1}, 1\}^\infty_{t=0}$ solve this problem. Then, the problem of the firm is equivalent to

$$\max_{y \in Y_t} f(k_t, y_{3t}) - q^*_{2t} k_t - q^*_{3t} y_{3t}$$

Taking FOCs with respect to k^*_t and y_{3t} respectively

$$q^*_{1t} f_k(k^*_t, 1) = q^*_{2t}$$

$$q^*_{1t} f_n(k^*_t, 1) = q^*_{3t}$$

Hence, the price of labor services must satisfy $f_n(k^*_t, 1) = q^*_{3t}/q^*_{1t}$ $\forall t$.

[1] This condition is misleading, since we don't have depreciation. In the more general case when $\delta \neq 1$, the condition is

$$\frac{q^*_{1t}}{q^*_{1t+1}} = 1 - \delta + \frac{q^*_{2,t+1}}{q^*_{1t+1}}$$

We know now how to characterize the sequence of prices at equilibrium in the growth model. The problem with the AD framework however, is that we have a triple infinite (!) number of prices. Together with the assumption that all trade takes place at $t = 0$, this implies that agents must know a triple infinite number of prices in order to solve their problem.

We want to depart from this assumption of all trading happening at the beginning of time, so we will define *sequential markets* and a corresponding *sequential markets equilibrium* (SME). Note that we would like to maintain existence, uniqueness and optimality of the equilibrium, so we would like $ADE \Leftrightarrow SME$.

1.2 Sequential Markets Equilibrium

- We need a spot market at every period of time where agents would be able to trade output, capital and labor services and a new good (which we will specify below) which are 'loans'.
- Agents must be able to move resources across time.

Clearly, the budget constraint will change from the previous setup. In ADE

$$\sum_{i=1}^{3}\sum_{t=0}^{\infty} q_{it}x_{it} \leq 0$$

In SME, we introduce the concept of 'loans' (l), to enable agents to move resources across time. Loans are rights to a R units of output/consumption tomorrow, in exchange of 1 unit of output/consumption today. So, the budget constraint becomes

$$-l_t R_t + l_{t+1} + \sum_{i=1}^{3} q_{it}x_{it} \leq 0 \qquad \forall t$$

Definition 1. *A Sequence of Markets Equilibrium is* $\{x_{it}^*, q_t^*, y_{it}^*, l_{t+1}^*, R_t^*\}_{t=0}^{\infty}$ *such that*

- *Agents maximize, i.e.*

$$\{x_{it}^*, l_{t+1}^*\} \in \arg\max_{x\in X} \sum_{t=0}^{\infty} \beta u[c_t(x)]$$

$$st$$

$$c_t + k_{t+1} + l^*_{t+1} = R^*_t l_t + q^*_{2t} k_t + q^*_{3t}$$
$$k_0, l_0 \quad given$$

- *Firms maximize*
- $x^* = y^*$ *(market clears)*
- $l^*_{t+1} = 0 \quad \forall t$ *(loan market clears)*

To show that $ADE \Leftrightarrow SME$, we need to check that allocations and choices of the agents in both worlds are the same. In the $SME \Rightarrow ADE$ direction, it's easy to see that if we have a SME, we can construct an ADE just by ignoring $\{l_{t+1}\}$ (it's zero at equilibrium anyway).

Conversely ($ADE \Rightarrow SME$), if we have an ADE, we need l^*_{t+1} and R^*_t to construct a SME. Again, given the condition for the clearing of the loans market, l^*_{t+1} comes trivially. For R^*_t, we use an arbitrage condition: since loans and capital perform the same function (move resources from one period of time to another), then their price should be the same. Specifically

$$R_t = \frac{q^*_{1t}}{q^*_{1,t+1}}$$

Finally, we have a close relationship between prices between both equilibriums. If $\{x^*, y^*, q^*\}$ is an ADE and $\{x^*, y^*, \widehat{q}^*, R^*, l^*\}$ is a SME, the following is true since at a SME, the budget constraint is priced with respect to output/consumption at each point of time

$$\widehat{q}^*_{it} = \frac{q^*_{it}}{q^*_{1t}} \quad \forall t$$

1.3 SME 'easy'

Now we will define a simpler version of the SME. Basically, we will simplify the definition of equilibrium by ignoring loans and using the properties of the production function

Definition 2. *A SMEE is* $\{c^*_t, k^*_{t+1}, w^*_t, R^*_t\}$ *such that*

- *Agents maximize:*

$$\{c_t^*, k_{t+1}^*\} \in \arg \max_{\{c_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u[c_t]$$

$$st$$

$$c_t + k_{t+1} = R_t^* k_t + w_t^*$$

$$k_0 \quad given$$

- *Firms maximize:*

$$\{k_{t+1}^*, 1\} \in \arg \max_{k_t, n_t} f(k_t, n_t) - R_t^* k_t - w_t^* n_t$$

- *Market clearing:*

$$c_t^* + k_{t+1}^* = f(k_t^*, 1)$$

Note that the last condition is redundant, because the production function is homogenous of degree one, i.e., production is exhausted in the payment to production factors.

After all this work, we still have the problem of how to calculate the equilibrium. From the FOCs we know that to get a solution, we have to solve a second order difference equation, with an initial conditions plus a transversality condition. Nevertheless, the Growth model has infinite dimensions, which complicate things a bit.

The next step is to reformulate the problem in a recursive form. This is much better, since we will be able to solve the problem recursively, that is, every new period, the agent faces the same problem.

2 Stochastic Processes

2.1 Markov Process

In this course, we will concentrate on Markov productivity shocks. Considering shocks is really a pain, so we want to use less painful ones. A Markov shock is a stochastic process with the following properties:

1. there are FINITE number of possible states for each time. More intuitively, no matter what happened before, tomorrow will be represented by one finite set.

2. the only thing that matters for the realization of tomorrow's shock is today's state. More intuitively, no matter what kind of history we have, the only thing you need to predict the realization of the shock tomorrow is TODAY's realization.

More formally, for each period, suppose either z^1 or z^2 happens [2]. Denote z_t is the state today and Z_t is the set of possible states today, i.e. $z_t \in Z_t = \{z^1, z^2\}$ for all t. Since the shock follows a Markov process, the state tomorrow will only depend on today's state. So let's write the probability that z^j will happen tomorrow, conditional on today's state being z^i as $\Gamma_{ij} = prob[z_{t+1} = z^j | z_t = z^i]$. Since Γ_{ij} is a probability, we know that

$$\sum_j \Gamma_{ij} = 1 \qquad for\ \forall i \tag{8}$$

Notice that a 2-state Markov process is summarized by 6 numbers: z^1, z^2, Γ_{11}, Γ_{12}, Γ_{21}, Γ_{22}.

The great beauty of using a Markov process is that we can use the explicit expression of probability for future events, instead of using the ambiguous operator called expectation, which very often people don't know what it means when they use it.

2.2 Representation of History

- Let's concentrate on a 2-state Markov process. In each period the state of the economy is $z_t \in Z_t = \{z^1, z^2\}$.
- Denote the history of events up to t (which one of $\{z^1, z^2\}$ happened from period 0 to t, respectively) by $h_t = \{z_1, z_2, ..., z_t\} \in H_t = Z_0 \times Z_1 \times ... \times Z_t$.
- In particular, $H_0 = \emptyset$, $H_1 = \{z^1, z^2\}$, $H_2 = \{(z^1, z^1), (z^1, z^2), (z^2, z^1), (z^2, z^2)\}$.
- Note that even if the state today is the same, past history might be different. By recording history of events, we can distinguish the two histories with the same realization today but different realizations in the past (think that the current situation might be "you do not have a girl friend", but we will distinguish the history where "you had a girl friend 10 years ago" and the one where you didn't

[2] Here we restrict our attention to the 2-state Markov process, but increasing the number of states to any finite number does not change anything fundamentally.

- Let $\Pi(h_t)$ be the unconditional probability that the particular history h_t does occur. By using the Markov transition probability defined in the previous subsection, it's easy to show that (i) $\Pi(h_0) = 1$, (ii) for $h_t = (z^1,\ z^1)$, $\Pi(h_t) = \Gamma_{11}$ (iii) for $h_t = (z^1,\ z^2,\ z^1,\ z^2)$, $\Pi(h_t) = \Gamma_{12}\Gamma_{21}\Gamma_{12}$.

February 1st, 2007

3 Stochastic Growth Model

With this, we can rewrite the growth model when these shocks affect the production function (usual convention in Macro). Preferences are given by the usual von Neumann-Morgenstern utility

$$u(x) = \sum_t \beta^t \sum_{h_t \in H_t} \pi(h_t)\, u\left[c_t(h_t)\right].$$

In an Arrow -Debreu world the constraint is[3]

$$\sum_t \sum_{h_t \in H_t} \sum_j p_t^j(h_t)\ \ x_t^j(h_t) \leq 0, \qquad \text{where} \qquad j = 1, 2, 3.$$

In a SM setting we need to give to the agent enough tools, so that she can consume different quantities in different states of the world. In other words, we have to make sure that whatever she was able to do in an AD setting, she will also be able to do it in the SM setting. To that end, we will introduce loans and state contingent claims (Also known as *Arrow* securities or bonds). For example, $b_t\left(h_{t-1}, z^i\right)$ is a claim that the agent bought in period $t-1$, and will pay 1 unit of consumption for sure if state i occurs. In the SM world, the budget constraint will be

$$\begin{gathered} c_t(h_t) + k_{t+1}(h_t) + \ell_t(h_t) + \sum_{z_{t+1}} q_t(h_t, z_{t+1}) b_{t+1}(h_t, z_{t+1}) = \\ k_t(h_{t-1})\ \ R_t^k(h_t) + \ell_{t-1}(h_{t-1})\ \ R_t^\ell(h_t) + w_t(h_t) + b_t(h_{t-1}, z_t), \end{gathered}$$

[3]The definition of the relevant commodity spaces and consumption/production sets is in the second problem set

where $q_t(h_t, z_{t+1})$ is the price of the state contingent claim that pays 1 in period $t+1$ if state z_{t+1} occurs.

From the definition of these assets, we can derive immediately a no-arbitrage condition for loans and the state contingent securities. On the one hand, if we save one unit of consumption today and get a loan, the gross return is given by

$$\frac{R^{\ell}_{t+1}(h_t)}{1}$$

, since tomorrow the loan will pay some interests. On the other hand, by using that same unit of consumption and investing in Arrow securities, we get a gross return of

$$\frac{\sum_{z_{t+1}} q_t(h_t, z_{t+1})\bar{b}}{\bar{b}} = \sum_{z_{t+1}} q_t(h_t, z_{t+1})$$

where $\bar{b}$ is the level of *certain* future consumption one can get by spending 1 unit of current consumption. Since these two ways of savings are the same, the no-arbitrage condition is

$$1 = R^{\ell}_{t+1}(h_t) \sum_{z_{t+1}} q_t(h_t, z_{t+1})$$

The importance of the no arbitrage conditions is that we can eliminate (shut down) some markets, since they can be perfectly replicated by other markets. In the example above, we can close the state contingent market, since we already have a loan market.

Furthermore, given our representative agent assumption, in equilibrium we don't need any additional markets, since

$$\ell^* = b^* = 0 \;\; \forall t, (h_t, z_{t+1})$$

We can also derive this no-arbitrage condition through first order conditions. Using the lagrangian of the problem (with multipliers λ for each t and h_t) and taking a FOC with respect to $b_t(h_t, z_{t+1})$

$$\begin{aligned} q_t(h_t, z_{t+1})\lambda_t(h_t) &= \lambda_{t+1}(h_t, z_{t+1}) \\ \Rightarrow q_t(h_t, z_{t+1}) &= \frac{\lambda_{t+1}(h_t, z_{t+1})}{\lambda_t(h_t)} \end{aligned} \tag{9}$$

Next, taking a first order condition with respect to $\ell_t(h_t)$ we get

$$\lambda_t(h_t) = \sum_{z_{t+1}} \lambda_{t+1}(h_t, z_{t+1}) R^{\ell}_{t+1}(h_t)$$

Since the return on loans is 'set' before the shock is known (hence, that return is NOT state-dependent), we can take R^{ℓ} outside the sum:

$$1 = R^{\ell}_{t+1}(h_t) \sum_{z_{t+1}} \frac{\lambda_{t+1}(h_t, z_{t+1})}{\lambda_t(h_t)}$$

Finally, using (9) we arrive to the same no-arbitrage condition as before:[4]

$$1 = R^{\ell}_{t+1}(h_t) \sum_{z_{t+1}} q_t(h_t, z_{t+1})$$

[4] In problem set # 2, you derived an analogous condition for the return on capital $R^k_t(h_t)$, but since this IS state dependent, the condition is slightly different:

$$1 = \sum_{z_{t+1}} q_t(h_t, z_{t+1}) R^k_{t+1}(h_t, z_{t+1})$$