Part I
Growth

Growth is a vast literature in macroeconomics, which seeks to explain some facts in the long term behavior of economies. The current section is an introduction to this subject, and will be divided in three sections. In the first section, we set forth the motivation for the theory: the empirical regularity which it seeks to explain. The second section is about exogenous growth models; that is, models in which an exogenous change in the production technology results in income growth as a theoretical result. Finally, the third section introduces technological change as a decision variable, and hence the growth rate becomes endogenously determined.

1 Facts in long-run macroeconomic data

1.1 Kaldor's stylized facts

The first five "facts" refer to the long run behavior of economic variables in a given economy, whereas the sixth one involves an inter-country comparison.

1) The growth rate of output g_y is constant over time.
2) The capital-labor ratio $\frac{K}{L}$ grows at a constant rate.
3) The capital-income ratio $\frac{K}{y}$ is constant.
4) Capital and labor shares of income are constant.
5) Real rates of return are constant.
6) Growth rates vary across countries, persistently.

1.2 Other facts

Besides these classical facts, there also other empirical regularities which growth theory must account for. These are:

(i) $\frac{Y}{L}$ very dispersed across countries.

(ii) The distribution of $\frac{Y}{L}$ does not seem to spread out (although the variance has increased somewhat).

(iii) Countries with low incomes in 1960 did not on average show higher subsequent growth (this phenomenon is sometimes referred to "no absolute (β) convergence").

(iv) There is, though, "conditional convergence": Within groups classified by 1960 human capital measures (such as schooling), 1960 savings rates, and other indicators, a higher "initial" income y_0 (in 1960) was positively correlated with a lower growth rate g_y.

This is studied by performing the "growth regression":

$$g_{y,\,i}^{1960-1990} = \alpha + \beta \cdot \log y_{0i} + \gamma \cdot \log edu_{0i} + \varepsilon_i \quad i = 1,\, ...,\, n$$

Then controlling for the initial level of education, the growth rate was negatively correlated with initial income for the period 1960-1990: $\widehat{\beta} < 0$. Whereas if the regression is performed without controlling for the level of education, the result for the period is $\widehat{\beta} = 0$ - "no absolute convergence", as mentioned above.

(v) (Foreign?) Trade volume seems to correlate positively with growth.

(vi) Demographic growth ("fertility") is negatively correlated with income.

(vii) Growth in factor inputs (capital K, labor L, land, ...) does not suffice in explaining output growth. The idea of an "explanation" of growth is due to Solow, who envisaged the method of "growth accounting". Based on a neoclassical production function

$$y = z \cdot F\,(K,\, L,\, ...)$$

the variable z captures the idea of technological change. If goods production is performed using a constant returns to scale technology, operated under perfect competition, then (by an application of the Euler Theorem) it is possible to estimate how much out of total production growth is due to each production factor, and how much to the technological factor z. The empirical studies have shown that the contribution of z (the "Solow residual") to output growth is very significant.

(viii) Workers tend to migrate into high-income countries.

2 Exogenous growth

In this section we will study the basic framework to model output growth, by introducing an exogenous change in the production technology, that takes place over time. Mathematically, this is just a simple modification of the standard neoclassical growth model that we have seen before (maybe we should say that we had been studying the standard neoclassical no-growth model!).

Two basic questions arise, one on the technique itself, and one on its reach. First, we may ask how complicated it will be to analyze the model. The answer is quite reassuring to you: it will just be a relatively easy transformation of material we have seen before. The second question is what is the power of this model: What types of technological change can be studied with these tools?

We will separate the issue of growth in two components. One is a technological component: is growth feasible with the assumed production technology. The second one is the decision making aspect involved: will a central planner choose a growing path? Which types of utility function allow what we will call a "balanced growth path"? Then this section of the course is organized in three sections. The first and second ones address the technological and decision making issues, respectively. In the third one, we will study a transformation to the exogenous growth model that will help us in the analysis.

2.1 The technology of exogenous growth

2.1.1 Feasibility of growth

Given the assumptions on the production technology, on the one hand, and on the source of technological progress, on the other, we want to analyze whether the standard neoclassical growth model is really consistent with sustained output growth. At least from the point of view of the production side: is sustainable output growth feasible? You probably guess the answer, but let us go into it in detail.

The standard case is that of "labor augmenting" technological change (à la Solow). The resource constraint in the economy is:

$$c_t + i_t = F_t(K_t, \underbrace{n_t}_{\text{hours}}) \quad = F(K_t, \underbrace{\gamma^t \cdot n_t}_{\text{efficiency units}})$$

where F represents a constant returns to scale production technology, and $\gamma > 1$. The capital accumulation law is

$$K_{t+1} = (1 - \delta) \cdot K_t + i_t$$

Sustained growth is possible, given the constant returns to scale assumption on F.

Our object of study is what is called <u>balanced growth</u>: all economic variables grow at <u>constant</u> rates (that could vary from one variable to another). In this case, this would imply that for all t, the value of each variable in the model is given by:

$$\left.\begin{aligned} y_t &= y_0 \cdot g_y^t \\ c_t &= c_0 \cdot g_c^t \\ K_t &= K_0 \cdot g_K^t \\ i_t &= i_0 \cdot g_i^t \\ n_t &= n_0 \cdot g_n^t \end{aligned}\right\} \begin{array}{l} \text{balanced growth path} \\ \text{all variables grow a constant rates} \\ \text{(that could be different)} \end{array}$$

This is the analogue of a steady state, in a model with growth.

Our task is to find the growth rate for each variable in a balanced growth path, and check whether such a path is consistent. We begin by guessing one of the growth rates, as follows. From the capital accumulation law

$$K_{t+1} = (1 - \delta) \cdot K_t + i_t$$

if both i_t and K_t are to grow at a constant rate, it must be the case that they both grow at the same rate. That is,

$$g_K = g_i$$

must hold.

By the same type of reasoning, from the resource constraint

$$c_t + i_t = F_t(K_t,\, n_t) = F(K_t,\, \gamma^t \cdot n_t) \equiv y_t$$

we must have that $g_y = g_c = g_i$.

Next, using the fact that F represents a constant returns to scale technology (hence it is an homogenous of degree one function), we have that

$$F(K_t,\, \gamma^t \cdot n_t) = \gamma^t \cdot n_t \cdot F\left(\frac{K_t}{\gamma^t \cdot n_t},\, 1\right)$$

Hence, since we have postulated that K_t and y_t grow at a constant rate, we must have that

$$\frac{K_t}{\gamma^t \cdot n_t} = 1$$

in addition, since the time endowment is bounded, actual hours can not growth beyond a certain upper limit (usually normalized to 1); hence $g_n = 1$ must hold.

But this results in $g_K = \gamma$, and all other variables also grow at rate γ. Hence, it is possible to obtain constant growth for all variables: A balanced growth path is technologically feasible.

2.1.2 The nature of technological change

¿From the analysis in the previous section, it seems natural to ask whether the assumption that the technological change is "labor augmenting" is relevant or not. First, what other kinds of technological change can we think of? Let us write the economy's resource constraint, with all the possible types of technical progress that the literature talks about:

$$c_t + \gamma_i^{-t} \cdot i_t = \gamma_z^t \cdot F\left(\gamma_K^t \cdot K_t,\, \gamma_n^t \cdot n_t\right)$$

and we have:

- γ_i^{-t}: Investment-specific technological change. You could think of this as the relative price of capital goods showing a long term decreasing trend, *vis a vis* consumption goods. In fact this has been measured in the data, and in the case of the US this factor accounts for 50% of growth **(??? Is this correct?)**.
- γ_z^t: Neutral (or Hicks-neutral) technological change.
- γ_K^t: Capital augmenting technological change.
- γ_n^t: Labor augmenting technological change.

The question is which ones of these γ's (or which combinations of them) can be larger than 1 on a balanced growth path. We can immediately see that if F is homogeneous of degree 1 (if production technology exhibits constant returns to scale) then the γ_z is redundant, since in that case we can rewrite:

$$\gamma_z^t \cdot F\left(\gamma_K^t \cdot K_t,\, \gamma_n^t \cdot n_t\right) = F\left[\left(\gamma_z \cdot \gamma_K\right)^t \cdot K_t,\, \left(\gamma_z \cdot \gamma_n\right)^t \cdot n_t\right]$$

As for the admissible values of the other γ's, we have the following result.

Theorem 1 *For a balanced growth path to hold, none of the shift factors γ (except γ_n) can be larger than 1, unless F is a Cobb-Douglass function.*

Proof. In one of the directions, the proof requires a partial differential equations argument which we shall not develop. However, we will show that if F is a Cobb-Douglass function then any of the γ can be larger than 1, without invalidating a balanced growth path as a solution.

If F is a Cobb-Douglass function, the resource constraint reads:

$$c_t + \gamma_i^{-t} \cdot i_t = \left(\gamma_K^t \cdot K_t\right)^{\alpha} \cdot \left(\gamma_n^t \cdot n_t\right)^{1-\alpha}$$

Notice that we can redefine:

$$\widehat{\gamma}_n \equiv \gamma_K^{\frac{\alpha}{1-\alpha}} \cdot \gamma_n$$

to rewrite the production function:

$$\left(\gamma_K^t \cdot K_t\right)^{\alpha} \cdot \left(\gamma_n^t \cdot n_t\right)^{1-\alpha} = K_t^{\alpha} \cdot \left(\widehat{\gamma}_n^t \cdot n_t\right)^{1-\alpha}$$

In addition, consider the capital accumulation equation:

$$K_{t+1} = (1-\delta) \cdot K_t + i_t$$

Dividing through by γ_i^t,

$$\frac{K_{t+1}}{\gamma_i^{t+1}} \cdot \gamma_i = (1-\delta) \cdot \frac{K_t}{\gamma_i^t} + i_t \cdot \gamma_i^{-t}$$

We can define

$$\begin{aligned} \widetilde{K}_t &\equiv \frac{K_t}{\gamma_i^t} \\ \widetilde{i}_t &\equiv \frac{i_t}{\gamma_i^t} \end{aligned}$$

and, replacing $\widetilde{K}_t$ in the production function, we obtain:

$$\begin{aligned} c_t + \widetilde{i}_t &= \left(\gamma_K^t \cdot \gamma_i^t \cdot \widetilde{K}_t\right)^{\alpha} \cdot \left(\gamma_n^t \cdot n_t\right)^{1-\alpha} \\ \widetilde{K}_{t+1} \cdot \gamma_i &= (1-\delta) \cdot \widetilde{K}_t + \widetilde{i}_t \end{aligned}$$

The model has been transformed into an equivalent system in which $\widetilde{K}_{t+1}$, instead of K_{t+1} is the object of choice (more on this below). Notice that since F is Cobb-Douglass, the γ's multiplying $\widetilde{K}_t$ can in fact be written as labor-augmenting technological drift factors. Performing the transformation, the rate of efficiency labor growth is:

$$\gamma_n \cdot \gamma_K^{\frac{\alpha}{1-\alpha}} \cdot \gamma_i^{\frac{\alpha}{1-\alpha}}$$

γ_i^{-t} can be ■

2.2 Choosing growth

The next issue to address is whether an individual who inhabits an economy in which there is some sort of exogenous technological progress, and in which the production technique is such that sustained growth is feasible, will choose a growing output path or not.

Initially, Solow overlooked this issue by assuming that capital accumulation rule was determined by the policy rule

$$i_t = s \cdot y_t$$

where the savings rate $s \in [0,\, 1]$ was constant and exogenous. It is clear that such a rule can be consistent with a balanced growth path. Then the underlying premise is that the consumers' preferences are such that they choose a growing path for output.

However, this is too relevant an issue to be overlooked. What is the generality of this result? Specifically, what are the conditions on preferences for constant growth to obtain? Clearly, the answer is that not all types of preferences will work. We will restrict attention to the usual time-separable preference relations. Hence the problem faced by a central planner will be of the form:

$$\max_{\{i_t,\, c_t,\, K_{t+1},\, n_t\}} \left\{ \sum_{t=0}^{\infty} \beta^t \cdot u\left(c_t,\, n_t\right) \right\}$$

$$\text{s.t.} \quad \begin{aligned} & c_t + i_t = F\left(K_t,\, \gamma^t \cdot n_t\right) \\ & K_{t+1} = i_t + (1-\delta) \cdot K_t \\ & K_0 \ \text{given} \end{aligned} \tag{CP}$$

For this type of preference relations, we have the following result:

Theorem 2 *Balanced growth* <u>*is possible*</u> *as a solution to the central planner's problem (CP) if and only if*

$$u\left(c,\, n\right) = \frac{c^{1-\sigma} \cdot v\left(1-n\right) - 1}{1-\sigma}$$

(where time endowment is normalized to 1 *as usual).*

Proving the theorem is rather endeavored in one of the two directions of the double implication, because the proof involves partial differential equations. Also notice we say that balanced growth is a <u>possible</u> solution. The reason is that initial conditions also have an impact on the resulting output growth. The initial state has to be such that the resulting model dynamics (that may initially involve non-constant growth) eventually lead the system to a balanced growth path (constant growth). Not any arbitrary initial conditions will satisfy this.

Comments:

1. $v\left(1-n\right) = A$ fits the theorem assumptions; hence non-valued leisure is consistent with balanced growth path.

2. What happens if we introduce a "slight" modifications to $u\left(c,\, n\right)$, and use a functional form like

$$u\left(c,\, n\right) = \frac{\left(c-\overline{c}\right)^{1-\sigma} - 1}{1-\sigma} \ ?$$

$\bar{c}$ can be interpreted as a minimum subsistence consumption level. When c gets large with respect to $\bar{c}$, risk aversion decreases. Then for a low level of consumption c, this utility function representation of preferences will not be consistent with a balanced growth path; but, as c increases, the dynamics will tend towards balanced growth. This could be an explanation to observed growth behavior in the early stages of development of poor countries.

2.3 Transforming the model

You should solve the exogenous growth model for a balanced growth path. To do this, assume that preferences are represented by the utility function

$$\frac{c^{1-\sigma} \cdot v(1-n) - 1}{1-\sigma}$$

You should take first order conditions of the central planner's problem (CP) described above using this preference representation. Next you should *assume* that there is balanced growth, and show that the implied system of equations can be satisfied.

After solving for the growth rates, the model can be transformed into a stationary one. We will do this for the case of labor-augmenting technology under the constant returns to scale. The original problem is

$$\max_{\{i_t,\, c_t,\, K_{t+1},\, n_t\}} \left\{ \sum_{t=0}^{\infty} \beta^t \cdot \frac{c^{1-\sigma} \cdot v(1-n) - 1}{1-\sigma} \right\}$$

$$s.t. \quad \begin{aligned} & c_t + i_t = \gamma^t \cdot n_t \cdot F\left(\frac{K_t}{\gamma^t \cdot n_t}, 1\right) \\ & K_{t+1} = i_t + (1-\delta) \cdot K_t \\ & K_0 \text{ given} \end{aligned} \tag{GM}$$

We know that the balanced growth solution to this Growth Model (GM) has all variables growing at rate γ, except for labor. We define transformed variables by dividing each original variable by its growth rate:

$$\begin{aligned} \widehat{c}_t &= \frac{c_t}{\gamma^t} \\ \widehat{i}_t &= \frac{i_t}{\gamma^t} \\ \widehat{K}_t &= \frac{K_t}{\gamma^t} \end{aligned}$$

and thus obtain the transformed model:

$$\max_{\{\widehat{i}_t,\, \widehat{c}_t,\, \widehat{K}_{t+1},\, n_t\}} \left\{ \sum_{t=0}^{\infty} \beta^t \cdot \frac{\widehat{c}^{1-\sigma} \cdot \gamma^{t\cdot(1-\sigma)} \cdot v(1-n) - 1}{1-\sigma} \right\}$$

$$s.t. \quad \begin{aligned} & \left(\widehat{c}_t + \widehat{i}_t\right) \cdot \gamma^t = \gamma^t \cdot n_t \cdot F\left(\frac{\widehat{K}_t \cdot \gamma^t}{\gamma^t \cdot n_t}, 1\right) \\ & \widehat{K}_{t+1} \cdot \gamma^{t+1} = \left[\widehat{i}_t + (1-\delta) \cdot \widehat{K}_t\right] \cdot \gamma^t \\ & K_0 \text{ given} \end{aligned}$$

Notice that we can write

$$\sum_{t=0}^{\infty} \beta^t \cdot \frac{\widehat{c}^{1-\sigma} \cdot \gamma^{t\cdot(1-\sigma)} \cdot v(1-n) - 1}{1-\sigma} = \sum_{t=0}^{\infty} \widehat{\beta}^t \cdot \frac{\widehat{c}^{1-\sigma} \cdot v(1-n) - 1}{1-\sigma} + \sum_{t=0}^{\infty} \widehat{\beta}^t \cdot \frac{1-\gamma^{-t\cdot(1-\sigma)}}{1-\sigma}$$

where

$$\widehat{\beta} = \beta \cdot \gamma^{(1-\sigma)}$$

Then we can simplify the γ's to get:

$$\max_{\left\{\widehat{i}_t, \widehat{c}_t, \widehat{K}_{t+1}, n_t\right\}} \left\{ \sum_{t=0}^{\infty} \beta^t \cdot \frac{\widehat{c}^{1-\sigma} \cdot v(1-n) - 1}{1-\sigma} + \sum_{t=0}^{\infty} \widehat{\beta}^t \cdot \frac{1-\gamma^{-t\cdot(1-\sigma)}}{1-\sigma} \right\}$$

$$s.t. \quad \begin{aligned} & \widehat{c}_t + \widehat{i}_t = n_t \cdot F\left(\frac{\widehat{K}_t}{n_t}, 1\right) \\ & \widehat{K}_{t+1} \cdot \gamma = \widehat{i}_t + (1-\delta) \cdot \widehat{K}_t \\ & K_0 \text{ given} \end{aligned} \tag{TM}$$

And we are back to the standard neoclassical "growth" model that we have been dealing with before. The only differences are that there is a γ factor in the capital accumulation equation, and the discount factor is modified.

We need to check the conditions for this problem to be well defined. This requires that $\beta \cdot \gamma^{1-\sigma} < 1$. Recall that $\gamma > 1$, and the usual assumption is $0 < \beta < 1$. Then:

1. If $\sigma > 1$, $\gamma^{1-\sigma} < 1$ so $\beta \cdot \gamma^{1-\sigma} < 1$ holds.
2. If $0 < \sigma < 1$, then for some parameter values of γ and β, we may run into a poorly defined problem.
3. If $\sigma > 2$, then we can afford to have $\beta > 1$!

Next we address the issue of the system behavior. If leisure is not valued and the production technology

$$f(k) \equiv F\left(\frac{K}{L}, 1\right) + (1-\delta) \cdot \frac{K}{L}$$

satisfies the Inada conditions:

- $f(0) = 0$
- $f'(\cdot) > 0$
- $f''(\cdot) < 0$
- $\lim_{k\to\infty} f'(\cdot) = 0$
- $\lim_{k\to 0} f'(\cdot) = \infty$

Then global convergence to steady state obtains for the transformed model (TM):

$$\begin{aligned} \lim_{t\to\infty} \widehat{c}_t &= \overline{\widehat{c}} \\ \lim_{t\to\infty} \widehat{i}_t &= \overline{\widehat{i}} \\ \lim_{t\to\infty} \widehat{k}_t &= \overline{\widehat{k}} \end{aligned}$$

But, construction of $\widehat{c}_t$, $\widehat{i}_t$, and $\widehat{k}_t$, this is equivalent to ascertain that the original variables c_t, i_t, and k_t grow at rate γ asymptotically.

Therefore with the stated assumptions on preferences and on technology, the model converges to a balanced growth path, in which all variables grow at rate γ. This rate is exogenously determined; it is a parameter in the model. That is the reason why it is called "exogenous" growth model.

3 Endogenous growth

The exogenous growth framework analyzed before has a serious shortfall: growth is not truly a result in such model. It is an assumption. However, we have reasons (data) to suspect that growth must be a rather more complex phenomenon than this long term productivity shift γ, that we have treated as somehow intrinsic to economic activity. In particular, rates of output growth have been very different across countries for long periods; trying to explain this fact as merely the result of different γ's is not a very insightful approach. We would prefer our model to produce γ as a result; thus, we look at endogenous growth models.

But what if the countries that show smaller growth rates are still in transition, and transition is slow? Could this be a plausible explanation of the persistent difference in growth? What does our model tell us about this? At least locally, the rate of convergence can be found from

$$\log y' - \log \overline{y} = \lambda \cdot (\log y - \log \overline{y})$$

where λ is the eigenvalue smaller than 1 in absolute value found when linearizing the dynamics of the growth model (around the steady state). Recall it was the root to a second degree polynomial. The closest λ is to 1 (in absolute value), the slowest the convergence. Notice that this equation can be rewritten to yield the growth regression:

$$\log y' - \log y = -(1-\lambda) \cdot \log y + (1-\lambda) \cdot \log \overline{y} + \alpha$$

where $-(1-\lambda)$ is the "β" parameter in the growth regressions, $\log y$ shows up as $\log y_0$; $(1-\lambda)$ is the γ, and $\log \overline{y}$ the residual z; and finally α (usually called γ_0) is the intercept that shows up whenever a technological change drift is added.

In calibrations with "reasonable" utility and production functions, λ tends to become small in absolute value - hence, not large enough to explain the difference in growth rates of Korea and Chad. In general, the less curvature the return function shows, the fastest the convergence. The extreme special cases are:

1. u linear $\Rightarrow \lambda = 0$ - immediate convergence
2. f linear $\Rightarrow \lambda = 1$ - no convergence

The more curvature in u, the less willing consumers are to see their consumption pattern vary over time - and growth is a (persistent) variation. On the other hand, the more curvature in f, the higher the marginal return on capital when the accumulated stock is small; hence the more willing consumers are to put up with variation in their consumption stream, since the reward is higher.

3.1 The AK model

Let us recall the usual assumptions on the production technology in the neoclassical growth model. We had that F was constant returns to scale, but also that the "per capital" production function f verified:

- $f(0) = 0$
- $f'(\cdot) > 0$
- $f''(\cdot) < 0$
- $\lim_{x \to 0} f'(\cdot) = \infty$
- $\lim_{x \to \infty} f'(\cdot) = 0$

With the resulting shape of the global dynamics (with a "regular" utility function):

Long run growth is not feasible. Notice that whenever the capital stock k exceeds the level k^*, then next period's capital will decrease: $k' < k$. In order to allow long run growth, we need the introduce at least some change to the production function: We must dispose of the assumption that $\lim_{x\to\infty} f'(\cdot) = 0$. f should not cross the 45^o line.

Then $\lim_{x\to\infty} f'(\cdot) > 0$ seems necessary for continuous growth to obtain. In addition, if we have that $\lim_{x\to\infty} f'(\cdot) = 1$ (that is, the production function is asymptotically parallel to the 45^o line), then exponential growth is not feasible - only arithmetic growth is. Then we must have $\lim_{x\to\infty} f'(\cdot) > 1$ for a growth *rate* to be sustainable through time.

The simplest way of having this, is assuming the production technology to be represented by a function of the form:

$$f(k) = A \cdot k$$

with $A > 1$. More generally, for any depreciation rate δ, we have that the return on capital is

$$\begin{aligned}(1-\delta)\cdot k + f(k) &= (1-\delta)\cdot k + A \cdot k \\ &= (1-\delta+A)\cdot k \\ &\equiv \widetilde{A}\cdot k\end{aligned}$$

so the requirement in fact is $A > \delta$ for exponential growth to be feasible.

The next question is whether the consumer will choose growth, and if so, how fast. We will answer this question assuming a CES utility function (needed for balanced growth), with non-valued leisure. The planner's problem then is:

$$\begin{aligned}U = &\max_{\{c_t,\, k_{t+1}\}_{t=0}^{\infty}} \left\{ \sum_{t=0}^{\infty} \beta^t \cdot \frac{c_t^{1-\sigma}}{1-\sigma} \right\} \\ s.t. \quad & c_t + k_{t+1} = A \cdot k_t\end{aligned}$$

where $\sigma > 0$. The Euler Equation is

$$c_t^{-\sigma} = \beta \cdot c_{t+1}^{-\sigma} \cdot A$$

The marginal return on savings is $R_t = A$, a constant. This seems like a price. Before, we the (gross) return on savings was

$$R = f_k + 1 - \delta$$

For example, in the Cobb-Douglass case it was

$$\alpha \cdot k^{1-\alpha}$$

which tends to 0 if k goes to infinity.

Back to the Euler Equation, we have that the growth rate of consumption must satisfy:

$$\frac{c_{t+1}}{c_t} = (\beta \cdot A)^{\frac{1}{\sigma}}$$

The savings rate is a function of (all) the parameters in utility and production - the return function. Notice that this implies that the growth rate is constant as from $t = 0$. There are no transitional dynamics in this model; the economy is in the balanced growth path from the start. There will be long run growth provided that

$$(\beta \cdot A)^{\frac{1}{\sigma}} > 1 \tag{PC I}$$

This does not quite settle the problem, though: an issue remains to be addressed. If the parameter values satisfy the condition for growth: is utility still bounded? Are the transversality conditions met? We must evaluate the optimal path using the utility function:

$$U = \sum_{t=0}^{\infty} \left[\beta \cdot \left[(\beta \cdot A)^{\frac{1}{\sigma}} \right]^{1-\sigma} \right]^t \cdot \frac{c_0^{1-\sigma}}{1-\sigma}$$

So the necessary condition is that

$$\beta \cdot \left[(\beta \cdot \alpha)^{\frac{1}{\sigma}} \right]^{1-\sigma} < 1 \tag{PC II}$$

The two Parameter Conditions (PC I) and (PC II) must simultaneously hold for a balanced growth path to obtain.

Remark 3 (Distortionary taxes and growth) *Notice that the competitive allocation in this problem equals the central planner's (Why?). Now suppose that we have the government levy a distortionary tax on (per capita) capital income and use the proceeds to finance a lump-sum transfer. Then the consumer's decentralized problem has the following budget constraint:*

$$c_t + k_{t+1} = (1 - \tau_k) \cdot R_t \cdot k_t + \tau_t$$

while the government's budget constraint requires that

$$\tau_k \cdot R_t \cdot k_t = \tau_t$$

This problem is a little more endeavored to solve due to the presence of the lump-sum transfers τ_t. Notwithstanding this, you should know that τ_k (the distortionary tax on capital income) will affect the long run growth rate.

Remark 4 (Explanatory power) *Is the model realistic? Let us address this issue by stating the assumptions and the results obtained:*

∗ **Assumptions** *The $A \cdot K$ production function could be interpreted as a special case of the Cobb-Douglass function with $\alpha = 1$ - then labor is not productive. However, this contradicts actual data, that shows that labor is a hugely significant component of factor input. Clearly, in practise labor is important. But this is not captured by the assumed production technology.*

We could imagine a model where labor "becomes" unproductive; for example assuming that

$$F_t(K_t, n_t) = A \cdot K_t^{\alpha_t} \cdot n_t^{1-\alpha_t}$$

then if $\lim_{t \to \infty} \alpha_t = 1$, we have asymptotic linearity in capital. But this is unrealistic.

∗ **Results** *The growth has become a function of underlying parameters in the economy, affecting preferences and production. Could then the dispersion in cross-country growth rates be explained by differences in these parameters? Country i's Euler Equation (with a distortionary tax on capital income) would be:*

$$\left(\frac{c_{t+1}}{c_t}\right)_i = \left[\beta_i \cdot A_i \cdot \left(1 - \tau_k^i\right)\right]^{\frac{1}{\sigma_i}}$$

But the problem with the AK model is that, if parameters are calibrated to mimic the data's dispersion in growth rates, the simulation results in too much divergence in output level. The dispersion in 1960-1990 growth rates would result in a difference in output levels wider than the actual.

Remark 5 (Transitional dynamics) *The AK model implies no transitional dynamics. However, we tend to see transitional dynamics in the data (recall the conditional convergence result in growth regressions).*

3.2 Romer's externality model

The intellectual precedent to this model is Arrow's *learning by doing* paper in the 1960s. The basic idea is that there are externalities to capital accumulation, so that individual savers do not realize the full return on their investment. Each individual firm operates the following production function:

$$F\left(K,\, L,\, \overline{K}\right) = A \cdot K^{\alpha} \cdot L^{1-\alpha} \cdot \overline{K}^{\rho}$$

where K is the capital operated by the firm, and $\overline{K}$ is the aggregate capital stock in the economy. We assume that $\rho = 1 - \alpha$; so that in fact a central planner faces an AK-model decision problem. Notice that if we assumed that $\alpha + \rho > 1$, then balanced growth path would not be possible.

The competitive equilibrium will involve a wage rate equal to:

$$w_t = (1 - \alpha) \cdot A \cdot K_t^{\alpha} \cdot L_t^{-\alpha} \cdot \overline{K}_t^{1-\alpha}$$

Let us assume that leisure is not valued and normalize the labor endowment L_t 1 in every t. Assume that there is a measure 1 of representative firms, so that the equilibrium wage must satisfy

$$w_t = (1 - \alpha) \cdot A \cdot \overline{K}_t$$

So notice that in this model, wage increases whenever there is growth, and the wage as a fraction of total output is substantial. The rental rate, meanwhile, is given by:

$$R_t = \alpha \cdot A$$

The consumer's decentralized Euler Equation will be, assuming CES utility:

$$\frac{c_{t+1}}{c_t} = \left[\beta \cdot (R_{t+1} + 1 - \delta)\right]^{\frac{1}{\sigma}}$$

So, substituting for the rental rate, we can see that the rate of change in consumption is given by:

$$g_c^{CE} = \left[\beta \cdot (\alpha \cdot A + 1 - \delta)\right]^{\frac{1}{\sigma}}$$

On the other hand, it is immediate that, since a planner faces an AK model, his chosen growth rate would be:

$$g_c^{CP} = \left[\beta \cdot (A + 1 - \delta)\right]^{\frac{1}{\sigma}}$$

Then $g_c^{CP} > g_c^{CE}$; a competitive equilibrium implements a lower than optimal growth rate, consistently with the presence of externalities to capital accumulation.

Remark 6 (Pros and cons of this approach) *The following advantages and disadvantages of this model can be highlighted:*

+ *Overcomes the "labor irrelevant" shortfall of the AK model.*
− *There is little evidence in support of a significant externality to capital accumulation. Notice that if we agreed for example that $\alpha = 1/3$, then the externality effect would be immense.*
− *The model leads to a large divergence in output levels, just as the AK.*

3.3 Lucas' human capital accumulation model

In the Lucas' model, plain labor in the production function is replaced by human capital. This can be accumulated, so the technology does not run into decreasing marginal returns. For example, in the Cobb-Douglass case, we have:

$$F(K,\,H) = A \cdot K^{\alpha} \cdot H^{1-\alpha}$$

There are two distinct capital accumulation equations:

$$\begin{aligned} H_{t+1} &= \left(1-\delta^{H}\right) \cdot H_t + I_t^{H} \\ K_{t+1} &= \left(1-\delta^{K}\right) \cdot K_t + I_t^{K} \end{aligned}$$

And the resource constraint in the economy reads:

$$c_t + I_t^{H} + I_t^{K} = A \cdot K_t^{\alpha} \cdot H_t^{1-\alpha}$$

Notice that, in fact, there are two assets: H and K. But there is no uncertainty; hence one is redundant. The return on both assets must be equal.

Unlike the previous model, in the current setup a competitive equilibrium does implement the central planner's solution (why can we say so?). The first order conditions in the latter read:

$$\begin{aligned} c_t &: \beta^{t} \cdot c_t^{-\sigma} = \lambda_t \\ K_{t+1} &: \lambda_t = \lambda_{t+1} \cdot \left[1-\delta^{K} + F_K\left(K_{t+1},\,H_{t+1}\right)\right] \\ H_{t+1} &: \lambda_t = \lambda_{t+1} \cdot \left[1-\delta^{H} + F_H\left(K_{t+1},\,H_{t+1}\right)\right] \end{aligned}$$

Which lead to two equivalent instances of the Euler Equation:

$$\frac{c_{t+1}}{c_t} = \left(\beta \cdot \left[1-\delta^{K} + F_K\left(\frac{K_{t+1}}{H_{t+1}},\,1\right)\right]\right)^{\frac{1}{\sigma}} \quad \text{(EE-K)}$$

$$\frac{c_{t+1}}{c_t} = \left(\beta \cdot \left[1-\delta^{H} + F_H\left(\frac{K_{t+1}}{H_{t+1}},\,1\right)\right]\right)^{\frac{1}{\sigma}} \quad \text{(EE-H)}$$

Notice that if the ratio $\dfrac{K_{t+1}}{H_{t+1}}$ remains constant through time, this delivers balanced growth. Let us denote

$$x_t \equiv \frac{K_t}{H_t}$$

and we have that:

$$1-\delta^{K} + F_K\left(x_t,\,1\right) = 1-\delta^{H} + F_H\left(x_t,\,1\right)$$

But then equilibrium in the asset market requires that $x_t = \overline{x}$ be constant for all t; and $\overline{x}$ will depend on δ^{H}, δ^{K}, and parameters of the production function F.

Example 7 *Assume that* $\delta^H = \delta^K$*, and* $F(K, H) = A \cdot K^\alpha \cdot H^{1-\alpha}$.

Then (EE-K) = (EE-H) requires that:

$$\alpha \cdot A \cdot x^{\alpha-1} = (1-\alpha) \cdot A \cdot x^\alpha$$

So

$$x = \frac{\alpha}{1-\alpha} = \frac{K_t}{H_t}$$

¿From $t = 1$ *onwards,* $K_t = x \cdot H_t$ *- it is like having the capital stock on the one side, and the state variable on the other. Then*

$$\begin{aligned} A \cdot K_t^\alpha \cdot H_t^{1-\alpha} &= A \cdot (x \cdot H_t)^\alpha \cdot H_t^{1-\alpha} \\ &= \widetilde{A} \cdot H_t \\ &= \widehat{A} \cdot K_t \end{aligned}$$

where $\widetilde{A} \equiv A \cdot x^\alpha$*, and* $\widehat{A} \equiv A \cdot x^{1-\alpha}$*. In any case, this reduces to an AK model !*

Remark 8 (Pros and cons of this approach) *We can highlight the following advantages and disadvantages of this model:*

+ *Labor is treated "seriously", and not resorting to "tricks" like externalities.*

− *The law of motion of human capital is too "mechanic-like":*

$$H_{t+1} = \left(1 - \delta^H\right) \cdot H_t + I_t^H$$

Arguably, knowledge might be bounded above at some point. On the other hand, this issue could be counter-argued by saying that H_t *should be interpreted as general formation (such as on-the-job training, etcetera), and not narrowly as schooling.*

− *This model implies divergence of output levels; it is an AK model in essence, as well.*

3.4 Romer's qualitative technological change

3.4.1 The model

Based on the Cobb-Douglass production function $F(K, L) = A \cdot K^\alpha \cdot L^{1-\alpha}$, this model seeks to make A endogenous. One possible way of modeling this would be simply to make firms chose the inputs knowing that this will affect A. However, if A is increasing in K and L, this would lead to increasing returns, since

$$A(\lambda \cdot K, \lambda \cdot L) \cdot (\lambda \cdot K)^\alpha \cdot (\lambda \cdot L)^{1-\alpha} > \lambda \cdot A \cdot K^\alpha \cdot L^{1-\alpha}$$

An alternative approach would have A being the result of an external effect of firm's decisions. But the problem with this approach is that we want A to be *somebody's* choice; hence an externality will not work. The way out of this A dilemma is to lift the assumption of perfect competition in the economy.

In the model to be presented, A will represent "variety" in production inputs. The larger A, the wider the range of available production (intermediate) goods. Specifically, in this economy capital and consumption goods are produced according to the function

$$y_t = L_t^\beta \cdot \int_0^{A_t} x_t^{1-\beta}(i)\, di$$

Where i is the type of intermediate goods, and $x_t(i)$ is the amount of good i used in production at date t. Therefore, there is a measure A_t of different intermediate goods. You may notice that the production function exhibits constant returns to scale.

The intermediate goods $x_t(i)$ are produced with capital goods using a linear technology:

$$\int_0^{A_t} \eta \cdot x_t(i)\, di = K_t$$

That is, η units of capital are required to produce 1 unit of intermediate good of type i, for all i.

The law of motion and resource constraint in this economy are the usual:

$$\begin{aligned} K_{t+1} &= (1-\delta) \cdot K_t + I_t \\ c_t + I_t &= y_t \end{aligned}$$

We will assume that an amount L_{1t} of labor is supplied to the goods production sector at time t. In addition, we assume that A_t grows at rate γ:

$$A_{t+1} = \gamma \cdot A_t$$

Q: Given this growth in A, is long run output growth feasible?

A: A key issue to answer this question is to determine the allocation of capital among the different types of intermediate goods. Notice that this decision is of a static nature: the choice at t has no (dynamic) consequences on the future periods' state.

So the production maximizing problem is to:

$$\begin{aligned} &\max_{x(i)} \left\{ L_{1t}^\beta \cdot \textstyle\int_0^{A_t} x_t^{1-\beta}(i)\, di \right\} \\ &s.t.\ \textstyle\int_0^{A_t} \eta \cdot x_t(i)\, di = K_t \end{aligned}$$

Since the objective function is concave, the optimal choice has $x_t(i) = x_t$ for all i. This outcome can be interpreted as a preference towards "variety" - as much variety as possible is chosen.

Replacing the optimal solution in the constraint:

$$\begin{aligned} \int_0^{A_t} \eta \cdot x_t\, di &= K_t \\ A_t \cdot x_t \cdot \eta &= K_t \qquad \text{((I))} \end{aligned}$$

And maximized production is:

$$\begin{aligned} y_t &= L^\beta \cdot \int_0^{A_t} x_t^{1-\beta}\, di \\ &= L^\beta \cdot A_t \cdot x_t^{1-\beta} \qquad \text{((II))} \end{aligned}$$

Using (I) in (II),

$$\begin{aligned} y_t &= L_{1t}^{\beta} \cdot A_t \cdot \left(\frac{K_t}{\eta \cdot A_t}\right)^{1-\beta} \\ &= \frac{L_{1t}^{\beta}}{\eta^{1-\beta}} \cdot A_t^{\beta} \cdot K_t^{1-\beta} \end{aligned}$$

Clearly A_t^{β} grows if A_t grows at rate γ. Then the production function is linear in the growing terms. Therefore the answer to our question is Yes: A balanced growth path is feasible; with K_t, y_t and A_t growing at rate γ.

So the next issue is how to determine γ, since we are dealing with an endogenous growth model. We will make the following assumption on the motion equation for A_t:

$$A_{t+1} = A_t + L_{2t} \cdot \delta \cdot A_t$$

Where L_{2t} denotes labor effort in research and development, and $L_{2t} \cdot \delta$ is the number of new "blueprints" that are developed at time t, as a consequence of this R&D. This motion equation resembles a learning by doing effect.

Exercise: Assume that leisure is not valued, and total time endowment is normalized to 1. Then the amount of labor effort allocated to the production and to the R&D sectors must satisfy the constraint:

$$L_{1t} + L_{2t} = 1$$

Assume that there is an individual who consumes the consumption goods c_t. Solve the planning problem to obtain γ. ■

3.4.2 The decentralized problem

We will work with the decentralized problem. We assume that there is perfect competition in the final output industry. Then a firm in that industry solves at time t:

$$\max_{x_t(i),\, L_{1t}} \left\{ L_{1t}^{\beta} \cdot \int_0^{A_t} x_t^{1-\beta}(i)\, di - w_t \cdot L_{1t} - \int_0^{A_t} q_t(i) \cdot x_t(i)\, di \right\}$$

Notice that the firm's is a static problem. w_t and $q_t(i)$ are taken as given. Equilibrium in the final goods market then requires that these are:

$$\begin{aligned} w_t &= \beta \cdot L_{1t}^{1-\beta} \cdot \int_0^{A_t} x_t^{1-\beta}(i)\, di \\ q_t(i) &= (1-\beta) \cdot L_{1t}^{\beta} \cdot x_t^{-\beta}(i) \qquad ((*)) \end{aligned}$$

As for the intermediate goods industry, instead of perfect, we will assume that there is *monopolistic competition*. There is only one firm per type i (a patent holder). Each patent holder takes the demand function for its product as given. Notice that $(*)$ is just the inverse of this demand function. All other relevant prices are also taken as given. In particular, the rental rate R_t, paid for the capital that is rented to consumers. Then the owner of patent i solves:

$$\pi(i) = \max_{K_t^i} \left\{ q_t(i) \cdot x_t(i) - R_t \cdot K_t^i \right\}$$
$$s.t.\ x(i) \cdot \eta = K_t^i$$

or equivalently, using (∗) and the constraint,

$$\pi\left(i\right)=\max_{K_t^i}\left\{\left(1-\beta\right)\cdot L_{1t}^{\beta}\cdot\left(\frac{K_t^i}{\eta}\right)^{1-\beta}-R_t\cdot K_t^i\right\}$$

The first order conditions for this problem are:

$$\left(1-\beta\right)^2\cdot L_{1t}^{\beta}\cdot\eta^{1-\beta}\cdot K_t^{i\ -\beta}=R_t$$

$\pi\left(i\right)>0$ is admissible ! The firm owns a patent, and obtains a rent from it. However, this patent is not cost free. It is produced by "R&D firms", who sell them to intermediate goods producers. Let p_t^P denote the price of a patent at time t.

Then ideas producers solve:

$$\max_{A_{t+1},\,L_{2t}}\left\{p_t^P\cdot\left(A_{t+1}-A_t\right)-w_t\cdot L_{2t}\right\}$$
$$s.t.\ A_{t+1}=A_t+L_{2t}\cdot\delta\cdot A_t$$

And we will assume that there is free entry in the ideas industry: hence, there must be zero profits from engaging in research and development. Notice that there is an externality (sometimes called "standing on the shoulders of giants"). The reason is that the decision involving the change in A, $A_{t+1}-A_t$, affects future production via the term $\delta\cdot A_{t+j}$ in the equation of motion for A_t. But this effect is not realized by the firm who chooses the change in A. Hence this is a second reason why the planner's and the decentralized problems will have different solutions (the first one was the monopoly power of patent holders).

The zero profit condition in the ideas industry requires that the price p_t^P be determined from the first order condition

$$p_t^P\cdot\delta\cdot A_t=w_t$$

where w_t is as determined in the market for final goods.

Once this is solved, if p_t^C denotes the date-0 price of consumption (final) goods at t, then we must have

$$p_t^P\cdot p_t^C=\sum_{s=t+1}^{\infty}\pi_s\left(i\right)\cdot p_s^C$$

As a result, nobody makes profits in equilibrium. The inventors of patents appropriate the extraordinary rents that intermediate goods producers are going to obtain from purchasing the rights on the invention.

3.4.3 Balanced growth path

Next we solve for a (symmetric) balanced growth path. We assume that all variables grow at (the same, and) constant rates:

$$\begin{aligned}K_{t+1}&=\gamma\cdot K_t\\A_{t+1}&=\gamma\cdot A_t\\c_{t+1}&=\gamma\cdot c_t\\L_{1t}&=L_1\\L_{2t}&=L_2\\w_{t+1}&=\gamma\cdot w_t\end{aligned}$$

With respect to the intermediate goods $x_t(i)$, we already know that an equal amount of each type of them is produced each period: $x_t(i) = x_t$. In addition, we have that this amount must satisfy:

$$A_t \cdot \eta \cdot x_t = K_t$$

Since both A_t and K_t (are assumed to) grow at rate γ, then x_t must remain constant for this equation to hold for every t. Hence,

$$x_t = x = \frac{K_t}{A_t \cdot \eta}$$

Then the remaining variables in the model must remain constant as well:

$$\begin{aligned} R_t &= R \\ \pi_t(i) &= \pi \\ p_t^P &= p^P \\ q_t(i) &= q \end{aligned}$$

It is up to you to solve this problem:

Exercise: Write down a system of n equations and n unknowns determining γ, L_1, L_2, etcetera. After that, compare to the planner's growth rate γ which you have already found. Which one is higher?

4 Concluding Remarks

4.1 Dealing with convergence

One of the key elements to test the explanatory power of both the exogenous and the endogenous growth models is their implications respecting convergence of growth rates among different countries. Recall that:

Exogenous growth	*vs*	Endogenous growth
$A \cdot K^{\alpha} \cdot L^{1-\alpha}$		$A \cdot K$
does not lead to divergence		leads to divergence in relative income levels

Is it possible to solve the riddle through appropriate calibration? Using $\alpha = 1/3$, the exogenous growth framework leads to too fast convergence. A "brilliant" solution is to set $\alpha = 2/3$. The closer to 1 α is set, the closer the exogenous growth model looks like the AK model.

But we are not so free to play around with α. This parameter can be measured from the data:

$$\alpha = \frac{K \cdot F_K}{y} = \frac{K \cdot R}{y}$$

A possible solution to this problem is to introduce a "mystery capital", so that the production function looks like:

$$y = A \cdot K^{\alpha} \cdot L^{\beta} \cdot S^{1-\alpha-\beta}$$

Or, alternatively introduce "human capital" as a third production factor, besides physical capital and labor:

$$y = A \cdot K^{\alpha} \cdot L^{\beta} \cdot H^{1-\alpha-\beta}$$

4.2 Dealing with returns on investment

We will explore the argument developed by Lucas [] to study the implications of the growth model in cross-country differences in rates of return on investment. This will allow us to study how actual data can be used to test implications of theoretical models.

There is a significant assumption made by Lucas: Suppose that it was possible to export US production technology (or "know how") to other countries. Then the production function, both domestically and abroad, would be

$$y = A \cdot K^{\alpha} \cdot L^{1-\alpha}$$

with a different level of K and L in each country, but the same A, α, and capital depreciation level δ. Then imagine a less developed country whose annual (per capita) output is a seventh of the US output:

$$\frac{y_{LDC}}{y_{US}} = \frac{1}{7}$$

Using per capita variables ($L_{US} = L_{LDC} = 1$), the marginal return on capital investment in the US is calculated as:

$$R_{US} = \alpha \cdot A \cdot K_{US}^{\alpha-1} - \delta$$

and the parameters α and δ take values of 1/3 and .1, respectively.

The net return on capital in the US can be estimated to be 6.5% per annum, so the gross rate is:

$$R_{US} = 1.065$$

Manipulating the Cobb-Douglass expression a little,

$$\alpha \cdot A \cdot K_{US}^{\alpha-1} = \alpha \cdot \frac{A \cdot K_{US}^{\alpha}}{K_{US}} = \alpha \cdot \frac{y_{US}}{K_{US}}$$

Now, what is the return on capital in the less developed country?

$$R_{LDC} = \alpha \cdot \frac{y_{LDC}}{K_{LCD}} - \delta$$

We have that

$$7 = \frac{y_{US}}{y_{LDC}} = \frac{A \cdot K_{US}^{\alpha}}{A \cdot K_{LDC}^{\alpha}} = \left(\frac{K_{US}}{K_{LDC}}\right)^{\alpha}$$

So

$$\begin{aligned} \frac{y_{LDC}}{K_{LDC}} &= \frac{7^{-1} \cdot y_{US}}{7^{-\frac{1}{\alpha}} \cdot K_{US}} \\ \frac{y_{LDC}}{K_{LDC}} &= 7^{\frac{1-\alpha}{\alpha}} \cdot \frac{y_{US}}{K_{US}} \end{aligned}$$

and, using $\alpha = 1/3$,

$$\frac{y_{LDC}}{K_{LDC}} = 7^2 \cdot \frac{y_{US}}{K_{US}}$$

Then

$$1.065 = \frac{1}{3} \cdot \frac{y_{US}}{K_{US}} - .1$$

We know from the data that

$$\frac{y_{US}}{K_{US}} = .495$$

Therefore,

$$\begin{aligned} \frac{y_{LDC}}{K_{LDC}} &= 49 \cdot \frac{y_{US}}{K_{US}} = 49 \cdot .495 \\ &= 24.255 \end{aligned}$$

Which implies that the (gross) rate of return on capital in the less developed country should be:

$$R_{LDC} = \frac{1}{3} \cdot 24.255 - .1 = 7.985$$

This is saying that <u>if the US production techniques could be exactly replicated</u> in less developed countries, the *net* return on investment would be 698.5%. This result is striking since if this is the case, then capital should be massively moving out of the US and into less developed countries. Of course, this riddle might disappear if only we let $A_{LDC} < A_{US}$.

5 References

Lucas, Robert E. Jr., Why doesn't capital flow from rich countries to poor countries?,