Note for ECON702 (Professor Jose-Victor Rios-Rull) (Spring 2002)

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1 Jan 22: Overview and Review of SPP of RA-NGM

1.1 Introduction

- What is an equilibrium?
- Loosely speaking, an equilibrium is a mapping from environments (preference, technology, information, market structure) to allocations.
- Mathematically (for George?), we can get an equilibrium by solving a system of equations. But just solving equations is not enough (for Victor?), at least for macroeconomics. We want to get an equilibrium which characterizes what we think to happen in a given environment.
- Thus existence and uniqueness is valuable. Otherwise, we have trouble to be able to say (or predict) what is likely to happen in a given environment.
- A sensible definition of an equilibrium is (i) agents optimize, and (ii) markets clear (actions of agents in the economy are compatible to each others)

1.2 The Road Map

- In the first two weeks with Randy, we learned how to solve Social planner's problem (SPP) of neoclassical growth model with representative agent (RA-NGM), using dynamic programming. Also we know that solution to SPP is Pareto Optimal (PO) in our model. The solution

*Email: makoto@ssc.upenn.edu. I thank Vivian Zhanwei Yue for valuable helps and comments. This note is still (or forever) incomplete, so please report any mistakes or comments if you benefit from this. Thank you.

of SPP can be interpreted as the allocation to be chosen if the God exists and has control over everything (by definition!) and is benevolent (maybe by definition). In other words, the solution does not predict what is going to happen in an environment.

- Other good things for solution to SPP is that, in RA-NGM, we know that (i) it exists and (ii) it's unique.

- Besides, we have two welfare theorems (FBWT, SBWT) from Dave's class. If we carefully define the environment, those two theorems guarantee (loosely) that (i) under certain conditions, Arrow-Debrew Competitive Equilibrium (ADE, or Walrasian equilibrium or valuation equilibrium) is PO, and (ii) also under certain conditions, we can construct an ADE from a PO allocation.

- Using those elements, we can argue that ADE exists and is unique, and we just need to solve SPP to derive the allocation of ADE, which is much easier task than solve a monster named ADE.

- But we have another problem: The market assumed in ADE is not palatable to us in the sense that it is far from what we see in the world. So, next, we look at an equilibrium with sequential markets (Sequential Market Equilibrium, SME). Surprisingly, we can show that, for our basic RA-NGM, the allocation in SME and the allocation of ADE turn out to be the same, which let us conclude that even the allocation of the equilibrium with sequential markets can be analyzed using the allocation of SPP.

- Lastly, we will learn that equilibrium with sequential markets with recursive form (Recursive Competitive Equilibrium, RCE) gives the same allocation as in SME, meaning we can solve the problem using our best friend = Dynamic Programming.

- (Of course, these nice properties are available for limited class of models. We need to directly solve the equilibrium, instead of solving SPP, for large class of interesting models. We will see that Dynamic Programming method is also very useful for this purpose. We will see some examples later in the course.)

1.3 Review of Ingredients of RA-NGM

1.3.1 Technology

- Represented by production function:

$$f : R^2_+ \to R_+ \quad \text{such that } y_t = f(k_t, n_t) \tag{1}$$

- We assume (i) Constant Returns to Scale (CRS, or homogeneous of degree one, meaning $f(\lambda k, \lambda n) = \lambda f(k, n)$), (ii) strictly increasing in both arguments, and ((iii) INADA condition, if necessary)

1.3.2 Preference

- We assume infinitely-lived representative agent (RA).[1]
- We assume that preference of RA is (i) time-separable (with constant discount factor $\beta < 1$), (ii) strictly increasing in both consumption and leisure, (iii) strictly concave
- Our assumptions let us use the utility function of the following form:

$$\sum_{t=0}^{\infty} \beta^t u(c_t, 1 - n_t) \tag{2}$$

Homework 1.1. *Define strict concavity of u(c,l)*

Homework 1.2. *Show that if u is strictly concave, (2) is also strictly concave.*

1.3.3 Allocation

- Initial capital stock k_0 is given.

1.4 Review of SPP of RA-NGM

1.4.1 The Problem

$$\max_{\{c_t, n_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t, n_t) \tag{3}$$

subject to[2]

$$k_{t+1} + c_t = f(k_t, n_t) + (1 - \delta) k_t \tag{4}$$

$$c_t, k_{t+1} \geq 0 \tag{5}$$

$$n_t \in [0, 1] \tag{6}$$

$$k_0 \text{ is given} \tag{7}$$

[1]For now, let's treat the economy as if there were only one agent in the economy. We might interpret it as the result of normalization (so the number of population is 1) of the economy with FINITE number of identical (sharing the same technology, preference, and allocation) agents. If we proceed to the economy with mass of zero measure agents, things will be not so trivial because changing allocation of one agent does not change the aggregate amount of resources in the economy (since, by assumption, measure of an agent is zero), but let's forget it for now.

[2]We can also define f (the production function) as including depreciation of capital. In the 1st class, Victor actually took this approach, but I modified the notation to make notation consistent across classes.

1.4.2 Property 1: Existence

- Use Weirstrass Theorem
- Need to show (i) maximand is continuous function and (ii) constraint set is compact (closedness and boundedness).
- Not go into details but be aware that commodity space is infinite dimensional space (so exactly the same argument as in 701 (where commodity space is finite dimensional space) is not valid here). In particular, need to define commodity space as a topological linear space with sup-norm (more later).

1.4.3 Property 2: Uniqueness

- Need (i) convex constraint set, and (ii) strictly concave function

Homework 1.3. *Prove it.*

1.4.4 Property 3: Pareto Optimality

- Trivial (if assume finite number of agents).

Homework 1.4. *Prove it.*

2 Jan 24: Principle of Optimality

2.1 Background

- We know how to solve SPP of RA-NGM.
- But what we want to know is equilibrium (price and allocation).
- If we can apply welfare theorems to the allocation of SPP, we can claim that "God's will realizes" and can analyze allocation of SPP instead of directly looking at an equilibrium allocation.
- In order to use the argument above, we formalize the environment of RA-NGM in the way such that we can apply welfare theorems. By using (i) existence of solution to SPP, (ii) uniqueness of solution of SPP, and (iii) welfare theorems, we can claim that ADE (i) exists, (ii) is unique, (iii) and PO.
- However, market arrangement of ADE is not palatable to us in the sense that set of markets that are open in the ADE is NOT close to the markets in our real world. In other words, there is notion of time in ADE: all the trades are made before the history begins and there is no more choices after the history begins.

- So we would like to proceed to the equilibrium concept that allows continuously open markets, which is SME and we will look at it closely next week.

2.2 Environment revisited

- For simplicity of notation, assume only one consumer and one producer from now on.
- Define commodity space as space of bounded real sequences with sup-norm $L = l_\infty^3$ (1=consumption goods, 2=labor, 3=capital goods).
- Define $Dual\ of\ L$. An elements of $Dual(L)$ is $p(x)$, which is a function from L into R.
- Define the consumption possibility set X as:

$$X = \{x \in L = l_\infty^3 : \exists \{c_t, k_{t+1}\}_{t=0}^\infty \geq 0 \text{ such that}$$

$$\begin{aligned} k_{t+1} + c_t &= x_{1t} + (1-\delta)k_t \quad \forall t \\ x_{2t} &\in [0,1] \quad \forall t \\ x_{3t} &\leq k_t \quad \forall t \\ k_0 &= \text{given}\} \end{aligned} \tag{8}$$

- Interpretation is that x_{1t}=received goods at period t, x_{2t}=labor supply at period t, x_{3t}=capital service at period t

Homework 2.1. *Show that X is a convex set.*

- Define the production possibility set Y as:

$$Y = \{y \in L : y_{1t} \leq F(y_{3t}, y_{2t}) \quad \forall t\} \tag{9}$$

- Interpretation is that y_{1t}=production at period t, y_{2t}=labor input at period t, y_{3t}=capital input at period t

Homework 2.2. *Show that Y (i) is convex, (ii) is closed, (iii) has an interior point.*[3]

[3] Free disposal assumption is sufficient to guarantee an existence of an interior point under our commodity space. For more details, see Harris' book, note 6 of Chapter 3, but we don't need to go into such details here.

2.3 Agents' Problem

- Consumer's problem is:

$$\max_{x \in X} U(x) = \max_{x \in X} \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - x_{2t}) \tag{10}$$

  subject to

$$p(x) \leq 0 \tag{11}$$

- Firm's problem is:

$$\max_{y \in Y} p(y) \tag{12}$$

  We assume the same properties to u, β, F, δ as in the last class. Also note $p(y)$ is a function from L into R.

2.4 ADE

- An Arrow-Debreu Competitive Equilibrium is a triad (p^*, x^*, y^*) such that
  1. x^* solves the consumer's problem.
  2. y^* solves the firm's problem.
  3. markets clear, i.e. $x^* = y^*$.

- Note that the price system (or valuation function) p^* is an element of $Dual(L)$ and not necessarily represented as a familiar "price vector".

- Note there are many implicit assumptions like (i) all the markets are competitive (agents are price taker), (ii) absolute commitment (economy with a lack of commitment is also a topic of macroeconomics, maybe from your 2nd year on), (iii) all the future events are known, with the probability of each events when trade occurs (before the history begins).

2.5 Welfare Theorems

Theorem 2.3. *(FBWT) If the preferences of consumers are nonsatiated ($\exists \{x_n\} \in X$ that converges to $x \in X$ such that $U(x_n) > U(x)$), an allocation (x^*, y^*) of an ADE (p^*, x^*, y^*) is PO.*

Theorem 2.4. *(SBWT) If (i) X is convex, (ii) preference is convex (for $\forall x, x' \in X$, if $x' < x$, then $x' < (1-\theta)x' + \theta x$ for any $\theta \in (0,1)$), (iii) $U(x)$ is continuous, (iv) Y is convex, (v)Y has an interior point, then with any PO allocation (x^*, y^*) such that x^* is not a satuation point, there exists a continuous linear functional p^* such that (x^*, y^*, p^*) is a Quasi-Equilibrium ((a) for $x \in X$ which $U(x) \geq U(x^*)$ implies $p^*(x) \geq p^*(x^*)$ and (b) $y \in Y$ implies $p^*(y) \leq p^*(y^*)$)*

Lemma 2.5. *If, for (x^*, y^*, p^*) in the theorem above, the budget set has cheaper point than x^* ($\exists x \in X$ such that $p(x) < p(x^*)$), then (x^*, y^*, p^*) is a ADE.*[4]

Homework 2.6. *Show that conditions for SBWT are satisfied in the PO allocation of RA-NGM.*

Now we established that the ADE of the RA-NGM exists, is unique, and is PO. The next thing we would like to establish is that the price system $p^*(x)$ takes the familiar form of inner product of price vector and allocation vector, which we will establish next.

2.6 Inner Product Representations of Prices

Let's start from the result.

Theorem 2.7. *(based on Prescott and Lucas 1972) If, in addition to the conditions to SBWT, $\beta < 1$ and u is bounded, then $\exists \hat{p}$ such that $(x^*, y^*, \hat{p})$ is a QE and*

$$\hat{p}(x) = \sum_{t=0}^{\infty} \sum_{i=1}^{3} \hat{p}_{it} x_{it} \tag{13}$$

i.e. price system has an inner product representations.

Remark 2.8. *Remember that most of the familiar period utility functions (CRRA (including log utility function), CARA) in macroeconomics do not satisfy the conditions, as the utility function is not bounded. There is a way to get away with it, but we you not need to go into details (for those interested, see Stokey, Lucas, and Prescott, Section 16.3, for example).*

Though it is not mentioned in the class, the result above is a special case of the more general theorem proved by Prescott and Lucas (1972). Before stating the theorem, let's define some notations. Let L^n be the subspace of L such that, for $x \in L^n$, $x = ((x_{11,} x_{21,x}31), (x_{12}, x_{22}, x_{32}), (x_{13}, x_{23}, x_{33}), ..., (x_{1n-1}, x_{2n-1}, x_{3n-1}), (0,0,0), (0,0,0), ...)$, i.e. $x_{it} = 0$ for $t \geq n$. Also Let x^n denote the projection of $x \in L$ on L^n.

Now we are ready to state the theorem in a more general form.

Theorem 2.9. *(Prescott and Lucas 1972) If (i) X is convex, (ii) preference is convex (these two conditions are same as those in the SBWT), (iii) for every n, $x^n \in X$ and $y^n \in Y$, (iv) if $x, x' \in X$ and $U(x) > U(x')$, then there exists and integer N such that, for $\forall n \geq N$, $U(x^n) > U(x')$, then, for a QE (x^*, y^*, p^*) with non-satiation point x^*, there exists $\hat{p}$ such that (1) $\hat{p}(x) = \lim_{n\to\infty} p(x^n)$ for a $p \in Dual(L)$, and (2) $(x^*, y^*, \hat{p})$ is a QE.*

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[4]Victor commented that for our basic RA-NGM, we do not need to worry about the difference between QE and ADE, since QE is ADE. To be precise, we need to check if the condition of this lemma is satisfied to treat QE as ADE. I am not sure what properties of the model guarantee that this lemma holds.

Remark 2.10. *The results of the theorem allows us to consider the price system of a QE as the limit of a price system of the finite commodity space and thus represent price system of a QE by inner product representations. Intuitively, the additional two conditions of the theorem ((iii) and (iv)) tell that (iii) truncated consumption or production allocation is also feasible, and (iv) truncation of the sufficiently "future" consumption does not change the preference relationship.*

Remark 2.11. *The conditions in the first theorem in this subsection are just the sufficient conditions in a particular environment for conditions in the second theorem.*

Now that we have the inner product representations of price system, we can solve the relative prices of the goods in this economy. In particular, we can derive the following relationships:

$$\frac{\hat{p}_{3t}}{\hat{p}_{1t}} = F_k(k_t^*, n_t^*) \tag{14}$$

$$\frac{\hat{p}_{2t}}{\hat{p}_{1t}} = F_n(k_t^*, n_t^*) = \frac{u_l(c_t^*, 1-n_t^*)}{u_c(c_t^*, 1-n_t^*)} \tag{15}$$

$$\frac{\hat{p}_{1t}}{\hat{p}_{1t+1}} = \frac{u_c(c_t^*, 1-n_t^*)}{\beta u_c(c_{t+1}^*, 1-n_{t+1}^*)} = 1-\delta+\frac{\hat{p}_{3t+1}}{\hat{p}_{1t+1}} \tag{16}$$

Homework 2.12. *Prove them.*[5]

3 Jan 29: From ADE to SME

3.1 The Road Map Today

- We established the equivalence between SPP allocation and ADE allocation using Welfare Theorems.
- But it is not sufficient, because the market arrangement in the ADE is not realistic.
- Today we will get another result, which connects SPP to an equilibrium with more reasonable market arrangement (SME).
- In the next class, we will see that we can use Dynamic Programming to solve SME (an associated equilibrium concept is Recursive Competitive Equilibrium, RCE).

[5] Prices of capital and labor inputs might turn out to be negative, depending on the way the budget constraint is defined. In order to derive the exact relationships in this case, we have to redefine the prices of capital and labor inputs as the negative of those ones. But the implications are essentially same.

3.2 Digressions in the Class

- Why we did not go from SPP to SME or RCE directly? Because Welfare Theorems are available only between SPP and ADE, though what we want is to derive equivalence between SPP allocation and SME (or RCE) allocation. For some particular environments, as the equivalence result between SPP allocation and RCE allocation is available, we can exploit the result and can argue directly that some RCE allocation is indeed PO.

- Do people maximize utility? Maybe. The important thing is that we do not have ”operational substitute” for utility maximization. Behavioral science people are considering the alternatives, like the model where agents have limited ability to process information, but there is no alternative for us which is sensible and gives us sharp prediction power on what is going happen in a given environment.

- Same thing can be said about Rational Expectation. We know that the assumption is not realistic (we know that there are infinite ways that agents being stupid). But again, there is no operational substitute. So we use it.

- Remember that we can make assumptions on environments where agents live, but we cannot tell what they do. In this sense, economists have only the limited power over their models. If we can tell what agents do, solving for SPP might be enough, but we need equilibrium concept because we want to know what agents do (or what's going to happen) in a given environment.

3.3 Consumer's Problem in ADE

$$\max_{x \in X} \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - x_{2t}) \tag{17}$$

subject to

$$\sum_{t=0}^{\infty} \sum_{i=1}^{3} \hat{p}_{it} x_{it} \leq 0 \tag{18}$$

(Note that we are using the result of Lucas and Prescott (1972) and representing the price system by inner product.) How many constraints do we have? Two. One is feasibility constraint ($x \in X$) and the other is budget constraint (18). But forget the feasibility constraint now. Often we can either (i) forget the condition, or (ii) show that it is not going to be binding. Let's concentrate on the budget constraint. There is only 1 budget constraint. Why? Because we make a choice only ONCE in AD world: all the trades are made at period 0, and after the history starts, all that agents can do is to follow what was promised (full commitment is assumed). But this is a weird market arrangement. To see the point this more clearly, imagine the decision of an agent who is going to be born in period t. At period 0, although the agents is not born yet, the agent also joins the market at period 0! At period 0, she trades (by solving the consumer's problem above), and she goes to limbo from period 0 (after trade) until period t-1, and she is born in period t. As we want the market arrangement of the model to be comparable to the one in the real world, this unrealistic assumption on market arrangement is not desirable. That is the motivation to consider Sequential Market Equilibrium (SME), where markets are open every period.

3.4 Various Market Arrangements

So we will look at SME. Two things are important here: (i) there are infinitely many markets in SME (because markets are open every period), which means that there are infinitely many budget constraints to be considered, (ii) an allocation in SME has to give as much utility as in ADE to agents in order to be PO. Otherwise, agents will choose to trade in AD markets, meaning SME doesn't work. Remember that we cannot force agents to do certain things.

Also note that there are many ways of arranging markets so that the equilibrium allocation is equivalent to that in ADE. We'll see two of them. Note that if the number of markets open is TOO FEW, we cannot achieve the allocation in the ADE (incomplete market). To the contrary, if the number of markets are TOO MANY, we can close some of the markets and still achieve the ADE allocation in this market arrangement. Also it means that there are many ways to achieve ADE allocation because some of the market instruments are redundant and can be substituted by others. If the number of markets are not TOO FEW nor TOO MANY, we call it JUST RIGHT.

Let's start from the world where agents can (i) lend and borrow (loans, l), or (ii) buy or sell capital (x_{2t}). The budget constraint for period t in such world is:

$$l_{t+1}+\hat{p}_{2t}x_{2t+1}+\hat{p}_{1t}x_{1t}\leq\hat{p}_{2t}x_{2t}+\hat{p}_{3t}x_{3t}+\hat{p}_{2t}(1-\delta)x_{2t}+l_t(1+r_t) \tag{19}$$

where x_{2t} is capital in period t, x_{1t} is consumption in period t, x_{3t} is labor input in period t, l_t is a loan (can be positive or negative) in period t, r_t is interest rate associated with loans at period t. In this world, capital can be sold or bought but not rented, i.e. there is no market for renting capital.

Another world we can think of is the one where agents can (i) lend and borrow (loans, l), or (ii) rent their capital (x_{2t}). The budget constraint associated with such world is:

$$l_{t+1}+\hat{p}_{1t}x_{1t}\leq\hat{p}_{2t}x_{2t}+\hat{p}_{3t}x_{3t}+l_t(1+r_t)$$

$$x_{2t}\leq k_t$$

$$k_{t+1}=\hat{p}_{2t}x_{2t}+\hat{p}_{3t}x_{3t}-\hat{p}_{1t}x_{1t}+(1-\delta)k_t$$

Note in this world capital is rented but is kept in your backyard at the end of each period, i.e., there is no market of selling or buying capital goods. You can check that these two budget constraints from two different market arrangements are equivalent.

Next thing we notice is, we can close the market of loans without changing the resulting allocation. This is because we need someone to lend you loans in order that you borrow loans, but there is only one agents in the economy. But surprisingly, we will see that even though there is no trade in certain markets in equilibrium, we can solve for prices in those markets, because prices are determined even though there is no trade in equilibrium, and agents do not care if actually trade occurs or not because they just look at prices in the market (having market means agents do not care about the rest of the world but the prices in the market). Using this technique, we can determine prices of all market instruments even though they are redundant in equilibrium. This is the virtue of Lucas Tree Model and this is the fundamental for all finance literature (actually, we can price any kinds of financial instruments in this way. we will see this soon.)

So... close the markets for loans. We have the following budget constraint for each period:

$$c_t + k_{t+1} = w_t n_t + [(1-\delta) + r_t]k_t \tag{20}$$

3.5 Consumer's Problem in SME

The problem is as follows:

$$\max_{\{c_t, n_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t, n_t) \tag{21}$$

subject to

$$c_t + k_{t+1} = w_t n_t + [(1-\delta) + r_t]k_t \qquad t = 0, 1, 2, ... \tag{22}$$

$$k_0 \text{ is given} \tag{23}$$

3.6 The Road Map

We want to show the following theorem:

Theorem 3.1. *(i) if (x^*, y^*, p^*) is an ADE, we can construct SME with (x^*, y^*), (ii) if $(\tilde{x}, \tilde{y}, \tilde{r}, \tilde{w})$ is a SME, we can construct ADE with $(\tilde{x}, \tilde{y})$.*

To derive the theorem, we take the following strategy[6]. To prove (i),

1. Assume that (x^*, y^*, p^*) is an ADE. We know some properties which (x^*, y^*, p^*) satisfies (remember the homework from the last class).

[6]Actually, I guess that we can show the equivalence by just showing that the maximand is the same (which is almost obvious) and the constraint set is the same, thus the solution should be the same. Later, when we want to show the similar equivalence result between RCE and ADE, we definitely need to solve FOC (because, in RCE, consumers' problem is characterized with Bellman equation and there is no way to compare the problem in RCE and the one in ADE directly).

2. Using the properties, construct a candidate for $(\tilde{r}, \tilde{w}) = \{\tilde{r}_t, \tilde{w}_t\}_{t=0}^{\infty}$
3. Verify that, given $(\tilde{r}, \tilde{w})$, (1) consumers in the SM world choose x^*, (2) firms choose y^*, (3) markets clear (which is trivial because of our choice of (x^*, y^*)), so $(x^*, y^*, \tilde{r}, \tilde{w})$ is a SME.

Similarly, to prove (ii),

1. Assume that $(\tilde{x}, \tilde{y}, \tilde{r}, \tilde{w})$ is a SME. We will know some properties which $(\tilde{x}, \tilde{y}, \tilde{r}, \tilde{w})$ satisfies (in proving (i))
2. Using the properties, construct a candidate for $p^* = \{\hat{p}_{1t}, \hat{p}_{2t}, \hat{p}_{3t}\}_{t=0}^{\infty}$
3. Verify that, given p^*, (1) consumers in the AD world choose $\tilde{x}$, (2) firms choose $\tilde{y}$, (3) markets clear (which is trivial because of our choice of $(\tilde{x}, \tilde{y})$), so $(\tilde{x}, \tilde{y}, p^*)$ is an ADE.

3.7 Proof of (i) of Theorem

Let's start from defining the SME.

Definition 3.2. *A Sequential Market Equilibrium (SME) is $\{\tilde{r}_t, \tilde{w}_t\}_{t=0}^{\infty}$, both positive, and $\{\tilde{c}_t, \tilde{n}_t, \tilde{k}_{t+1}, \tilde{y}_t\}_{t=0}^{\infty}$ such that (1) given $\{\tilde{r}_t, \tilde{w}_t\}_{t=0}^{\infty}$, $\{\tilde{c}_t, \tilde{n}_t, \tilde{k}_{t+1}\}_{t=0}^{\infty}$ solves the consumer problem, (ii) given $\{\tilde{r}_t, \tilde{w}_t\}_{t=0}^{\infty}$, $\{\tilde{y}_t, \tilde{n}_t, \tilde{k}_t\}_{t=0}^{\infty}$ solves the producer problem (see below), (iii) markets clear ($y_t = c_t + k_{t+1}$ for $\forall t$)*

The producer's problem is for all $t = 0, 1, 2, ...$

$$\max_{\{y_t, n_t, k_t\}} \{y_t - w_t n_t - r_t k_t\} \tag{24}$$

subject to

$$y_t \leq F(k_t, n_t) \tag{25}$$

Notice there is a cheating. Why? The firm's problem should be the intertemporal one but we simplify it by splitting across time. We can do it because there is NO dynamic links to firms problem.

Following the solution strategy described above, let's pick up candidate for $\{\tilde{r}_t, \tilde{w}_t\}_{t=0}^{\infty}$ (of course, we are going to show that the candidate actually supports SME).

$$\begin{aligned} \tilde{r}_t &= F_k(x_{2t}^*, x_{3t}^*) \\ \tilde{w}_t &= F_n(x_{2t}^*, x_{3t}^*) \end{aligned} \tag{26}$$

We are going to show that, given $\{\tilde{r}_t, \tilde{w}_t\}_{t=0}^{\infty}$ defined above, (x^*, y^*) solves agents' problem[7]. To show this, we will show that necessary and sufficient conditions for the solution of the agents' problems in SME are satisfied by (x^*, y^*). So, the first step is to derive necessary and sufficient conditions for agents' optimization problems. Let's start from the easy one, the firm's problem. The necessary and sufficient conditions for firm's optimization problem are, for $\forall t$:

$$\begin{aligned} \tilde{r}_t &= F_k(k_t, n_t) \\ \tilde{w}_t &= F_n(k_t, n_t) \end{aligned} \tag{27}$$

Comparing (26) and (27), we know that $\{x_{2t}^*, x_{3t}^*\} = \{k_t, n_t\}$ is optimal for firm. For consumer's problem, following First Order Conditions are necessary and sufficient[8].

$$\frac{\beta^t u_c(c_t, n_t)}{\beta^{t+1} u_c(c_{t+1}, n_{t+1})} = (1+\delta) + \tilde{r}_t \tag{28}$$

$$\frac{u_l(c_t, n_t)}{u_c(c_t, n_t)} = \tilde{w}_t \tag{29}$$

Again, comparing (14), (15), (16), (26), (28) and (29), we can see that $\{x_{1t}^*, x_{3t}^*\} = \{\tilde{c}_t, \tilde{n}_t\}$ satisfies the necessary and sufficient conditions for consumer's optimal choice. I will show briefly how to derive (28) and (29) in the next section.

3.8 Deriving FOC of Consumer's Problem (with Lagrangian)

As usual, define Lagrangian:

$$L = \sum_{t=0}^{\infty} \beta^t \left[u(c_t, n_t) - \lambda_t (c_t + k_{t+1} - w_t n_t - (1 - \delta + r_t) k_t) \right] \tag{30}$$

Take First Order Conditions:[9]

$$\begin{aligned} \text{With respect to } c_t &: \quad \beta^t (u_c(c_t, n_t) - \lambda_t) = 0 \\ \text{With respect to } k_{t+1} &: \quad -\beta^t \lambda_t + \beta^{t+1} \lambda_{t+1} (1 - \delta + r_{t+1}) = 0 \\ \text{With respect to } n_t &: \quad \beta^t (u_l(c_t, n_t) - \lambda_t w_t) = 0 \\ \text{With respect to } \lambda_t &: \quad c_t + k_{t+1} - w_t n_t - (1 - \delta + r_t) k_t = 0 \end{aligned} \tag{31}$$

We just need to play with these equations to derive (28) and (29).

[7] Alternatively, we can define $\{\tilde{r}_t, \tilde{w}_t\}_{t=0}^{\infty}$ using p^*. In this case, we define $\tilde{r}_t = \frac{\hat{p}_{1t}}{\hat{p}_{1t+1}} + \delta - 1$, $\tilde{w}_t = \frac{\hat{p}_{2t}}{\hat{p}_{1t}}$ and the rest are the same.

[8] Note that we do not go into details but necessary and sufficient conditions of optimality must include Transversality Condition (TVC), and showing that TVC is satisfied by (x^*, y^*) is not so trivial (I guess).

[9] Of course we need assumptions which guarantee that the solution is interior.

4 Jan 31: Stochastic Model

4.1 Review on Market Arrangement

In the world with (i) capital market (you can sell or buy the capital), and (ii) loans (you can borrow or lend consumption goods with others), the budget constraint for the representative consumer can be written as follows[10]:

$$l_{t+1} + c_t + k_{t+1} = (1 + r_t)k_t + w_t n_t + l_t(1 + R_t^l) \tag{32}$$

Note that market clear condition for loans market is

$$\sum_{agent} l_t = 0 \qquad for\ \forall t \tag{33}$$

Since we have only one agent (representative agent), we know that, in equilibrium:

$$l_t = 0 \qquad for\ \forall t \tag{34}$$

This conjecture enables us to write budget constraint without including loans market. But the important things here are (i) we allow agents to trade (but trade does not occur), and (ii) because of that, we can price the market instruments (like loans) even though they are not traded. In this case, we know from non-arbitrage condition,

$$r_t = R_t^l \qquad for\ \forall t$$

as long as (i) the choice of agents is interior (not corner solution), and (ii) the constraint for both assets are same.

4.2 The Plan

We are going to do the same things we have done with deterministic RA-NGM with stochastic RA-NGM. After finishing this, we are going to define and analyze Recursive Competitive Equilibrium.

4.3 On Shock and History

4.3.1 Possible Shocks

Examples of possible shocks in RA-NGM are:

- Shock to productivity (technology).
- Shock to depreciation (technology, shock to mice!!)

[10] In this class, think that return on capital is net basis, i.e. depreciation is included in the return for capital.

- Shock to preference.

We will concentrate on shocks to productivity, which is the most popular kind in NGM (you can go to RBC!).

4.3.2 Markov Process

In this course, we will concentrate on Markov productivity shock. Considering shock is really a pain, so we want to use less painful one. Markov shock is a stochastic process with the following properties:

1. there are FINITE number of possible states for each time. More intuitively, no matter what happened before, tomorrow will be represented by one of a finite set.
2. what only matters for the realization tomorrow is today's state. More intuitively, no matter what kind of history we have, the only thing you need to predict realization of shock tomorrow is TODAY's realization.

More formally, for each period, suppose either z^1 or z^2 happens[11] [12]. Denote z_t is the state of today and Z_t is a set of possible state today, i.e. $z_t \in Z_t = \{z^1, z^2\}$ for all t. Since the shock follow Markov process, the state of tomorrow will only depend on today's state. So let's write the probability that z^j will happen tomorrow, conditional on today's state being z^i as $\Gamma_{ij} = prob[z_{t+1} = z^j | z_t = z^i]$. Since Γ_{ij} is a probability, we know that

$$\sum_j \Gamma_{ij} = 1 \qquad for\ \forall i \tag{35}$$

Notice that 2-state Markov process is summarized by 6 numbers: z^1, z^2, Γ_{11}, Γ_{12}, Γ_{21}, Γ_{22}.

Homework 4.1. *Compute (i) conditional mean, (ii) unconditional mean, (iii) conditional variance, (iv) unconditional variance, (v) average duration of each state, of the Markov process above.*

The great beauty of using Markov process is we can use the explicit expression of probability of future events, instead of using weird operator called expectation, which very often people don't know what it means when they use.

[11] In this class, superscript denotes the state, and subscript denotes the time.

[12] Here we restrict our attention to the 2-state Markov process, but increasing the number of states to any finite number does not change anything fundamentally.

4.3.3 Representation of History

- Let's concentrate on 2-state Markov process. In each period, state of the economy is $z_t \in Z_t = \{z^1, z^2\}$.
- Denote the history of events up to t (which of $\{z^1, z^2\}$ happened from period 0 to t, respectively) by $h_t = \{z_1, z_2, ..., z_t\} \in H_t = Z_0 \times Z_1 \times ... \times Z_t$.
- In particular, $H_0 = \emptyset$, $H_1 = \{z^1, z^2\}$, $H_2 = \{(z^1, z^1), (z^1, z^2), (z^2, z^1), (z^2, z^2)\}$.
- Note that even if the state today is the same, past history might be different. By recording history of event, we can distinguish the two histories with the same realization today but different realizations in the past (think that the current situation might be "you do not have a girl friend", but we will distinguish the history where "you had a girl friend 10 years ago" and the one where you didn't (tell me if it is not an appropriate example...).)
- Let $\Pi(h_t)$ be the unconditional probability that the particular history h_t does occur. By using the Markov transition probability defined in the previous subsection, it's easy to show that (i) $\Pi(h_0) = 1$, (ii) for $h_t = (z^1, z^1)$, $\Pi(h_t) = \Gamma_{11}$ (iii) for $h_t = (z^1, z^2, z^1, z^2)$, $\Pi(h_t) = \Gamma_{12}\Gamma_{21}\Gamma_{12}$.

Homework 4.2. *Verify that* $\sum_{h_3 \in H_3} \Pi(h_3) = 1$.

4.4 SPP,ADE, and SME in a Stochastic RA-NGM

4.4.1 Big Picture

- Now we have Nature, who decides the realization of productivity shock every period. Even God cannot control it. In this sense, our God is a kind of medium-sized God.
- Social Planner's Problem (the benevolent God's choice) in this world is a state-contingent plan, i,e, optimal consumption and saving (let's forget about labor-leisure choice in this section for simplicity [13]) choice for all possible nodes (imagine the nodes of a game tree. we need to solve optimal consumption and saving for each node in the tree).
- Notice that the number of nodes for which we have to solve for optimal consumption and saving is countable. This feature allows us to use the same argument as the deterministic case to deal with the problem. The only difference is that for deterministic case, the number of nodes is equal to number of periods (which is infinite but countable), but here the number of nodes is equal to the number of date-events (which is also infinite but countable).
- More mathematically, the solution of the problem is the mapping from the set of date-events (which is specified by history) to the set of feasible consumption and saving.

[13] Or just assuming the consumers do not value leisure (drop leisure from utility function) is enough to let agents work as much as possible in this world.

4.4.2 The SPP and ADE

$$\max_{\{k_{t+1}(h_t),c_t(h_t)\}} \sum_{t=0}^{\infty} \beta^t \sum_{h_t \in H_t} \Pi(h_t) u(c_t(h_t)) \tag{36}$$

subject to

$$(1-\delta)k_t(h_{t-1}) + F[z_t, k_t(h_{t-1}), 1] = c_t(h_t) + k_{t+1}(h_t) \tag{37}$$

$$k_0 \qquad \text{given} \tag{38}$$

Couple of comments:

- Here capital in indexed by the time it is used. k_t is a mapping from h_{t-1} because the amount of capital used today is determined yesterday. Alternatively, you can index capital by the time when the amount is chosen, but the former notation is the tradition and more common so we use the former notation. Anyway it is just a matter of notation.
- An assumption here is leisure is not valued by consumer so time of consumer is inelastically supplied for working.
- Measurability (very loosely) means whether an object is known when agents make their choice. Choice of agents must not depend on an object which agents do not know when they make choices.

Let's denote the solution as $x^* = \{c_t^*(h_t), k_{t+1}^*(h_t)\}$. It's easy to show that (i) the utility function is strictly concave, (ii) the constraint set is convex, (iii) commodity set is same as deterministic case. Using these properties, we can show (i) existence of the solution, (ii) uniqueness of the solution, (iii) FBWT (ADE is PO), (iv) SBWT (PO allocation can be supported as an ADE), (v) price system has a nice inner product representation (Lucas and Prescott (1972))[14], (vi) some equations (derived from FOC of SPP) which characterize the ADE allocation (remember (14), (15), and (16)).

The consumer's problem in ADE is:

$$\max_{x \in X} \sum_{t=0}^{\infty} \beta^t \sum_{h_t \in H_t} \Pi(h_t) u(c_t(h_t)) \tag{39}$$

[14] Remember the deterministic version of Lucas-Prescott Theorem. For the stochastic model, we need two additional assumptions, corresponding (iii) and (iv) of the deterministic one. Very loosely, we need additionally, that (iii) and (iv) of the deterministic one hold for truncation with respect to certain history when probability of occurrence of the history with truncation is sufficiently small. For more details, see Lucas-Prescott paper or Harris (p62-64).

subject to

$$\sum_{t=0}^{\infty}\sum_{h_t\in H_t}\sum_{i=1}^{3}\hat{p}_{it}(h_t)x_{it}(h_t)\leq 0 \tag{40}$$

Homework 4.3. *Derive the corresponding equations of (14), (15), and (16) for stochastic economy.*

4.4.3 ADE and SME

What is $\hat{p}_{it}(h_t)$ in the previous section? It is prices of goods in different date-event (for example, price of apple is different depending on whether it is raining today or not). The budget constraint above implies that you can freely transfer goods from one date-event to another (for example, you can transfer consumption from (tomorrow if it is raining) to (tomorrow if it is not raining).

Now what we want is to define an equilibrium with sequence of markets (SME) which gives agents the same welfare as ADE (remember, otherwise, the agents start trading in AD way to get higher utility). In the deterministic world, sequence of markets only need to enable agents to transfer consumption goods from one period to another, and giving one-period loans to agents is enough (because agents can use the one-period loans successively to transfer consumption goods from one period to another period even though the distance between the two periods are more than one). In the stochastic world, agents have to be able to transfer consumption goods across different realization of events, in addition to across time. Arrow Security (of course invented by Ken Arrow), enables agents to do this[15].

For example, let's consider the world with 2-states Markov shock. You need to have at least two assets, one gives consumption goods in one state tomorrow, and the other gives consumption goods in the other state tomorrow, to transfer goods across states tomorrow[16]. You can think that various insurances are real world counterparts of Arrow Securities. Examples are the followings[17]:

- Health insurance is a state contingent security, which gives you money if you are sick and go to doctor tomorrow, and gives you no money otherwise.
- House insurance is a state contingent security, which gives you money if your house burns down tomorrow, and no money otherwise.
- Death insurance (or annuity) is a state contingent security, which gives someone you designated money if you die tomorrow, and no money otherwise (in our model, agents are

[15] If you want to learn about Arrow Security seriously, see books on General Equilibrium, for example Mas-Colell, Whinston, and Green (p699-).

[16] Of course, there are many other combinations of assets to achieve the same result. We are just looking at the easiest set of assets.

[17] Please do not care about ”money” in the following examples. We do not have money in our model but we can think that we receive some consumption goods from these insurance contract instead of money.

immortal. But if you imagine that you, your parents, your spouse, your kids,... consist a single agent (dynasty), life insurance is also received by the same agent (dynasty)).

Now, let's denote the price of an Arrow Security after history h_t which gives you a single consumption good in the next period if state is z tomorrow as $q_t(h_t, z)$. Using this, budget constraint for the representative agent in SME world is:

$$\begin{aligned} & k_t(h_{t-1})[1+r_t(h_t)] + w(h_t) + d_t(h_t) \\ = \quad & c_t(h_t) + k_{t+1}(h_t) + \sum_{z^i \in Z} q(h_t, z^i) d_{t+1}(h_t, z^i) \end{aligned} \tag{41}$$

Notice that equilibrium condition for market for Arrow Securities are

$$\sum_{agents} d_{t+1}(h_t, z^i) = 0 \quad \forall t, h_t, z^i \tag{42}$$

Again since we know that there is only one representative agent in the world, there is no trade of Arrow Securities in this world: even when an agent want to buy an Arrow Security, there is no one who sells it. So again, you can close down the markets for Arrow Securities and still get the Pareto Optimality of SME. But remember that the important thing is that market is available to agents, but it's just no trade occurs in equilibrium. If we do not have the markets for Arrow Securities without knowing that there is no trade for them in equilibrium, we are in the world with incomplete markets and we are not sure if we can achieve Pareto Optimal allocation in SME. We need to give agents the markets for Arrow Securities, no matter if trade occurs or not in equilibrium, to make sure that the allocation in SME is PO.

In the same way as the loans in the deterministic world (see the first thing we studied today), we can price the Arrow Securities, even though there is no trade in equilibrium. This can be done using FOC of consumer's problem as in the similar way as we have done with deterministic world.

Homework 4.4. *Derive explicit expressions for $q(h_t, z_{t+1})$.*

5 Feb 5: Recursive Competitive Equilibrium

5.1 Review of SME of Stochastic RA-NGM

The consumer's problem in the SME is:

$$\max_{\{k_{t+1}(h_t), c_t(h_t)\}} \sum_{t=0}^{\infty} \beta^t \sum_{h_t \in H_t} \Pi(h_t) u(c_t(h_t)) \tag{43}$$

subject to

$$\begin{aligned} & k_t(h_{t-1})[1+r_t(h_t)] + w(h_t) + d_t(h_t) \\ = \quad & c_t(h_t) + k_{t+1}(h_t) + \sum_{z^i \in Z} q(h_t, z^i) d_{t+1}(h_t, z^i) \end{aligned} \tag{44}$$

and

$$k_0 \text{ given} \tag{45}$$

Suppose that there are 2 possible states for each period $\{z^1, z^2\}$. How many markets do we have for each period? Four: (i) Consumption goods (note as there is a linear technology which enables agents to costlessly transform between capital and consumption, so there is one market for both consumption goods and capital goods), (ii) labor service, (iii) Arrow Security market for z^1 (paid if state of next period is z^1), and (iv) Arrow Security market for z^2. Thus we need four market clear conditions (of course you can apply Walras' Law and eliminate one). Two of them are from factor market clearing conditions, and other two are like follows:

$$\sum_{agent} d_{t+1}(h_t, z^i) = 0 \qquad \forall z^i \in \{z^1, z^2\} \tag{46}$$

Notice that the economy we have now is a representative agent economy, meaning that we can think as if we had only one agent in the economy. Therefore, the condition above is equivalent to:

$$d_{t+1}(h_t, z^i) = 0 \qquad \forall z^i \in \{z^1, z^2\} \tag{47}$$

Exploiting the property, we know that the allocation in the equilibrium in the economy with Arrow Securities turn out to be the same as the one without. But, we can solve the prices of Arrow Securities, using the equilibrium allocation of the economy without Arrow Securities. If you look at the resulting equations representing the prices of the Arrow Securities, in general we realize that (i) the bond associated with z^i which has higher probability is more expensive, (ii) the bond associated with z^i where consumption is valued highly is more expensive. These are intuitive. Compare the prices of (i) the bond which gives you one unit of consumption good if Japan wins the World Cup, and (ii) the bond which gives you one unit of consumption good if France (or maybe Brazil) wins the World Cup. The price of the first bond is expected to be higher (maybe in Japan the opposite might occur...). Next, compare the prices of (i) the bond which gives you an umbrella if tomorrow is a sunny day, and (ii) the bond which gives you an umbrella if tomorrow is a rainy day. The price of the latter bond is expected to be higher.

In addition, you can use the same technique to solve prices of any kinds of bonds, and options (American, or European). It is the beauty of Lucas 1978 Econometrica paper. You will see this more in the future lecture.

5.2 Big Picture: Where Do We Stand Now.

- In Randy's class, we learned that a Sequential Problem of SPP can be solved using Dynamic Programming.
- We will see that we can use the Dynamic Programming technique to solve an equilibrium. We will use the same technique as we solve the SPP, but do not mix up the SPP and equilibrium.

- First, we defined the SPP of RA-NGM, and showed the equivalence between an allocation of SPP and an allocation of ADE, using Welfare Theorems. So far, two Welfare Theorems are the only tools for us to connect equilibrium and SPP.
- Second, we showed that ADE can be represented as SME, where the market arrangements are more palatable.
- Third (from today), we will see that SME is equivalent to RCE.
- For the problem so far we have, since we have the Welfare Theorems, we do not need to directly solve the equilibrium, because we know that allocation of SPP can be supported as an equilibrium and it is unique, meaning the SPP allocation is the only equilibrium.
- But if (i) assumptions of Welfare Theorems do not hold or (ii) we have more than one agent, thus we have many equilibrium depending on the choice of the Pareto weight in the Social Planner's Problem, we no longer can follow the same argument, and we need to solve the equilibrium directly. Since (i) solving ADE is "almost impossible", (ii) solving SME is "very hard", but (iii) solving RCE is "possible", RCE is important for analyzing this class of economies, where Welfare Theorems fail to hold.
- In ADE and SME, sequences of allocations and prices characterize the equilibrium, but in RCE, what characterize the equilibrium are functions from state space to space of controls and values.

5.3 Sequential and Recursive representation in SPP

Remember that we showed the equivalence of the following two problems (forget about initial conditions etc...):

1. $$\max_{\{k_{t+1},c_t\}_{t=0}^{\infty}} \sum_t \beta^t u(c_t) \tag{48}$$

   subject to

   $$(1-\delta)k_t + f(k_t) = c_t + k_{t+1} \tag{49}$$

2. $$V(k) = \max_{c,k'}[u(c) + \beta V(k')] \tag{50}$$

   subject to

   $$(1-\delta)k + f(k) = c + k' \tag{51}$$

From the next section, we are going to do the same thing for equilibrium.

5.4 Sequential and Recursive representation in equilibrium

Remember that the consumer's problem in SME is as follows:

$$\max_{\{k_{t+1},c_t\}_{t=0}^{\infty}} \sum_t \beta^t u(c_t) \tag{52}$$

$$c_t + k_{t+1} = w_t + [1+r_t]k_t \tag{53}$$

How to translate the problem using recursive formulation? First we need to define the state variables. state variables need to satisfy the following criteria:

1. PREDETERMINED: when decisions are made, the state variables are taken as given.
2. It must MATTER for decisions of agents: there is no sense of adding irrelevant variables as state variable.
3. It VARIES across time and state: otherwise, we can just take it as a parameter.

This is one of the most important thing in the whole course: be careful about the difference between aggregate state and individual state. Aggregate state is not affected by individual choice. But aggregate state should be consistent with the individual choice (we will consider the meaning of "consistency" more formally later), because aggregate state represents the aggregated state of individuals. In particular, in our RA-NGM, as we have only one agent, aggregate capital turns out to be the same as individual state in equilibrium, but this does not mean that the agent decide the aggregate state or the agent is forced to follow the average behavior, but rather the behavior of the agent turns out to be the aggregate behavior, in equilibrium.

Also note that prices (wages, and rental rates of capital) is determined by aggregate capital, rather than individual capital, and since individual takes aggregate state as given, she also takes prices as given (because they are determined by aggregate state). Again, the aggregate capital turns out to coincide with the individual choice, but it is not because of the agent's choice, rather it is the result of consistency requirement.

One notational note. Victor is going to use a for individual capital and K for aggregate capital, in order to avoid the confusion between K and k. But the problem with aggregate and individual capital is often called as "big-K, small-k" problem, because the difference of aggregate capital and individual capital is crucial. So for our case, the counterpart is "big-K, small-a" problem.

Having said that we guess that candidates for state variables are $\{K, a, w, r\}$. But we do not need $\{r, w\}$. Why? Because they are redundant: K is the sufficient statistics to calculate $\{r, w\}$ and K is a state variable, we do not need $\{r, w\}$ as state variables[18].

[18] Of course, you can construct an equilibrium using r or w, but these turns out to be the redundant.

Now let's write the representative consumer's problem in the recursive way.

$$V(K,a;G^e) = \max_{c,a'}[u(c) + \beta V(K',a';G^e)] \tag{54}$$

subject to

$$c + a' = w + [1+r]a \tag{55}$$

$$w = w(K) = F_L(K,1) = F(K,1) - KF_K(K,1) \tag{56}$$

$$r = r(K) = F_K(K,1) - \delta \tag{57}$$

$$K' = G^e(K) \tag{58}$$

Couple of comments:

- All the variables in the maximand (in the problem above: $[u(c) + \beta V(K',a';G^e)]$) have to be either (i) a state variable (so an argument of V(.)), (ii) a choice variable (so appear below max operator), (iii) or defined by a constraint, in order for the problem to be well defined. In the case above, note (i) c is a choice variable, (ii) K' is defined by (58) (which we will discuss below), (iii) a' is defined by (55), (iv) the variables in (55) (especially r and w) are also defined by constraints, which only contains state variables (K), thus we know that the problem is well defined.

- Again, prices $\{r,w\}$ are functions of aggregate variables, so agents have to take them as given. Note that this is because individual is measure zero, by assumption (so, although we are dealing with representative agent, at the same time we assume that agents are measure zero and have no power to affect aggregate state of the world, hence prices).

- (58) might look strange, but without it the problem is not well defined. In other words, we have to allow agents to make ”belief” or ”forecast” or ”expectations” about the future state of the world, to solve the problem, because agents need to make expectations about the return to capital in the next period to make consumption - saving choice.

- What can be the arguments of G^e function? individual variables $\{c,a,a'\}$ cannot be, because by assumptions, individual agents have no power to affect the aggregate state of world. $\{r,w\}$ cannot be if K is an argument, because K is a sufficient statistics for prices. Thus, we know that K is the only argument of G^e function.

- We index the value function with G^e because the solution of the problem above depends on the choice of G^e. But what is ”appropriate” G^e? This is revealed when we see the definition of an equilibrium below.

- Notice that $\{K, w(K), r(K), G^e(K)\}$ are enough to generate all future prices if today's aggregate capital is K.

Homework 5.1. *Prove properties of V(.) function.*

Now, Let's define the Recursive Competitive Equilibrium[19]:

Definition 5.2. *A Recursive Competitive Equilibrium is* $\{V^*(.), g^*(.), G^*(.), r(.), w(.)\}$ *such that*

1. *Given* $\{G^*(.), r(.), w(.)\}$, $\{V^*(.), g^*(.)\}$ *are characterized by the optimal decisions of the consumers, i.e.:*

$$V^*(K, a, G^*) = \max_{c, a'}[u(c) + \beta V(K', a'; G^*)]$$

subject to (55), (56), (57), and $K' = G^*(K)$, *and*

$$a' = g^*(K, a; G^*) \in \arg\max(\textit{the same problem})$$

2. $\{r(.), w(.)\}$ *are characterized by the optimal decisions of firms.*
3. $G^*(K) = g^*(K, K; G^*)$

Some comments on the third condition. The third condition means that if a consumer turns out to be average this period (her individual capital stock is K, which is aggregate capital stock), the consumer will choose to be average in the next period (she chooses $G^*(K)$, which is a belief on the aggregate capital stock in the next period if today's aggregate capital stock is K). You can interpret this condition as "consistency" condition, because this condition guarantees that in an equilibrium, individual choice turns out to be consistent with the aggregate law of motion.

Homework 5.3. *(for the next class) Define the Recursive Competitive Equilibrium for the economy with labor-leisure choice (leisure is valued by the consumer).*

6 Feb 7: Applications of RCE

6.1 RCE for the Economy with Endogenous Labor-Leisure Choice

Let's start from SME.

Definition 6.1. *A SME is a set of sequences* $\{c_t^*, n_t^*, a_{t+1}^*, w_t^*, r_t^*\}$ *such that:*

[19] I added the conditions from the firms' optimization problem as it's more familiar definition. But Victor seems to prefer treating the firm's problem in an implicit way to ease the notation. So I will follow his convention from the next definition of RCE. For reference, compare this definition with the one in the next class (with labor-leisure choice).

1. *Given* $\{w_t^*, r_t^*\}$, $\{c_t^*, n_t^*, a_{t+1}^*\}$ *solves the consumer's problem below:*

$$\max_{\{c_t^*, n_t^*, a_{t+1}^*\}} \sum_{t=0}^{\infty} \beta^t u(c_t, n_t)$$

subject to

$$c_t + a_{t+1} = a_t(1 + r_t^*) + w_t^* n_t$$

a_0 *is given*

2. $\{w_t^*, r_t^*\}$ *are given by the marginal products of factors at* $\{c_t^*, n_t^*, a_{t+1}^*\}$.

A couple of remarks:

Remark 6.2. *The condition 2. is derived from the firm's optimization problem, but since the firm's problem is static one and not interesting, we directly write the implications of the firm's optimization problem instead of writing the formal problem of firms. In the most of the course, we follow this convention.*

Remark 6.3. *The definition above doesn't include "consistency" (or "market clearing" condition as a particular form of consistency condition) explicitly, but note that it is implicitly considered. Since we assume that the technology is CRS and strictly increasing in both arguments, period utility function is strictly concave, and there is only one representative consumer and representative firm, optimal choice of the consumer and the firm guarantees market clearing.*

Now let's define the RCE of this economy[20].

First is question, as always, is what are the state variables? Of course, K and a. What else? That's it. Why? Because, possible candidate, N, is not predetermined when the consumer wakes up in the morning of period t: Rather, the aggregate labor supply is determined by the agents in the economy THIS PERIOD. Though the N share the same property as K in that they are aggregate and cannot be influenced by tiny tiny agent in the economy, but the difference is that K is predetermined while N is not.

The problem of consumer is as follows:

$$V(K, a; H^e, N^e) = \max_{c,n,a'} \{u(c, n) + \beta V(K', a'; H^e, N^e)\} \tag{59}$$

subject to

$$c + a' = [1 + r(K, N)]a + w(K, N)n \tag{60}$$

[20]As I mentioned in the note of the last class, Victor prefer to reduce notation by implicitly considering the firm's problem if possible. So I follow the convention. Also I change the expectation function for aggregate capital in the next period from G to H because we want to use G for government related function.

$$K' = H^e(K) \tag{61}$$

$$N = N^e(K) \tag{62}$$

And the solutions are:

$$a' = h(K, a; H^e, N^e) \tag{63}$$

$$n = n(K, a; H^e, N^e) \tag{64}$$

Note that the consumer sees K, r, w in the morning when they get up and make a guess for K' and N. But because of the assumptions of the technology, the guess for N is "really accurate". So, in that sense, there seems to be no gain from introducing a expectation function for aggregate labor supply, but treating N in the same way as K' makes the definition of RCE cleaner. Now comes the definition of RCE:

Definition 6.4. *A RCE is a set of functions* $\{V^*(.), H^*(.), N^*(.), h^*(.), n^*(.)\}$ *such that*

1. *Given* $H^e(.) = H^*(.)$ *and* $N^e(.) = N^*(.)$, *then* $\{V^*(.), h^*(.), n^*(.)\}$ *solves the consumer's problem.*

2.
$$H^*(K) = h^*(K, K; H^*, N^*) \tag{65}$$
$$N^*(K) = n^*(K, K; H^*, N^*) \tag{66}$$

Note that $N^e(.)$ is easily guessed by consumers, because consumers know K, $r(K,N)$, $w(K,N)$. Using this property, we can think the equilibrium in a different way: the equilibrium can be interpreted as (i) consumers make a guess for equilibrium prices when they make decisions, and (ii) the forecast prices are consistent with the actual prices realized. Formal definition of RCE based on this idea is as follows:

The consumer solves:

$$V(K, a; H^e, w^e, r^e) = \max_{c,n,a'} \{u(c,n) + \beta V(K', a'; H^e, w^e, r^e)\} \tag{67}$$

subject to

$$c + a' = [1 + r^e(K)]a + w^e(K)n \tag{68}$$

$$K' = H^e(K) \tag{69}$$

And the solution is:

$$a' = h(K, a; H^e, w^e, r^e) \tag{70}$$

$$n = n(K, a; H^e, w^e, r^e) \tag{71}$$

And the definition of RCE is:

Definition 6.5. *A RCE is a set of functions* $\{V^*(.), H^*(.), w^*(.), r^*(.), h^*(.), n^*(.)\}$ *such that*

1. *Given* $H^e(.) = H^*(.)$, $w^e(.) = w^*(.)$, *and* $r^e(.) = r^*(.)$, *then* $\{V^*(.), h^*(.), n^*(.)\}$ *solves the consumer's problem.*

2.

$$H^*(K) = h^*(K,K;H^*,w^*,r^*) \tag{72}$$

$$w^*(K) = w^*(K,K;H^*,w^*,r^*) \tag{73}$$

$$r^*(K) = r^*(K,K;H^*,w^*,r^*) - \delta \tag{74}$$

Homework 6.6. *Consider a RA-NGM without labor-leisure choice. Show the equivalence between SME and RCE.*

Hint 6.7. *You need to show two ways: (i) RCE is SME, and (ii) SME is RCE. The former is easier than the latter. For this class, just do the former. You can show by construction. Suppose we have a RCE. Using* a_0 *(given) and* $H^*(K)$*, we can derive a whole sequence of* $\{k_t, c_t\}_{t=0}^{\infty}$*. Using the constructed sequences of allocation, we can construct sequence of prices* $\{r_t, w_t\}_{t=0}^{\infty}$*. Remember that we have necessary and sufficient conditions for SME. we just need to show that the necessary and sufficient conditions are satisfied by the constructed sequences.*

6.2 A Digression on Notation

Notation is important. Especially for macro guys who have to deal with lots of parameters, aggregate and individual state variables and control variables. Use of good (in the sense that following the convention) notations help your explanation. Bad notations would make you waste your time in explaining notational things when you present your paper. Here are some conventions on notations:

- Never use Greek letters for choice variables. Very often, Roman alphabets are used for controls and states, while Greek letters are used for parameters and endogenous functions.
- Use small letters for individual states and choices, and capital letters for aggregate states. If possible, it is kind to use the same alphabet for aggregate (capital letter) and individual (small letter) states.
- Always follow the same order when you list up things. Example of conventional order is (i) aggregate exogenous variable (following exogenous law of motion), (ii) aggregate endogenous variable, (iii) individual variables.

6.3 RCE for non-PO economies

6.3.1 Introduction

What we did with RCE so far can be claimed to be irrelevant. Why? Because, since the Welfare Theorems hold for these economies, equilibrium allocation, which we would like to investigate, can be solved by just solving SPP allocation. But RCE can be useful for analyzing much broader class of economies, many of them is not PO (where Welfare Theorems do not hold). That's what we are going to do from now. Let's define economies whose equilibria are not PO, because of distortions to prices, heterogeneity of agents, etc.

6.3.2 The Government

What is the government? It is an economic entity which takes away part of our income and uses it. The traditional (or right-wing) way of thinking of the role of the government is to assume that the government is taking away part of our disposable income and holds a party, which we do not enjoy (do not earn utility from the party). Or often we describe as "throwing away into ocean". If you are left-wing person, you might think that we value what the government does, as we do for our own consumption. Or the government holds a party which we can enjoy. Even if you think in that way, as long as we assume that utility is separable between consumption (and labor) and government expenditure, the effect of the government is the same, since the prices and behavior of consumers are not affected by the expenditure of the government (of course, prices are distorted by taxes, in the same way as "throwing into ocean" case). We might assume that the government expenditure is NOT separable with consumption and try to analyze the effect of the government expenditure, but we do not do this in this class.

Another thing we need to think at this stage is the constraint of the government. For now, we assume that (i) the government is restricted by period-by-period budget constraint (so the government cannot run deficit nor surplus). Of course we can analyze the government with intertemporal budget constraint, but we do not do this, to keep our analysis simple (remember that we need some kind of Non-Ponzi Scheme constraint to prohibit the government from accumulating public debt infinitely).

The last thing we need to consider here is the form of tax. We can assume capital income tax, labor income tax, or general income tax. For each tax, we can assume constant marginal tax rate, or progressive (or regressive) marginal tax rate. For now, let's choose the simplest case: constant general income tax.

6.3.3 RCE with Government

Let's forget about the labor-leisure choice. And let's define the consumer's problem with both right wing (government expenditure is not values) and left wing (government expenditure is values) assumptions, one by one. For right winger, the consumer's problem is as follows:

$$V(K,a;H^e,G^e) = \max_{c,a'}[u(c) + \beta V(K',a';H^e,G^e)] \tag{75}$$

subject to

$$c + a' = a + [w(K) + r(K)a](1-\tau) \tag{76}$$

$$K' = H^e(K) \tag{77}$$

$$G = G^e(K) \tag{78}$$

To change the problem to the version of left-winger, we just need to use $u(c,G)$ instead of $u(c)$ as a period utility function. Couple of remarks:

Remark 6.8. *Notice that the economy does not achieve Pareto Optimality, since (i) the allocation from equilibrium with government is different from the allocation from the equilibrium without government, and (ii) we know that the latter allocation is unique PO allocation. This is because the prices are distorted by the existence of the income tax and thus allocations are affected.*

Remark 6.9. *As mentioned earlier, in the left-winger case, if we assume $u(c,G) = u(c) + v(G)$, i.e. utility function is separable between (private) consumption and public expenditure, the solutions of the problems are going to be the same between left and right.*

The solution of the problem is:

$$a' = h(K,a;H^e,G^e) \tag{79}$$

Now define the RCE of the economy:

Definition 6.10. *A RCE is a set of functions $\{V^*(.),\, H^*(.),\, G^*(.),\, h^*(.)\}$ such that*

1. *Given $H^e(.) = H^*(.)$, $G^e(.) = G^*(.)$, then $\{V^*(.),\, h^*(.)\}$ solves the consumer's problem.*
2. $$H^*(K) = h^*(K,K;H^*,G^*) \tag{80}$$

$$\begin{aligned} G^*(K) &= [r(K)K + w(K)]\tau \\ &= \tau[F(K,1) - \delta K] \end{aligned} \tag{81}$$

Remark 6.11. *Notice that the depreciation is not taxed, i.e. the tax base is the national income, not GDP.*

Homework 6.12. *Consider an economy with labor-leisure choice and an arbitrary tax regime with progressive marginal tax rate $\tau(y)$ ($\tau'(y) \geq 0$) (for example, imagine general income tax with progressive marginal tax rate, i.e. $\tau(y) = \tau(r(K)a + w(K)n)$). Define the RCE for this economy.*

6.4 The Topics Ahead

We are going to talk about some extensions. An example is the economy with government where government can accumulate debt. Also we will see a RCE for the stochastic economy. After the exam (BidTerm), we will see the economies with heterogeneous agents, incomplete markets, etc.

7 Feb 12: Economy with Government Bond

7.1 Introduction

Now assume that the government taxes labor income and issues debt to pay for a constant stream of government expenditures $\bar{G}$. This economy is more complicated and tricky than the previous economy (where the amount of the government expenditure is equal to the tax income) Why?

- When the government issues debt, period by period government budget constraint is balanced, like in the economy we discussed in the last class.
- Government budget constraint will not be satisfied automatically in defining equilibrium.
- Tax policy, that is represented by a function τ, should depend on state of the economy. In particular, since the government always spends a constant expenditure, (i) the government will retire the debt that was issued before when it has a higher revenue, and , (ii) the government will issue more debt when it has a lower revenue.

The tricky part of the problem is to ensure that the government budget constraint is satisfied in the sense of present value. In other words, we want to rule out the insufficient taxation when debt keeps growing. We call such situation as "snowball effect" or "Ponzi scheme".

7.2 Define RCE

7.2.1 State variables

- Aggregate state variable: K, B. K is the aggregate capital in the economy. B is the government debt stock.

  Government debt here is one period debt in the form of discount bond. Government sell bond today at price q and promise to repay one unit of good tomorrow.
- Individual state variables: a. a is a total asset holding of the agent.

Representative agent only cares about the value of her asset holding, not the composition of her asset portfolio. So, in defining RCE, we only need one state variable for the asset, not both physical capital holding k and financial asset b.In doing so, one equilibrium condition is implied:

physical capital holding k and financial asset b bear the same rate of return. This condition holds because they are perfect substitutes, and by No Arbitrage argument.

Remark 7.1. *Making the space of state variables is small very important for computational purpose: this will save a lot of time and help avoiding colinearization.*

7.2.2 Household's problem

$$V(K,B,a) = \max_{c,a'} u(c) + V(K',B',a') \tag{82}$$

subject to

$$\begin{aligned} c + a' &= (1+r)a + w(1-\tau) \\ K' &= \Phi(K,B) \\ B' &= \Psi(K,B) \\ w &= F_L(K,1) \\ r &= F_K(K,1) - \delta \\ \tau &= \tau(K,B) \end{aligned} \tag{83}$$

And the solutions are:

$$a' = h(K,B,a) \tag{84}$$

There are different ways of writing an equilibrium. Some are long and tedious, but here we are using short cut in the following sense. The functional form of w and r are given explicitly by marginal product of labor and marginal product of labor capital minus depreciation. So in the definition of equilibrium, we do not need to write out firm's problem.

Household needs to know B because B will affect future prices. In our problem, law of motion for K, B and future taxes τ depend on B, so future prices are affected by B. Why in this problem household expects K' and B' to evolve according to Φ and Ψ? We set it so and this is true in RCE.

There is no government expenditure in household's problem, because household does not care G, rather G will affect individual problem indirectly through B and τ.

7.2.3 Definition of RCE

Definition 7.2. *Given $\tau(K,B)$, a RCE is a set of functions $\{V^*, \Phi^*, \Psi^*, a^*, q^*\}$ such that*

1. *(Household's optimization) Given $\{\Phi^*, \Psi^*\}, \{h^*, V^*\}$ solve the household problem.*

2. *(Consistency)*

$$\Phi^*(K,B) + q^*(K,B)\Psi^*(K,B) = h^*(K,B,K+\frac{B}{1+r}) \tag{85}$$

where r is defined in the household problem.

3. *(No Arbitrage Condition)*

$$q^*(K,B) = \frac{1}{1+F_K(\Phi^*(K,B),1)-\delta} \tag{86}$$

4. *(Government Budget Constraint at Present Value)*

$$q^*(K,B)\Psi^*(K,B) = \bar{G} + B - \tau(K,B)F_L(K,1) \tag{87}$$

5. *(No Ponzi Scheme Condition)* $\exists \underline{B}$ *and* $\overline{B}$, *such that* $\forall K \in [0,\overline{\overline{K}}]$, $B \in [\underline{B},\overline{B}]$

$$\Psi^*(K,B) \in [\underline{B},\overline{B}],\ \Phi^*(K,B) \in [0,\overline{\overline{K}}] \tag{88}$$

$q(K,B)$ is the present value of government debt. In other words, q is today's price of tomorrow's debt and it is the value of B before interest. Please recall the price for Arrow security. q is the relative price of consumption between today and tomorrow.

$\frac{1}{1+r}$ is yesterday's price of today's debt (notice that r is the rate of return today), where q is today's price of tomorrow's debt. Therefore, though $\frac{1}{1+r}$ is analogous to q, they are not the same. Only at steady state, $\frac{1}{1+r} = q$. To see this more clearly, let's see the budget constraint in the sequential market equilibrium:

$$c_t + a_{t+1} = a_t(1+r_t) + w_t(1-\tau_t) \tag{89}$$

In the budget constraint, r_t is the compensation for giving up one unit of consumption in the last period. Similarly, saving additional one unit of a_{t+1} gives the rate of return r_{t+1}, and q is associated with this.

Homework 7.3. *Consider an economy with the government. Suppose agents do not value leisure and there is no stochastic shock. Period utility function is given by* $u(c)$. *Production function is* $F(K,N)$. *Government taxes labor income and issues debt to pay for a constant stream of government expenditures* $\bar{G}$. *Government debt is issued at face value with a stream of interest rate* $\{r_{B,t}\}$. *Given a tax regime with labor income tax rate function* τ, *define a RCE for this economy.*

Hint 7.4. *One individual state variable suffices for household's problem because again, by no arbitrage argument, both physical capital and financial asset have the same rate of return.*

In our example in class, bonds are issued as discount bond. And in the homework, bonds are interest bonds. To illustrate the difference, let's compare a dollar in a saving account and a dollar invested into a discount bond.

	today	tomorrow
saving account	\$1	$\$(1+r')$
discount bond	\$1 (buy $\frac{1}{q}$ bonds at price q)	$\$\frac{1}{q}$

Remark 7.5. *Recall Walras's law: Suppose there are n goods in the market. If (n-1) markets are cleared, by budget constraint, the market clearing condition for the last good (good n) is also satisfied. We can use the similar argument here: In our example, if RCE exists, budget constraint of the government is automatically satisfied because of the household's budget constraint and aggregate law of motion for aggregate asset holding.*

7.2.4 Discussion on Non Ponzi Scheme Condition

Why do we need Non Ponzi Scheme condition in a definition of RCE? If we only use the present value government budget constraint, for some tax regime, the government could tax too little or too much, and the government debt stock would keep increasing or decreasing without limit. We would like to eliminate such possibility. That's why we need a compact set to set bound on the government bond stock.

Recall in the standard growth model, there is also one implicit constraint on capital: $K > 0$ and $\exists \overline{\overline{K}}$ such that capital always is in the compact set of $[0, \overline{\overline{K}}]$. So, very strictly, the Bellman equation in the standard growth model is:

$$V(K) = \max_{c \geq 0, 0 \leq K \leq \overline{\overline{K}}} \{u(c) + \beta V(K')\}$$

subject to

$$c + K' = F(K)$$

and

$$\exists \overline{\overline{K}} \text{ such that } F(\overline{\overline{K}}) < \overline{\overline{K}}.$$

If we assume Inada's condition on production function, the above condition for capital is assured by the coverture. Here the Non Ponzi Scheme Condition does the same thing in this economy. For example, if the government keeps taxing too little, then the government debt stock will grow like a snowball and eventually exceed $\overline{B}$, which implies that this cannot be an equilibrium.

Homework 7.6. *(Economy with Externality) Assume that agents in the economy care about other's leisure. That is, the preference is given by:*

$$u(c, 1-n, 1-N)$$

where $L = (1-N)$ is the aggregate leisure and u is increasing in all arguments.
Assume there is no shock and no government. Production function is $F(K,N)$. Define a RCE for this economy. Compare the equilibrium allocation with the solution of SPP and explain why.

Hint 7.7. *When agent makes her optimal decision, she treats aggregate leisure as given. But social planner only faces aggregate variables and decides allocation. So, there will be difference in taking FOC.*

8 Feb 14: Asset Pricing

8.1 More words on RCE with government

Is there always a RCE, given $\tau(K,B)$ and $\overline{G}$? To answer this question, let us start from the similar question without government debt. If for a given τ_1, there exists a RCE $\{V^*,...\}$, for $\tau_2 = \frac{1}{2}\tau_1$ does a RCE exist? The answer is YES. Why? The government spends whatever it collects from tax and there is no government debt. The government spending of the economy with tax rate τ_1, and τ_2 are:

$$\begin{aligned} G_1(K) &= \tau_1[F(K,N(K)) - \delta K] \qquad (90) \\ G_2(K) &= \tau_2[F(K,N(K)) - \delta K] \end{aligned}$$

As is clear now, level of tax rate is irrelevant to the existence of the associated RCE.

Next consider the economy with government debt and exogenously given government expenditure. If tax rate is too high, government debt stock will keep decreasing without lower bound, whereas if tax rate is too low, government debt stock will keep growing without upper bound. Only with right tax rate $\tau(K,B)$ can B remain bounded. Therefore, tax regime $\tau(K,B)$ is required to be an appropriate one. $\tau(K,B)$ can be of various functional forms, but it is unique in the sense that if $\tau(K,B)$ is right tax regime for a given government expenditure, $\lambda\tau(K,B)$ is not.

We can see the similar argument in finding the solution of the optimal capital stock path. Euler equation in the standard growth model is a second order difference equation with only one initial condition like the following:

$$\begin{aligned} 0 &= \varphi(k_t, k_{t+1}, k_{t+2}) \qquad (91) \\ & k_0 \text{ given} \end{aligned}$$

We should choose k_1 which does not violate transversality condition, i.e. $\{k_t\}$ does not diverge. Recall No Ponzi Scheme Condition in our definition of RCE for the economy with government debt. That condition has to be satisfied in order that the tax regime $\tau(K,B)$ is feasible.

8.2 The Road Ahead

Today we will study a simple model which will is a very powerful tool for solving asset prices and thus actively used in finance. From the next class, we will talk about the model with multiple agents.

8.3 Lucas Tree Model (Lucas 1978)

8.3.1 The Model

Suppose there is a tree which produces random amount of fruits every period. We can think of these fruits as dividends and use d_t to denote the stochastic process of fruits production. Further, assume d_t follows Markov process. Formally:

$$d_t \sim \Gamma(d_{t+1} = d_i \mid d_t = d_j) = \Gamma_{ji} \tag{92}$$

Let h_t be the history of realization of shocks, i.e., $h_t = (d_0, d_1, ..., d_t)$. Probability that certain history h_t occurs is $\pi(h_t)$.

Household in the economy consumes the only good, which is fruit. With usual assumption on preference retained, consumers maximize:

$$\sum_t \beta^t \sum_{h_t \in H_t} \pi(h_t) u(c_t) \tag{93}$$

Since we assume representative agent in the economy, and there is no storage technology, in an equilibrium, the representative household eats all the dividends every period. So the lifetime utility of the household will be:

$$\sum_t \beta^t \sum_{h_t \in H_t} \pi(h_t) u(d_t) \tag{94}$$

Now suppose that the household is given some STUFF at period 0 and there exists a market to trade fruits. It's trivial to guess that the equilibrium allocation will be an autarky (almost by definition), but the key thing is to find the price which can support the equilibrium allocation of autarky.

Define the household's problem.

$$\max_{\{c(h_t)\}_{t=0}^{\infty}} \sum_t \beta^t \sum_{h_t \in H_t} \pi(h_t) u(c_t(h_t)) \tag{95}$$

subject to

$$\sum_t \beta^t \sum_{h_t \in H_t} p(h_t) c(h_t) = STUFF \tag{96}$$

and

$$p_0 = 1 \tag{97}$$

Note that we are considering the Arrow-Debreu market arrangement, with consumption goods in period 0 as a numeraire.

8.3.2 First Order Condition

Take first order condition of the above maximization problem:

$$FOC\ \ c(h_t) \qquad \frac{p(h_t)}{p_0} = p_t(h_t) = \frac{\beta^t \pi(h_t) u'(c(h_t))}{u'(c(h_0))} \tag{98}$$

By combining this FOC with the following equilibrium condition:

$$c(h_t) = d_t \ \forall t, h_t \tag{99}$$

We get the expression for the price of the state contingent claim in the Arrow-Debreu market arrangement.

$$p_t(h_t) = \frac{\beta^t \pi(h_t) u'(d(h_t))}{u'(d(h_0))} \tag{100}$$

8.3.3 Price the tree

Now we can compute the mysterious STUFF which satisfies the budget constraint.
What is the STUFF? STUFF is the sufficient amount to buy fruits in every period in every contingency from time 0 on, measured in period 0 consumption good. We can Imagine that the STUFF is a TREE, which bears fruits.
Tree in this model is a package of a stream of good. In asset pricing,

the price of an asset = value of all the things that the asset entitles you to get.

Therefore, the formula to compute q_t =the price of tree at period 0 is:

$$q_0 = \sum_t \sum_{h_t \in H_t} p_t d_t = \sum_t \sum_{h_t \in H_t} \frac{\beta^t \pi(h_t) u'(d(h_t))}{u'(d(h_0))} d(h_t) \tag{101}$$

8.3.4 Sequential Market

In sequential market, the household can buy and sell fruits in every period, and the tree (the asset). To consider the trade of the asset, let s_t be share of asset and q_t be the asset price at period t. The budget constraint at every time-event is then:

$$q_t s_{t+1} + c_t = s_t(q_t + d_t) \tag{102}$$

Thus, the consumer's optimization problem turns out to be:

$$\max_{\{c_t(h_t), s_{t+1}(h_t)\}_{t=0}^{\infty}} \sum_t \beta^t \sum_{h_t \in H_t} \pi(h_t) u(c_t(h_t)) \tag{103}$$

subject to

$$q_t(h_t)s_{t+1}(h_t) + c_t(h_t) = s_t(h_{t-1})[q_t(h_t) + d_t] \tag{104}$$

Again, from first order condition, we can derive q_t, which is the price of one tree after history h_t in terms of consumption goods at node h_t. To solve the problem, construct Lagrangian as follows:

$$L: \quad \sum_t \beta^t \sum_{h_t \in H_t} \pi(h_t)[u(c_t(h_t)) - \lambda_t(h_t)\{s_t(h_{t-1})[q_t(h_t) + d_t] - q_t(h_t)s_{t+1}(h_t) + c_t(h_t)\}] \tag{105}$$

Note that there are many ways to write equivalent Lagrangians. In the case above, the sequence of Lagrange multipliers is $\{\beta^t\pi(h_t)\lambda_t\}$. We write it in this way to simplify expressions of the first order conditions. First order conditions are:

$$FOCw.r.t.c_t(h_t) \qquad u'(c_t(h_t)) = \lambda_t(h_t) \tag{106}$$

$$FOC\ w.r.t.\ s_{t+1}(h_t) \qquad \pi(h_t)\lambda_t(h_t)q_t(h_t) = \beta \sum_{h_{t+1}|h_t} \pi(h_t)\lambda_{t+1}(h_{t+1})[q_{t+1}(h_{t+1}) + d_{t+1}(h_{t+1})] \tag{107}$$

Recall, d_t follows a Markov process,

$$\pi(h_{t+1}) = \pi(h_t)\Gamma_{ij} \text{ where } d_t(h_t) = d_i,\ d_{t+1} = d_j \tag{108}$$

so, combine (106) and (107), we get:

$$u'(c_t(h_t))q_t(h_t) = \beta \sum_j \Gamma_{ij} u'(c_{t+1}(h_{t+1}))[q_j + d_j] \tag{109}$$

In equilibrium, $c_t(h_t) = d_t(h_t)$. Let's pick $d_t(h_t) = d_i$, then,

$$u'(d_i)q_i = \beta \sum_j \Gamma_{ij} u'(d_j)[q_j + d_j] \tag{110}$$

From this equation, we can see that (i) the price of asset is also Markovian, and (ii) the marginal utility today is equal to marginal utility tomorrow weighted by prices at each node. To see them, look below:

$$q_i = \beta \sum_j \Gamma_{ij} \frac{u'(d_j)}{u'(d_i)}[q_j + d_j] \tag{111}$$

$$u'(d_i) = \beta \sum_j \Gamma_{ij} u'(d_j) \frac{[q_j + d_j]}{q_i} \tag{112}$$

In order to solve for the prices of q_i, we need to solve the system of equations that consists of (112) for each i. Now, suppose that the dividend process is not Markovian. We can still get price of tree in terms of h_t good as follows:

$$q_t(h_t) = \frac{\sum_{\tau=t+1}^{\infty} \sum_{h_\tau | h_t} p(h_\tau) d_t(h_\tau)}{p(h_t)} \tag{113}$$

or

$$p(h_t) q_t(h_t) = \sum_{\tau=t+1}^{\infty} \sum_{h_\tau | h_t} p(h_\tau) d_t(h_\tau) = \sum_{h_{t+1} | h_t} p(h_{t+1}) [d_{t+1}(h_{t+1}) + q_{t+1}(h_{t+1})] \tag{114}$$

Remark 8.1. *The prices in the sequential market must be normalized by $p(h_t)$ because otherwise, we get the time 0 value of the bundle of good. Keep in mind the difference and transformation between time 0 price and date-event price. $p(h_t)$ is time 0 ADE price of the consumption goods in date-event h_t from static market, and $q(h_{t+1})$ is date-event h_t price for asset (tree) in the date-event h_t in the sequential market.*

8.3.5 Pricing an Arbitrary Asset

Because in a complete market any asset can be reproduced by buying and selling contingent claims at every node, we can use this model as a powerful asset pricing formula. For example, discount bond is a promise to pay one unit of good tomorrow no matter what happens. To reproduce bond, it suffices to buy one unit of state contingent claim at every node in the next period. Therefore, at h_t, the price of bond is:

$$p^b(h_t) = \frac{\sum_{h_{t+1} | h_t} p(h_{t+1})}{p(h_t)} \tag{115}$$

Consols is a promise to pay one unit of good forever from now on. Thus its price is:

$$p^{consol}(h_t) = \frac{\sum_{\tau=t}^{\infty} \sum_{h_\tau | h_t} p(h_\tau)}{p(h_t)} \tag{116}$$

Consider a one period call option, which is a right to buy one share of a tree at the fixed price (exercise price) $\bar{q}$. The price of this option is:

$$p^{o,\bar{q}}(h_t) = \frac{\sum_{h_{t+1} | h_t} p(h_{t+1}) [q(h_{t+1}) - \bar{q}] 1_{[q(h_{t+1}) - \bar{q}] > 0}}{p(h_t)} \tag{117}$$

where 1 is an indicator function (see the note of the next class).

Homework 8.2. *Price a two period option.*

9 Feb 19: Economy with Two Types of Agents

9.1 Review of Asset Price Model

9.1.1 General Principle

Consider asset prices in a model with uncertainty. In general, we only need to know prices of consumption at each node to price any kinds of assets and options. Once we have the prices of consumption, all we need to calculate a price of an asset is to calculate the amount of consumption gain entitled to the holder of the asset at each node, multiplied by the price of consumption at the node, and sum them up across all the nodes.

If the asset gives a stream of dividends like the Lucas tree, the price of the asset (the tree) is the sum of the amount of future dividends given to the holder of the tree at each node, multiplied by the price of consumption at each node. Similarly, the price of an option is a sum of gain under the option at each node (considering the decision of whether to exercise the option or not), multiplied by the price of consumption at each node.

In addition, if we assume that the stochastic process is Markov, we can show (by guess and verify) that the prices of assets are also Markov. But note that it is just the special case: anyway what we need is the price of consumption at each node, and Markov just simplifies the calculation with its simple stochastic structure.

9.1.2 Two Period Option

To see that we can price any kinds of assets or options using this principle, let's price two periods option. Option here is the RIGHT to buy a consumption goods at a negotiated price. When we talk about multiple period options, we have to be aware the difference between American and European option. American option can be exercised AT ANY TIME before its maturity. On the contrary, European option can be exercised ONLY AT ITS MATURITY. But the principle to price them is same. By the way, notice that American option is always more expensive than its European counterpart, because American option contains more options to its holders.

Here let's price two period American and European options at a node h_t. As a set up, assume that the set of the possible aggregate shock contains two elements. Start from h_t, possible nodes in the next periods are h^1_{t+1} and h^2_{t+1}. In the two period ahead, there are four possible nodes, h^1_{t+2}, h^2_{t+2}, h^3_{t+2}, h^4_{t+2}, where h^1_{t+2}, and h^2_{t+2} can be reached only from h^1_{t+1}.

Firstly, remember the price of an one period option at the node $h_{t+1} : p^o(h_t)$ with negotiated price $\bar{q}$. This is:

$$p^{o1}(h_t) = \sum_{h_{t+1}|h_t} [q(h_{t+1}) - \bar{q}] 1_{[q(h_{t+1})-\bar{q}]>0} \frac{p(h_{t+1})}{p(h_t)} \tag{118}$$

where $1_{[\text{expression}]}$ is an indicator function that takes value of 1 if the [expression] is true and 0 if false, and $p(h_t)$ is the price of consumption goods at node h_t. You can also use χ for an indicator function.

Price of an European option (option which can be exercised ONLY in the two period ahead), which is just the natural extension of this one period option, is as follows:

$$p^{o2}(h_t) = \sum_{h_{t+2}|h_t} [q(h_{t+2}) - \bar{q}] 1_{[q(h_{t+2})-\bar{q}]>0} \frac{p(h_{t+2})}{p(h_t)} \tag{119}$$

Price of an American option is a little bit more tricky:

$$p^{oa2}(h_t) = \sum_{h_{t+1}|h_t} \max\left\{p^{o1}(h_{t+1}), [q(h_{t+1}) - \bar{q}]\right\} \frac{p(h_{t+1})}{p(h_t)} \tag{120}$$

In the period t+1, a holder can either (i) exercise the option (and then the option expires), or (ii) keep the option to the next period (in this case, the option is exactly the same as the one period option bought in the period t+1).

9.1.3 Final Remark

In this fashion, we can price any kinds of assets or options. For example, you can easily price future transaction[21]. This is basically finance guys are doing during while their life. They are just solving the price, without solving the allocation (because of RA assumption, we do not need to solve the asset portfolio of agents, which are the same in equilibrium).

9.2 NGM with Two Types of Agents

9.2.1 Introduction

From now, we will see what happens if we relax the assumptions we made for our standard RA-NGM. As a first step, we drop the assumption of representative agent. In particular, we start from assuming that there are two types of agents in the economy.

9.2.2 Environment

There are two types of agents. Let's call the two types as type A and type B. Since there are infinite identical agents for each type, each agent is a price taker. Measure of the agents of type A and type B are the same. Without loss of generality, we can think of the economy as the one with two agents, both of whom are price takers.

[21] Future transaction is a contract to buy or sell a goods in a negotiated period at a negotiated price. The difference from option is that you MUST perform the transaction, no matter whether you want to do or not. Naturally, option contract is more expensive, as you are given an option not to exercise.

Preference is standard: separable in time and realization of shock. Leisure is not valued. More formally, life time utility of each agent is:

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \tag{121}$$

The only difference between type A and B is the production technology which is available to agents. Agents of type A can use technology of type A, and same for type B. More specifically, the production technology for type $i \in \{A, B\}$ can be expressed as:

$$y = z^i F(k) \tag{122}$$

where F satisfies standard assumptions (strictly increasing in k, strictly concave in k). Assume that the final goods are same for both types of agents and can be traded even between different types of agents.

Let $z = (z^A, z^B)$ and $z \in Z = \{z^1, z^2, ..., z^{nz}\}$. Each element of Z contains a pair of productivities assigned to type A and B. Assume that z follows a finite state Markov process, and denote the probability that z' occurs conditional on z as $\Gamma_{zz'}$. Also let $h_t = (z_0, z_1, ..., z_t)$, i.e. history of realization of shocks up to period t.

9.2.3 SPP

The problem is NOT "Solve the SPP" here. Our problem is "Solve the SPP's". Why? Since we have two types of agents, There are many solutions of SPP, corresponding to many possibilities of relative Pareto weights attached to both types of agents (note that only the relative weights matter). In other words, all the points on the Pareto frontier are the solutions to SPP. We will solve ALL of them by finding solutions for all possible Pareto weights.

Let (λ^1, λ^2) be the set of Pareto weights attached to each type of agents. Then, the SPP is:

$$\max_{\{c_t^i(h_t), k_{t+1}^i(h_t)\}} \sum_i \lambda^i E \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t^i(h_t)) | z_0 \right\} \tag{123}$$

subject to

$$\sum_i k_{t+1}^i(h_t) + \sum_i c_t^i(h_t) = \sum_i z_t^i F(k_t^i(h_{t-1})) \quad \forall t, h_t \tag{124}$$

Couple of remarks below:

Remark 9.1. *If we assume autarky, i.e. agents can only live on their own production, (note that this is an assumption on technology), the feasibility constraint would be:*

$$k_{t+1}^i(h_t) + c_t^i(h_t) = z_t^i F(k_t^i(h_{t-1})) \quad \forall i, t, h_t \tag{125}$$

Which set-up gives higher welfare to agents? The model with trade. Why? The constraints characterized by (124) nests those characterized by (125). I.e., allocations that are feasible in the autarky economy are always feasible in the economy with trade. Comparing the constraint set is a common technique to compare welfare.

Remark 9.2. *We can assign different production function F to each type of agents. In this case, we need to index production function by the type of agents that use the technology. In particular, we need to use F^i instead of F.*

Remark 9.3. $\lambda^A = \lambda^B$ *does not guarantee the SPP allocation where $u_A = u_B$. This constraint guarantees that the slope of the Pareto frontier curve is -1. In order to guarantee $u_A = u_B$., either (i) solving the SPP with additional condition $u_A = u_B$, or (ii) change the maximand of SPP as*

$$\max u_A u_B$$

might be sufficient (prove by yourself!).

Remark 9.4. *In GE course, we used Brauer's or Kakutani's Fixed Point Theorem to find an equilibrium, by solving the prices. In our NGM, where (i) competitive equilibrium is PO, and (ii) the number of agents are small, Negishi method can be used instead of those FPT to find an equilibrium more easily. The essence of Negishi method is to transform the problem of finding prices to that of finding Pareto weights. You will see more on this method later.*

Remark 9.5. *Most of international economics is just concentrating on solving the SPP of the model with two agents. Two agents represents two countries. Most of the models in international economics satisfy Welfare Theorems, so we do not need to care about decentralization (equilibrium) and can concentrate on SPP.*

9.2.4 RCE

Denote $K = (k^A, k^B)$. Is $(z, K) = (z^A, z^B, k^A, k^B)$ sufficient as a set of aggregate state variables? No. Why? Since we have heterogeneity among agents (remember there are two types of agents), the state contingent claims (remember that we need to allow) are actually traded! So the aggregate state of the economy must contain the financial positions of each type of agent. In other words, now there is a difference among agents so they want to insure by trading state contingent claims each other.

Having said that, let l^A and l^B be the balance of state contingent claims for agents of type A and B and $L = (l^A, l^B)$. Aggregate state of the economy is $(z, K, L) = (z^A, z^B, k^A, k^B, l^A, l^B)$. Notice that[22]

$$l^B = -l^A$$

[22] Victor treated L as a scalar by exploiting $l^B = -l^A$ in class.

Let the individual state as (k^i, l^i). Agents' problem is[23]:

$$V^i(z,K,L,k^i,l^i) = \max_{c^i,k^{i\prime},l^{i\prime}(z')} \left\{ u(c^i) + \beta \sum_{z'} \Gamma_{zz'} V^i(z',K',L'(z'),k^{i\prime},l^{i\prime}(z')) \right\} \tag{126}$$

subject to

$$\sum_{z'} l^{i\prime}(z')q(z') + k^{i\prime} + c^i = z^i F(k^i) + l^i \tag{127}$$

$$q(z') = Q(z')(z,K,L) \tag{128}$$

$$K' = G(z,K,L) \tag{129}$$

$$L'(z') = H(z')(z,K,L) \tag{130}$$

Solutions are:

$$k^{i\prime} = k^i(z,K,L,k^i,l^i) \tag{131}$$

$$l^{i\prime}(z') = l^i(z')(z,K,L,k^i,l^i) \tag{132}$$

Now we are ready to define a RCE.

Definition 9.6. *A RCE is a set of functions* $\{V^i(.),\ G(.),\ H(z')(.),\ Q(z')(.),\ k^i(.),\ l^i(z')(.)\}$ *such that*

1. *(Optimal decision of agents)* $\{V^i(.),\ k^i(.),\ l^i(z')(.)\}$ *solves the agent's problem.*
2. *(Market clearing of claims)* $Q(z')(.)$ *is determined such that:* [24]

$$l^A(z')(z,K,L,k^A,l^A) + l^B(z')(z,K,L,k^B,l^B) = 0 \tag{133}$$

3. *(Consistency)*

$$G(z,K,L) = \begin{pmatrix} k^A(z,K,L,K^A,L^A) \\ k^B(z,K,L,K^B,L^B) \end{pmatrix} \tag{134}$$

$$H(z')(z,K,L) = \begin{pmatrix} l^A(z')(z,K,L,K^A,L^A) \\ l^B(z')(z,K,L,K^B,L^B) \end{pmatrix} \tag{135}$$

[23] I index all the individual variables by agent's type i, whereas Victor suppressed for notational simplicity.
[24] Victor included this condition as a part of consistency condition.

10 Feb 26: Economy with Heterogeneous Agents (1)

10.1 Introduction

So far, the type of agents does not change over time. In this case, especially, if the number of type of agents is small, it's easy to keep track of all the types, and so is to define an equilibrium (remember the case of two types in the last class). But from now, we consider the economies with (i) many agents who are very different to each other (many types), and (ii) agents change their types over time.

Since agents might trade each other, we need to keep track of the aggregate state of the world (remember the aggregate state variables in the economy with two types of agents). There are two ways to do it. One is "Spanish Interior Minister way". People in the economy are given identification number and you record the types of agents according to the number. But this way is not efficient, because the id number does not tell the properties of agents: we use the id numbers just to keep track of individuals. So we take the second way. We are not going to keep track of agents by id numbers given to each agent but we use MEASURE. To further proceed, we need some knowledge on thew measure theory, so let's study it briefly, and after that we will see how measure theory is useful for our purpose.

10.2 Practical Introduction to Measure Theory

10.2.1 Intuition

Measure theory can be understood nicely by comparing to weight. Measure is useful in literally measuring a mass in a mathematically consistent way, which is similar to the way of weighting a mass. Therefore, intuitively the following properties are expected to be satisfied by measure:

1. $measure(nothing) = 0$
2. if $A \cap B = \emptyset$, $measure(A + B) = measure(A) + measure(B)$

These properties are intuitive with weight. The weight of nothing is zero. If a body is 200 pounds, and you chop off a hand from the body (you know whose body it is) and put the hand and the rest of the body together on the scales, they must weight 200 pounds (forget about the blood or anything like that!). Now consider an economy with many agents. The measure of nobody in the economy is zero. If a measure of the total population is normalized to one, and you take away the rich people from the population and measure the sum of rich people and the rest of the population, they must have measure one.

10.2.2 Definitions

Definition 10.1. *For a set A, $\mathcal{A}$ is a set of subsets of A.*

Definition 10.2. *σ-algebra $\mathcal{A}$ is a set of subsets of A, with the following properties:*

1. $A, \emptyset \in \mathcal{A}$
2. $B \in \mathcal{A} \Rightarrow B^c \in \mathcal{A}$ *(closed in complementarity)*
3. *for* $\{B_i\}_{i=1,2...}$, $B_i \in \mathcal{A} \Rightarrow [\cap_i B_i] \in \mathcal{A}$ *(closed in countable intersections)*

The intuition of the property 2 of σ-algebra is as follows. If we chop off a hand from a body, and if the hand is an element of $\mathcal{A}$, the rest of the body is also an element of $\mathcal{A}$. Soon we will define measure as a function from σ-algebra to a real number Then the property of σ-algebra implies that if we can measure the chopped hand, we can measure also the rest of the body.

Homework 10.3. *Prove that countable unions of elements are also an element of σ-algebra.*

Examples of σ-algebra are the follows:

1. Everything (all the possible subsets of a set A)
2. $\{\emptyset, A\}$
3. $\{\emptyset, A, A_{1/2}, A_{2/2}\}$ where $A_{1/2}$ means the lower half of A (imagine A as an closed interval on $\mathcal{R}$).
4. $\{\emptyset, A, A_{1/4}, A_{2/4}, A_{3/4}, A_{4/4}, A^c_{1/4}, A^c_{2/4}, A^c_{3/4}, A^c_{4/4}, A_{1,2/4}, A_{1,2/4}, A_{1,3/4}, A_{1,4/4}, A^c_{1,2/4}, A^c_{1,3/4}, A^c_{1,4/4}\}$

Remark 10.4. *A convention is (i) use small letters for elements, (ii) use capital letters for sets, (iii) use "fancy" letters for set of subsets.*

Look at the examples of 3 and 4. Imagine you are given $a \in A$. If the only information we can get with respect to a is whether a is included in an element of $\mathcal{A}$ or not, it is true that we have richer information on a with σ-algebra 4 than 3 because, with 4, we can know a is included in which of $A_{1/4}, A_{2/4}, A_{3/4}, A_{4/4}$, where with 3, we only know a is included in which of $A_{1/2}, A_{2/2}$. In this sense, σ-algebra is similar to the notion of information.

Definition 10.5. *A measure is a function $x : \mathcal{A} \to \mathcal{R}_+$ such that*

1. $x(\emptyset) = 0$
2. *if* $B_1, B_2 \in \mathcal{A}$ *and* $B_1 \cap B_2 = \emptyset \Rightarrow x(B_1 \cup B_2) = x(B_1) + x(B_2)$ *(finite additivity)*
3. *if* $\{B_i\}_{i=1}^{\infty} \in \mathcal{A}$ *and* $B_i \cap B_j = \emptyset$ *for all* $i \neq j \Rightarrow x(\cup_i B_i) = \sum_i x(B_i)$ *(countable additivity)*

In English, countable additivity means that measure of the union of countable disjoint sets is the sum of the measure of these sets.

Definition 10.6. *Borel-σ-algebra is (roughly) a σ-algebra which is generated by a family of open sets.*

Remember the discussion on the information. Since Borel-σ-algebra contains all the subsets generated by intervals, you can recognize any subset of a set, using Borel-σ-algebra. In other words, Borel-σ-algebra corresponds to the complete information.

Remark 10.7. *You might find that σ-algebra is similar to topology. Topology is also a set of subsets, but its elements are open intervals and it does not satisfy closedness in complementarity (complement of an element is not an element of topology). Very roughly, the difference implies that topologies are useful in dealing with continuity and σ-algebra is useful in dealing with measure.*

Definition 10.8. *Probability (measure) is a measure such that $x(A) = 1$*

10.3 Introduction to the Economy with Heterogeneous Agents

10.3.1 Environment

The economy is populated by a mass of farmers, each of whom lives in an island separately. Every period farmer wakes up and receives $s = stuff$. This is an endowment. So this is an endowment economy (no production). Assume that farmers cannot send or receive its endowments each other, i.e. all of them are in autarky. Endowment s follows a Markov chain. Formally,

$$s \sim \Gamma_{ss'} \text{ where } s \in \{s^1, s^2, \ldots, s^{ns}\}$$

There is a storage technology. In other words, a farmer can store part of its endowment to use in the future. In addition, if a farmer stores q units of stuff in period t, she will receive 1 unit of stuff in period t+1. You can interpret this as a farmer can consume the seeds or keep them in her backyard. In the latter case, she will receive a harvest in the next period. Or, you can understand that the (constant) real interest rate out of the saving is $\frac{1}{q} - 1$. With this storage technology, the flow budget constraint of each farmer is:

$$s + a = c + qa'$$

where c is consumption today, a is saving from the past, s is endowment today, and a' is the saving for tomorrow on.

What does a farmer do? A farmer chooses a path of consumption and saving conditional on a history of shock realizations and her optimal behavior is represented by the following optimization problem:

$$\max_{\{a_{t+1}(h_t)\}_{t=0}^{\infty}} E_0 \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t(h_t)) \mid s_0 \right\} \tag{136}$$

subject to

$$
\begin{aligned}
c_t(h_t) + qa_{t+1}(h_t) &= s_t + a_t(h_{t-1}) \\
a_t(h_t) &\geq 0 \\
& a_0 \text{ given}
\end{aligned} \tag{137}
$$

10.3.2 Argument on Set of Possible Saving Level

This problem is similar to NGM but there is a big difference. In NGM, there is a production technology but here there isn't. This feature is important in order to make sure that the state space for an agent is compact, which is a necessary condition to have well-defined recursive formulation of the problem. In NGM, the production technology and assumption of Inada conditions for production function guarantees that the level of capital stock in equilibrium stays in a certain interval. Why? Because of the curvature in production function, saving too small is not optimal since the marginal productivity of capital goes up without bound and saving too much is not optimal either because the marginal productivity of capital approaches to zero. However, in this problem, there is no such mechanism. There is no trivial upperbound and lowerbound.

As for the lowerbound, we assume that there is no technology which allows negative amount of saving (it sounds natural. Can you imagine that an agent keeps -1 unit of capital in her backyard?). We assume that Mother Nature says you cannot! As for the upperbound, we use the following theorem without formally proving it:

Theorem 10.9. *if $\beta < \frac{1}{1+r} = q$, then $\exists \bar{a}$ such that, if $a_0 < \bar{a}$, $a_t < \bar{a}$ for $\forall t$.*

Large β means that you are patient (you do not discount future much). On the other hand, small β means you are impatient. If you are impatient enough compared with the returns from the storage technology, gains from saving disappear eventually, and you stop saving more and start dissaving.

10.3.3 Recursive Formulation

Once we know that this condition holds, we can make sure that, if all agents have initial asset $a_0 < \bar{a}$, $a_t < \bar{a}$ for $\forall t$. Combined with the lowerbound imposed by the Mother Nature, we can make sure that $a_t \in [0, \bar{a}] \equiv A_a$.Thus we are sure that the asset space and the choice of the saving level of an agent is compact set and can formulate the problem recursively as follows:

$$
V(s,a) = \max_{a' \in A_a = [0,\bar{a}]} u(c) + \beta \sum_{s'} \Gamma_{ss'} V(s', a') \tag{138}
$$

subject to

$$
c + qa' = s + a \tag{139}
$$

Homework 10.10. *You should be able to show the equivalence of the solutions of the recursive problem and the sequential problem.*

Homework 10.11. *You should be able to show that the functional operator defined by (138) is a contraction, using Blackwell's sufficient conditions.*

In this economy, the type of a farmer is (s,a). Suppose we want to store the information of all the farmers in the economy. How? As we have discussed, there are two ways. One is give an identification number to each farmer and store (s,a) of each farmer. The second way is to use measure.

10.3.4 Use Measure for this Economy

Define $A_a = [0,\bar{a}]$ and $S = \{s^1, s^2, ..., s^{ns}\}$ (assume that $s^1 < s^2 < ... < s^{ns}$), and $A = A_a \times S$. Define Borel-σ-algebra $\mathcal{A}$ of the set A and probability measure $x : \mathcal{A} \to \mathcal{R}_+$. Defining x as a probability measure means that the total population is normalized to one. Using measure, we can represent various statistics with very simple form. The followings are the examples:

1. The total population:

$$\int_A dx = x(A) = 1$$

   The intuition of this operator is that you pick up a certain type (s,a) and checks the number of agents who have this type, and sum this up across all the possible types.

2. The proportion of agents in the economy who get the worst shock s^1:

$$\int_A 1_{[s=s^1]} dx$$

   where $1_{[\text{logical expression}]}$ is an indicator function which takes value one if the [logical expression] is true and zero if false.

3. Total wealth is:

$$\int_A a \, dx$$

   Notice that,since the size of population is normalized to one, total wealth is same as the average wealth. This is why we use this normalization.

4. Median wealth a^m is a solution to:

$$\frac{1}{2} = \int_A 1_{[a>a^m]} dx$$

Homework 10.12. *Derive an expression for the wealth held by richest 1% of agents.*

11 Feb 28: Economy with Heterogeneous Agents (2)

11.1 Discussion on the Notion of Complete Markets

We know that if the market is complete, the equilibrium allocation is PO. But what we are going to do is to close a market for insuring idiosyncratic risk. Why do we do this? To see why, consider the economy where agents have the same preference, and have access to the same technologies, and the market is complete. What happens in this economy? We know that in equilibrium:

$$\frac{p(h_t)}{p(h_0)} = MRS^i(h_t, h_0) \quad \forall i \text{ (i.e. for all agents)} \tag{140}$$

This is equivalent to:

$$\frac{u'^i(c(h_t))}{u'^i(c(h_0))} = \frac{u'^j(c(h_t))}{u'^j(c(h_0))} \tag{141}$$

or

$$\frac{u'^i(c(h_t))}{u'^j(c(h_t))} = \frac{u'^i(c(h_0))}{u'^j(c(h_0))} \tag{142}$$

Assume that the preference is represented by CRRA period utility function[25], which is

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma} \tag{143}$$

With CRRA period utility function, (142) is:

$$\frac{c^i(h_t)}{c^j(h_t)} = \frac{c^i(h_0)}{c^j(h_0)} \tag{144}$$

What does this mean? This means that the ratio of consumption between any two agents in the economy is constant over event-time. In other words, there is no mobility in the economy: if an agent consumes twice as much as another agent in a certain period, she consumes twice as much forever! Remember that this occurs despite the heterogeneity of agents (agents are facing an individual uncertainty). In the economy with complete market, the agents can insure their idiosyncratic risk each other and, in the end, what only matters is the initial difference among agents.

However, we know that this is NOT consistent with what we see in the real world. This is the motivation that we depart from the world with complete markets. However you have to be

[25] CRRA (Constant Relative Risk Aversion) utility function has the following properties: (i) it's popular!, (ii) coefficient of relative risk aversion is constant (by definition!), and (iii) because of the property, there is no level effect, i.e. change of unit or absolute level does not affect the slope of MRS. Also note that this utility function allows balanced growth path thanks to this property, precisely because having a balanced growth path requires neutrality with respect to normalization.

cautious how we depart from the complete markets world, because we CANNOT tell agents to take or not to take certain actions. As economists, the discipline is that we can set up the environment (preference, technology, endowment, information) but after we set up the environment, what we can do is to let agents behave freely. If signing contracts each other to insure against risks is optimal behavior of agents, we just let them do it. We observe the outcome where agents do whatever they want in a given environment, and we analyze the outcome. Therefore, on the one hand, we would like to close certain markets, but on the other hand we have to be cautious in doing this. We have to make sure why we can do it. Possible arguments which support closure of certain markets are (i) impossibility of writing certain state contingent contracts (you cannot write all the possible events in the future!), (ii) slavery is prohibited (so you cannot sell your time), (iii) realization of your idiosyncratic shock cannot be observed by others, etc.

11.2 Measure Theory Continued

11.2.1 Why We Use Measure: Revisited

In addition to the fact that we can use concise notation of various statistics by using measure for keeping record of the agents' states, another virtue of using measure is that we can exploit the Law of Large Numbers. Since we have mass of agents (let's assume that the size of population is normalized to one), if the probability of receiving a certain shock is π, exactly π agents receive the shock in the economy. In other example, if all the agents flip the coin, exactly 50% of agents get tails and 50% get heads.

11.2.2 Some More Examples of Statistics

1. Wealth held by rich 1% of agents:
$$\int_A a\ 1_{[a \geq a^{99}]} dx$$
where a^{99} is a solution to:
$$\frac{1}{100} = \int_A 1_{[a > a^{99}]} dx$$

2. The proportion of agents in the economy who get the worst shock s^1 both today and tomorrow:
$$x(S_b \times A_a)\Gamma_{s^1 s^1} = \Gamma_{s^1 s^1} \int_A 1_{[s=s^1]} dx$$

How about the proportion of agents who are in the bottom half of wealth distribution both today and tomorrow? We cannot get this number at this point because tomorrow's wealth is the choice of agents. In other words, we need to know the saving decision of the agents who are in the bottom half of wealth distribution today. We will construct a version of Γ matrix for wealth, which is constructed from the decision rule of agents. We will call it transition function. To this end, let's start from some theoretical preparation.

11.2.3 Some Definitions

Definition 11.1. *For a set A and a σ-algebra $\mathcal{A}$, a function $f : A \to \mathcal{R}$ is measurable if $\forall c \in \mathcal{R}$, $\{a : f(a) \leq c\} \in \mathcal{A}$.*

Definition 11.2. *A transition function $Q : A \times \mathcal{A} \to \mathcal{R}$ such that:*

1. *$\forall \bar{B} \in \mathcal{A}$, $Q(.,\bar{B}) : A \to \mathcal{R}$ is measurable,*
2. *$\forall \bar{a} \in A$, $Q(\bar{a},.) : \mathcal{A} \to \mathcal{R}$ is a probability measure.*

Q function is a probability that a type a agent ends up in the type which belongs to B. We need to have this kind of function because the probability that a type a agent ends up in a type a' is zero, as long as there is no mass at a' (which is usually the case). It's easy to see that Q function can be understood as a generalized version of Γ matrix. Once we get this transition function, we can calculate the statistics like the proportion of agents who stay in the bottom half of wealth distribution.

11.2.4 Transition Function

Let's derive the transition function at first and discuss its properties. Suppose we have the optimal decision rule of an agent: $a' = g(s,a)$. Also we have the Markov transition matrix for endowment process $\Gamma_{ss'}$. Using these, the transition function is constructed as follows (note that B_s and B_a are the projections of B over the spaces S and A):

$$Q(s,a,B) = 1_{[g(s,a) \in B_a]} \sum_{s' \in B_s} \Gamma_{ss'} \tag{145}$$

Homework 11.3. *Verify that Q constructed as above is actually a transition function (you just need to show that Q constructed as above satisfies the conditions of a transition function).*

Pick up one measure $x(B)$. This specifies the distribution of types of agents. Using transition function Q, distribution in the next period: $x'(B)$ can be expressed as follows:

$$x'(B) = \int_A Q((s,a),B)dx \tag{146}$$

Homework 11.4. *Define three period updating operator Q^3.*

Why we often look at steady state of the NGM? Two answers: (i) laziness of economists, (ii) it's globally stable, i.e. no matter from where you start, asymptotically you will converge to the steady state. So if you assume that the initial state of the economy is not in the steady state, but there has been a long time since the economy started operating, steady state might be a good approximation of where the economy is now. It's good if we have the similar state for the economy we are looking at. Of course there is no steady state, because the fates (states) of individual agents change over time. Otherwise, we come back to the complete markets world, which we know is contradictory to what we see in the real world. It happens that, under a certain set of conditions, we have such state. We call it stationary distribution. If the economy is in the stationary distribution, individual state changes over time, but as a whole, the distribution of types (states) of agents do not change over time. In the mathematical expression, stationary distribution is x^* such that:

$$x^*(B) = \int_A Q((s,a),B)dx^* \quad \forall B \in \mathcal{A} \tag{147}$$

You can compare to a steady state condition:

$$k^* = f(k^*) \tag{148}$$

Conditional on a certain set of conditions, (i) x^* exists, (ii) it's unique, and (iii) it's globally stable (no matter from which distribution x the economy start, the economy asymptotically goes to x^*). So we can use x^* exactly for the same reason as we use steady state.

What is the set of conditions? In the case of the NGM, as long as the optimal decision saving function crosses the 45 degree line twice (once at zero capital stock), there are two steady states typically, and the one (which is not zero capital stock) is globally stable. For the case of the economy we are looking at, the sufficient conditions for the existence of such x^* is not so trivial. The conditions are shown in Hopenhayn and Prescott (Econometrica 1992)[26] but you do not need to know the details. Very roughly, the most important condition is the "American Dream and American Nightmare" condition. This means, that no matter your initial type is, there is a sufficiently large probability of going to any different type in the sufficiently near future. In other words, even if you are born in a poorest family, there is a sufficiently large probability that you will be like Bill Gates. And, even if you are born as a kid of Gates family, there is a sufficiently large probability that you will lose everything and sleep on the street in the near future. More seriously, the condition is called Monotone Mixing Condition. The transition function has to be such that it allows a sufficient mixing of all type of agents in order to have a stationary distribution.

What kind of transition function DOES NOT satisfy the conditions? Let's consider by the matrix, for simplicity. Example which violate the conditions are:

1.
$$\begin{bmatrix} \Gamma_{11} & \Gamma_{12} & 0 & 0 \\ \Gamma_{21} & \Gamma_{22} & 0 & 0 \\ 0 & 0 & \Gamma_{33} & \Gamma_{34} \\ 0 & 0 & \Gamma_{43} & \Gamma_{44} \end{bmatrix}$$

   For this matrix, there is no mixing between the agents start from type 1 or 2 and agents start from type 3 or 4. So the initial condition matters.

[26] Hopenhayn, H. A., and E. C. Prescott (1992), "Stochastic Monotonicity and Stationary Distributions for Dynamic Economies", Econometrica, 60-6, 1387-1406. The main result is Theorem 2 in page 1397.

2.

$$\begin{bmatrix} \Gamma_{11} & 0 & 0 & 0 \\ \Gamma_{21} & \Gamma_{22} & 0 & 0 \\ \Gamma_{31} & \Gamma_{32} & \Gamma_{33} & 0 \\ \Gamma_{41} & \Gamma_{42} & \Gamma_{43} & \Gamma_{44} \end{bmatrix}$$

With this transition matrix, there is no way to end up in state 4 if you start from state 1. Notice that even though this matrix DOES NOT satisfy the monotone mixing condition, there is a stationary distribution associated with this transition matrix: everybody eventually converges to type 1. But this degenerate distribution is not interesting. Roughly, sparse matrix is not nice for our purpose. Sparse matrix means that there is no transition between certain types, which is against the spirit of the monotone mixing.

11.2.5 The Preview

Now we are ready to analyze the economy with many farmers who can trade with each other. What we are going to do is to give each farmers initial wealth, let them consume, save, or trade for a while and see that is happening in the economy. Stationary distribution must realize there. This is what we will look at.

12 March 5: Economy with Heterogeneous Agents (3)

12.1 What Are We Doing?

We are constructing a model economy with many agents. Why are we interested in this? As we are doing macroeconomics, we are interested in macroeconomic aggregates (e.g., fluctuation of aggregate production) but disaggregation might be helpful in deepening our understanding of the aggregate behavior of the economy. It might seem that we are studying similar thing as applied microeconomics, but we are more concerned with aggregate consistency in our analysis. We keep the discipline of general equilibrium: prices are determined so that markets clear.

Why are we studying measure theory? Because (i) it's a very effective way of storing information on aggregate state of the economy, (ii) we can use Law of Large Numbers: while each individual fate is random (individual are facing uncertainty), aggregate economy is not random (there is no aggregate uncertainty). This feature makes our life so much easier.

12.2 What Have We Learned?

Let x is a probability measure (function from Borel-σ-algebra to [0,1]) and Q is a mapping from a space of x into itself and is characterized by a transition function. We learned that, under a certain condition ("American dream and American nightmare" condition), there exists a unique x^* such that:

$$x^* = \lim_{n\to\infty} Q^n(x_0) \quad \forall x_0 \tag{149}$$

Note that this implies that:

$$x^* = Qx^* \tag{150}$$

So you can consider x^* as a fixed point of the transition function. This means that, for arbitrary element of Borel-σ-algebra, B, $x^*(B)$, which is the measure of agents that have the type belonging to B, doesn't change over time.

Homework 12.1. *Show that, if the endowment process is* $\{0,1,0,1,0,1,...\}$ *(i.e., if the state today is 0, state tomorrow is 1 with probability 1 and vice versa), stationary distribution does not exist.*

12.3 On Various Statistics on Inequality and Mobility

Let's study how to measure the inequality and mobility of this economy. To begin with, what is a statistic? A statistic is a function from data to $\mathcal{R}$.

12.3.1 Inequality

How to measure inequality? The first thing we need to specify is "inequality of what?". "What" can be wealth, earning, income, etc. Below are the examples which are used to measure inequality of these.

1. **Coefficient of variation**, which is $\frac{\text{Standard Deviation}}{\text{Mean}}$. Note this is desirable than Standard Deviation, because it is independent of the unit. Imagine we compare the inequality of income between Italy and US. Even if the real inequality is the same, the Standard Deviation of income across Italian people measures in Lira is much larger than the Standard Deviation of income across US people measured in US dollar, because of the difference of the denomination. By dividing by mean, we can avoid such trouble.

2. **Lorenz curve**. Here is how to construct Lorenz of income: First sort the agents by their income. And arrange the agents on the horizontal axis [0,1]. The agent with lowest income comes on the point 0 and the one with highest income comes on the point 1. The vertical axis [0,1] represents the proportion of total income owned by agents. Specifically, a point on the income Lorenz curve (x,y) represents that fraction of agents in the interval [0,x] owns fraction y of total income of the economy. By definition, the Lorenz curve crosses the point (0,0) because no agent owns nothing. Similarly, the Lorenz curve crosses the point (1,1) because all agents in the economy own all of the income in the economy. Note that The Lorenz curve never crosses the 45 degree line by construction, unless everybody has the same amount (in this case Lorenz curve is exactly the 45 degree line). If you are considering income, and there is nobody with negative income in the economy, the Lorenz curve does not have a negative slope in [0,1]. Suppose you are considering the Lorenz of the net asset. In this case the poorest agents have usually negative net assets, and the asset Lorenz has a negative slope as long as the corresponding agents have negative net asset.

3. **GINI index**. GINI index is calculated by:

$$\frac{\text{Area enclosed by Lorenz curve and 45 degree line}}{\text{Triangle made by connecting the points (0,0), (1,0), (1,1)}}$$

   Since the denominator is 0.5 by definition, GINI index is equal to two times the numerator of the above expression. If the data concerned is always positive, the upperbound of GINI index is 1. Otherwise, GINI index can be bigger than 1.

4. **Histogram**. Can be a good approximation of density function. But be careful in choosing the distance between grids. The message that the histogram delivers might change a lot depending on the choice of the distance between grids.

12.3.2 Mobility

Having x^* is not enough to analyze mobility: we need both x^* and Q. Be careful in choosing the right period when dealing with mobility. Mobility over a week is much smaller than the mobility over a year. Statistics that are useful in analyzing mobility are the followings:

1. **Autocorrelation**. Actually, if the transition is expressed as a Markov matrix, autocorrelation is the 2nd largest eigenvalue of the transition matrix.

   Homework 12.2. *Show that the largest eigenvalue is 1 with Markov matrix.*

2. **Persistence matrix**. This is the approximation of transition matrix. Suppose there are two states for earning shock $\{s^1, s^2\}$. then, divide the asset space into n intervals. As an example take n=2: $\{A^1, A^2\}$. Next, type of agents are classified into one of 4 combinations of $\{(s^1,A^1), (s^1,A^2), (s^2,A^1), (s^2,A^2)\}$. Define these four states as $\{y^1, y^2, y^3, y^4\}$Transition matrix is then 4 by 4 matrix like follows:

$$\begin{bmatrix} \pi^{11} & \pi^{12} & \pi^{13} & \pi^{14} \\ \pi^{21} & \pi^{22} & \pi^{23} & \pi^{24} \\ \pi^{31} & \pi^{32} & \pi^{33} & \pi^{34} \\ \pi^{41} & \pi^{42} & \pi^{43} & \pi^{44} \end{bmatrix}$$

   where each element π^{ij} represents the measure of agents that are type y^i in a period and type y^j in the next period.

   Homework 12.3. *Show that all the elements of persistence matrix add up to 1.*

3. **Joint distribution over a and a'**. This is the function $y : \mathcal{A} \times \mathcal{A} \rightarrow [0,1]$. By construction, this statistic is the one that uses the information completely, because the transition is Markov.

4. **Halflife**. For example pick up an initial asset level a_0. Suppose that the mean asset in the stationary distribution is a^*. Pick up the point $\hat{a}$ which is the middle point between these two, i.e. $\hat{a} = a_0 + 0.5(a^* - a_0)$. Suppose that the initial measure x_0 is such that mass of agent (measure 1) are given a_0 in the initial period and distribution of s is the same as the stationary distribution of s. Halflife T is the first period where $\int_A a dQ^T(x_0) \geq \hat{a}$. In words, T is the first period where the average asset holding of agents who are given asset a_0 in the initial period, exceed $\hat{a}$. You can also understand this using probability. T is the first period that the expected asset of an agent, who started with an initial asset a_0 exceed $\hat{a}$. These two interpretations are the same thing because there is no aggregate uncertainty when there is the mass of agents in the economy (remember Law of Large Numbers).
Related to this, note that:

$$\lim_{T\to\infty} \int_A a dQ^T(x_0) = a^*$$

because

$$\lim_{T\to\infty} Q^T(x_0) = x^* \quad \text{for any } x_0$$

In other words, as time passes by, the initial condition becomes more and more irrelevant. In the limit (period ∞), initial condition is forgotten.

12.4 Some Extensions of the Basic Model with Heterogeneous Agents

Suppose that the idiosyncratic shock is NOT the earning shock, as we assumed so far, but preference shock, instead. In this case, roughly, the recursive representation of the agent problem is like follows[27]:

$$V(s,a) = \max_{c,a'} \left\{ u(s,c) + \beta \sum_{s'} \Gamma_{ss'} V(s',a') \right\} \tag{151}$$

subject to

$$c + a'q = e + a$$

where e is the earning which is constant every period. $u(s,c)$ is a general expression. This can be $su(c)$. So if the realization of s is large, you feel hunger, i.e. you put relatively high value in current consumption and vice versa.

What if the realization of the shock affects both utility and earning. An example is a health shock. If you are hit by some bad health problem, you might need to spend a lot today, at the same

[27] The value function does not include the aggregate state here, but for now forget about the aggregate state. You will see the reason later.

time your earning might decrease because of your health problem. We can formulate the shock into our basic framework as follows:

$$V(s,a) = \max_{c,a'} \left\{ u(s,c) + \beta \sum_{s'} \Gamma_{ss'} V(s',a') \right\} \tag{152}$$

subject to

$$c + a'q = e(s) + a$$

12.5 Heterogeneous Agents Economy with Trade[28]

Consider an economy with many agents. Now there is no storage technology (if you keep a coconut in your backyard, it will be rotten tomorrow. Another way of understanding it is that the rate of return for storage technology $\frac{1}{q}$ is 0). Instead, trade between agents is available. What kind of securities will be traded? Of course, if we do not impose any restrictions, agents trade Arrow securities and achieve PO allocation. Once agents achieve PO allocation, it's the end of the story. So let's make some assumptions so that state contingent claims cannot be traded. Cole and Kocherlakota (1997) shows that under some conditions including that the individual shock is only privately observable, the best thing that agents can do is to trade non-state contingent claims among themselves to smooth consumption over time (not across states)[29]. Let's assume this. Now the market arrangement includes only the non-state contingent claims.

There is another question. We know that it is important to have compact asset space to recursively formulate the problem. For the economy with storage shock, the lower bound was set by the Mother Nature such that $a \geq 0$. For the economy with borrowing and lending, there is no natural lowerbound, which means that an agent can run a Ponzi Scheme (keep borrowing without bound). What can we do to avoid Ponzi Scheme? We can achieve this by assuming that (i) the government shoots you if you default, (ii) thus you get a period utility of minus infinity if you default, (ii) you have to pay back your debt every period. Thanks to these assumptions, there is a maximum amount of debt so that you do not have possibility of defaulting. How to calculate this. The debt limit, let's call $\underline{a}$ has to satisfy the following equation, where q is a price of borrowing and lending, and s^1 is the lowest possible earning in the next period:

$$0 + \underline{a}q = s^1 + \underline{a} \tag{153}$$

What does it mean? The asset level $\underline{a}$ has to satisfy that, if your debt position is $\underline{a}$ and you draw the worst possible earning tomorrow, you can still enjoy a nonnegative consumption, by borrowing

[28]Original paper of this model is Hugget, Mark (1993), "The Risk-Free Rate in Heterogeneous -Agent Incomplete-Insurance Economies", Journal of Economic Dynamics and Control 17, 953-969. We will see some variations of this model, for which, Chapters 13-14 of Sargent and Ljungqvist will be useful as a side reading..

[29]Cole, H. L., and Kocherlakota, N. (1997), "A Microfoundation for Incomplete Security Markets," Federal Reserve Bank of Minneapolis Working Paper 577.

again up to the level $\underline{a}$. Solution of this equation is:

$$\underline{a} = \frac{s^1}{q-1} = \frac{s^1}{\frac{1}{1+r}-1} \tag{154}$$

Since $q < 1$, $\underline{a}$ is negative.

Fix q for now. The only difference from our previous problem of many Robinson Crusoes is that the lowerbound of asset can be negative, instead of zero. Therefore, we know that the solution exists and can be represented by $a' = g(s,a;q)$. From this decision rule and Markov transition matrix of the shock process, we can construct the transition function Q and associated stationary distribution x^* (of course we need to make sure that this exists). Now we are ready to define the stationary equilibrium. Note, however, that everything so far depends on the choice of q. We will define the value of q which is consistent (i.e. clear the loan market).

Definition 12.4. *A stationary equilibrium for the loan economy is a set* $\{q^*, x^*(q^*), Q(s,a,B;q^*), g(s,a;q^*)\}$ *such that*

1. *(Agent Optimization) Given* q^*, $g(s,a;q^*)$ *solves the agent's problem.*
2. *(Consistency)* $Q(s,a,B;q^*)$ *is a transition matrix associated with* $\Gamma_{ss'}$ *and* $g(s,a;q^*)$.
3. *(Stationarity)* x^* *is the unique stationary distribution associated with* $Q(s,a,B;q^*)$.
4. *(Market clear)*

$$\int a dx^*(q^*) = 0$$

How to prove existence of the equilibrium? We take the following steps (more in the next class):

1. Given q, we know that the agent's problem is well-defined and the solution $g(s,a;q)$ exists.
2. We know how to construct $Q(s,a,B;q)$ from $\Gamma_{ss'}$ and $g(s,a;q^*)$.
3. We know (make sure!) that the unique x^* exists.
4. We can calculate the aggregate demand of asset: $\int a dx^*(q)$. We want that this is zero.

So basically we are solving one equation $\int a dx^*(q^*) = 0$ with one unknown q^*.

13 March 7: Economy with Heterogeneous Agents (4)

13.1 Stationary Feature of Huggett Model

The model we studied in the last class, which is an economy with heterogeneous agents, where the source of the heterogeneity is uninsurable idiosyncratic earning shock. Note that we defined only a STATIONARY equilibrium, meaning that we can analyze only the situation where the initial measure x_0 coincides with the stationary distribution x^*. Note that this is a subset of equilibria of this economy. Why we are restricting our focus on this subset of equilibria? Because it's too messy to analyze the other equilibria. To understand it better, let's look at an NON-STATIONARY equilibrium.

13.2 Non-Stationary Equilibrium

Let's keep all the assumptions of Huggett model, and suppose the initial measure x_0 (initial type distribution) is not x^*. Since there is no aggregate uncertainty[30], the agents know how x_t evolves, and using this information, agents know the path of the price of the loans: $\{q_t(x_0)\}_{t=0}^{\infty}$[31]. Given this sequence of prices, the problem of an individual agent whose initial state is (s_0, a_0) is:

$$\max_{\{c_t(h_t), a_{t+1}(h_t)\}} \sum_t \beta^t \sum_{h_t} \Pi(h_t) u(c_t(h_t)) \tag{155}$$

subject to

$$\begin{aligned} c_t(h_t) + a_{t+1}(h_t) q_t(x_0) &= a_t(h_{t-1}) + s(h_t) \quad \forall t, h_t \\ & a_0, s_0 \textit{ given} \end{aligned} \tag{156}$$

where $h_t = \{s_0, s_1, ...\}$ is a private history of an agent and $s(h_t)$ indicates the endowment of the agent in the current period. Agents only care about their private history of shocks and not the aggregate state because there is no uncertainty at the aggregate level. The solution of the problem is:

$$\begin{aligned} c_t(h_t) &= c_t(s, a; x_0) \\ a_{t+1}(h_t) &= g_t(s, a; x_0) \end{aligned} \tag{157}$$

Note that we could write the solution as:

$$\begin{aligned} c_t(h_t) &= c_t(h_t; x_0) \\ a_{t+1}(h_t) &= g_t(h_t; x_0) \end{aligned} \tag{158}$$

[30] Remember that given an initial condition (x_0), we know precisely how the aggregate economy evolves over time, using transition function Q. There is no uncertainty at the aggregate level, because of the Law of Large Numbers, though at the individual level there is a large uncertainty.

[31] Note that $\lim_{t\to\infty} q_t(x_0) = q^*$ for $\forall x_0$, because of the global stability of x^*.

But we do not because of the Markov structure of the problem: agents only take into account current (s,a), instead of the whole private history, in making decisions. Now we are ready to define a nonstationary equilibrium of this economy.

Definition 13.1. *A competitive equilibrium is a set of sequences* $\{q_t(x_0),\ c_t(s,a;x_0),\ g_t(s,a;x_0),\ Q_t(s,a,B;x_0),\ x_t\}$ *such that*

1. *(Agent Optimization) Given* $q_t(x_0)$*,* $c_t(.)$ *and* $g_t(.)$ *solves the agent's problem.*
2. *(Consistency)* $Q_t(.)$ *is constructed from* $g_t(.)$ *and* $\Gamma_{ss'}$*.*
3. *(Updating)* $\{x_t\}$ *is recursively constructed by:*

$$x_{t+1}(B) = \int_A Q_t(s,a,B;x_0)dx_t$$

4. *(Market Clear)*

$$\int a dx_t = 0 \quad \forall t$$

It's a horribly complicated problem. In the Huggett model (stationary equilibrium), we only need to solve for single q, by solving one equation with one unknown, basically. But here we have to solve infinitely many equations with infinitely many unknowns. To see more clearly, consider how to find an equilibrium. Let's use "guess and verify" method. The procedure is as follows.

1. Given x_0, guess a path of prices of loans $q = \{q_t(x_0)\}_{t=0}^{\infty}$.
2. Given q, we can solve the agent's problem. The solution is $c_t(h_t) = c_t(s,a;x_0,q)$ and $a_{t+1}(h_t) = g_t(s,a;x_0,q)$[32].
3. Construct $Q_t(s,a,B;x_0,q)$ from $\Gamma_{ss'}$ and $g_t(s,a;x_0,q)$.
4. Construct $x_t(x_0,q)$ using $Q_t(.)$.
5. For each period t, we can calculate the aggregate demand for loans, using the following formula:

$$\int a dx_t(x_0,q)$$

   If this is equal to zero, for all t, your guess is correct, i.e. it is verified that the guess of q actually consists an equilibrium.

[32]I added an extra argument q to the solution to make it clear that the solution is conditional on the guess of the sequence of prices of loans.

Remember that the condition in the procedure 5 has to be satisfied for ALL periods. So it's really a tough job.[33].

13.3 Stationary Economy with Production (Aiyagari,QJE1994)[34]

Let's go back to the stationary equilibrium. But now let's add a production technology to the model. Now our model is a growth model with many agents. The details of the environment is as follows:

1. **Preference**: Utility is separable in time and event. The period utility of an agent is $u(c)$. An agent values consumption but not leisure. Standard assumptions for $u(.)$ function applies. An agent discounts future utility with the constant discount factor β.

2. **Endowment**: Every period, each agent receives s efficiency units of labor. This shock follows a Markov process with a Markov transition matrix $\Gamma_{ss'}$. You can think of it as the hours that agents can use for either leisure or labor. Since agents do not value leisure, they just use all of their efficiency units as a labor supply. Note that, since we will concentrate on the stationary equilibrium, the aggregate amount of efficiency unit of labor, which is:

$$L = \int_A s dx^*(q^*) \tag{159}$$

   is constant over time (remember there is no aggregate uncertainty). In other words, aggregate labor supply in this economy is constant. There is one consumption good which can either be consumed or costlessly transformed into capital and stored as a capital and lent it to the firm in the next period (depreciation rate is $\delta \in [0,1]$). The stock of capital of each agent is a, and the aggregate stock of capital is:

$$K = \int_A a dx^*(q^*) \tag{160}$$

   Note again that this is constant over time (because we are concentrating on stationary equilibrium).

3. **Technology**: There is a representative firm with standard CRS technology. The production function is $f(K,L)$. The firm uses aggregate labor L and capital K as inputs and produce

[33](You do not need to understand this) One very good (and very often used) technique to deal with this problem is to assume that after sufficiently long periods, the economy will actually in the stationary equilibrium. For example, let's suppose that the economy will be in the stationary equilibrium after 200 periods. So we know the initial q_0, and q_{200} because $q_{200} = q^*$ by assumption. So we just need to guess the sequence of $\{q_t\}_{t=1}^{199}$. Our problem is now 199 equations with 199 unknowns (looks still hard but much better than infinite equations). In addition, usually constructing $\{q_t\}_{t=1}^{199}$ by assuming uniform convergence to q^* works.

[34]Aiyagari, Rao (1994), "Uninsured Idiosyncratic Risk and Aggregate Savings," Quarterly Journal of Economics, 109-3, 659-684.

consumption goods. Note that the wage and capital rental rate which clear the market are:

$$\begin{aligned} w^* &= f_2(K,L) \\ r^* &= f_1(K,L) \end{aligned} \tag{161}$$

Considering the depreciation, q^* is:

$$q^* = \frac{1}{f_1(K,L)+1-\delta} \tag{162}$$

If we assume complete markets, because agents are risk averse, agents trade Arrow securities to insure their earning shocks away and the model collapses to our familiar representative agent model[35]. If we assume that the earning risks cannot be insured (i.e., the agents cannot trade state contingent securities), agents are expected to save a part of their earning in the form of capital[36] in order to ”prepare for the bad time in the future”. The saving for ”preparing for the bad time in the future” is what we call ”precautionary saving”. In the economy with complete markets there is no precautionary saving, because there is no such risk (agents end up receiving the same amount by trading Arrow securities). The question that Aiyagari tries to answer with his model is how large this precautionary saving is. And his answer is at most 3% increase of the aggregate saving rate.

Let’s consider a little bit more why the aggregate capital stock in the economy with uninsured risk is larger than the capital stock in the economy with complete markets. In the steady state of the complete market world, the following equation is satisfied:

$$\frac{1}{\hat{q}} = f_1(\hat{K},L) + (1-\delta) = \frac{1}{\beta} \tag{163}$$

where $\hat{K}$ is the steady state capital stock in the complete market world. If we consider a graph whose horizontal axis represents aggregate capital stock K and vertical axis represents $\frac{1}{q}$, (163) means that $\hat{K}$ is determined at the crossing point of $\frac{1}{q} = f(K,L)+(1-\delta)$ (aggregate capital demand curve) and $\frac{1}{q} = \frac{1}{\beta}$ (aggregate capital supply curve). For the economy with uninsured idiosyncratic shock, the aggregate capital demand curve is the same but the aggregate capital supply curve is not the same. We know that when $\frac{1}{q} \to \frac{1}{\beta}$ from below ($q \to \beta$ from above), aggregate capital supply increases without bound, and if $\frac{1}{q}$ is low enough, this is going to be zero[37]. So the aggregate capital stock in the economy with uninsured earning shock is determined by the crossing point of the two

[35] We need to let agents ex-ante identical, i.e. give the same initial state to all the agents.

[36] Agents should be able to lend and borrow as in the previous model, but the rate of return of these two ways of saving should be equal in equilibrium (which is a consequence of the non arbitrage condition). Therefore, even if we assume lending and borrowing among agents, in addition to saving capital, we can analyze the economy just to allow agents to have a negative capital stock but not to borrow and lend each other. So the problem of whether we allow agents to borrow and lend comes down to the choice of lowerbound of individual capital stock.

[37] (You do not need to understand this.) Note that we need to make sure the continuity of aggregate capital supply function in order to argue the equilibrium q^* and K^*. Also note that, since there is no way to have monotonicity of the aggregate capital supply curve, we it’s impossible to get uniqueness result for this kind of models.

curves. Notice that because of the difference of the shape of the aggregate capital supply curve, $K^* > \hat{K}$[38].

Homework 13.2. *Define Stationary RCE for Aiyagari's economy with uninsured idiosyncratic shock.*

13.4 Heterogeneous Agents Economy with Aggregate Shock

In the previous economies, there was no aggregate uncertainty. In the stationary equilibrium, agents know that the measure of agents x^* does not change over time. In the nonstationary economy, the measure of agents x_t changes over time but the path is deterministic because there is no shock to transition function $Q(.)$. This feature made our life so much easier. Especially, in the stationary equilibrium, as the aggregate capital stock and labor supply is constant over time, prices (r^* and w^*) are constant as well. Since agents only look at prices in making decisions, the agent's just need to look at one price for each good, instead of looking at aggregate state of the economy (this is why we did not include aggregate state variables in agent's problem in Aiyagari's and Huggett's economies).

But there is a price for this simplification. We cannot discuss a fluctuation of the aggregate economy (e.g., business cycle). Also we cannot derive a risky asset price (because there is no risky asset, i.e. an asset whose return is under the influence of an aggregate shock). So we might want to be able to deal with the economies with aggregate uncertainty. But if the measure of agents changes over time and prices change too, aggregate state variable possibly have to include the measure of agents, which is an infinite dimensional object, and it's very very difficult. But several ways to get away with this problem has been introduced.

1. Approximate the measure of agents. This technique was introduced by Krusell and Smith (1998)[39]. This will be explained in the class shortly briefly. This is one of the frontier of this field.

2. Assume that prices are not affected by the measure of agents. There are several ways to do this. One is a small open economy assumption. Another is assuming that the government can determine the prices. Let's study this example in the next section.

13.5 Economy with Moody Government (1)

13.5.1 Environment

1. **Preference**. Utility is separable in time and event. The period utility of an agent is $u(c)$. Agent values consumption but not leisure. Standard assumptions for $u(.)$ function applies. Agent discounts future utility with the constant discount factor β.

[38] Definitely it's easier to understand it with the graph. See your note or the original paper of Aiyagari.

[39] Krusell, Per and Smith, Anthony (1998), "Income and Wealth Heterogeneity in the Macroeconomy," Journal of Political Economy, 106-5, 867-896.

2. **Endowment**. Every period, each agent receives s efficiency units labor. Agents can consume it (as a leisure) or use it as a labor input. Since agents do not value leisure, agents use all of their efficiency units as labor supply. s follows a finite state Markov process with a Markov transition matrix $\Gamma_{ss'}$. There is one consumption good. Agents can consume it or leave it to the government (or you can consider banks, which are the agents of the government). Individual agent does not have a storage technology but the government has.

3. **Government**. The government does the following: (i) it collects constant lump-sum taxes from all the agents (the amount τ) every period, (ii) it pays $[1+R(z)]$ for each unit of saving that an agent leaves to the government in the previous period. The rate of return for saving $R(z)$ is determined by the mood of Mr. Greenspan. His mood follows a finite state Markov process with a Markov transition matrix $\Pi_{zz'}$ (maybe he forgets the old history and just remembers what happened yesterday). (iii) the government receives capital from agents and return it, with returns, in the next period. (iv) the government expends $G(x,z)$. This is calculated as a residual of the activities above. In other words, the government collects taxes, receives savings, pays the returns to savings in the previous periods, and throws all the remaining money into the sea. Assume that $R(z)$ is sufficiently low for $\forall$z such that there always exists an upperbound for agent's saving.

13.5.2 Equilibrium

The agent's problem is as follows:

$$V(z,s,a) = \max_{c,a'} \left\{ u(c) + \beta \sum_{z'} \Pi_{zz'} \sum_{s'} \Gamma_{ss'} V(z',s',a') \right\} \tag{164}$$

subject to

$$c + a' = a[1+R(z)] + s - \tau \tag{165}$$

The solution of the problem is

$$\begin{aligned} a' &= g(z,s,a) \\ c &= c(z,s,a) \end{aligned} \tag{166}$$

Next, let's define the policy of the government and equilibrium associated with a policy.

Definition 13.3. *A policy is* $\{R(z), \Pi_{zz'}, \tau, G(x,z)\}$

Definition 13.4. *A policy is feasible is there exists an equilibrium associated with the policy.*

Definition 13.5. *An equilibrium is* $\{V(z,s,a), g(z,s,a), Q(z,s,a,B), \{R(z), \Pi_{zz'}, \tau, G(x,z)\}\}$ *such that*

1. $V(z,s,a)$ *and* $g(z,s,a)$ *solves the agent's problem.*

2. $Q(.)$ *is constructed from* $g(.)$, $\Pi_{zz'}$ *and* $\Gamma_{ss'}$.

3. $x(B)$ *evolves according to the following law of motion:*

$$x'(B) = \int_A Q(z,s,a,B)dx$$

4. *(Government Budget Constraint)*

$$G(x,z) + \int_A a[1+R(z)]dx = \tau + \int_A g(z,s,a)dx$$

A couple of remarks:

Remark 13.6. *The last equation is a government budget constraint. The terms in the left hand side are the expenditures in the current period and those in the right hand side are the revenues.* τ *is not in an integral because it's a lump-sum tax and the total population is normalized to 1.*

Remark 13.7. *In order for an equilibrium to exist,* $G(x,z)$ *has to be nonnegative for all* (x,z)*. Otherwise the economy wide feasibility constraint is violated.*

Remark 13.8. x *is not included in the equilibrium because there is no stationary equilibrium or something like that.* x *just evolves according to the history of realizations of "moods".*

Remark 13.9. *We can compare various policies or discuss the optimal policy, by comparing the welfare of agents under various policies.*

13.6 Economy with Moody Government (2)[40]

The paper by Diaz-Gimenez, Prescott, Fitzgerald, and Alvarez (1992) can be understood as a small extension of the model we have seen in the previous section. The difference is the asset structure: There are two kinds of assets in their model, namely T-bill and money. T-bills carry higher rate of return but the unit of T-bills are much larger than money. For example, though real rate of return of T-bills is 4-5%, minimum amount of T-bills that you can buy is 10,000 US dollars. The paper tries to explain by this lumpiness why people use money even though T-bills have much higher rate of return.

Let's look at the problem of an agent:

$$V(z,s,a,b) = \max_{\substack{a' \in \{0,1,2,\ldots\} \\ b' \in \{0,100,200,\ldots\}}} \left\{ u(c) + \beta \sum_{z'} \Pi_{zz'} \sum_{s'} \Gamma_{ss'} V(z',s',a',b') \right\} \tag{167}$$

[40] Diaz-Giménez, Javier, Prescott, Edward C., Fitzgerald, Terry J., and Alvarez, Fernando (1992), "Banking in Computable General Equilibrium Economies," Journal of Economic Dynamics and Control, 16, 533-559.

subject to

$$c + a' + b' = a[1 + r(z)] + b[1 + R(z)] + s \tag{168}$$

where a, b are the stock of money and stock of T-bills, respectively[41].

In this model, wealth rich agents can store their asset by T-bills, whereas wealth poor agents can use only money for saving. This is the same effect as the creditcard has: If only the wealth rich agents can use creditcards, they can avoid inflation tax by using creditcards, whereas the poor, who do not have a creditcard, might suffer from inflation. The inflation (=negative rate of return of money) has real effect on the economy. In another sense, the government can collect seignorage by changing the rate of return of money. The formula of seignorage is:

$$seignorage = \int_A \{g(z,s,a,b) - a[1 + r(z)]\}\, dx$$

where $g(.)$ is an optimal decision rule for a'.

14 March 19: Transition and Policy Analysis

14.1 Big Picture

The models of heterogeneous agents economy that we have seen are:

1. Economy with farmers living in separated islands (so there is no trade nor equilibrium in the economy)
2. Economy with households who can borrow and lend each other but there is no production (Huggett's economy). Remember that the market clear condition of loans is:

   $$\int adx^*(q^*) = 0 \tag{169}$$

   where q^* is a market clearing rate of return of loans, $x^*(q^*)$ is the stationary distribution of agent types conditional on the rate of return of loans q^*.
3. Economy with households who can rent capital to the representative firm and gain rate of returns of capital (Aiyagari's economy). Remember that the market clear condition of capital rental market is:

   $$\int adx^*(q^*) = K^* \tag{170}$$

   where q^* is a market clearing rate of return of capital $(= \frac{1}{1+r^*})$, $x^*(q^*)$ is the stationary distribution of agent types conditional on the rate of return of capital q^*, and K^* is the steady state level of aggregate capital, which satisfies $r^* = F_K(K^*)$.

[41] Since the constraint set of agents is no longer a convex set, standard dynamic programming technique does not apply here, but at least it is easy to find a solution using computer.

In these economies, we restrict our attention to the subset of equilibria where aggregate state of the economy is constant (stationary economy). By this restriction, aggregate state of the economy ($x =$distribution of agent types) does not matter for agent's optimization problem, and thus is excluded from the state variables of individual agent's problem.

We might want to consider an aggregate dynamics of the economy, which is an impossible task for the economies above. One way of aggregate dynamics is to consider an aggregate shock (let's call it z). But, to do this analysis, we need to deal with $q(z,x)$, i.e. we need to solve for the price that is a function of aggregate state of the economy. How we can do this? We have seen that there are the following two tricks to do it nicely.

1. Assume that the price does not depend on the distribution of the agent types, i.e. $q(z)$. An example of this approach is the model where government (or moody Greenspan) decides the prices.

2. Approximate the type distribution x by some statistics (e.g. finite number of moments). We will see this in the next class.

There is another way of considering an aggregate dynamics of the economy with heterogeneous agents. Suppose we have a guess on the sequence of future prices from period t on: $\vec{q}_t = \{q_\tau\}_{\tau=t}^{\infty}$. Conditional on $\vec{q}_t$, we can formulate the agent's problem as follows:

$$V_t(s,a;\vec{q}_t) = \max_{c,a'} u(c) + \beta \sum_{s'} \Gamma_{ss'} V_{t+1}(s',a';\vec{q}_{t+1}) \tag{171}$$

subject to

$$\begin{aligned}
&c + a'q_t = s + a \\
&\vec{q}_{t+1} = F(\vec{q}_t) \\
&s_0, a_0, \vec{q}_0 \text{ given}
\end{aligned}$$

where $F(.)$ is a forward operator (the only thing Victor learned in econometrics class) which does:

$$F(\{q_t, q_{t+1}, q_{t+2}, ...\}) = \{q_{t+1}, q_{t+2}, q_{t+3}, ...\}$$

The solution of this problem is:

$$a_{t+1} = a_t(a_t, s_t; \vec{q}_t) \tag{172}$$

Using this and $\Gamma_{ss'}$, we can construct a transition function $Q_t(s,a,B;\vec{q}_t)$ and associated transition operator T such that:

$$\begin{aligned}
x_{t+1}(\vec{q}_{t+1}) &= T(x_t(\vec{q}_t) \\
&= \int Q_t(s,a,B;\vec{q}_t) dx_t(\vec{q}_t)
\end{aligned} \tag{173}$$

Given x_0 and $\vec{q}_0$, we can construct recursively the whole sequence of $\{x_t(B;\vec{q}_t)\}_{t=0}^{\infty}$. Our guess of $\vec{q}_t$ is the right one if market is cleared for all periods, i.e.:

$$\int a dx_t(\vec{q}_t) = 0 \qquad \forall t = 0,1,2,... \tag{174}$$

Therefore, the problem is to solve a system of infinite equations with infinite unknowns. This is a horrible problem.

So what can we do to make our life easier? We can use the property of global stability. This means that no matter what x_0 is, the distribution of the economy asymptotically converges to its stationary distribution x^*. In other words, for practical purpose, we may well assume that the economy is in its stationary distribution after sufficiently long periods. Or we could say that, after a sufficiently long periods, the economy is actually in its stationary distribution with a certain precision[42].

In particular, let's assume that after $n = 200$ periods, the economy is actually in its stationary distribution. Also assume that the value in the stationary equilibrium is $V^*(a,s;q^*)$ (we know how to solve this). Then we have:

$$V_{200}(s,a;\vec{q}_{200}) = V^*(a,s;q^*) \tag{175}$$

Having this value at hand, the problem of agents in period 199 is:

$$V_{199}(s,a;\vec{q}_{199}) = \max_{c,a'} u(c) + \beta \sum_{s'} \Gamma_{ss'} V^*(s',a';q^*) \tag{176}$$

subject to

$$c + a' q_{199} = s + a \tag{177}$$

We can go backward to period 0, recursively. To solve this, make a guess for $\{q_0, q_1, ..., q_{199}\}$ and solve the household's problem, check if market clearing conditions for period $0 \sim 199$ are satisfied. If so, done. If not, make a new guess and do the same thing, and keep doing this until we reach to the "correct" guess. As is clear now, the problem is now a system of 200 equations with 200 unknowns, which is much tractable problem than the previous one.

14.2 The Road Ahead

Why we talked about this dynamic path problem? Because it's useful for policy analysis. Suppose we have two choices of policies: A and B, how can you compare the policies? Solving stationary equilibrium associated to each policy and compare the welfare of agents in the two stationary equilibria is wrong. Because the problem is NOT "whether you would like to be born in an economy with policy A or rather be born in an economy with policy B", but "if you are living in an economy

[42] Victor calls it "Thick Point Theorem".

with policy A, would you support the change of policy to policy B or rather stay with policy A." In other words, you should not compare the policy with different initial condition. Another example is the following: if you compare the economic policy of Indonesia and that of US, merely comparing the economic performance or welfare of people in both countries is not appropriate as a policy analysis. You have to do either of the following two:

1. Compare (i) the welfare of US people under US economic policy and (ii) the welfare of US people under Indonesian economic policy
2. Compare (i) the welfare of Indonesian people under US economic policy and (ii) the welfare of Indonesian people under Indonesian economic policy.

Another example is: suppose Victor has two policies in the way he teaches: (i) teach in Spanish or (ii) teach in English. You can compare the two policies. However, if the choices are (i) Victor teaches in Spanish but gives you 2 million dollars, and (ii) Victor gives you nothing and teaches in English, still you can compare the two but the meaning of comparison is different from the original one.

OK. Enough for simple examples. Let's see some more serious examples.

14.3 Example (1): Subsidy to Saving

Suppose a representative agent NGM. There are two choices of policies:

1. Subsidizing saving: Agents receive subsidy to their saving with constant marginal subsidy rate τ.
2. No subsidy.

Of course, we know that the allocation under policy 2 is PO and the allocation under policy 1 is not, because there is a distortion to the price for saving.

The recursive formulation of the agent's problem under policy 1 is as follows:

$$V(K) = \max_{C,K'} u(C) + \beta V(K') \tag{178}$$

subject to

$$C + K' = (1+\tau) f(K) - T + (1-\delta)K \tag{179}$$

where T is a tax by the government. Sequential formulation of the problem is as follows:

$$\max_{\{C_t,K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(C_t) \tag{180}$$

subject to the same budget constraint. Now, take FOC with respect to K', which is:

$$-u'(C_t) = \beta[(1+\tau)f'(K_{t+1}) + 1 - \delta]u'(C_{t+1}) \tag{181}$$

Therefore, steady state capital stock level K^* satisfies:

$$\frac{1}{\beta} = (1+\tau)f'(K^*) + 1 - \delta \tag{182}$$

Similarly, the FOC for the steady state capital under policy 2, K^{**} satisfies (note that the condition is the same as the one that we can derive from SPP):

$$\frac{1}{\beta} = f'(K^{**}) + 1 - \delta \tag{183}$$

It is easy to show that $K^* > K^{**}$. But does it matter for our comparison between policies? We cannot compare the welfare of agents in the different steady state equilibria under different policies in order to compare the policies themselves. An example of a right way of comparison is the following: Suppose agents are living in a steady state economy under policy 2 (steady state capital stock level K^{**}). You can ask agents whether we would like to allow government to subsidize saving or we would like to stay with the current policy (no subsidy to saving). We know that, if the policy changes to 1, capital stock of the economy will eventually increase to K^*, but we need also look at the welfare of the agents on the transition path of the economy, in order to compare the welfare of agents under different policies.

14.4 Example (2): Unemployment Insurance

Consider the heterogeneous agent economy with the following features:

- No aggregate uncertainty
- Idiosyncratic earning shock s can be either e (employed) or u (unemployed) and follows a Markov process with a Markov transition matrix $\Gamma_{ss'}$.
- Employed (= working) agents can supply one unit of labor, and unemployed agents can engage in the home production (imagine they grow potatoes in their backyards...) and receive a constant endowment $\bar{u}$.
- Agents value consumption but not leisure.
- Standard Cobb-Douglas production technology.
- Government has the power to implement unemployment insurance[43]. Government does it by collecting taxes from working agents and distribute the proceeds to the unemployed agents.

[43] According to Victor, people in Minnesota calls this kind of assumption as ”chicken paper” because the background story is (i) the government can grow chickens, (ii) agents cannot, (iii) so the government grows chickens (???).

- Consider only the stationary equilibrium.

Let's start from defining policy and feasible policy:

Definition 14.1. *A policy is a pair* $\{\tau, \theta\}$.

Definition 14.2. *A feasible steady state policy is a pair* $\{\tau, \theta\}$ *such that* $\exists K^*$ *such that there exists an stationary equilibrium associated with* $\{K^*, \tau, \theta\}$ *(conditions for equilibrium are specified below).*

The agent's problem is:

$$V(s, a; K, \tau, \theta) = \max_{c, a'} \left[u(c) + \beta \sum_{s'} \Gamma_{ss'} V(s', a'; K, \tau, \theta) \right] \tag{184}$$

subject to

$$\begin{aligned} c + a' &= a(1+r) + w(1-\tau) \quad \text{if } s = e \\ c + a' &= a(1+r) + \bar{u} + \theta \qquad \text{if } s = u \\ w &= F_2(K, \bar{N}) \\ r &= F_1(K, \bar{N}) - \delta \\ \bar{N} &= \int_{S \times A} 1_{s=e} dx \end{aligned} \tag{185}$$

Note that (i) the problem is conditional on policy, which is characterized by $\{\tau, \theta\}$, (ii) the problem is also conditional on K, since K is sufficient to determine prices (note aggregate labor supply is assumed to be constant). The optimal decision rule for this problem is:

$$a' = g(s, a; K, \tau, \theta) \tag{186}$$

Steady state equilibrium conditions are:

$$\int a dx^*(K^*, \tau, \theta) = K^* \tag{187}$$

$$\theta(1 - \bar{N}) = \tau w \bar{N} \tag{188}$$

Homework 14.3. *Define formally a stationary RCE for this economy, given a policy.*

Now consider two different policies $\{\tau_1, \theta_1\}$ and $\{\tau_2, \theta_2\}$. For each policy, there is an associated K^*. Now how can we compare the two economies under different policies? It's not so trivial because aggregate states of the economy are different, and you need to also consider the welfare of agents on the transition path. Let me give an example of how to compare the welfare of agents under different policies[44].

[44] This is an additional topic. Victor did not talk these details in class.

1. Suppose initially agents live in an economy under policy $\{\tau_1, \theta_1\}$. Let the stationary distribution of agent types be x^1.
2. You can calculate average welfare of agents by taking weighted sum of values of agents, using x^1 as weights. Let's call it V^1.
3. Suppose the economy suddenly switches to policy $\{\tau_2, \theta_2\}$. Let the stationary distribution of the agent types under this policy be x^2. We know that, although the initial distribution of agent types is x^1, the distribution eventually converges to x^2.
4. We can solve the equilibrium on the deterministic transition path, by assuming that the economy actually converges to x^2 in sufficiently long periods (remember the technique we have studies at the beginning of this class).
5. A value function of the problem, $V_0(s,a;\vec{q}_0)$ is the value of an agent with type (s,a), in period 0, under the new policy $\{\tau_2, \theta_2\}$. Note that this value takes into account the transition to the new stationary distribution.
6. You can calculate average welfare of agents by taking weighted sum of $V_0(s,a;\vec{q}_0)$, using x^1 as weights in order to give the same initial condition. It's important to use x^1 as the weight. Let's call it V^2.
7. We can compare V^1 and V^2. This is the answer to the following question: Suppose agents are living in the economy under policy $\{\tau_1, \theta_1\}$. If, one day, the government switches the policy to $\{\tau_2, \theta_2\}$, does it increase the average welfare of agents?

15 March 21: Q&A and Krusell and Smith (JPE 1998)

15.1 Question 1: How to Construct Q and x'

Suppose, as usual, individual type is $(s,a) \in S \times A$. Suppose that $S = \{b,m,g\}$. Pick up a B^1, a subset of $S \times A$. For example, pick up

$$B^1 = \{(s,a) \mid s \in \{b,m\} \text{ and } a \leq \hat{a}\} \tag{189}$$

Transition function Q with a given B^1 is a probability that an agent of type (s,a) will end up in the types belonging to B^1. Using math, this is:

$$\begin{aligned} Q(s,a,B^1) &= \Gamma_{sb} 1_{g(s,a) \leq \hat{a}} + \Gamma_{sm} 1_{g(s,a) \leq \hat{a}} \\ &= \sum_{s' \in \{b,m\}} \Gamma_{ss'} 1_{g(s,a) \leq \hat{a}} \end{aligned} \tag{190}$$

For a general B, transition function is:

$$Q(s,a,B) = \sum_{s'} \Gamma_{ss'} 1_{[(s',g(s,a)) \in B]} \tag{191}$$

Now consider how the transition function is used to update the distribution.

[Step1] Suppose that, in the current period, all agents have the same $(\hat{s},\hat{a})$, i.e. $x(\hat{s},\hat{a}) = 1$. We know that all agents choose the same $\hat{a}' = g(\hat{s},\hat{a})$, but agents draw different s', so not all agents will be of the same type in the next period. Since there are only agents of type $(\hat{s},\hat{a})$ in this period, transition function Q, given $(\hat{s},\hat{a})$ is the following[45]:

$$Q(\hat{s},\hat{a},B) = \sum_{s' \in B_s} \Gamma_{ss'} 1_{[g(\hat{s},\hat{a}) \in B_a]} \tag{192}$$

What does this represent? This Q describe the proportion of agents in the next period who have the type (s,a) that belongs to B. In other words, given this $(\hat{s},\hat{a})$, Q itself represents the distribution in the next period.

[Step2] Consider a more general case. Suppose agents are different in types in a current period. Pick up agents of a certain type $(\hat{s},\hat{a})$. The proportion of agents who have this type is $x(\hat{s},\hat{a})$ (which is not necessarily one, of course). For these agents, transition function given $(\hat{s},\hat{a})$, $Q(\hat{s},\hat{a},B)$ represents the proportion of agents who were of type $(\hat{s},\hat{a})$ and ends up in the types that belong to B. In other words, for the agents whose measure if $x(\hat{s},\hat{a})$ in a certain period, $Q(s,a,B)$ describes how many of them ends up in the types belonging to B.

[Step3] In order to construct the type distribution of all the agents in the economy, we have to know the type distribution in the next period for agents of all types in the current period. In other words, in order to construct distribution in the next period x', we need to sum up the transition function over the types in the current period. Using math, it is:

$$x'(B) = \int_{S \times A} Q(s,a,B) dx \tag{193}$$

15.2 Question 2: Capital Stock of Aiyagari's Economy and Representative Agent Economy

In Aiyagari (1994), he compares the aggregate capital stock level of (i) complete market economy and (ii) incomplete market economy at their stationary equilibria.

For complete market economy, we know that a competitive equilibrium allocation is PO (from FBWT), and everybody consumes the same amount. This is roughly because, if the initial condition is the same, everybody faces the following FOC

$$\frac{p_t}{p_0} = \frac{u'(c_t)}{u'(c_0)} \tag{194}$$

and the prices that everybody are facing are the same, implying that $\{c_t\}_{t=0}^{\infty}$ are the same for everybody. Thus, stationary equilibrium capital stock level is equivalent to the steady state capital

[45] B_s is the projection of B over S. Note that we can split B into B_s and B_a because they are independent. If not, we need to use more general expression as we have just seen above.

stock level of the corresponding representative agent economy. We know that the steady state capital stock is characterized by:

$$\begin{aligned} \frac{1}{\beta} &= r^* && (195) \\ &= f_K(K^*, N) + 1 - \delta && (196) \end{aligned}$$

For incomplete market economy with production (Aiyagari's model), we learned how to solve the model. Assume, for simplicity, agents cannot hold negative capital stock level. Suppose the stationary distribution of agent types is x^*. In order to consider the equilibrium pair (r^{**}, K^{**}), first note that the following condition must be satisfied:

$$r^{**} = f_K(K^{**}, N) + 1 - \delta \tag{197}$$

How about the agent's side? Suppose $r = -(1-\delta)$, i.e. $(r + 1 - \delta) = 0$. Then, no matter how much capital you save, you get nothing in the next period. So nobody would save capital, i.e. $\int adx^* = 0$. Suppose r is approaching $\frac{1}{\beta} - 1 + \delta$ from below, i.e. $(r + 1 - \delta)$ is approaching to $\frac{1}{\beta}$ from below, we learned that optimal saving level of an individual agent in not going to be bounded. Equivalently, $\lim_{r \to [\frac{1}{\beta} - 1 + \delta]_-} \int adx^* = \infty$. The intuition behind this result is there is no cost of transferring consumption from now to any period in the future, so agents would like to save more no matter how much they own now. In addition, there is a result that tells that the aggregate capital stock derived from individual agent's problem is continuous, and typically increasing[46], we can connect the two extreme cases to represent the agent's aggregate saving decision. Finally, an equilibrium pair (r^{**}, K^{**}) is determined as the crossing point of (197) and agent's aggregate saving function.

Since agent's aggregate saving decision function always stays below $\frac{1}{\beta}$ if we draw a graph with r as a vertical axis and K as a horizontal axis, we know that (i) $K^{**} > K^*$ (i.e. aggregate saving level is higher in incomplete market economy because there is a precautionary saving motive), (ii) $r^{**} < r^*$ (because the aggregate saving demand is higher in incomplete market economy).

15.3 Question 3: Why x is defined as a function of q in the Huggett Model?

The answer will be clear if we see how to find an equilibrium for the Huggett model. The procedure is as follows:

1. Make a guess for q (equilibrium price of loans).
2. Given, q, we can formalize the individual agent's problem. The solution is characterized by $V(s, a; q)$ and $g(s, a; q)$.

[46] You do not need to understand the formal proof of these properties for now.

3. Using $g(s,a;q)$ and $\Gamma_{ss'}$, we can construct the transition function $Q(s,a,B;q)$.

4. We can find a stationary distribution $x(q)$ associated with $Q(s,a,B;q)$. Here we define x as a function of q because so far $\{V,g,Q,x\}$ are constructed conditional on a guess for q.

5. If $x(q)$ satisfies:

$$\int ax(q) = 0$$

this q is an equilibrium q.

6. Now consider the case where $q \to 0$, then agents would like to lend without bound, so $\int ax(q) \to +\infty$

7. In the opposite case, i.e. $q \to \beta$, the agents would want to borrow without bound, so $\int ax(q) \to -\infty$

8. Since there is a result saying that $\int ax(q)$ is continuous, Mean Value Theorem tells that $\exists q^*$ such that $\int ax(q^*) = 0$ (remember that this is the market clearing condition because there is no storage technology in this world, so all that agents can do is to make a contract which enables agents to transfer endowment in the future, but this does not change the aggregate amount consumed in a certain period.)

9. So we can call it q^*, and we can call the associated x as $x^*(q^*)$.

15.4 Krusell and Smith (JPE 1998)[47]

Let's briefly discuss the essence of their paper. Consider an economy with heterogeneous agents with production and without labor-leisure choice (exogenous labor supply). In this model, equilibrium rate of return of capital at period t is represented by:

$$r_t = F_K(K_t, \bar{N}) - \delta \tag{198}$$

Since everything except for K_t is constant, we can consider r_t as a function of aggregate capital stock at period t; K_t. In other words, aggregate capital stock K_t is a sufficient statistic for TODAY's prices. Or we could say that agents only need to know K_t to know the price of TODAY.

How about the TOMORROW's price? We know that:

$$r_{t+1} = F_K(K_{t+1}, \bar{N}) - \delta \tag{199}$$

So, K_{t+1} is a sufficient statistic for TOMORROW's prices. If, in addition, K_t is a sufficient statistics for K_{t+1}, as agents can forecast K_{t+1} using K_t, and then use K_{t+1} to forecast r_{t+1}, K_t will be a sufficient statistic for tomorrow's prices.

[47] Krusell, Per and Smith, Anthony, Jr. (1998), "Income and Wealth Heterogeneity in the Macroeconomy", Journal of Political Economy, 106-5, 867-896.

But... does it hold for our economy? To answer this question, we can use the following proposition:

Proposition 15.1. *K_t is a sufficient statistic for K_{t+1} if and only if the optimal decision rule $a' = g(.,a)$ is linear.*

Homework 15.2. *Prove this.*

If the decision rule satisfies the condition above, agents only need to know K_t to forecast future prices, so even if the economy is not in the stationary distribution and distribution, aggregate capital stock, and prices, are changing, agents only need to know K_t to forecast future prices. This implies that we only need to include K_t as an aggregate state variable for individual agent's problem. But what if the condition is not satisfied? Then the individual agent's problem have to contain the distribution of agent types in the aggregate state variables, which makes the problem very hard to compute, even with computers. The contribution of Krusell and Smith is that they showed, even for this complicated case, we can handle the individual agent's problem by approximating the distribution of agent types with a small set of statistics (usually first moment of cross sectional distribution of capital stock will do a pretty much good job).

They use this technique to answer to the following question. Representative agent NGM is basically the island with one Robinson Crusoe living on the island. And people have been considering that the interactions of agents in an economy might be important to explain the aggregate behavior of the economy (like business cycle), which is neglected in representative agent NGM. However, Krusell and Smith showed that the aggregate fluctuation of the economy with many heterogeneous agents is not so different from its representative agent counterpart, by comparing the aggregate behavior of a representative agent economy and an economy with heterogeneous agents.

16 March 28: Economy with One-Side Lack of Commitment (1)[48]

16.1 Outline

What is the most important (worst) thing we have done with the model up to the last class? We exogenously closed markets for state contingent loans and thus prevented exogenously the economy from collapsing to the representative agent economy. From now, we do not do this. Instead, we will assume more on what agents can do and what agents can see.

In particular, we will study two classes of models. The first class is the models with lack of commitment. In the world without commitment, the contract among agents need to be self-enforceable. Otherwise, agents will just quit the contract and walk away. The second class is the models with unobservable individual shock. In other words, there is asymmetric information or incomplete information in the model.

[48] The source of this part is the updated Chapter 15 of Tom Sargent's Recursive Economic Theory.

16.2 The Model

This is an endowment economy (no production). There is no storage technology. Consider a Mexican village (why?). The village is populated by mass of agents, who receive $y_s \in \{y_1, y_2, ..., y_S\}$ every period. y is iid. The probability that certain y_s realizes is Π_s. h_t is a history of shocks up to period t, i.e. $h_t = \{y_0, y_1, y_2, ..., y_t\}$[49]. In the Arrow Debreu world (complete markets and commitment), agents insure their shocks each other and the first best allocation (all agents consume the same amount in every time-event) is achieved and this is the end of the story. Agents achieve the first best allocation by signing a contract which tells that agents with high endowments (lucky agents) transfer some of their endowments to agents with lower endowments (unlucky agents), every period.

However, can this contract be sustainable if agents can break the contract and walk away, in particular, if we assume that an agent has to live in autarky if she ever breaks the contract? If agents are not subject to commitment, very lucky agents with very high endowments might want to walk away with the high endowments in hand, rather than observe the contract and give part of the endowments to other agents. Therefore, the question here is "What kind of allocation can we achieve under the lack of commitment?".

To make the following analysis simple, we will assume that there is a "moneylender" in the village. She is a risk neutral agent with the ability of lending or borrowing with the outside world at a constant rate $\frac{1}{\beta}$[50]. Moneylender can offer a contract for each agent and is subject to commitment. Note that this model is one-sided commitment model: an agent can walk away from a contract but moneylender cannot. Also note that there is a gain from trade in this economy because moneylender is risk neutral and thus does not care the timing of consumption, whereas all the other agents are risk averse and thus care the timing of consumption. Also assume that agents cannot trade each other, then we only need to analyze the relationship between the moneylender and an agent.

Let's define what moneylender can do. Moneylender can offer a contract to an agent. And the contract is the following object:

Definition 16.1. *A contract is $f = \{f_t\}_{t=0}^{\infty}$ where f_t is a function from h_t to $[0, c_{\max}]$.*

Notice that the domain of the contract is a set of all possible histories, which is a monster. And the sequence of events are as follows:

1. At the beginning of a period t, the moneylender offers a contract f to an agent.
2. The agent decides whether to sign the contract or not.
3. Today's shock y_t realizes.

[49]This subscript represents time, while the subscript in the definition of the set of y represents state.

[50]Equating time preference rate for normal agents and the interest rate for moneylender simplifies the analysis a lot, but this assumption does not change the result of the model in a fundamental way.

4. If the agent signed the contract and observes it, she gives y_t to the moneylender and receives $c_t = f_t(h_{t-1}, y_t)$. If the agent signed the contract and decided not to observe it, she consumes y_t this period and cannot enter a contract in the future, i.e. she has to live in autarky in the future[51]. If the agent did not sign the contract, she just consumes y_t (she is in autarky from the beginning).

16.3 Problem of the Moneylender (1)

In this model, problem of the moneylender is to find an optimal contract that maximizes her utility. To analyze this, let's define a expected discounted sum of utility of the moneylender, conditional on a contract f:

$$P(f) = E_0 \sum_{t=0}^{\infty} \beta^t (y_t - c_t) = E_0 \sum_{t=0}^{\infty} \beta^t (y_t - f_t(h_{t-1}, y_s)) \tag{200}$$

Notice that the contract f has to satisfy the condition that the agent voluntarily observes the contract. This is represented by a Participation Constraint (or Incentive Compatibility Constraint). To write the PC, let's start from writing the value of autarky for an agent (before observing the realization of the shock), which is the reservation value for the agent.

$$V_{AUT} = \sum_{t=0}^{\infty} \beta^t \sum_{s} \Pi_s u(y_s) \tag{201}$$

If an agent breaks a contract after observing y_s (her shock today), since she has to live in autarky in the future from next period on, her value is the following:

$$u(y_s) + \beta V_{AUT} \tag{202}$$

Using them, we can write PC, as follows:

$$u(f_t(h_t)) + \beta \sum_{j=1}^{\infty} \beta^{j-1} \sum_{s} \Pi_s u(f_{t+j}(h_{t+j})) \geq u(y_s) + \beta V_{AUT} \tag{203}$$

Moneylender's problem is to maximize (200) subject to (203). However, this problem is a horrible one. We need to solve optimal $f_t(h_t)$ for all the possible nodes! However, it turns out that we do not need to use the whole history to define the optimal $c_t(h_t)$. Instead, we only need to remember one number: the promised value of the moneylender. Therefore, the optimal contract of the moneylender is characterized by just a function from two numbers (promised value and current shock) to consumption. This is the beauty of this approach. We will see this approach in the next section.

[51] We do not consider renegotiation here.

16.4 Problem of the Moneylender (2)

Let v_t be the promised value for an agent. Using this and y_t, realization of the shock in the current period, let's define the contract as follows:

$$\begin{aligned} c_t &= g(v_t, y_t) \\ v_{t+1} &= l(v_t, y_t) \end{aligned} \tag{204}$$

This means that if an agent was promised the value v_t and drew y_t in period t, she gives the moneylender y_t, receives $c_t = g(v_t, y_t)$ and the moneylender promises $v_{t+1} = l(v_t, y_t)$ from the next period on, to her.

Note that using $l(v_t, y_t)$, we can summarizes the history of shock realization by just one number v_t. To understand this, see the following:

$$\begin{aligned} v_t &= l(v_{t-1}, y_{t-1}) \\ &= l(l(v_{t-2}, y_{t-2}), y_{t-1}) \\ &= l(l(l(v_{t-3}, y_{t-3}), y_{t-2}), y_{t-1}) \\ &\vdots \\ &= l(l(...l(l(v_0, y_0), y_1), y_2)..., y_{t-2}), y_{t-1}) \\ &\equiv l_{t-1}(v_0, y_0, y_1, y_2, ..., y_{t-1}) \end{aligned} \tag{205}$$

Now we are ready to define the problem of the moneylender using recursive formula. Firstly, let's define the value of moneylender if she promised v to an agent by $P(v)$. $P(v)$ can be defined recursively by using the following Bellman-like functional equation:

$$P(v) = \max_{\{c_s, \omega_s\}_{s=1}^S} \sum_s \Pi_s [(y_s - c_s) + \beta P(\omega_s)] \tag{206}$$

subject to

$$u(c_s) + \beta\omega_s \geq u(y_s) + \beta V_{AUT} \quad \forall s \tag{207}$$

$$\sum_s \Pi_s [u(c_s) + \beta\omega_s] \geq v \tag{208}$$

Notice that there are $1 + S$ constraints.

16.5 Characterization of the Optimal Contract

In order to characterize the optimal contract, construct a Lagrangian.

$$\begin{aligned} P(v) = \max_{\{c_s, \omega_s, \lambda_s\}_{s=1}^S, \mu} & \sum_s \Pi_s [(y_s - c_s) + \beta P(\omega_s)] \\ & + \mu \left[\sum_s \Pi_s [u(c_s) + \beta\omega_s] - v \right] + \sum_s \lambda_s [u(c_s) + \beta\omega_s - u(y_s) - \beta V_{AUT}] \end{aligned} \tag{209}$$

First order conditions are the followings:

$$
\begin{aligned}
&FOC_c && \Pi_s = (\lambda_s + \mu\Pi_s)u'(c_s) && (210)\\
&FOC_\omega && -\Pi_s P'(\omega_s) = \mu\Pi_s + \lambda_s && (211)\\
&FOC_\mu && \sum_s \Pi_s[u(c_s) + \beta\omega_s] \geq v && (212)\\
&FOC_\lambda && u(c_s) + \beta\omega_s \geq u(y_s) + \beta V_{AUT} && (213)
\end{aligned}
$$

In addition, Envelope Theorem tells that:

$$P'(v) = -\mu \tag{214}$$

Homework 16.2. *Prove the Envelope Condition above.*

Homework 16.3. *Show that P(v) is decreasing.*

Homework 16.4. *Show that P(v) is concave.*

17 April 2: Economy with One-Side Lack of Commitment (2)

17.1 A Remark on the Results in the Last Class

Remark 17.1. *$P(v)$ is not always positive. It can be negative. To see this, solve for $\tilde{v}$, the promised value of always giving the agent y_S (highest possible endowment). This is calculated by the following:*

$$\tilde{v} = \sum_{t=0}^{\infty} \beta^t u(y_S)$$

Then the value of the moneylender who promised $\tilde{v}$ is as follows:

$$P(\tilde{v}) = \sum_t \beta^t \sum_s \Pi_s(y_s - y_S)$$

This is trivially strictly negative (as long as there is a positive probability of realization of $y_s < y_S$).

17.2 Interpreting FOCs

The FOCs we derived in the last class are:

$$
\begin{aligned}
&FOC_c && \Pi_s = (\lambda_s + \mu\Pi_s)u'(c_s) && (215)\\
&FOC_\omega && -\Pi_s P'(\omega_s) = \mu\Pi_s + \lambda_s && (216)\\
&FOC_\mu && \sum_s \Pi_s[u(c_s) + \beta\omega_s] \geq v && (217)\\
&FOC_\lambda && u(c_s) + \beta\omega_s \geq u(y_s) + \beta V_{AUT} && (218)
\end{aligned}
$$

Notice that FOC represents that marginal cost is equal to marginal benefit at the optimal. For example, as for (215), the LHS represents the cost of increasing promised consumption if s occurs. This is multiplied by Π_s because this event occurs at this probability. The RHS of the equation tells the benefit of increasing one unit of c_s. There are two benefits. It would be easier to satisfy Participation Constraint if c_s is increased. This is represented by $\lambda_s u'(c_s)$. Simultaneously, it would be easier to achieve promised value if c_s is increased. This is represented by $\mu \Pi_s u'(c_s)$. FOC tells that the optimal c_s is determined such that the marginal cost of changing c_s balances the marginal benefit of doing this.

How about (216)? the LHS represents the cost of additionally guaranteeing one unit of value in the future. The RHS represents the benefit of additionally guaranteeing one unit of value in the future. There are two parts, again. It would make it easier to satisfy Participation Constraint ($\lambda_s \omega_s$) and simultaneously make it easier to satisfy promised value ($\mu \Pi_s$).

OK. Enough for interpretation of FOCs. Using (215) and (216), we can derive:

$$u'(c_s) = \frac{-1}{P'(\omega_s)} \tag{219}$$

What does this mean? This means that MRS of an agent ($\frac{u'(c_s)}{\beta}$) must be equal to MRT of moneylender ($\frac{-1}{\beta P'(\omega_s)}$).

17.3 Characterizing the Optimal Contract

We will characterize the optimal contract by considering the two cases: (i) $\lambda_s > 0$ and (ii) $\lambda_s = 0$. Firstly, if $\lambda_s > 0$, we have the following equations:

$$u'(c_s) = \frac{-1}{P'(\omega_s)} \tag{220}$$

$$u(c_s) + \beta \omega_s = u(y_s) + \beta V_{AUT} \tag{221}$$

Note that this is a system of two equations with two unknowns (c_s and ω_s). So these two equations characterize the optimal contract in case $\lambda_s > 0$. In addition, we can find the following properties by carefully observing the equations:

Remark 17.2. *The equations don't depend on v. Therefore, if a Participation Constraint is binding, promised value does not matter for the optimal contract.*

Remark 17.3. *From the first order condition with respect to ω_s, $P'(\omega_s) = P'(v) - \frac{\lambda_s}{\Pi_s}$, where $\frac{\lambda_s}{\Pi_s}$ is positive. Besides, we know that P is concave. This means that $v < \omega_s$. In words, if a Participation Constraint is binding, the moneylender promises more than before for future.*

Remark 17.4. *We know that $\omega_s > v > V_{AUT}$. Therefore, $u(y_s) > u(c_s)$, which means $y_s > c_s$.*

Let's consider the second case, where $\lambda_s = 0$. In this case, the equations that characterize the optimal contract are:

$$P'(v) = P'(\omega_s) \tag{222}$$

$$u'(c_s) = \frac{-1}{P'(\omega_s)} \tag{223}$$

We can derive the following implications:

Remark 17.5. $\omega_s = v$*, i.e. the moneylender promises the same as in the last period, if the Participation Constraint is not binding.*

Remark 17.6. c_s *is the same for all* s*. For all* s *such that the Participation Constraint is not binding, the moneylender offers the same consumption and promised future value.*

Combining all the results we have got, we can characterize the optimal contract as follows:

1. Let's fix v_0. We can find a $y_s(v_0)$, where for $\forall y_s \leq y_s(v_0)$, the participation constraint is not binding. And vice versa.

2. The optimal contract that the moneylender offers to an agent is the following:

3. If $y_t \leq y_s(v_0)$, the moneylender gives $(v_0, c(v_0))$. Both of them are the same as in the previous period. In other words, the moneylender offers the agent the same insurance scheme as before.

4. If $y_t > y_s(v_0)$, the moneylender gives $(v_1, c(y_s))$, where $v_1 > v_0$ and c doesn't depend on v_0. In other words, the moneylender promises larger value to the agent to keep her around.

5. So the path of consumption and promised value for an agent is increasing with steps.

Homework 17.7. *Solve for* c_S *and* ω_S *(capital S!).*

Remark 17.8. *In our model, since the moneylender offers a contract to an agent, the moneylender takes all the gains from trade out of the contract. If we change the model such that the agent can offer take-it-or-leave-it offer to the moneylender, now the agent can takes all the gains from trade out of the contract. Of course, we have infinite patterns in between. The set of feasible combination of values of agent and moneylender is similar concept as the contract curve. Which point realizes depends on the assumption on the bargaining process of the two agents.*

Remark 17.9. *As the agent draws good shocks, the amount that the agent receives from the moneylender increases, and the moneylender runs a deficit in the later stages, but the moneylender received more than she paid in the earlier stages and ran a surplus. This is why the moneylender can earn positive profit in total, even though the amount the she pays to the agent is increasing.*

18 April 4: Economy with Two-Side Lack of Commitment (1) [52]

18.1 Plan Ahead

The following four topics will be covered in the remaining classes.

1. Economy with two-side lack of commitment. Today and the next class.

2. Epstein-Zin recursive utility function. CRRA utility function is widely used in macroeconomics, primarily because this utility function is unit independent, implying that this utility allows the balanced growth path, and it's simple. However, this utility function put three pieces of information into one parameter: σ. These three are (i) coefficient of relative risk aversion (attitude towards uncertainty), (ii) intertemporal elasticity of substitution, and (iii) value of life. As for (iii), since the range of CRRA utility function is positive if $\sigma < 1$ and negative if $\sigma > 1$ (and it collapses to log utility function if $\sigma = 1$), and we usually put the value of being dead as zero, we need to be careful when we allow agents to die or to have children, because the level of utility matters in these cases. Epstein-Zin recursive utility function allows us to disentangle these three parameters separately, while retaining the virtue of CRRA function. Therefore, this utility function is becoming more and more popular in recent years.

3. Model of fertility choice (how many kids do you have?)

4. OLG. As you have seen in Randy's class, OLG has a lot of troublesome properties (e.g., money has raison d'etre.). But it is definitely useful in applications. So we will learn how to use OLG positively in applications.

18.2 The Model

This is a democratic world, i.e. all the agents are the same ex-ante. Let's assume that it is an economy with two brothers. Both of them are not subject to commitment. In other words, the two can sign a contract, but either of them can walk away if he does not feel like observing it.

This is an endowment economy (no production). No storage technology. Endowment is represented by $(y_s^1, y_s^2) \in Y \times Y$, where y_s^i is an endowment to brother i. $s = (y_s^1, y_s^2)$ follows a finite state Markov process with Markov transition matrix $\Gamma_{ss'}$. History up to period t is $h_t = \{(y_0^1, y_0^2), (y_1^1, y_1^2), (y_2^1, y_2^2), ..., (y_t^1, y_t^2)\}$. Probability of history h_t at period 0 is $\Pi(h_t)$. The expected life time

[52] The source of this model is Attanasio, Orazio, and Rios-Rull, Jose-Victor (2000), "Consumption smoothing in island economies: Can public insurance reduce welfare?", European Economic Review, 44-7, 1225-1258. You can download the paper from Victor's HP for this course or HP of ScienceDirect (offering electric form of recent articles in EER).

utility of a brother i is:

$$E_0\left\{\sum_{t=0}^{\infty}\beta^t\sum_{h_t}\Pi(h_t)u(c_i(h_t))\right\} \tag{224}$$

Brothers can sign a contract but are assumed to be able to break it and walk away. In that case, the brothers are assumed to be forced to live in autarky forever (no renegotiation considered). Let's define the value that the brother i lives in autarky after history h_t on by $\Omega_i(h_t)$. This is expressed as follows:

$$\Omega_i(h_t) = \sum_{r=0}^{\infty}\beta^{r-t}\sum_{h_r}\Pi(h_r|h_t)u(y_i(h_t)) \tag{225}$$

18.3 First Best allocation

What is the first best allocation? We can derive it by solving the SPP with arbitrary weights λ_i to brothers, as follows:

$$\max_{\{c_i(h_t)\}_{\forall h_t,\forall i}}\sum_{i=1}^{2}\lambda_i\sum_{t=0}^{\infty}\sum_{h_t}\beta^t\Pi(h_t)u(c_i(h_t)) \tag{226}$$

subject to the resource constraint:

$$c_1(h_t)+c_2(h_t) = y_1(h_t)+y_2(h_t) \qquad \forall h_t \tag{227}$$

Lagrangian for this problem is as follows:

$$L = \sum_{i=1}^{2}\lambda_i\sum_{t=0}^{\infty}\sum_{h_t}\beta^t\Pi(h_t)u(c_i(h_t)) + \sum_{t=0}^{\infty}\sum_{h_t}\gamma(h_t)[y_1(h_t)+y_2(h_t)-c_1(h_t)-c_2(h_t)]$$

Take FOCs with respect to $c_i(h_t)$, $i=1,2$. Then we get:

$$\lambda_1\beta^t\Pi(h_t)u'(c_1(h_t)) = \gamma(h_t) \tag{228}$$

$$\lambda_2\beta^t\Pi(h_t)u'(c_2(h_t)) = \gamma(h_t) \tag{229}$$

Combining these two yields:

$$\frac{\lambda_1}{\lambda_2} = \frac{u'(c_2(h_t))}{u'(c_1(h_t))} \tag{230}$$

Besides, if we assume CRRA utility function, it is equivalent to:

$$\frac{c_2(h_t)}{c_1(h_t)} = \text{constant} \tag{231}$$

This implies that, regardless of the node (history), ratio of consumption between the brothers are constant.

Homework 18.1. *Show that the first best allocation is characterized by a similar formula as this one for the economy with one-side commitment.*

18.4 Constrained Optimal Allocation

If there is no commitment technology, the first best allocation may not be feasible. Imagine that you get very good shock today. Then it might not be optimal for you to share a part of your endowment today with your brother. You might feel like walking way and enjoy what you got. In this situation, what kind of allocation we can achieve? Now, the social planner's problem is the one with the first best plus the following participation constraint, which guarantees that none of the brothers would walk away.

$$\sum_{r=0}^{\infty} \beta^{r-t} \sum_{h_r} \frac{\Pi(h_r)}{\Pi(h_t)} u(c_i(h_t)) \geq \Omega_i(h_t) \qquad \forall h_t, \forall i$$

Notice that the contract is considered AFTER the brothers observe today's shock in this model, contrary to the model in the last class.

Again, there are many constraints: two (because we have two brothers) for each node! So again, our problem is how to handle it? We will see that recursive formulation works pretty well here.

19 April 8: Epstein-Zin Recursive Utility (1) (by Jesus)[53]

19.1 Background: Theory

Consider the following preference:

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{232}$$

Let L, L', L'' be lotteries over $C = \{c_0, c_1, \ldots\}$. We would like to define a utility function over the lotteries to consider many interesting problems in macroeconomics. By far the most popular utility functional form is given by the expected utility function. This is given by the Expected Utility Theorem. Remember that the theorem says that if a preference satisfies (i) the independence axiom (i.e., $L \succsim L' \Longleftrightarrow \alpha L + (1-\alpha)L'' \succsim \alpha L' + (1-\alpha)L''$ for $\forall \alpha \in [0,1]$), and (ii) the continuity axiom, (i.e., $\{\alpha \in [0,1] : \alpha L + (1-\alpha)L' \succsim L''\}$ and $\{\alpha \in [0,1] : \alpha L + (1-\alpha)L' \precsim L''\}$ are closed), then there exists an expected utility which represents the preference.

However, it's better if we can use larger class of utility functions than the expected utility functions. Epstein and Zin (1989) showed a set of conditions under which larger class of utility functions exist with a recursive form. This is what we are going to learn.

[53]The reference is Epstein, Larry G., and Zin, Stanley E. (1989), "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework," Econometrica, 57-4, 937-969.

19.2 Background: Application

19.2.1 Equity Premium Puzzle

However, a question is ”why do we need such wider class of utility functions?” There are ample examples that standard period utility function with expected utility form is not sufficient to analyze some questions in macroeconomics. A famous example is the equity premium puzzle.

CRRA utility function is very popular in macroeconomics, because this is unit independent. Because of the unit independency, we do not need to care about the units, and what’s more, there exists a balanced growth path in the neoclassical growth models. However, one of the defects of the CRRA utility function with expected utility form is that for this utility function coefficient of relative risk aversion and elasticity of intertemporal substitution are represented in one parameter: σ. But these two are the different things, and economists started thinking that this property of the CRRA utility function produces misleading results for some questions in macroeconomics. One of such examples is the equity premium puzzle. This puzzle basically says that standard representative agent neoclassical growth model with CRRA utility function with ”normal” parameter values fails to explain the huge difference between risky stock returns and riskless bond in US. For example, Dimson, Marsh, and Staunton (2002) reported that the average annual real returns of equity (over 1900-2000) is 6.7%, while the average annual returns of risk-free[54] T-bill over the same period is 0.9%. So the risk premium is around 6% annually. Of course, equity premium puzzle depends on many assumptions, as I listed above, so there are many other assumptions which might cause the problem. But if we change only σ to match this high equity premium, it is known that we need $\sigma = 20 - 50$. In other words, people have to very very risk averse to hold T-bills regardless of the huge difference in average return.

19.2.2 Consistent parameter values

However, this huge value of σ is not consistent with other implication of the model. To see this point, let’s see what kind of parameter values are needed for the model to be consistent with the data. For simplicity, think about the deterministic growth model. We know that the Euler Equation is the following:

$$u'(c_t) = \beta(1+r)u'(c_{t+1}) \tag{233}$$

Suppose that the economy is on the balanced growth path, i.e. $c_{t+1} = (1+g)c_t$, then:

$$u'(c_t) = \beta(1+r)u'((1+g)c_t) \tag{234}$$

Assume CRRA functional form:

$$(1-\sigma)c_t^{-s} = \beta(1+r)(1-\sigma)c_t^{-s}(1+g)^{-\sigma} \tag{235}$$

[54]We ignore the inflation risk here. If we consider the inflation risk, T-bill is also risky unless it is inflation adjusted (and it is the case).

Clean up the terms, we get:

$$\beta(1+r) = (1+g)^{\sigma} \tag{236}$$

What are the values of r and g that data suggests? Long-run average of interest rate is around 4% p.a. and long-run trend growth rate is around 1.9%. Plugging these values into the equation above, we get:

$$1.04\beta = 1.019^{\sigma} \tag{237}$$

Or, by taking log:

$$\log 1.04 + \log\beta = \sigma \log 1.019 \tag{238}$$

Or approximation yields:

$$0.04 + \log\beta = 0.019\sigma \tag{239}$$

Let's interpret this equation as a constraint from NGM on the values of (σ, β). If $\sigma = 2$, this equation tells that β should be 0.998. If $\sigma = 20$, this equation tells that β should be 1.40. Remember that the value of σ which is consistent with the large equity premium is the order of 20. But if we assume that $\sigma = 20$, we need to have a very high β. Indeed, we need to let the representative agent to "love the future". Why this happens? Because elasticity of intertemporal substitution, which is the inverse of the coefficient of relative risk aversion, is very very low with $\sigma = 20$. In other words, when we force the representative agent to hate the risk, at the same time we need to force her to love the future!

This is a kind of digression but suppose that σ and β are the same for all countries. Then, (236) says that in a country with high growth rate (high g), interest (r) should be high, too. But this is inconsistent with the data. This argument is also an evidence that the standard representative agent neoclassical growth model can be misleading for some problems that we are interested in.

19.3 The Set-up

Let $X, B(X), M(X)$ be a metric space, Borel-σ-algebra defined on X, and a probability measure on X. Let $y \in \mathcal{R}_+^\infty$ is a particular consumption path, i.e., $y = (c_0, c_1, c_2, \ldots)$. $d \in D$ represents a lottery over C. Let $m \in M(D)$, i.e. m is a probability measure over the set of possible consumption paths. Then we can write $d = (c, m)$. In words, a lottery d is a set of sure current consumption (c) and a probability measure over the possible consumption paths from the next period on (m).

For any $b \geq 1$ and $l > 0$, define the followings:

$$Y(b; l) = \{(c_0, c_1, c_2, \ldots) \in \mathcal{R}_+^\infty : \sup \frac{c_t}{b^t} < l\}, b < \infty \tag{240}$$

$$Y(\infty; l) = \mathcal{R}_+^\infty \tag{241}$$

$$M^b(\mathcal{R}^\infty) = \bigcup_{l>0} M(Y(b;l)) \tag{242}$$

Intuitively, $Y(b;l)$ is a set of consumption paths that does not explode too fast. The notion of "too fast" is characterized with $b$and l. Or we can interpret this restriction on Y be a certain kind of boundedness. $M(Y(b;l))$ is a measure of consumption paths that do not explode too fast. Finally, $M^b(\mathcal{R}^\infty)$ is a union of $M(Y(b;l))$ with all the possible value of l. In other words, $M^b(\mathcal{R}^\infty)$ is a measure of consumption paths that do not explode too fast with "too fast" is defined only with b (and not l); a certain consumption path y is counted in $M^b(\mathcal{R}^\infty)$ if y is bounded for some value of l, and conditional on b.

A bit more definition.

$$\begin{aligned} D(b) &= \{d \in D : d_1 = (c_0, m_1), m_1 \in M^b(\mathcal{R}^\infty)\} && (243) \\ D(b,l) &= \{d \in D : d_1 = (c_0, m_1), m_1 \in M(b;l)\} && (244) \end{aligned}$$

19.4 Recursive Preference

In general, if the future consumption path is stochastic, the (expected) utility function is a function from the probability measure on the (bounded) possible consumption path to a real number. Specifically:

$$V : D(b) \to \mathcal{R} \tag{245}$$

If the utility function can be represented by the following formula, we call that the utility is recursive:

$$V(c_0, m_1) = W(c_0, \mu(V(c_1, m_2))) \tag{246}$$

where $W : \mathcal{R}_+^2 \to \mathcal{R}_+$ is an aggregator function and μ is an operator which gives a certainty equivalence. Specifically, μ is a function from the distribution on a random variable to a real number. An property of μ function is that $\mu(\delta_x) = x$ where δ_x is a random variable with probability 1 on $x \in \mathcal{R}_+$. You can interpret μ as a kind of expectation operator.

Homework 19.1. *Show that the standard expected utility is recursive in this sense, where W is just a weighted sum.*

19.5 The Theorem

The question here is under what kind of assumptions there exists utility function (245) satisfying (246). This is exactly the theorem states:

Theorem 19.2. *(Esptein and Zin (1989), Theorem 3.1, details omitted) If (i) W is the following CES form:*

$$W(c,z) = [c^\rho + \beta z^\rho]^{\frac{1}{\rho}} \qquad 0 \neq \rho < 1, 0 < \beta < 1$$

and (ii) μ satisfies 1st and 2nd order stochastic dominance, then there exists a solution V which satisfies (246).

Very intuitively, 1st and 2nd stochastic dominance is defined as follows:

1. 1st order stochastic dominance: Let $F(x), G(x)$ be the CDFs. Then G first order stochastically dominates F if $F(x) \geq G(x), \forall x$. This is a way of saying that G delivers more than F.
2. 2nd order stochastic dominance: Let $F(x), G(x)$ be the CDFs. Then G first order stochastically dominates F if $\int udG \geq \int udF$ for all nondecreasing concave u. This is a stronger way of saying that G delivers more than F.

Just for your reference, precise assumption on μ in the theorem is the following: if $\{p_n\}$ and p are in $M([0,b])$, then

(a) $\lim \int f dp_n = \int f dp$ for $\forall f : \mathcal{R}_+ \to \mathcal{R}_+$ increasing $\Rightarrow \lim \mu(p_n) = \mu(p)$

(b) if $\limsup \int f dp_n \leq \int f dp$ $\forall f : \mathcal{R}_+ \to \mathcal{R}_+$ increasing $\Rightarrow \lim \mu(p_n) \leq \mu(p)$

19.6 Examples

An example of μ and V which are supported by the theorem is the following. This is the standard expected utility function.

$$\mu(p) = \left(\int x^{\rho} dp(x)\right)^{\frac{1}{\rho}} \tag{247}$$

$$V(c_0, m_1) = \left(c_0^{\rho} + E_{m_1} \sum_{t=1}^{\infty} \beta^t c_t^{\rho}\right)^{\frac{1}{\rho}} \tag{248}$$

Note that the following recursive utility function of Kreps and Porteus is another example. Notice that the standard expected utility form is the special case of this function where $\alpha = \rho$.

$$\mu(p) = \left(\int x^{\rho} dp(x)\right)^{\frac{1}{\rho}} \tag{249}$$

$$V(c_0, m_1) = \left(c_0^{\rho} + \beta [E_{m_1} V^{\alpha}(c_1, m_2)]^{\frac{\rho}{\alpha}}\right)^{\frac{1}{\rho}} \tag{250}$$

20 April 10: Epstein-Zin Recursive Utility (2) (by Jesus)

20.1 Recover the Sequential Form

Take the standard expected utility we saw in the last class.

$$\mu(p) = \left(\int x^{\rho} dp(x)\right)^{\frac{1}{\rho}} \tag{251}$$

$$V(c_0, m_1) = \left(c_0^{\rho} + E_{m_1} \sum_{t=1}^{\infty} \beta^t c_t^{\rho}\right)^{\frac{1}{\rho}} \tag{252}$$

For this class of function, we can recover the sequential representation of the utility function by recursively plugging V and μ into W. Let's do it.

$$\begin{aligned}
V(c_0, m_1) &= W(c_0, \mu(V(c_1, m_2))) \qquad (253)\\
&= [c_0^{\rho} + \beta\mu(V(c_1, m_2))^{\rho}]^{\frac{1}{\rho}}\\
&= [c_0^{\rho} + \beta\left(\int V(c_1, m_2)^{\rho} dp\right)^{\frac{\rho}{\rho}}]^{\frac{1}{\rho}}\\
&= [c_0^{\rho} + \beta\int V(c_1, m_2)^{\rho} dp]^{\frac{1}{\rho}}\\
&= [c_0^{\rho} + \beta\int \left([c_1^{\rho} + \beta\int V(c_2, m_3)^{\rho} dp]^{\frac{1}{\rho}}\right)^{\rho} dp]^{\frac{1}{\rho}}\\
&= [c_0^{\rho} + \beta\int [c_1^{\rho} + \beta\int V(c_2, m_3)^{\rho} dp] dp]^{\frac{1}{\rho}}\\
&= [c_0^{\rho} + \beta\int c_1^{\rho} dp + \beta^2\int V(c_2, m_3)^{\rho} dp]^{\frac{1}{\rho}}\\
&= [c_0^{\rho} + \beta\int c_1^{\rho} dp + \beta^2\int c_2^{\rho} dp + \beta^3\int V(c_3, m_4)^{\rho} dp]^{\frac{1}{\rho}}\\
&\vdots\\
&= \left[c_0^{\rho} + E\sum_{t=1}^{\infty} \beta^t c_t^{\rho}\right]^{\frac{1}{\rho}}
\end{aligned}$$

Since $\frac{1}{\rho}$ at the outside of the square brackets is just a monotone transformation and thus irrelevant for utility, we can say that we recovered the standard sequential form of expected utility function. But this recovery can be done just for the expected utility function and it is impossible to derive the clean sequential form in the same way as here for more general recursive utility function, for example, Kreps and Porteus type. In other words, the theorem says that, even when we cannot recover the simple sequential utility function, do not worry about it, because as long as W and μ satisfies certain conditions we saw above, it is guaranteed that we have a well behaved utility function which is characterized recursively by W and μ.

20.2 An Example: Application to RA-NGM

Consider a representative agent neoclassical growth model. Agent has a wealth A_t at the beginning of period t and can invest this asset into N different assets. Return for these N assets at period t is represented as a following vector:

$$R_t = (R_{1t}, R_{2t}, R_{3t}, ..., R_{Nt}) \tag{254}$$

Let

$$\omega_t = (\omega_{1t}, \omega_{2t}, \omega_{3t}, ..., \omega_{Nt}) \tag{255}$$

represent the proportion the wealth of the agent that is invested to each asset. In other words, $\{\omega_t\}$ is an asset portfolio of the agent at period t. Of course, $\sum_{j=1}^{N} \omega_{jt} = 1$. Since the return from asset j is $R_{jt}\omega_{jt}$, the wealth of the agent in the next period is expressed as follows:

$$A_{t+1} = (A_t - c_t) \sum_{j=1}^{N} R_{jt}\omega_{jt} \tag{256}$$

Use the following Esptein and Zin recursive utility form to define a value function:

$$J(A_t) = \max_{c_t, \omega_t} \left\{ c_t^{\rho} + \beta \left[E J^{\sigma}(A_{t+1}) \right]^{\frac{\rho}{\sigma}} \right\}^{\frac{1}{\rho}} \tag{257}$$

Note that ρ controls elasticity of intertemporal substitution and σ controls degree of risk aversion. Is is shown that this operator is a contraction mapping (weighted contraction). So we have a unique fixed point $J(A)$.

Another good thing for this problem is that the portfolio is time independent, i.e. the agent invests the same proportion to each asset in every period[55]. So we can define:

$$A_{t+1} = (A_t - C_t) \sum_{j=1}^{N} R_{jt}\omega_{jt} = (A_t - c_t) M_t \tag{258}$$

where M_t is a scalar. With this property, we can guess that the value function has the following linear form:

$$J(A') = \Phi(A_{t-1}) A_t \tag{259}$$

We will plug in this guess and have to verify this guess later (though we will not do it). Anyway, now the value function equation is:

$$J(A_t) = \max_{c_t} \left\{ c_t^{\rho} + \beta \left[E \Phi^{\sigma}(A_t)(A_t - C_t)^{\sigma} \right]^{\frac{\rho}{\sigma}} \right\}^{\frac{1}{\rho}} \tag{260}$$

[55]This is a popular property in dinky finance, but not in economics. We need to neglect labor income, or assume that earning follows iid process to get this property. And (according to Victor) this is against the spirit of economics.

Note (i) we neglect ω from optimization problem, assuming that we have plugged in the optimal portfolio, and (ii) $\frac{1}{\rho}$ outside the brackets does not matter for optimization problem. So forget $\frac{1}{\rho}$. And the optimization problem is:

$$\max_{c_t} c_t^{\rho} + \beta \left[E \left[\Phi(A_t) M_t \right]^{\sigma} (A_t - C_t)^{\sigma} \right]^{\frac{\rho}{\sigma}} \tag{261}$$

Now take FOC with respect to C. I omit the details of the calculations but at the end of the day, we get:

$$E \left[\beta \left(\frac{C_{t+1}}{C_t} \right)^{\rho-1} M_t \right]^{\frac{\sigma}{\rho}} = 1 \tag{262}$$

Notice that if $\rho = \sigma$, we recover the standard Euler Equation.

21 April 9: Economy with Two-Side Lack of Commitment (2)

21.1 A Comment on Recursive Utility

Suppose a two period model. The standard expected utility with CRRA period utility function is the following:

$$\frac{c_0^{1-\sigma}}{1-\sigma} + \beta E \left\{ \frac{c_1^{1-\sigma}}{1-\sigma} \right\} \tag{263}$$

One example of Epstein and Zin recursive utility the following:

$$\left[\left\{ \frac{c_0^{1-\sigma}}{1-\sigma} \right\}^{\rho} + \beta E \left\{ \frac{c_1^{1-\sigma}}{1-\sigma} \right\}^{\rho} \right]^{\frac{1}{\rho}} \tag{264}$$

Note that since there are only two periods, difference between the expected utility function and Epstein and Zin type utility function is only the way of aggregation between consumption today and consumption tomorrow. We can see that in the expected utility, elasticity of intertemporal substitution (EIS) and risk aversion are controlled by only one parameter σ, whereas in the latter utility, EIS is controlled by ρ and risk aversion is controlled by σ.

Homework 21.1. *Derive EIS and coefficient of relative risk aversion of both utility functions.*

21.2 Sequential Representation of Constrained SPP

Consider the social planner's problem, with participation constraints. The planner maximize the welfare of the brothers (weighted sum of utility of brothers) by determining allocation, but the planner is constrained in that the allocation is not allowed to violate participation constraints.

Remember that the Lagrangian for the unconstrained SPP consists of the welfare of brothers and a feasibility constraint. Lagrangian for the constrained SPP consists of the welfare of the two brothers, a feasibility constraint, and participation constraints. Number of participation constraints is the number of agents (2) times the number of nodes (infinity). Specifically, the Lagrangian associated with the planner's problem is the following:

$$\begin{aligned}
&\sum_{t=0}^{\infty}\beta^t \sum_{h_t\in H_t}\Pi(h_t)\sum_{i=1}^{2}\lambda_i u(c_i(h_t)) \qquad (265)\\
&+\sum_{t=0}^{\infty}\beta^t \sum_{h_t\in H_t}\Pi(h_t)\sum_{i=1}^{2}\mu_i(h_t)\left[\sum_{r=t}^{\infty}\beta^{r-t}\sum_{h_r}\Pi(h_r|h_t)u(c_i(h_r))-\Omega_i(h_t)\right]\\
&+\sum_{t=0}^{\infty}\sum_{h_t\in H_t}\gamma(h_t)\left[\sum_{i=1}^{2}c_i(h_t)-\sum_{i=1}^{2}y_i(h_t)\right]
\end{aligned}$$

where $\Omega_i(h_t)=\sum_{r=0}^{\infty}\beta^{r-t}\sum_{h_r}\Pi(h_r|h_t)u(y_i(h_t))$ is the value of being in autarky from period t on. In the expression above, the second line is the participation constraints, and the third line is the feasibility constraint. Notice that each $c_i(h_t)$ appears once in the first line, infinite times in the participation constraints, and once in the feasibility constraint.

21.3 Recursive Representation of Constrained SPP

We would like to transform this problem into the recursive form, because it would be easier to solve the optimal allocation with a computer. For the case without PC, transformation into the recursive representation is easy. But for the problem with PC, the transformation is not so trivial. We will show how to transform the sequential problem with PC into recursive representation.

The important thing is that we can rewrite the Lagrangian as follows

$$\begin{aligned}
&\sum_{t=0}^{\infty}\sum_{h_t}\sum_{i=1}^{2}\beta^t\Pi(h_t)\left\{M_i(h_{t-1})u(c_i(h_t))+\mu_i(h_t)[u(c_i(h_t))-\Omega_i(h_t)]\right\} \qquad (266)\\
&+\sum_{t=0}^{\infty}\sum_{h_t\in H_t}\gamma(h_t)\left[\sum_{i=1}^{2}c_i(h_t)-\sum_{i=1}^{2}y_i(h_t)\right]
\end{aligned}$$

where

$$\begin{aligned}
M_i(h_{-1}) &= \lambda_i \qquad (267)\\
M_i(h_t) &= M_i(h_{t-1})+\mu_i(h_t)
\end{aligned}$$

Homework 21.2. *Verify the transformation.*

What does this $M_i(h_t)$ means? This is a kind of a Pareto weight attached to each brother. This weight changes according to the history of shocks. In particular, notice that $\mu_i(h_t)\geq 0$ and >0 if the corresponding participation constraint is binding. The weight attached to one brother increases if the participation constraint for that brother binds. In this case, the planner need to increase the value for this brother to keep the brother with the insurance scheme. This result is clear if we take the FOCs and manipulate them. By playing with FOCs, we get:

$$\frac{u'(c_1(h_t))}{u'(c_2(h_t))} = \frac{M_2(h_{t-1})+\mu_2(h_t)}{M_1(h_{t-1})+\mu_1(h_t)} \tag{268}$$

This implies that the consumption allocated to brother 1 is increased if participation constraint for this brother binds (i.e., $\mu_1(h_t) > 0$).

Homework 21.3. *Verify the result.*

21.4 Recursive Formulation

Our goal is make the problem recursive, which is very nice when we work with computer. To do this, we need to find a set of state variables which is sufficient to describe the state of the world. Firstly, define:

$$\phi_i(h_i) = \frac{\mu_i(h_t)}{M_i(h_t)} \tag{269}$$

$$x(h_t) = \frac{M_2(h_t)}{M_1(h_t)} \tag{270}$$

Then we have

$$x(h_t) = \frac{1-\phi_1(h_t)}{1-\phi_2(h_t)} x(h_{t-1}) \tag{271}$$

We are going to use x as a state variable and (271) as law of motion for x. In every period, there are three possibilities,

1. Participation constraint is not binding for either 1 or 2. Then $x(h_t) = x(h_{t-1})$.
2. Participation constraint is binding for brother 1, but not for brother 2, i.e. $\phi_1(h_t) > 0, \phi_2(h_t) = 0$. Then, $x(h_t) < x(h_{t-1})$, i.e. relative weight to brother 2 is decreased.
3. Participation constraint is binding for brother 2, but not for brother 1, i.e. $\phi_1(h_t) = 0, \phi_2(h_t) > 0$. Then, $x(h_t) > x(h_{t-1})$, i.e. relative weight to brother 2 is increased.

Homework 21.4. *Prove that at most one participation is binding, .i.e.* $\phi_1(h_t) > 0, \phi_2(h_t) > 0$ *never occurs.*

22 April 11: Economy with Two-Side Lack of Commitment (3)

22.1 Review of Epstein and Zin recursive utility

Define a consumption path as follows:

$$\begin{aligned} c^0 &= (c_0, c_1, c_2, \ldots) \\ c^t &= (c_t, c_{t+1}, c_{t+2}, \ldots) \end{aligned} \tag{272}$$

The utility function is a function from consumption sequence to a real number. If the utility is time separable, the utility function can be expressed as:

$$V(c^0) = u(c_0) + \beta V(c^1) \tag{273}$$

If the utility is not time separable, the utility function can be expressed as:

$$V(c^0) = W(c_0, \mu(V(c^1))) \tag{274}$$

What the theorem tells is that if W is CES form and μ is well behaved, the utility function (function from a consumption sequence to a real number) can be well represented by a recursive form with W and μ. Using this result, we can write a value function (function from a state variable to a real number) with W and μ as follows:

$$\Omega(a) = \max_{c,a'} \left\{ u^{\rho}(c) + \beta \left[\int \Omega^{\sigma}(a') dF(a') \right]^{\frac{\rho}{\sigma}} \right\}^{\frac{1}{\rho}} \tag{275}$$

It turns out that this class of utility is useful in applications in macroeconomics.

22.2 Recursive Formulation of the Economy with Two Brothers

State variables are endowment: $y = (y_1, y_2)$ and weight to brother 2: x. Define the value function as follows:

$$V = \{(V_0, V_1, V_2) \text{ such that } V_i : X \times Y \to \mathcal{R}, \, i = 1, 2, V_0(x, y) = V_1(x, y) + x V_2(x, y)\} \tag{276}$$

What we are going find is the fixed point of the following operator (operation is defined later):

$$T(V) = \{T_0(V), T_1(V), T_2(V)\} \tag{277}$$

Firstly, we solve the problem without ICs (incentive compatibility constraints). Notice that if the solution of the problem without ICs does not violate the ICs, the solution from the problem without ICs is also the solution of the problem with ICs. This is because the constraint set of the problem with ICs is a subset of the constraint of the problem without ICs. Given the value V, the recursive SPP without ICs is the following[56]:

$$\Phi(y,x;V) = \max_{c_1,c_2} u(c_1) + xu(c_2) + \beta \sum_{y'} \Gamma_{yy'} V_0(y',x) \tag{278}$$

subject to

$$c_1 + c_2 = y_1 + y_2 \tag{279}$$

Notice that an argument in V_0 is x, not x'. This is because $x' = x$ if none of the PCs is not binding, and we are assuming it. FOCs of the problem are:

$$\begin{aligned} u'(c_1) &= \lambda \\ xu'(c_1) &= \lambda \end{aligned}$$

Combining these two yields:

$$\frac{u'(c_1)}{u'(c_2)} = x \tag{280}$$

Let the optimal decision rule of this problem as:

$$\begin{gathered} \tilde{c}_1(y,x;V) \\ \tilde{c}_2(y,x;V) \end{gathered}$$

The next step is to check if the optimal decision rule violates PC or not. Equivalently, we will check the following:

$$u(\tilde{c}_1(y,x;V)) + \beta \sum_{y'} \Gamma_{yy'} V_1(y',x) \quad \lesseqqgtr \quad u(y_1) + \beta \sum_{y'} \Gamma_{yy'} \Omega_1(y') \qquad (PC_1) \tag{281}$$

$$u(\tilde{c}_2(y,x;V)) + \beta \sum_{y'} \Gamma_{yy'} V_2(y',x) \quad \lesseqqgtr \quad u(y_2) + \beta \sum_{y'} \Gamma_{yy'} \Omega_2(y') \qquad (PC_2) \tag{282}$$

As we have argued in the previous class, there are three possibilities here (remember that ICs are not binding simultaneously for the two brothers):

1. Participation constraint is not binding for either 1 or 2. Then $x(h_t) = x(h_{t-1})$. In addition,

$$T_0(V) = \Phi(y,x;V) \tag{283}$$

$$T_i(V) = u(\tilde{c}_i(y,x;V)) + \beta \sum_{y'} \Gamma_{yy'} V_i(y',x) \qquad i = 1,2 \tag{284}$$

[56]Although Victor used V^n to make updating process explicit, I do not use here. Instead, I denote updated value by $T(V)$

2. Participation constraint is binding for brother 1, but not for brother 2, i.e. $\phi_1(h_t) > 0, \phi_2(h_t) = 0$.

3. Participation constraint is binding for brother 2, but not for brother 1, i.e. $\phi_1(h_t) = 0, \phi_2(h_t) > 0$.

If the case is the first one, the operation is done. we finished updating the value function. For the other two cases, we need to solve for the problem where $x' \neq x$. Since the last two cases are symmetrical, let's suppose that PC for brother 1 is binding and solve the problem. We need to solve the following system of equations in this case.

$$\begin{aligned} c_1 + c_2 &= y_1 + y_2 && (285)\\ u(c_1) + \beta \sum_{y'} \Gamma_{yy'} V_1(y', x) &= u(y_1) + \beta \sum_{y'} \Gamma_{yy'} \Omega_1(y') && (286)\\ x' &= \frac{u'(c_1)}{u'(c_2)} && (287) \end{aligned}$$

This is a system of three equations with three unknowns (x', c_1, c_2). Let the solution of this problem be $(\hat{x}', \hat{c}_1, \hat{c}_2)$. Having them, we can define the operator as follows:

$$\begin{aligned} T_i(V) &= u(\hat{c}_i) + \beta \sum_{y'} \Gamma_{yy'} V_i(y', \hat{x}') \qquad i = 1,2 && (288)\\ T_0(V) &= T_1(V) + x T_2(V) && (289) \end{aligned}$$

How does the optimal policy look like? See Figure 1.

We know that Pareto Optimal allocation satisfies:

$$\frac{u'(c_1)}{u'(c_2)} = \text{constant} \tag{290}$$

If we assume CRRA utility function and set constant=x and take log of both sides, this is equivalent to:

$$\log c_2 = \log c_1 + \frac{1}{\sigma} \log x \tag{291}$$

Therefore we can draw this equation as a line labelled as "First Best" in the graph above. In this case, consumption for brother 2 is given by the crossing point of the given x and this line. However, in the economy with Participation Constraints, the consumption induced by this given x violates PCs. In this case, x is adjusted such that the new consumption does not violate the PCs. This is described in the graph above. Two horizontal lines labelled as "PC_1 Binding" and "PC_2 Binding" represent PC for brother1 and 2. These lines move according to the current x and the endowment of each brother in the current period. For example, the consumption for brother 2 implied by the

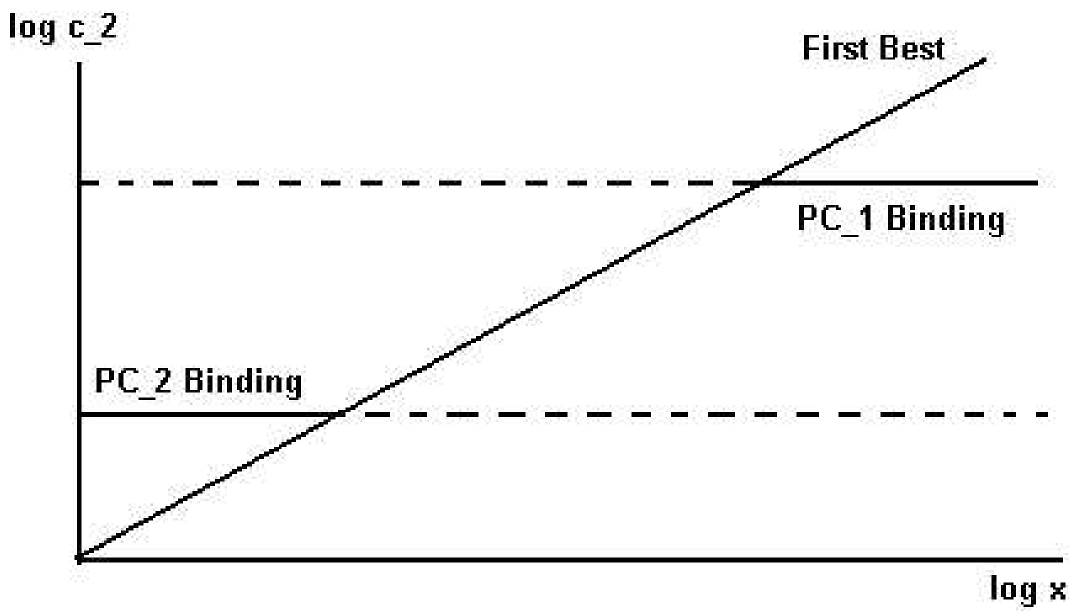

Figure 1: **Optimal Consumption for Brother 2.**

current x and the first best line is higher than the line ”PC_1 Binding”, the planner has to give brother 1 more consumption to keep him around, thus consumption for brother 2 is decreased up to the line ”PC_1 Binding” and x is adjusted. The opposite thing happens in the case where the consumption for brother 2 is too low such that the PC for brother 2 is binding. So, the consumption for brothers are determined on the increasing line and the value x but the value of x changes over time when the implied consumption violates one of PCs.

Before closing, let’s see the welfare implication of the model. The welfare of brothers in this world is described by Figure 2. Because of the lack of commitment problem, brothers cannot achieve the first best, but the brothers can achieve higher utility than the autarky (depicted as omega 1 and 2), because of the gains from trade. The constrained Pareto frontier in this model is the arc that is surrounded by autarky values for two brothers, and is placed inside the Pareto frontier for the first best. Again, there is no theory about which point on the constrained Pareto frontier is realized in equilibrium. It depends on the negotiation process between the two brothers. But according to the history of endowment shocks, the point moves along the arc. For example, let’s suppose the allocation is characterized by the point P1 in a certain period. As long as no PCs is binding, the allocation stays the same, and the point on the arc that characterizes the allocation also stays P1. However, if PC for brother 1 is binding, consumption for brother 1 is increased, and the point moves from P1 to P2. In the opposite case, the point moves from P1 to P3.

Final question with this model is ”how to implement this allocation?” or ”Is there any equilibrium that supports this allocation?”. The answer is yes. How? Think of this model as a repeated

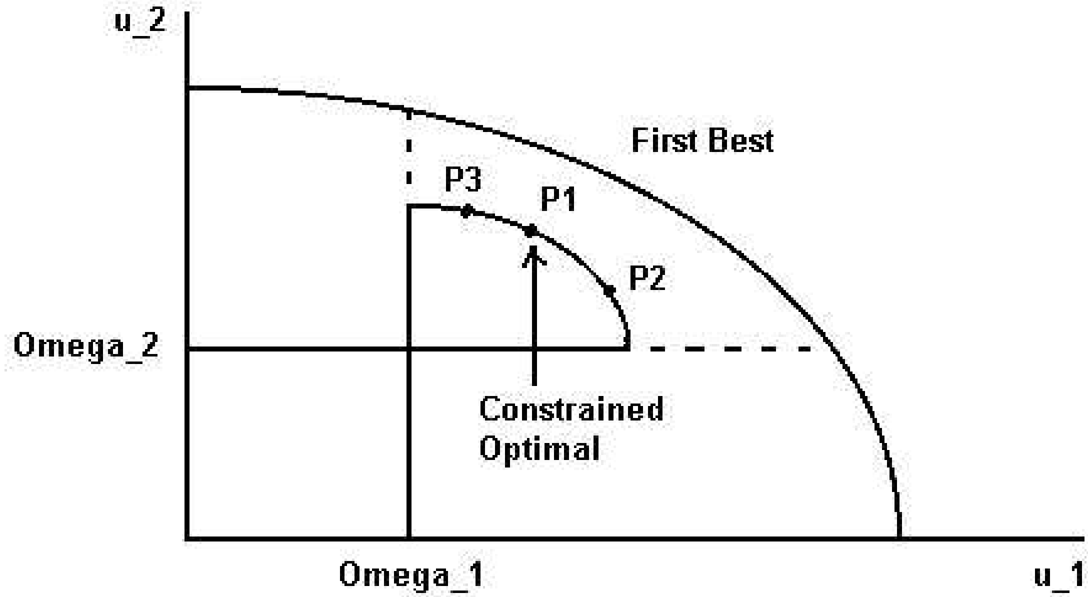

Figure 2: **Pareto Frontiers.**

game. And define the strategy as follows: keep accepting the contract characterized here until the other guy walks away. If the other guy walks away, go to autarky forever. We can construct a Nash equilibrium by assigning this strategy to both of the brothers.

23 April 15: Economy with Lack of Observability

23.1 Cross-Sectional Distribution of Consumption of the Economy with One-Side Lack of Commitment

Remember the economy with one-side lack of commitment, i.e. the economy with moneylender and an agent. Consider how the cross-sectional distribution of consumption of agents evolves over time. To start with, remember how the optimal contract look like. It looks like Figure 3:

Suppose, in period 0, all the agents are given initially the same promised value, which is equal to the value of autarky. All agents are proposed the same contract, because v is common. But the actual consumption and the promised value for the next period is different according to the realization of the shock. As we can see in the figure above, agents with low endowment are given the same consumption and same promised value. In addition, the value promised is the same as was promised initially, which is the value of autarky. The agents with high endowment are given higher consumption and higher promised value, according to how large is the endowment this period. What happens in the next period? Now the agents face different contract according to the promised value in period 0. Look at Figure 4:

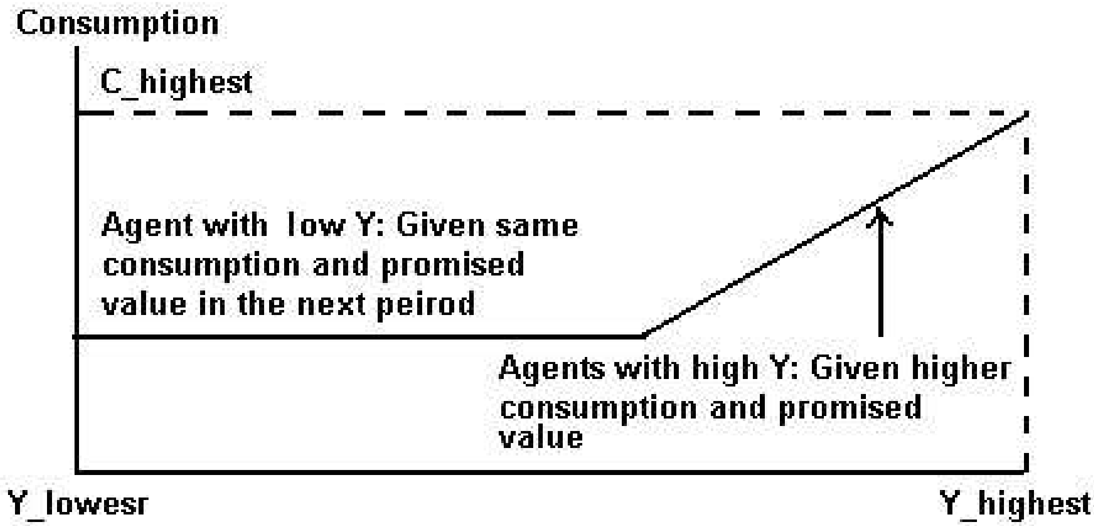

Figure 3: **Consumption in the Optimal Contract.**

Agents who were promised the same value in period 0 are facing the optimal contract depicted as S_0 in the figure. This is the same schedule as in the previous figure. To the contrary, agents who got high endowment in period 0 and thus were given higher promised value in period 0 are facing S_1 now. That is located higher than S_0 in the flat part but share the same slope. Agents who were promised even higher value are facing S_2. In any case, if the agents receive the endowment that corresponds to the slope part of the optimal contract in period 1, they will be given a higher consumption in period 1 and promised value for period 2.

Now, we can answer the following questions with respect to the cross-sectional distribution of consumption:

1. What happens to the cross-sectional distribution of consumption?: it starts from equally distributed (corresponds to the same value across agents), but begins to be dispersed as agents receive different endowments, but all the agents eventually receive the highest endowment and receive the highest consumption.

2. Is the limit distribution stationary?: Yes. Once all the agents start receiving the highest consumption, this consumption level is sustained.

3. Is it globally stable?: Yes. The limit distribution is the same for any initial distribution.

4. How is the individual fate?: Individual fate also converges to the highest consumption.

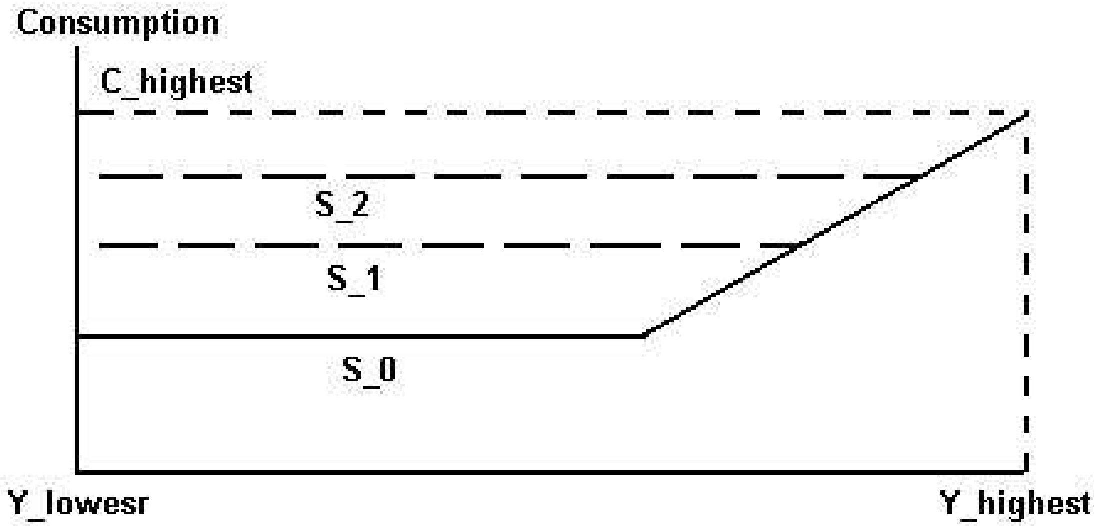

Figure 4: **Optimal Contract in the Subsequent Periods.**

Remark 23.1. *Notice that it is difficult to establish the property with respect to the variances, though, intuitively, we can imagine that the variance starts from zero, increases for a while and again decreases to zero.*

Remark 23.2. *If the utility function is linear in consumption, there is no gain from trade, so nobody will sign a contract. That's the end of the story.*

Remark 23.3. *We are not sure about the sign of $P(v)$. It depends on the environment. Suppose, if $y_s = 0$ or close, and utility of zero consumption is minus infinity, gain from trade (insurance) is very large for agent (because agents really want to avoid zero consumption), and even $P(v^{highest})$ is bigger than zero, i.e. the moneylender can earn profit even if an agent reaches to the highest consumption. In this case, average endowment of the agents is located above the highest consumption in the previous figures.*

Homework 23.4. *Show that highest consumption level is lower than the highest endowment.*

23.2 Cross-Sectional Distribution of Consumption of the Economy with Two-Side Lack of Commitment

Remember the economy with two brothers. Assume that the gain from trade (insurance) is not so large such that the first best allocation is not feasible (brothers have an incentive to walk away if the contract is determined according to the first best. Or in other words, PC is sometimes binding). Then x (the relative weight to brother 2) will oscillate according to the combination of the shock realizations. Also remember that, if one of PC is binding, the consumption allocation and x' is determined irrelevant to the current x (or history in the past, in general). Therefore, if we have many pairs of brothers in the economy, after a certain periods, all the initial x will become irrelevant. x of each individual pair still oscillates but the cross-sectional distribution of x (hence the distribution of pairs of consumption) will become constant (law of large numbers prevails here). Now let's answer the same questions as for the one-side commitment economy:

1. What happens to the cross-sectional distribution of consumption?: Let's suppose that initial x is same for all the pairs. Initially all the pairs have the same ratio of consumption. But the ratio changes over time as each pair gets different shocks. The cross-sectional distribution will be stabilized after a while.
2. Is the limit distribution stationary?: Yes.
3. Is it globally stable?: Yes.
4. How is the individual fate?: Individual fate moves ups and downs, though the distribution of the individual fates are becoming stable.

23.3 Economy with Lack of Observability[57]

Consider again the economy with a moneylender and an agent. No production. No storage technology. No trade among agents. There is no commitment problem. Agents cannot walk away the contract. Neither the moneylender. But the problem is that the endowment of the agents cannot be observed. In other words, this model is a model with private information.

Suppose there is a contract that tells that the moneylender gives to the agents if agent reports that she is hungry and receives from her if she reports that she is not. If the moneylender cannot observe her true state, she can cheat by telling that she is hungry, everyday. How can we avoid this? Intuitively, we can avoid this situation by putting costs for telling that ”I am hungry”.

This is a mechanism design problem. Therefore, as usual in this field, we can use revelation principle, and the problem is to find an optimal contract under the following mechanism:

1. The agent and the moneylender sign a contract, which is a function from promised value and reported endowment (can be the different from her true type).
2. After observing the endowment, the agent reports her endowment to the moneylender.
3. The agent receives consumption depending on her reported type.

[57] The source is again updated Chapter 15 of Sargent's Recursive Macroeconomic Theory, page 459-.

Under some conditions, the optimal contract is the one where agents are induced to always tell the truth (her true type) to the moneylender[58]. So let's define the problem recursively.

$$P(v) = \max_{\{b_s,\omega_s\}_{s=1}^{S}} \sum_{s=1}^{S} \Pi_s[-b_s + \beta P(\omega_s)] \tag{292}$$

subject to

$$\sum_{s=1}^{S} \Pi_s[u(y_s+b_s)+\beta\omega_s] \geq v \tag{293}$$

$$u(y_s+b_k)+\beta\omega_k \leq u(y_s+b_s)+\beta\omega_s \qquad \forall s,k \tag{294}$$

where b_s is the transfer of consumption which depends on the reported endowment and promised value. The first constraint is the constraint of promise keeping and the second one tells that for any realization of the shock, telling the truth (s) is better than telling any other type (k). Therefore, the number of truth telling constraints is S^2. It is immediate to see that the following constraint in the Sargent's book is exactly the same as the truth telling constraint above.

$$c_{sk} \equiv u(y_s+b_s)+\beta\omega_s - u(y_s+b_k)\beta\omega_k \geq 0 \qquad \forall s,k \tag{295}$$

What are the properties of $P(v)$?

1. $P(v)$ is decreasing in v. It's trivial, because the constraint set of the problem above is getting smaller as v gets bigger.

2. $P(v)$ is bounded above. It is because the problem without truth telling constraint is bounded above and the problem with truth telling constraint must have lower value than this. This is a common way of proving boundedness. $P(v)$ is also bounded below because, intuitively, the moneylender will never give much more than v to agents.

How does the optimal contract look like? The crucial point is to punish agents for saying that she is unhappy (low endowment today) to induce her not to lie. So the consumption of a particular agent is decreasing over time. Only the agents who did not tell that they are unhappy (and actually they are happy) can keep high consumption. So the distribution is going be like (i) agent who have been lucky receive high consumption, and (ii) unlucky agents consume less and less. And asymptotically, all the agents and up consuming the lowest consumption level. Note that this is obviously not a good theory of explaining the transition of cross sectional distribution of consumption. At least we want the consumption increasing over time on average.

[58] Do not worry about details of the conditions.

23.4 OLG without Trouble (1): Why Trouble?

OLG is a model with troubles. Why? Remember the trick of the hotel with infinite rooms, which you must have heard from Randy's class: even if you are told from a hotel that the rooms are occupied, you can get your room by moving a guy in the room 1 to room 2, moving a guy in the room 2 to room 3...

What happens with the OLG is a similar thing. Consider a two period (young and old) OLG. Endowment is 3 for the young guys and 1 for the old guys. Because of the curvature of the utility function, you want to delay part of your consumption in the young period. How to achieve this? If an old agent has a paper called money and can get one unit of consumption good by trading her money for consumption goods, by saying to the young guy that the young guy can do the same transaction when she gets old, agents can achieve consumption of two goods in both young and old period.

Why this is possible? This is because agents want to delay consumption. In the case of our example, the source of the problem is:

$$u(c_1-1,c_2+1) > u(c_1,c_2) \tag{296}$$

Or in general,

$$u_2(c_1,c_2) > u_1(c_1,c_2) \tag{297}$$

In words, net benefit of eating one unit more in the next period and consuming one unit less in this period is positive. Therefore, in order to avoid the trouble of OLG, we need to have:

$$\frac{u_1(c_1,c_2)}{u_2(c_1,c_2)} > 1$$

Notice that the empirical counterpart of $\frac{u_1}{u_2}$ is the rate of return in the aggregate economy[59]. Since in US, this is $7-8\%$ annually, this condition seems to hold for US economy (and of course, same thing can be said also for other countries), and we can forget the negative side of OLG.

24 April 16: OLG without Trouble (2)

24.1 Why Trouble? (Again)

Consider a OLG model with 2 periods of life. The endowment is 3 for the young and 1 for the old. Preference of an agent is:

$$u(c_1,c_2) = \log c_1 + \log c_2 \tag{298}$$

[59]Remember the Euler Equation for the standard growth model. It is $\frac{\beta u'(c')}{u'(c)}(1+r)=1$. And in our current argument β is a part of the utility function, so we can see that $\frac{u_1}{u_2}=(1+r)$.

where c_1, c_2 are the consumption of the young and the old period. What is the non monetary equilibrium (NME) for this economy.? In the NME, all the agents consume their endowment, i.e. $c_1 = 3, c_2 = 1$ (Why?). What is the price supporting this allocation? A price of the consumption in period $t+1$ over the price of consumption good in period t that supports non trade allocation is 3. This is because

$$\frac{p_{t+1}}{p_t} = \frac{u'_{t+1}}{u'_t} = \frac{u'_{c_2}}{u'_{c_1}} = \frac{c_1}{c_2} = \frac{3}{1} = 3 \tag{299}$$

Is this an Arrow Debreu equilibrium? If so, since the utility function satisfies nonsatiation, we can use First Basic Welfare Theorem and conclude that this allocation is PO. However, if so, it's impossible to have a monetary equilibrium (ME), which we know that is Pareto superior than the allocation in NME. As we can imagine from this argument, NME is NOT an Arrow Debreu equilibrium. Why? The price system is not a continuous bounded function. To see this, let's write the price of consumption goods in terms of period 0 consumption good price in NME. It is:

$$P = \{3^t\}_{t=0}^{\infty} \tag{300}$$

Obviously the pricing function constructed from this price sequence is not a bounded function. Let's link the result here to the discussion in the last class. If $\frac{u_2(c_1,c_2)}{u_1(c_1,c_2)} > 1$, NME is not an ADE, and thus FBWT cannot hold with NME, and finally ME might be Pareto superior to NME.

24.2 The Basic Model of OLG

Suppose that agents can live up to period I but there is a probability that agents die before age I (early death). Agents are born with zero asset and can save. The rate of return from saving is assumed to be $[1+r]$. The wage rate per efficiency unit is w. Agents have a limited amount of time (normalized to one) and can allocate the time to either (i) work or (ii) enjoy leisure. Efficiency units which an agent can supply by working for a unit time changes as the agent grows older. This is captured by ε_i, where i is the age of the agent. The problem of the agent is as follows:

$$\max_{\{a_{i+1},c_t,n_i\}_{i=1}^{I}} \sum_{i=1}^{I} \beta_i s^i u_i[c_i, 1-n_i] \tag{301}$$

subject to

$$\begin{aligned} a_1 &= 0 && (302)\\ a_{i+1} + c_i &= a_i[1+r] + w\varepsilon_i[1-n_i] && (303)\\ s^i &= \Pi_{j=0}^{i-1} s_j && (304) \end{aligned}$$

Notice that β_i is not β^i. Time discount factor can be different according to age. Maybe young agents discount future more (NOW is the important time for the young) and adult agents discount future less (considering the future more than kids). Different β_i can capture these. s_j is a probability of surviving between age j and $j+1$. Therefore, s^i is a probability that an agent survives up to the age i. $s_I = 0$, to limit the lifetime. If we assume that $s_i = 1$, $i = 0, 1, ..., I-1$, it means that there is no early death (all the agents survive until the age of I with probability one), and we can drop s from the model (we will do this later).

24.3 Analysis of Early Death

Notice that in the above expression, we implicitly assume that the value of life is zero. Otherwise, we need to add the value of dying to the utility of the agent. This implies that, if we use CRRA preference, we need to have $\sigma > 1$. Otherwise, utility of living agent is always negative and all the agents would like to commit suicide if allowed! So assume that $\sigma > 1$. If there is a positive probability of early (unexpected) death, we need to decide what to do with the assets of these agents of early death. There are three alternatives:

1. Allow agents to insure against death risk. It is what the annuity market is doing in the real life. Suppose mass of agent (measure 1) of the same age i want to insure against the death risk. Because of the LLN, we (and the agents) know that s_i agents are going to survive until next period. So, if they insure each other, one unit of saving gives the agent $\frac{[1+r]}{s_i}$ in the next period. Why? $[1+r]$ is just a rate of return of saving. $\frac{1}{s_i}$ is the benefit from the death insurance. Because only s_i agents will be alive next period, each agent just needs to save $\frac{1}{s_i}$ to get one unit in the next period, if we assume that the asset of the dead agents are distributed equally. Therefore, if we allow agents to insure against death risk, the budget constraint will be modified as follows:

$$a_{i+1}s_i + c_i = a_i[1+r] + w\varepsilon_i[1-n_i] \tag{305}$$

   By the way, why agents want to buy life insurance? In this model, life insurance does not mean anything, because getting some goods when you die does not increase your utility at all. We have to model family to consider the life insurance. Agents buy life insurance because they care the other family members when they die.

2. Government can tax away all the assets of agents with early death. In this case, we need to assume what the government does with this income. The government can throw it into the sea, or can make a lump-sum transfer.

3. The last option is so called Pharaoh assumption: the assets of the dead agents are buried with them!

Remark 24.1. *Considering the difference of s_i of agents of different type (education, marital status, gender) is important in macroeconomics. There is a data that married males have 5 years longer life and married women's life is a couple of weeks shorter. Also there is a data that college graduates live 7 years longer than the agents without college degree. The result of macroeconomic analysis would be very different depending on what kind of theory we have behind these data.*

24.4 Labor Earning

What is a good theory on ε? If we look at the average wage per hour at the different age ($w\varepsilon_i$), the wage per hour increases with age, peaks at around 40, and slowly decreases until the retirement.

Since w is assumed to be same for all agents, we need a theory that explains the difference in ε to replicate the hump shape of the average wage profile. What kind of theory do we have? There are two ways, in general:

1. Take $\{\varepsilon_i\}$ as exogenous; i.e., assuming that the young agents are useless because they are young.
2. Human capital theory. Assume that the difference in capital stock between the young agents and the old agents yields the difference in ε. There are three branches:
   (a) Learning-by-doing: assume that agents accumulate human capital (ε) by working. Agents learn something which enhances their human capital stock while they are working. Imagine an interns of doctor. The young doctors learn how to do operations by actually working at hospitals. This idea is represented by:

   $$\varepsilon_{i+1} = \varphi_i(\varepsilon_i, n_i)$$

   where n_i is hours worked of agents of age i. φ is indexed by i because learning ability can be different depending on age.

   (b) Learning-by-not-doing: assume that agents accumulate human capital by actually learning (which is different from working or enjoying leisure). This idea is represented by:

   $$\varepsilon_{i+1} = \varphi_i(\varepsilon_i, l_i)$$

   where l_i is the time spent on learning, which is different from working or enjoying leisure. Agents allocate their time in learning to accumulate human capital.

   (c) Education: the difference from learning models above is that most of education is acquired in the early stage of life. Keane and Ken Wolpin (REStat1994)[60] showed that 90% of people's fate is determined before age 16, by using structurally estimated model of the career choice.

24.5 Constructing Recursive Problem (1): Stationary Equilibrium

Let's define a stationary equilibrium in the recursive way. Stationary equilibrium means that the prices: r and w do not change over time. Firstly, let's define the problem of an agent of age i,conditional on $\mathcal{K}$, which is a capital labor ratio. Since r and w are functions of $\mathcal{K}$, we only need to record $\mathcal{K}$ instead of keeping track of prices. Individual agent's problem is:

$$V_i(a;\mathcal{K}) = \max_{c,n,a'} \left\{ u(c, 1-n) + \beta V_{i+1}(a';\mathcal{K}) \right\} \tag{306}$$

[60] Keane , M., and Wolpin, K. (1994), "The Solution and Estimation of Discrete Choice Dynamic Programming Models by Simulation: Monte Carlo Evidence,' Review of Economics and Statistics, 76-4, 648-672.

subject to

$$
\begin{aligned}
c+a' &= a[1+r(\mathcal{K})]+w(\mathcal{K})n\varepsilon_i && (307)\\
n &\in [0,1] && (308)\\
V_{I+1} &= 0 && (309)\\
a_1 &= 0 && (310)
\end{aligned}
$$

Solution of the problem is sequences $\{a_{i+1}, c_i, n_i\}_{i=1}^{I}$ Now we are ready to define a stationary equilibrium.

Definition 24.2. *A stationary equilibrium is a set of allocations $\{a^*_{i+1}, c^*_i, n^*_i\}_{i=1}^{I}$, a pair of prices r^* and w^*, and $\mathcal{K}$ such that:*

1. *(Agents' optimization) Given prices r^* and w^*, $\{a^*_{i+1}, c^*_i, n^*_i\}_{i=1}^{I}$, solves the optimization problem of agents.*
2. *(Firm's optimization) Prices r^* and w^* are determined competitively.*
3. *(Consistency)*

$$\frac{\sum_{i=1}^{I} a^*_i}{\sum_{i=1}^{I} \varepsilon_i n^*_i} = \mathcal{K}$$

Remark 24.3. *This model has been a workhorse of public finance. Why? Because this model allows agents of different types to have different wages and allows a life cycle behavior. Examples of how to use this model are as follows:*

1. *We can used this model to answer how much taxes to be collected from different types of agents in the economy. We can set different ε_i for agents of different types (age, education, race, etc). For example, what type of agent prefers general income tax (forget details like progressiveness)? It's college graduates, because for high earning agents it is easier to smooth consumption with income tax than consumption tax.*
2. *If we can extend the model to have multiple goods, we can analyze the effect of excise taxes (examples are taxes on tobacco, alcohol, gasoline, lottery, etc.) on the distribution of income. Excise taxes are the ones to the poor people, because the poor people expends more on these.*
3. *We can incorporate the social security into this kind of model. Who likes social security? College graduates, whites, females like it most, because they live longer and thus receive more social security benefits.*

Remark 24.4. *How to incorporate retirement in this economy? Exogenous retirement is simple. Add an assumption that $\varepsilon_i = 0$, $\forall i \geq 66$ (retirement age in assumed to be 65). If we consider* *a voluntary retirement, how can we explain the fact that people retire at a certain age, instead of keeping on working less hours in the whole life? Two possible explanations are (i) wage is nonlinear function of hours, i.e. 100 people work 40 hours each in a week is different from 1,000 people work 4 hours each in a week, (ii) there is a "preparation" cost for working, like dressing up before going to office or commuting*[61].

Remark 24.5. *We can also a version of this model with a kind of balanced growth path. We can apply exactly the same technique to detrend the NGM.*

24.6 Constructing Recursive Problem (2): Non-Stationary Equilibrium

Let's define an equilibrium which is not restricted to stationary one. Prices can change over time. The recursive formulation of the agent's problem is as follows:

$$V_i(a,K;G,H) = \max_{c,n,a'} \left\{ u(c,1-n) + \beta V_{i+1}(a',K';G,H) \right\} \tag{311}$$

subject to

$$\begin{aligned} c + a' &= a[1+r(K,N)] + w(K,N)\varepsilon_i n & (312)\\ n &\in [0,1] & (313)\\ V_{I+1} &= 0 & (314)\\ a_1 &= 0 & (315)\\ K_0 &= 0 & (316)\\ K' &= G(K) & (317)\\ N &= H(K) & (318) \end{aligned}$$

Notice that K is a vector:

$$K = [K_1, K_2, ..., K_I]'$$

And $G(K)$ is:

$$\begin{bmatrix} K_1' \\ K_2' \\ \vdots \\ K_I' \end{bmatrix} = G(K) = \begin{bmatrix} G_1(K) \\ G_2(K) \\ \vdots \\ G_I(K) \end{bmatrix} \tag{319}$$

The solution of this problem is

$$\begin{aligned} a_i' &= g_i(a,K;G,H) & (320)\\ c_i &= c_i(a,K;G,H) & (321)\\ n_i &= h_i(a,K;G,H) & (322) \end{aligned}$$

Now we are ready to define a nonstationary recursive equilibrium.

[61] Victor has a paper which deals with a nonlinear wage function.

Definition 24.6. *A nonstationary equilibrium is a set of functions* $\{V_i^*(.),\, g_i^*(.),\, c_i^*(.),\, n_i^*(.)\}_{i=1}^I$, $G^*(K)$, $H^*(K)$, $r^*(K,N)$, $w^*(K,N)$ *such that:*

1. *(Agent's optimization) Given* $G^*(K), H^*(K), r^*(K,N)\; w^*(K,N)$, $\{V_i^*(.), g_i^*(.), c_i^*(.), n_i^*(.)\}_{i=1}^I$ *solves the agents' problem.*

2. *(Firm's optimization)* $r^*(K,N)$ *and* $w^*(K,N)$ *are determined competitively.*[62]

3. *(Consistency)*

$$\begin{aligned} H^*(K) &= \sum_{i=1}^{I} h_i^*(K_i,K;G^*,H^*)\varepsilon_i \\ G_i^*(K) &= g_i^*(K_i,K;G^*,H^*) \qquad \forall i \end{aligned}$$

24.7 Model of Human Capital

The problem of agents in the economy where human capital is accumulated by learning by doing is as follows:

$$V_i(a,\varepsilon) = \max_{c,n,a',\varepsilon'} \left\{ u(c,1-n) + \beta V_{i+1}(a',\varepsilon') \right\} \tag{323}$$

subject to

$$\begin{aligned} c+a' &= a[1+r]+wn\varepsilon && (324) \\ \varepsilon' &= \varphi_i(\varepsilon,n) && (325) \\ n &\in [0,1] && (326) \\ V_{I+1} &= 0 && (327) \\ a_1 &= 0,\ \varepsilon_1 = given && (328) \end{aligned}$$

Notice that the accumulated human capital stock is now a state variable. Also I put ε' as a choice variable to make the definition of the equilibrium easier[63].

Homework 24.7. *Define a stationary equilibrium of this economy. Also define a stationary equilibrium of the economy with learning by not doing.*

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[62] In case of Cobb Douglas production function, $r^*(K,N) = \alpha \left(\frac{\sum_{i=1}^I K_i}{N}\right)^{\alpha-1}$ and $w^*(K,N) = (1-\alpha)\left(\frac{\sum_{i=1}^I K_i}{N}\right)^{\alpha}$.

[63] Remember what I did in the last review session. We can construct a agent's problem without using ε' as a choice variable, but anyway we need to have a sequence $\{\varepsilon_i\}$ to state the consistency condition in the definition of the equilibrium.

25 April 18: Sequential Representation of an Equilibrium for OLG Economy

In the last class, we defined an equilibrium of OLG using recursive representation. Of course, we can do it in sequential representation. Let's do it. The only thing we need to take care of is how to keep track of time. There are two conventions in how to keep track of time:

1. c_{it} represents a consumption of age i agent that is born in period t.
2. c_{it} represents a consumption of age i agent at period t.

Let's drop labor leisure choice from the basic model in the last class, for simplicity, and define a nonstationary equilibrium using the first convention.

Definition 25.1. *A nonstationary equilibrium for an OLG economy is a sequence of prices $\{r_t^*, w_t^*\}_{t=0}^{\infty}$ and a sequence of allocations $\{c_{it}^*, a_{it}^*\}_{t=0}^{\infty}$ such that:*

1. *(Agent's optimization) Given prices, $\{c_{it}^*, a_{it}^*\}_{t=0}^{\infty}$ is a solution to the following optimization problem for $t = 0, 1, ...$*[64]*:*

$$\max_{\{c_{it}^*, a_{it}^*\}_{t=0}^{\infty}} \sum_{i=0}^{I} \beta_i u_i(c_{it}) \tag{329}$$

subject to

$$c_{it} + a_{i+1,t} = a_{it}[1 + r_{t+i}] + w_{t+i}\varepsilon_i \tag{330}$$

2. *(Consistency)*[65] *for all $t = 0, 1, ...$*

$$f_1\left(\sum_{i=0}^{I} a_{i,t-i}, \sum_{i=0}^{I} \varepsilon_i\right) - \delta = r_t^* \tag{331}$$

$$f_2\left(\sum_{i=0}^{I} a_{i,t-i}, \sum_{i=0}^{I} \varepsilon_i\right) = w_t^* \tag{332}$$

If we take the second convention, the definition is as follows:

Definition 25.2. *A nonstationary equilibrium for an OLG economy is a sequence of prices $\{r_t^*, w_t^*\}_{t=0}^{\infty}$ and a sequence of allocations $\{c_{it}^*, a_{it}^*\}_{t=0}^{\infty}$ such that:*

[64] There is no convention on whether the agents are born at the age of 0 or 1. I assume 0 to keep consistency with what VIctor wrote on the blackboard. But it's a detail, which you do not need to worry at this stage.

[65] This includes both consistency condition and firm's optimality condition.

1. *(Agent's optimization) Given prices, $\{c_{it}^*, a_{it}^*\}_{t=0}^{\infty}$ is a solution to the following optimization problem for $t = 0, 1, \ldots$*[66]*:*

$$\max_{\{c_{it}^*, a_{it}^*\}_{t=0}^{\infty}} \sum_{i=0}^{I} \beta_i u_i(c_{i,t+i}) \tag{333}$$

subject to

$$c_{it} + a_{i+1,t+1} = a_{it}[1 + r_t] + w_t \varepsilon_i \tag{334}$$

2. *(Consistency)*[67] *for all $t = 0, 1, \ldots$*

$$f_1\left(\sum_{i=0}^{I} a_{i,t}, \sum_{i=0}^{I} \varepsilon_i\right) - \delta = r_t^* \tag{335}$$

$$f_2\left(\sum_{i=0}^{I} a_{i,t}, \sum_{i=0}^{I} \varepsilon_i\right) = w_t^* \tag{336}$$

Notice that the difference is the subscripts for allocations.

[66] There is no convention on whether the agents are born at the age of 0 or 1. I assume 0 to keep consistency with what Victor wrote on the blackboard. But it's a detail, which you do not need to worry at this stage.

[67] This includes both consistency condition and firm's optimality condition.