1 Jan 28: Overview and Review of Equilibrium

1.1 Introduction

- What is an equilibrium (EQM)?
  - Loosely speaking, an equilibrium is a mapping from environments (preference, technology, information, market structure) to allocations.
  - Equilibrium allows us to characterizes what happens in a given environment, that is, given what people like, know, have...
- Two requirements of an equilibrium is (i) agents optimize. They do as best as possible. and (ii) actions of agents in the economy are compatible to each others. Examples of equilibrium concepts: Walrasian equilibrium.
- Properties of equilibrium concepts: existence and uniqueness.
  - We can prove existence of equilibrium by constructing one. We need uniqueness because otherwise, we do not have sharp prediction on what is likely to happen in a given environment.
  - Pareto optimality is not necessary property of equilibrium. Neither is tractability.

1.2 Arrow-Debreu Competitive Equilibrium

- What is Arrow-Debreu Competitive Equilibrium (ADE)?
  - All trades happen at time 0.
  - Perfect commitment
  - Everything is tradable and things are traded conditional on date and event.
  - Agents are price takers.
- Agents' Problem

  $$\max_{x \in X} U(x) \tag{1}$$

  subject to

  $$p(x) \leq 0 \tag{2}$$

  $p : S \to R$ is a continuous, linear function. $S \supset X$ may include infinity dimensional subjects. Thus, p is a linear function, but not necessary a vector.

This problem can be applied to infinite horizon and/or stochastic environment.

In this problem, the single objective function is $U(x)$, and restriction is $x \in X$ and $p(x) \leq 0$. As commitment is perfect in X and trades happen at time 0, people do whatever they choose at time 0.

- Properties of ADE
  - Existence. $CE(\mathcal{E}) \neq \phi$.Existence is relatively easy to get. For uniqueness, we need more to get sufficient condition.
  - Pareto optimality. $CE(\mathcal{E}) \subset PO(\mathcal{E})$ when there is no externalities and non-satiation for utility function. But we need to be careful here. Given agents, endowment, preferences, technology and information structure, elements of competitive equilibrium are (i) price system and (ii) allocation. But $CE(\mathcal{E}) \subset PO(\mathcal{E})$ is only for allocation, which means allocation from such competitive equilibrium is Pareto optimal.
  - From $CE(\mathcal{E})$ to $PO(\mathcal{E})$, three things are required: (1) find price to get CE. (ii) right redistribution (transfer t) to give agents enough resources for the allocation. (iii) free disposal to ensure the quasi equilibrium is a true equilibrium.
  - Second Basic Welfare Theorem. Proof uses separation theorem, for which the sufficient condition is (i) nonempty convex set.(ii) interior point.

    For any allocation $x \in PO(\mathcal{E})$, $\exists p$, such that (p, t, x) is quasi equilibrium with transfers (QET).

1.3 The Road Map

- In the first two weeks with Randy, we learned how to solve Social planner's problem (SPP) of neoclassical growth model with representative agent (RA-NGM), using dynamic programming. Also we know that solution to SPP is Pareto Optimal (PO) in our model. Other good things for solution to SPP is that, in RA-NGM, we know that (i) it exists and (ii) it's unique.

- Besides, we have two welfare theorems (FBWT, SBWT) from Dave's class. If we carefully define the environment, those two theorems guarantee (loosely) that (i) under certain conditions, Arrow-Debreu Competitive Equilibrium (ADE, or Walrasian equilibrium or valuation equilibrium) is PO, and (ii) also under certain conditions, we can construct an ADE from a PO allocation.

- Using those elements, we can argue that ADE exists and is unique, and we just need to solve SPP to derive the allocation of ADE, which is much easier task than solve a monster named ADE.

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- Using those elements, we can argue that ADE exists and is unique, and we just need to solve SPP to derive the allocation of ADE, which is much easier task than solve a monster named ADE.
- But we have another problem: The market assumed in ADE is not palatable to us in the sense that it is far from what we see in the world. So, next, we look at an equilibrium with sequential markets (Sequential Market Equilibrium, SME). Surprisingly, we can show that, for our basic RA-NGM, the allocation in SME and the allocation of ADE turn out to be the same, which let us conclude that even the allocation of the equilibrium with sequential markets can be analyzed using the allocation of SPP.
- Lastly, we will learn that equilibrium with sequential markets with recursive form (Recursive Competitive Equilibrium, RCE) gives the same allocation as in SME, meaning we can solve the problem using our best friend = Dynamic Programming.
- (Of course, these nice properties are available for limited class of models. We need to directly solve the equilibrium, instead of solving SPP, for large class of interesting models. We will see that Dynamic Programming method is also very useful for this purpose. We will see some examples later in the course.)

1.4 Review of Ingredients of RA-NGM

Technology

- Representative agent's problem.

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{3}$$

subject to

$$k_{t+1} + c_t = f(k_t) \tag{4}$$

$$c_t, k_{t+1} \geq 0 \tag{5}$$

$$k_0 \text{ is given} \tag{6}$$

There are many variations of this problem, including models with distortion, stochastic environment. In writing such optimization problem, you should always specify control variable, initial condition.

Solutions is a sequence $\{c_t, k_{t+1}\}_{t=0}^{\infty} \in l_{\infty}$.

- Existence of a solution: We use Maximum theorem to prove existence. Sufficient condition to use Maximum theorem: (i) maximand is continuous function and (ii) constraint set is compact (closedness and boundedness). We assume u is continuous, f is bounded.
- Uniqueness: Sufficient condition includes: (i) convex constraint set, and (ii) strictly concave function. We assume u is strictly concave and f is concave
- Characterization of the solution: If u and f are differentiable and $\{c_t^*, k_{t+1}^*\}_{t=0}^{\infty}$ is the unique solution, the following condition has to be satisfied.

$$u'(c_t) = \beta f'(k_{t+1}) u'(c_{t+1}) \tag{7}$$

  And to rule out corner solution, Inada condition is assumed.

Homework 1.1 *Derive* (7)

(7) can be rewritten as

$$u'(f(k_t) - k_{t+1}) = \beta f'(k_{t+1}) u'(f(k_{t+1}) - k_{t+2}) \tag{8}$$

This is a second order difference equation. We need two initial conditions to pin down the entire sequence. We have got one initial capital, we have to look for k_1 that does not go out of track. Therefore, to solve the problem as an infinite sequence is difficult. Now let's look at another way of solving it as you seen in Randy's class.

1.5 Dynamic Programming

Define $V(k)$ to be the highest utility of the agent by doing right things in life. The Bellman equation is

$$V(k) = \max_{\{c,k'\}} u(c) + \beta V(k') \tag{9}$$

subject to

$$c + k' = f(k) \tag{10}$$

The solution is

$$k' = g(k) = \arg\max u(c) + \beta V(k') \tag{11}$$

Using $g(.)$, we can construct the sequence of $\{k_{t+1}\}$.

$$\begin{aligned} k_0 &= g(k_{-1}) \\ k_1 &= g(k_0) = g^2(k_{-1}) \\ &... \end{aligned}$$

We can show that the sequence constructed this way using Bellman equation satisfies the first order condition (7)

- F.O.C. of Bellman equation

$$V(k) = \max_{\{c,k'\}} u(f(k) - k') + \beta V(k') \tag{12}$$

is

$$-u'(f(k) - k') + \beta V_{k'}(k') = 0 \tag{13}$$

We know $V(.)$ is differentiable.

$$\begin{aligned} V_k(k) &= \frac{\partial}{\partial k}\{u[f(k) - g(k)] + \beta V(g(k))\} \\ &= [f_k(k) - g_k(k)]\, u_c[f(k) - g(k)] + \beta g'(k)\, V'(g(k)) \\ &= u_c f_k + \underbrace{g_k[-u_c + \beta V_c(g(k))]}_{=0 \text{ from FOC above}} \\ &= u_c f_k \end{aligned}$$

So,

$$-u_c + \beta f_k(k')\, u_c(c') = 0 \tag{14}$$

Therefore, we got the same FOC as what we have before.

$$-u_c + \beta f_k(k')\, u_c(c') = 0 \tag{15}$$

2 Jan 30: Neoclassical Growth Model

2.1 Review of growth model

- The model we studied last class is

$$\max_{\{c_t,k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{16}$$

  subject to

$$k_{t+1} + c_t = f(k_t, 1) \tag{17}$$

$$c_t, k_{t+1} \geq 0 \tag{18}$$

$$k_0 \text{ is given} \tag{19}$$

  We usually assume u and f are strictly concave, f is bounded. Then, there exists a unique solution which can be characterized by first order condition.

  - First Order Condition is a necessary but not sufficient condition for the optimal solution. Remember, the FOC is actually a second order difference equation. With only 1 initial condition k_0, there can be many sequences indexed by k_1. And the sequence may end up to be negative or infinity, which is not even feasible for this problem.
  - We can also get the optimal solution$\{c_t^*, k_{t+1}^*\}_{t=0}^{\infty}$ from Bellman equation as we see in Randy's class.

- Bellman equation for the growth model

$$V(k) = \max_{\{c,k'\}} u(c) + \beta V(k') \tag{20}$$

  subject to

$$c + k' = f(k, 1) \tag{21}$$

  The solution is

$$k' = g(k) \tag{22}$$

The solution is a fixed point of an functional operator, which is a contraction. Using $g(.)$, we can construct the sequence of $\{k_{t+1}\}.k_0, k_1 = g(k_0)\ldots$

We can show that the sequence constructed this way using Bellman equation satisfies the first order condition (7)

-   - We can go back and forth between these two forms of problem. One way is to construct $\{k_{t+1}\}$ by $g(.)$. The other direction is to see the sequence $\{k_{t+1}\}$ satisfies $k_{t+1}^* = g(k_t)$.

2.2 Social Planner's Problem

- We can write the growth model as a social planner's problem (SPP). We assume the economy is populated by a huge number of identical agents. The social planner is like the God who tells people what they should do.
  - Properties of the solution to SPP:

    (1) Pareto optimal. It is a one-line proof: if the solution to SPP is not optimal, there is better allocation to make everyone happier.

    (2) Uniqueness. The social planner treats everyone the same. So the SPP is symmetric and we get unique solution. Since all the Pareto optimal allocation are solution to the SPP, PO($\mathcal{E}$) is unique.

- Road map: What we want to know is equilibrium (price and allocation). If we can apply welfare theorems to the allocation of SPP, we can claim that "God's will realizes" and can analyze allocation of SPP instead of directly looking at an equilibrium allocation. In order to use the argument above, we formalize the environment of RA-NGM in the way such that we can apply welfare theorems. By using (i) existence of solution to SPP, (ii) uniqueness of solution of SPP, and (iii) welfare theorems, we can claim that ADE (i) exists, (ii) is unique, (iii) and PO. However, market arrangement of ADE is not palatable to us in the sense that set of markets that are open in the ADE is NOT close to the markets in our real world. In other words, there is notion of time in ADE: all the trades are made before the history begins and there is no more choices after the history begins. So we would like to proceed to the equilibrium concept that allows continuously open markets, which is SME and we will look at it closely next week.

2.3 Arrow-Debreu Equilibrium (ADE)

- Elements of ADE are commodity space, consumption possibility space, production possibility space and preference set.
- Commodity space:

  Commodity space is a topological vector space S which is space of bounded real sequences with sup-norm.S includes everything people trade, which are sequences. Agents have time and rent it to firm (labor services), owe capital and rent it to firm (capital services) and buy stuff to consume some and save some for the future. Hence, $S = l_\infty^3$. $s_t = \{s_{1t}, s_{2t}, s_{3t}\}_{t=0}^\infty$, which are goods, labor services and capital services, respectively.
- Consumption possibility set X.

$X \subset S$ and

$$X = \{x \in S = l_\infty^3 : \exists \{c_t, k_{t+1}\}_{t=0}^\infty \geq 0 \text{ such that}$$

$$\begin{aligned} k_{t+1} + c_t &= x_{1t} + (1-\delta)k_t \qquad \forall t \\ x_{2t} &\in [0,1] \qquad \forall t \\ x_{3t} &\leq k_t \qquad \forall t \\ k_0 &= \text{given}\} \end{aligned} \tag{23}$$

Interpretation is that x_{1t}=received goods at period t, x_{2t}=labor supply at period t, x_{3t}=capital service at period t. $k_{t+1} + c_t = x_{1t} + (1-\delta)k_t$ comes from real accounting. Note: capital and capital service are not the same thing. Think of the difference between a house and to rent a house.

- Preference $U : X \to R$.

$$U(x) = \sum_{t=0}^{\infty} \beta^t u(c_t(x)) \tag{24}$$

c_t is unique given x because each x implies a sequence $\{c_t, k_{t+1}\}_{t=0}^\infty$. If $x_{3t} = k_t$, $c_t = x_{1t} + (1-\delta)x_{3t} - x_{3t+1}$.

- Production possibility set Y.

Firm's problem is relatively simple as firm do not have intertemporal decision. Firms just rent production factors and produce period by period.

$Y = \Pi_{t=0}^\infty \widehat{Y}$:

$$Y = \Pi_{t=0}^\infty \widehat{Y}_t : \widehat{Y}_t = \{y_{1t}, y_{2t}, y_{3t} \geq 0 : y_{1t} \leq f(y_{3t}, y_{2t})\} \tag{25}$$

Interpretation is that y_1=production at period t, y_{2t}=labor input at period t, y_{3t}=capital input at period t.

  - Note: We did not use the convention in general equilibrium that input is negative and output is positive.

    Implicitly, we assume firm is constant return to scale. So, we do not need to worry about industrial organization.

- Price.

A price is a continuos and linear function $q : S \to R$. $q \neq 0$ and $q \in S^*$. S^* is a separating hyperplane in separation hyperplane theorem.

  - Continuity: for $s^n \to s$, $\Rightarrow q(s^n) \to q(s)$

- Linearity: $q\left(s^1 + s^2\right) = q\left(s^1\right) + q\left(s^2\right)$
- $q\left(x\right)$ may not be represented
- Inner product representation: $S = l_{\infty}^3$, one candidate space for q is l_1. l_1 is space of sup-norm bounded sequences. If $\{z_t\} \in l_1, \sum_t^{\infty} |z_t| < \infty$. Then, for $s \in S$, $z \in l_1$, $\sum_t z_t s_t < \infty$. We use p to denote such price function. Not all $q\left(x\right)$ may not be represented in this inner product form. But we will see one theorem about inner product representation of price.
- If $p \in l_{1,3}$, we can write price as $p\left(s\right) = \sum_{t=0}^{\infty} p_{1t}s_{1t} + p_{2t}s_{2t} + p_{3t}s_{3t} < \infty$. Note here, the prices of labor service and capital service are negative, as we make input factor to be positive.

Homework 2.1 *$s \in S$, assume $s_{1t} = s_1, s_{2t} = s_2, s_{3t} = s_3, \forall t$ (that is we consider steady state, for simplicity), show that for any $r > 0$, the discounting $\left\{\frac{1}{(1+r)^t}\right\}_{t=0}^{\infty}$ is a price vector in $l_{1,3}$.*

- Define an ADE:

  An Arrow-Debreu Competitive Equilibrium is a triad (p^*, x^*, y^*) such that

  1. x^* solves the consumer's problem.

     $$x^* \in \arg\max_{x \in X} U(x) \tag{26}$$

     subject to

     $$q\left(x\right) \leq 0 \tag{27}$$

  2. y^* solves the firm's problem.

     $$y^* \in \arg\max_{y \in Y} p(y)$$

  3. markets clear, i.e. $x^* = y^*$.

  - Note that the price system (or valuation function) p^* is an element of $Dual(L)$ and not necessarily represented as a familiar "price vector".
  - Note there are many implicit assumptions like (i) all the markets are competitive (agents are price taker), (ii) absolute commitment (economy with a lack of commitment is also a topic of macroeconomics, maybe from your 2nd year on), (iii) all the future events are known, with the probability of each events when trade occurs (before the history begins).

2.4 Welfare Theorems

Theorem 2.2 *(FBWT) If the preferences of consumers U are locally nonsatiated ($\exists \{x_n\} \in X$ that converges to $x \in X$ such that $U(x_n) > U(x)$), then allocation (x^*, y^*) of an ADE is PO.*

Homework 2.3 *Show U is locally nonsatiated.*

Theorem 2.4 *(SBWT) If (i) X is convex, (ii) preference is convex (for $\forall x, x' \in X$, if $x' < x$, then $x' < (1-\theta)x' + \theta x$ for any $\theta \in (0,1)$), (iii) $U(x)$ is continuous, (iv) Y is convex, (v)Y has an interior point, then with any PO allocation (x^*, y^*) such that x^* is not a satiation point, there exists a continuous linear functional q^* such that (x^*, y^*, p^*) is a Quasi-Equilibrium with transfer.*

- We can get rid of transfers in this economy. Everyone is the same, so, given $q(x_i) \leq t_i$, $\sum_i t_i = 0$, $\Rightarrow$ $t_i = 0$ for all i.
- Quasi equilibrium and true equilibrium.

  Quasi equilibrium is allocation from cost minimization problem. That is, (a) for $x \in X$ which $U(x) \geq U(x^*)$ implies $q^*(x) \geq q^*(x^*)$ and (b) $y \in Y$ implies $q^*(y) \leq q^*(y^*)$.

  If, for (x^*, y^*, q^*) in the theorem above, the budget set has cheaper point than x^*, that is, $\exists x \in X$ such that $q(x) < q(x^*)$,

  then (x^*, y^*, p^*) is a ADE.

Homework 2.5 *Show that conditions for SBWT are satisfied in the PO allocation of RA-NGM.*

Now we established that the ADE of the RA-NGM exists, is unique, and is PO. The next thing we would like to establish is that the price system $q^*(x)$ takes the familiar form of inner product of price vector and allocation vector, which we will establish next.

2.5 Inner Product Representations of Prices

Theorem 2.6 *(based on Prescott and Lucas 1972) If, in addition to the conditions to SBWT, $\beta < 1$ (or some analog stochastic version about state) and u is bounded, then $\exists p^* \in l_{1,3}$ such that (x^*, y^*, p^*) is a QE.*

That is, price system has an inner product representations.

Remark 2.7 *For OLG (overlapping generation model), there may be no enough discounting. We will see how it works in that case.*

Remark 2.8 *Actually, for now, the condition $\beta < 1$ is what we need to know as you can see on class. For bounded utility function, remember that most of the familiar period utility functions (CRRA (including log utility function), CARA) in macroeconomics do not satisfy the conditions, as the utility function is not bounded. There is a way to get away with it, but we you not need to go into details (for those interested, see Stokey, Lucas, and Prescott, Section 16.3, for example).*

- Now the agent's problem can be written as

  $$x^* \in \arg\max_{x \in X} U(x) \tag{28}$$

  subject to

  $$\sum_{t=0}^{\infty} p_{1t}s_{1t} + p_{2t}s_{2t} + p_{3t}s_{3t} = 0 \tag{29}$$

- In ADE: all the trades are made before the history begins and there is no more choices after the history begins. However, market arrangement of ADE is not palatable to us in the sense that set of markets that are open in the ADE is NOT close to the markets in our real world. We will allow people to trade every period and use sequence of budget constraints in agent's problem. Next week we will proceed to the equilibrium concept that allows continuously open markets, which is sequential market equilibrium.