1 April 21

1.1 One Sided Lack of Commitment (II) (continued)

Last class we showed that the promise keeping constraint is always binding but that the participation constraint may or may not be binding. For shocks that are above a certain value, the participation constraint will be binding and for shocks that are less, it will not be binding.

Homework. *Show that the following holds:*
Participation constraint is not binding for $\widetilde{y}_s \Rightarrow$ *Participation constraint is not binding* $\forall y_s$ *such that* $y_s \leq \widetilde{y}_s$
Participation constraint is binding for $\widetilde{y}_s \Rightarrow$ *Participation constraint is binding* $\forall y_s$ *such that* $y_s \geq \widetilde{y}_s$

Cross sectional analysis with lots of granddaugthers:

First Period:

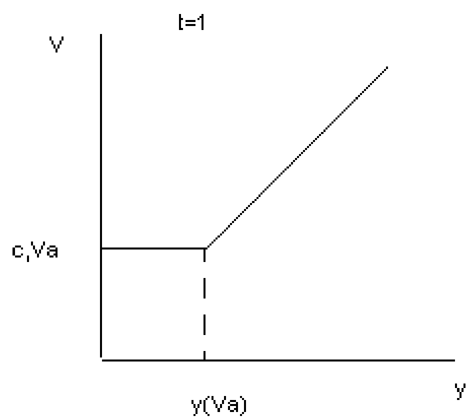


Figure 1:

The grandmother starts out by guaranteeing all of them the autarky value, V^A. But then once the shocks are realized, things will change. The following is what happens to the granddaughters according to the shocks they get:

- The granddaugthers who get bad shocks (those who get shocks less than the critical value $y^*(V^A)$):

The grandmother will give them what they were promised, so that the unlucky ones will remain with promised utility V^A. For the unlucky grand-daugthers, the grandmother does not need to increase their promised utility because their outside opportunity is not better than the deal she's already offering them.

- The granddaughters who get good shocks (those whoe get shocks above the critical value $y^*(V^A)$) :

  The grandmother will have to increase their promised utility. Otherwise, the granddaughters will not be willing to stay since their outside opportunity ($u(y_s) + \beta V^A$) is better than the deal she's offering them. Therefore, the grandmother will have to give them just enough promised utility ω_s and consumption c_s such that it gives the granddaughters what they would get if they left. This promised utility should therefore satisfy:

$$u(y_s) + \beta V^A = u(c_s) + \beta \omega_s$$

Note that $y^*(V^A)$ is the lowest endowment value such that the granddaughters are willing to stay with c_s and V^A. Any shocks to endowment that are higher than this critical value will make the granddaughthers willing to go so that their promised utility needs to be increased to keep them around.

Second period:

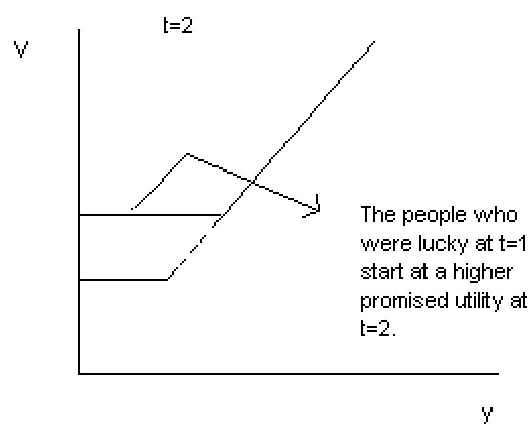


Figure 2:

- The granddaughters who were lucky in the previous period:

They will all start at a higher promised utility. A fraction of them who are lucky again in the second period will get higher promised utilities and the remaining unlucky ones will have the same promised utility as in the previous period.

- The granddaugthers who were unlucky in the previous period:

  They will all start at V^A again. A fraction of them who are unlucky again this period will get V^A again and the remaining who are lucky this period will get a higher promised utility.

You can see the pattern here: The average consumption over these two periods will go up because the grandmother gives more when the granddaughters get lucky. So as time passes, things are getting worse for the grandmother because she keeps having to promise more. On the other hand, over time, the granddaughters are doing better (As long as they had one good shock in the past, their promised utility is higher)

The grandmother is willing to sign anything with $P(V) \geq 0$. When she offers only the autarky value, with $P(V^A)$ she gets all the gains from trade.

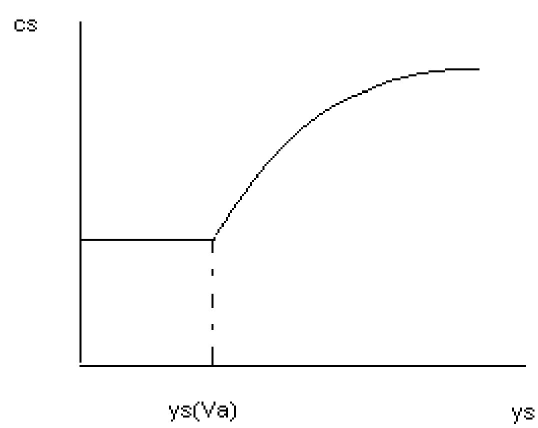


Figure 3:

In the above graph, you can see that the consumption stays the same until the critical value, $y_s^*(V^A)$. However, once the granddaughter gets a shock higher than that the grandmother has to increase the granddaughter's consumption. But an important thing to notice here is that the consumption is growing at a slower rate than endowment. To see why this is true, recall that u is concave so that the marginal utility of the consumer is decreasing. Therefore, for a certain increase in y_s , c_s will increase by less because the promised utility ω_s is also increasing (recall the equation $u(y_s)+\beta V^A = u(c_s)+\beta\omega_s$). A "stupid" thing to do for the grandmother would be to set $c_s = y_s$. It would be "stupid", because she can give less now and more in the future; so she can manage to keep the granddaughter around by offering some consumption less than y_s.

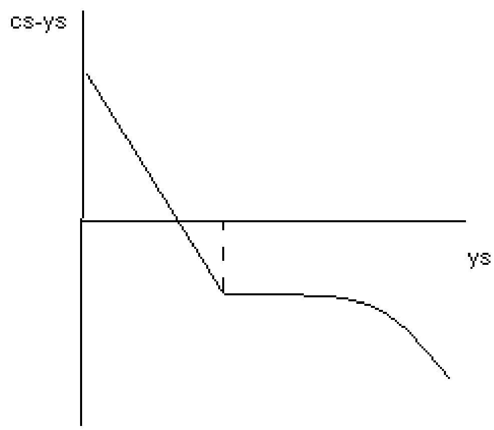


Figure 4:

In the beginning the grandmother gets a lot by guaranteeing the granddaughter a steady flow of utility, but then later she starts giving. This is because as time passes by, what she needs to promise increases because the granddaughter has good shocks (each shock increases the level of promised utility for good). In other words, the grandmother is at first a net receiver and then a net giver.

1.2 Economy with Two Sided Lack of Commitment

1.2.1 The Model

- Two brothers, A and B, and neither of them has access to a commitment technology. In other words, the two can sign a contract, but either of them can walk away if he does not feel like observing it.
- This is an endowment economy (no production) and there is no storage technology. Endowment is represented by $(y_s^A, y_s^B) \in Y \times Y$, where y_s^i is the endowment of brother i. $s=(y_s^A, y_s^B)$ follows a Markov process with transition matrix $\Gamma_{ss'}$.

1.2.2 First Best Allocation

We will derive the first best allocation by solving the social planner's problem:

$$\max_{\{c_i(h_t)\}_{\forall h_t, \forall i}} \lambda^A \sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t) u(c^A(h_t)) + \lambda^B \sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t) u(c^B(h_t))$$

subject to the resource constraint:

$$\sum_i c^i(h_t) - y^i(h_t) = 0 \qquad \forall h_t \quad \text{w/ multiplier } \gamma(h_t)$$

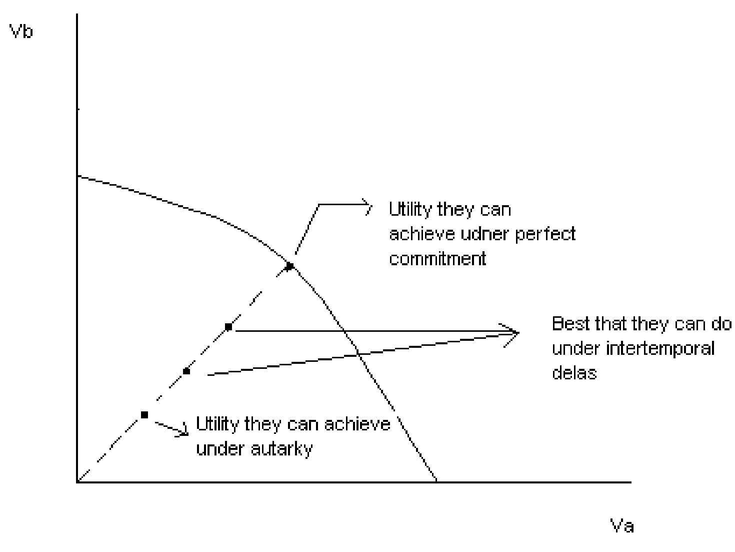
The First Order Conditions are:

$$\begin{aligned} FOC(c^A(h_t)) &: \lambda^A \beta^t \Pi(h_t) u'(c^A(h_t)) - \gamma(h_t) = 0 \\ FOC(c^B(h_t)) &: \lambda^B \beta^t \Pi(h_t) u'(c^B(h_t)) - \gamma(h_t) = 0 \end{aligned}$$

Combining these two yields:

$$\frac{\lambda^A}{\lambda^B} = \frac{u'(c^A(h_t))}{u'(c^B(h_t))}$$

Homework. *Show an implication of the above First Order Conditions under CRRA.*



The first best allocation will not be achieved if there is no access to a commitment technology. Therefore, the next thing we should do is look at the problem the planner is faced with in the case of lack of commitment. Due to lack of commitment, the planner needs to make sure that at each point in time and in every state of the world, h_t, both brothers prefer what they receive to autarky. Now we will construct the problem of the planner adding these participation constraints to his problem.

1.2.3 Constrained Optimal Allocation

The planner's problem is:

$$\max_{c^A(h_t), c^B(h_t)} \lambda^A \sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t) u(c^A(h_t)) + \lambda^B \sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t) u(c^B(h_t))$$

$$\sum_i c^i(h_t) - y^i(h_t) = 0 \qquad \forall h_t \quad \text{w/ multiplier } \gamma(h_t)$$

$$\sum_{r=t}^{\infty} \beta^{r-t} \sum_{h_r} \Pi(h_r|h_t) u(c^i(h_r)) \geq \Omega_i(h_t) \qquad \forall h_t, \forall i \quad \text{w/ multiplier } \mu_i(h_t)$$

where $\Omega_i(h_t) = \sum_{r=0}^{\infty} \beta^{r-t} \sum_{h_r} \Pi(h_r|h_t) u(y_i(h_t))$ (the autarky value)

- How many times does $c^A(h_{17})$ appear in this problem? Once in the objective function, once in the feasibility constraint, and it appears in the participation constraint from period 0 to period 16.
- We know that the feasibility constraint is always binding so that $\gamma(h_t) > 0 \quad \forall h_t$. On the other hand the same is not true for the participation constraint.
- Both participations cannot be binding but both can be nonbinding.
- Define $M_i(h_{-1}) = \lambda^i$

  and $M_i(h_t) = \mu_i(h_t) + M_i(h_{t-1})$

  (We will use these definitions for the recursive representation of the problem in the next class)

2 April 22

2.1 Recursive Representation of the Constrained SPP

We want to transform this problem into the recursive, because it would be easier to solve the optimal allocation with a computer. Now we will show how to transform the sequential problem with the participation constraints into its recursive representation.

Before we do this transformation, first recall the Lagrangian associated with the sequential representation of the social planner's problem:

$$\lambda^A \sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t) u(c^A(h_t)) + \lambda^B \sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t) u(c^B(h_t))$$
$$+ \sum_{t=0}^{\infty} \beta^t \sum_{h_t \in H_t} \Pi(h_t) \sum_{i=1}^{2} \mu_i(h_t) \left[\sum_{r=t}^{\infty} \beta^{r-t} \sum_{h_r} \Pi(h_r|h_t) u(c^i(h_r)) - \Omega_i(h_t) \right]$$
$$+ \sum_{t=0}^{\infty} \sum_{h_t \in H_t} \gamma(h_t) \left[\sum_{i=1}^{2} c_i(h_t) - \sum_{i=1}^{2} y_i(h_t) \right]$$

Note that here the Lagrangian multiplier associated witih the participation constraint for brother i after history h_t is $\beta^t \Pi(h_t) \mu_i(h_t)$.

Now we will use the definitions from the previous class (for $M_i(h_t)$) to rewrite the above Lagrangian in a more simple form,

Collect terms and rewrite,

$$\sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t) \sum_{i} \left\{ \lambda^i u(c^i(h_t) + \mu_i(h_t) \left[\sum_{r=t}^{\infty} \beta^{r-t} \sum_{h_r} \Pi(h_r|h_t) u(c^i(h_r)) - \Omega_i(h_t) \right] \right\}$$
$$+ \sum_{t=0}^{\infty} \sum_{h_t \in H_t} \gamma(h_t) \left[\sum_{i=1}^{2} c_i(h_t) - \sum_{i=1}^{2} y_i(h_t) \right]$$

Note that, $\sum_{r=t}^{\infty} \beta^{r-t} \sum_{h_r} \Pi(h_r|h_t) u(c^i(h_r)) - \Omega_i(h_t) = u(c^i(h_t)) + \sum_{r=t+1}^{\infty} \beta^{r-t} \sum_{h_r} \Pi(h_r|h_t) u(c^i(h_r)) - \Omega_i(h_t)$,

and that $\Pi(h_r|h_t)\Pi(h_t) = \Pi(h_r)$ so using these, rewrite as,

$$\sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t) \sum_{i} \left\{ \lambda^i u(c^i(h_t) + \mu_i(h_t) u(c^i(h_t)) \right\}$$
$$+ \sum_{t=0}^{\infty} \sum_{h_r} \sum_{i} \mu_i(h_t) \left[\sum_{r=t+1}^{\infty} \beta^r \sum_{h_r} \Pi(h_r) u(c^i(h_r)) - \Omega_i(h_t) \right]$$
$$+ \sum_{t=0}^{\infty} \sum_{h_t \in H_t} \gamma(h_t) \left[\sum_{i=1}^{2} c_i(h_t) - \sum_{i=1}^{2} y_i(h_t) \right]$$

Collect the terms of $u(c^i(h_r)$,

$$\sum_{t=0}^{\infty}\beta^t\sum_{h_t}\Pi(h_t)\sum_{i}\left\{\left[\lambda^i+\sum_{r=0}^{t-1}\mu_i(h_r)\right]u(c^i(h_t)+\mu_i(h_t)\left[u(c^i(h_t)-\Omega_i(h_t)\right]\right\}$$
$$+\sum_{t=0}^{\infty}\sum_{h_t\in H_t}\gamma(h_t)\left[\sum_{i=1}^{2}c_i(h_t)-\sum_{i=1}^{2}y_i(h_t)\right]$$

Introduce the variable $\mathrm{M}_i(h_t)$ and define it recursively as,

$$\begin{aligned} M_i(h_t) &= M_i(h_{t-1})+\mu_i(h_t) \\ M_i(h_{-1}) &= \lambda^i \end{aligned}$$

where $\mathrm{M}_i(h_t)$ denotes the Pareto weight plus the cumulative sum of the Lagrange multipliers on the participation constraints at all periods from 1 to t.

So rewrite the Lagrangian once again as,

$$\sum_{t=0}^{\infty}\beta^t\sum_{h_t}\Pi(h_t)\sum_{i}\left\{M_i(h_{t-1})u(c^i(h_t)+\mu_i(h_t)\left[u(c^i(h_t)-\Omega_i(h_t)\right]\right\}$$
$$+\sum_{t=0}^{\infty}\sum_{h_t\in H_t}\gamma(h_t)\left[\sum_{i=1}^{2}c_i(h_t)-\sum_{i=1}^{2}y_i(h_t)\right]$$

Now we are ready to take the First Order Conditions:

$$\frac{u'(c^A(h_t))}{u'(c^B(h_t))}=\frac{M_A(h_{t-1})+\mu_A(h_t)}{M_B(h_{t-1})+\mu_B(h_t)}$$

$$\left[\sum_{r=t}^{\infty}\beta^{r-t}\sum_{h_r}\frac{\Pi(h_r)}{\Pi(h_t)}u(c^i(h_r))-\Omega_i(h_t)\right]\mu_i(h_t)=0$$

$$\sum_{i=1}^{2}c_i(h_t)-\sum_{i=1}^{2}y_i(h_t)=0$$

Now let's characterize the solution using log utility:

Here x=$\frac{M_B}{M_A}$ so that x denotes the weight on person B. The above graph summarizes what happens to the participation constraints of each brother according to how lucky they get. You can see that, what happens to one brother affects the other, this is because if one brother gets lucky the planner needs to give him more and thus increasing his relative weight with respect to the other brother which translates into a worse deal for the other brother.

There are two kinds of periods here: Periods where nothing happens so that the ratio of the consumption of the brothers stay the same, and periods where one of them gets a good shock and his deal gets bettter whereas for the other it worsens.

Now let's look at the Pareto frontier and analyze where the solution of this problem lies:

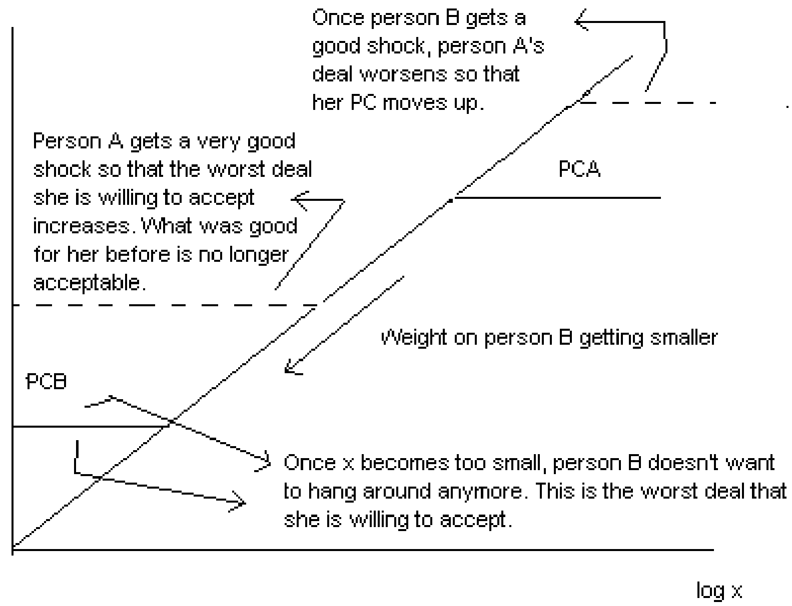


Figure 5:

2.2 Recursive Formulation

Our goal is make the problem recursive, which is very nice when we work with computer. To do this, we need to find a set of state variables which is sufficient to describe the state of the world. We are going to use x as a state variable.So the state variables are the endowment: $y = (y^A, y^B)$ and weight to brother 2: x. Define the value function as follows:

$$V = \{(V_0, V_A, V_B) \text{ such that } V_i : X \times Y \to \mathcal{R},\ i = 1, 2, V_0(x, y) = V_A(x, y) + xV_B(x, y)\}$$

What we are going find is the fixed point of the following operator (operation is defined later):

$$T(V) = \{T_0(V), T_1(V), T_2(V)\}$$

Firstly, we will ignore the participation constraints and solve the problem:

$$\max_{c_A, c_B} u(c^A(y, x)) + xu(c^B(y, x)) + \beta \sum_{y'} \Gamma_{yy'} V_0(y', x)$$

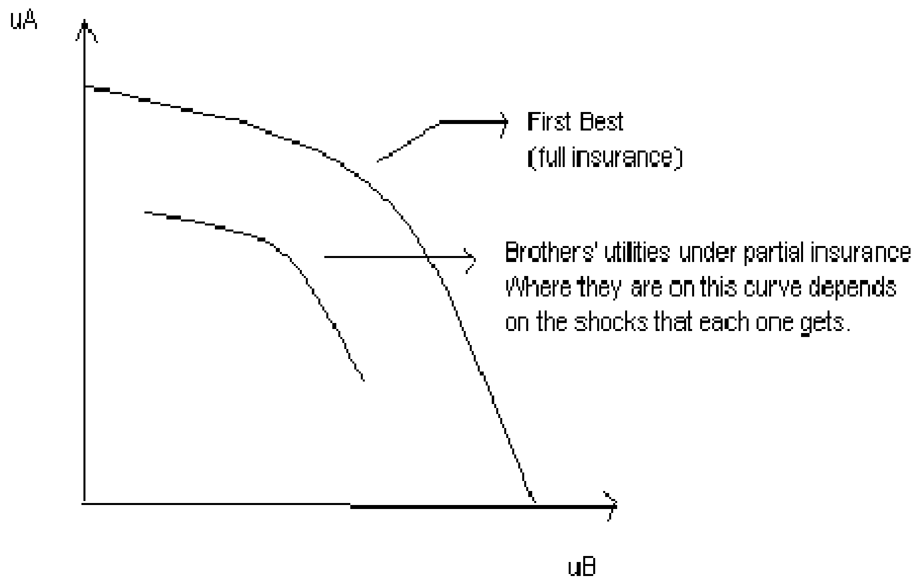


Figure 6:

subject to

$$c^A + c^B = y^A + y^B$$

First Order Conditions yield:

$$\frac{u'(c_A)}{u'(c_B)} = x$$

Second, we will check the participation constraints. There are two possibilities here:

1. Participation constraint is not binding for either 1 or 2. Then set $x(h_t) = x(h_{t-1})$. In addition,

$$\begin{aligned} V_0^N(y,x) &= V_0(y,x) \\ V_i^N(y,x) &= u(c^i(y,x)) + \beta \sum_{y'} \Gamma_{yy'} V_i(y',x) \end{aligned}$$

2. Participation constraint is not satisfied for one of the brothers (say A).

This means that agent A is getting too little. Therefore, in order for the planner to match the outside opportunity that A has, he needs to change x so that he guarantees person A the utility from going away. We need to solve the following system of equations in this case:

$$\begin{aligned} c^A + c^B &= y^A + y^B \\ u(c^A) + \beta \sum_{y'} \Gamma_{yy'} V_A(y', x) &= u(y_A) + \beta \sum_{y'} \Gamma_{yy'} \Omega_A(y') \\ x' &= \frac{u'(c_A)}{u'(c_B)} \end{aligned}$$

This is a system of three equations and three unknowns. Denote the solution to this problem by,

$$\begin{gathered} c^A(y,x) \\ c^B(y,x) \\ x'(y,x) \end{gathered}$$

So that,

$$\begin{aligned} V_0^N(y,x) &= V_A^N(y,x) + xV_B^N(y,x) \\ V_i^N(y,x) &= u(c^i(y,x)) + \beta \sum_{y'} \Gamma_{yy'} V_i(y', x'(y,x)) \end{aligned}$$

Thus we have obtained T(V)=V^N. And the next thing we need to do is find V* such that T(V*) = V^*.

Final question with this model is "how to implement this allocation?" or "Is there any equilibrium that supports this allocation?". The answer is yes. How? Think of this model as a repeated game. And define the strategy as follows: keep accepting the contract characterized here until the other guy walks away. If the other guy walks away, go to autarky forever. We can construct a Nash equilibrium by assigning this strategy to both of the brothers.

2.3 Some words on Economics

We use a model for answering two types of questions:

1. What accounts for........ (For example, why do we have GDP fluctuations, what accounts for these fluctuations? Possible explanations are that people are moody which is modelled with including shocks to preferences, or that nature is moody, or that the government is moody, etc.)
2. What if.........

What economists do is compute the model and compare that with data. But the important thing is to notice that whether the data generated from the model matches perfectly with the real data is not relevant. We are interested in which dimensions the data and the model generated data match and don't match with each other.

What is econometrics?

1. Descriptive Statistics
2. Estimation, $\widehat{\theta}$

   What econometricians do is look for parameters that will generate data that is close to real. But they assume that such parameters actually exist!

3. Truth

   Hypothesis Testing. The downside of the concept of hypothesis testing is that, it tests whether the model should be rejected or not. But all models are false. Whether they are the perfect match to the real or not is not relevant. What is relevant is whether they are useful or not and whether we are able to answer questions such as what is the impact of increasing social security benefits is good or not, etc.

3 April 23

3.1 Time Inconsistency

The problem of time inconsistency of an optimal policy is the following: The best thing to do at period t+1 when decided at period t is not the same as the best thing to do in period t+1 when decided at t+1. Recall the smoking example. You decide that your next cigarette will be your last one and that you will never smoke again. You are making this decision not anticipating the fact that after that supposedly last next cigarette, you'll want another one. Quitting after the next cigarette might be the optimal decision for you, but once that next cigarette is finished and gone and it's time to forget about smoking once and for all, you no longer want to go ahead with what you initially planned. An example that is more relevant why we economists care about time inconsistency is the following: Think of a government that has access to a commitment technology. Due to the Chamley & Judd result, we know that the optimal tax policy is to tax capital very high in the beginning and then never to tax it again. This is in order to minimize the distortionary effects that cpaital taxation has. Now think about what would happen if this government no longer had access to a commitment technology. The optimal plan which is to tax capital high in the beginning and never again after that is not time consistent in this case. This is because at the first period, the government will indeed tax capital high as prescribed by the optimal plan but then once we are in the later periods, the government will want to tax capital instead of following what its plan was. In other words, if the government is allowed to change its optimal behaviour and optimize at any history, it won't follow the initial plan. This is a problem because when we are looking for optimal fiscal policies, we need to make sure that they are time consistent, in other words that the optimal fiscal plan constitutes a subgame perfect equilibrium, so that it's the optimal thing to do after any history.

Most maximization problems are time consistent. For example, in the Solow growth model we have time consistency.

There are two different types of time inconsistency:

1. Time inconsistent due to inconsistent perferences:

   The agent's discount factor might change over time.The smoking example is a problem of time inconsistency dye to inconsistent preferences.

2. Time inconsistent due to changing constraints (consistent preferences)

   Here there is no problem with preferences, they are consistent. On the other hand, over time the constraint that the agent is subject to changes. For example, think of the two brothers 'economy. Increasing the consumption of the brothers at period 17 is good for the planner right now (at period 0), because not only does it increase the utility of the brothers at period 17 but also it relaxes the participation constraints between periods 0 to 16. But once period 16 actually comes, the planner does not care about all those benefits anymore, because it does not impute any value on relaxing teh participation constraints from the previous periods, they are already gone, relaxing them does him no good at that point in time. Thus, giving more consumption at period 17 is not good anymore for the planner, although it was good at period 0.

   The time inconsistency problem associated with the optimal fiscal policy is in this category. Although the government is benevolent, his constraints which are the first order conditions of the private sector, are changing. Therefore, the optimal thing to do for the government is changing also. So in order to find a time consistent optimal plan, we need to find such a plan that the government never wants to deviate from its prescribed strategies.

   How does the literature deal with this problem? They either ignore it, focus on trigger type of equilibria or think of the players in the next period as different guys and look for a fixed point of the policy rule (so that given that the future governments are going to follow this particular policy rule, the current government ends up actually choosing that same particular policy rule as its optimal plan).

3.2 Overlapping Generations Model

The basic differences between a pure exchange economy with infinitely lived agents and the OLG model is that in the OLG model, competitive equilibria may not be Pareto optimal and that money may have positive value.

Consider an economy where agents live for two periods. Each time period a new generation is born. Let (e^t, e^{t+1}) denote a generation's endowment and (c^t, c^{t+1}) denote their consumption in the first and second periods of their lives. Suppose $e^t = 3$ and e^{t+1}=1.

Homework. *Show that each agent chooses to consume his endowment at each period of their lives, i.e. that* $c^t = 3$ *and* $c^{t+1} = 1$.

The problem that we run into with the Pareto optimality of the equilibrium allocations in OLG is that the prices are growing too fast in so that the value of the endowments end up being not bounded.

Now consider the space $\mathcal{C}$ with only a finite number of elements that are not 0. And take $\mathcal{X} \subset \mathcal{C}$ and let the prices that support the autarky in OG be defined on this commodity space. Then we can say that $\mathcal{C} \subset AD$. This is an Arrow-Debreu equilibrium and it is Pareto optimal because the only better allocation is the young giving to the old and that is not feasible.

Now let's consider a particular OG model with different elements like marital status, gender, health, etc. in it.

Index agents by age and gender.

g: gender

z: marital status

h: health

e: education

η : love

κ : effort

The agent's value function is the following:

$$V_{i,g}(z,h,e,\eta,a,\kappa) = \max_{z' \in B^*(z), \kappa'} \quad \text{u(c,z)} + \beta_i \sum_{\eta', h'} \Gamma_{\eta' h', \eta h} V_{i+1,g}(z', h', e', \eta', a', \kappa')$$

subject to

$$ra + w(e,l) + w^*(.) = c + a'$$

4 April 24

4.1 OLG model with some detail

- We will use this model to understand what changing social security benefits does to skills. In other words, we are interested in seeing the effects of policy on human capital accumulation.

- We will abstract from endogenous mortality (people cannot affect their survival rate)
- γ_i : probability of surviving for an agent of age i.
- We will also abstract from cross sectional differences, i.e. we will not allow people of the same age group differ from each other.
- We will let agents work even if they are retired.
- l: hours worked

The individual state variables are: i (age), h (human capital), b (accumulated level of benefits) and a

The problem of the agent if retired is as follows:

$$V_i(h,a,b) = \max_{l,c,y} \text{u(c,1-l)} + \beta\gamma_i V_{i+1}(h',a',b')$$

subject to

$$\begin{aligned}
a' &= \frac{y}{\gamma_i} \quad \text{(death insurance, return of which is } \frac{1}{\gamma_i}\text{)} \\
b' &= b \quad \text{(benefits stay the same once retired)} \\
h' &= (1-\delta)h \quad \text{(Nothing to learn after retired, h decreases)} \\
c + y' &= hw(l)l + (1+r)a + l(b,i)
\end{aligned}$$

The problem of the agent if not retired is as follows:

$$V_i(h,a,b) = \max_{l,c,y} \text{u(c,1-l)} + \beta\gamma_i V_{i+1}(h',a',b')$$

subject to

$$\begin{aligned}
a' &= \frac{y}{\gamma_i} \\
b' &= \psi(b,i,h,w(l),l) \quad \text{(benefits accumulating)} \\
h' &= \chi(h,i,n) \\
c + y' &= hw(l)l(1-\tau) + (1+r)a + l(b,i)
\end{aligned}$$

Homework. *Suppose that after $81,000 , you don't pay social security. Prove that in this model nobody will ever make exactly $81,000.*