1 Feb 2

1.1 Review

- We established the equivalence between ADE and SPP. Unlike the SPP allocation, ADE can kind of tell us what happened on the world as people act optimally and in a compactible way.
- ADE exists. And ADE allocation is optimal by FBWT. SBWT tells us the any SPP allocation can be obtained from a QET. And there are three key points about QET.
  - With identical agents, transfer is zero
  - If there is a cheaper point, quasi equilibrium is a true equilibrium
  - Price may have an inner product representation given the condition in Prescott and Lucas (1972) is satisfied
- ADE definition: $\exists\ p^*$ such that

$$x^* = \arg\max_{x \in X} U(x) \tag{1}$$

subject to

$$p^*(x) \leq 0 \tag{2}$$

or

$$\sum_{t=0}^{\infty} p_{1t}^* x_{1t} + p_{2t}^* x_{2t} + p_{3t}^* x_{3t} = 0 \tag{3}$$

1.2 Prices in Arrow-Debreu Competitive Equilibrium

- How to get p^*?

  The basic intuition is from college economics: Price is equal to marginal rate of substitution. Remember, p_{1t}^* is the price of consumption good at period t in terms of consumption good at time 0.

  - Denote λ as Lagrangian multiplier associated with (3) .Normalize price of time 0 consumption to be 1. From first order condition, we can get

$$\frac{\partial U(x^*)}{\partial x_{1t}} = \lambda p_{1t}^* \text{ for } t \geq 1 \tag{4}$$

$$\frac{\partial U(x^*)}{\partial x_{10}} = \lambda \tag{5}$$

Therefore

$$p^*_{1t} = \frac{U_{1t}(x^*)}{U_{10}(x^*)} \tag{6}$$

Price of time t consumption good is equal to marginal rate of substitution between consumption at time t and consumption at time 0. ADE allocation x^* can be solved from SPP. As the functional form $U(.)$ is known, we can construct price series p^*_{1t}.

– To get price of labor service p^*_{2t} (related to wage), we have to look at the firm's problem under current setting. Consumer's problem says nothing about wage because leisure is not valued in utility function, although usually the marginal rate of substitution between consumption and leisure is a natural candidate.

– Firm's problem

$$y^*_t \in \arg\max_{y_t} p^*_{1t}y_{1t} + p^*_{2t}y_{2t} + p^*_{3t}y_{3t} \tag{7}$$

subject to

$$y_{1t} = f(y_{3t}, y_{2t}) \tag{8}$$

Denote λ_t as Lagrangian multiplier associated with (8)First order condition:

$$\begin{aligned} p^*_{1t} &= \lambda_t \\ p^*_{2t} &= -\lambda_t f_L(y^*_{3t}, y^*_{2t}) \end{aligned}$$

Thus, we can construct price p^*_{2t}. We have

$$\frac{p^*_{2t}}{p^*_{1t}} = -f_L(y^*_{3t}, y^*_{2t}) \tag{9}$$

And the wage rate at period t is price of labor service at time t in terms of time t consumption good.

– Capital service price and arbitrage.

$$\frac{p^*_{3t}}{p^*_{1t}} = -f_k(k^*_t, n^*_t) \tag{10}$$

Homework 1.1 *show (10) (note: there is no δ in this condition. And please relate it to (11))*

No Arbitrage: One freely tradable good can only have one market price. Two identical ways of transferring resource have to be priced at same level.

How people can move one unit of resources from t to t+1? There are two ways. one is to sell one unit x_{1t} at time t and get p^*_{1t}, then at time t+1, agents can get $\frac{p^*_{1t}}{p^*_{1t+1}}$ unit of consumption good x_{t+1}. The other way is to save one more unit of capital k_{1t+1}, at time t+1, agents can get rental of the additional unit of capital service and also non-depreciation part of k_{1t+1}. The relative price of doing so is $(1-\delta)+\frac{p^*_{3t+1}}{p^*_{1t+1}}$.

Hint 1.2 *The reason why we can see* $(1-\delta)$ *is the following: In ADE foc, all the arguments are* $x's$ *and* $y's$. *To relate to* k_{1t+1}, *we make use of consumption possibility space definition,* $k_{t+1}+c_t = x_{1t}+(1-\delta)k_t$ *and impose* $x_{3t} = k_t$. *Therefore, the benefit to save one more unit of capital is* $(1-\delta)+\frac{p^*_{3t+1}}{p^*_{1t+1}}$. *For details, see homework solution.*

From no arbitrage argument, we know

$$\frac{p^*_{1t}}{p^*_{1t+1}} = (1-\delta)+\frac{p^*_{3t+1}}{p^*_{1t+1}} \tag{11}$$

- In sum, we can solve SPP to get allocation of ADE, and then construct price using FOC of household and firm's problem.

1.3 Sequential Market Arrangements

- So far, rational expectation does not really apply. Agents do not need perfect forecasting as all the trades are decided at time 0. In agent's optimization problem, there is only one static constraint. At time t comes, people just execute their decision for this date.

- ADE allocation can be decentralized in different trade arrangement.

  - We will look at SME. Two things are important here: (i) Allocation in equilibrium with sequential market arrangements cannot Pareto-optimally dominate ADE allocation. (ii) But with various market arrangement, the allocation may be worse than ADE allocation. Say, labor market can be shut down or other arrangement to make people buy and trade as bad as it happens. (This topic is about endogenous theory of market institution).

- Sequence of Markets.

  With sequential markets, people have capital k_t, and rent it to the firm at rental $(1+r_t)$. People have time 1 and rent it to firm at wage w_t. They also consume c_t and save k_{t+1}. Agents can also borrow and lending one period loan l_{t+1} at price q_t. Then the budget constraint at time t is

$$k_t(1+r_t)+w_t+l_t=c_t+k_{t+1}+q_tl_{t+1} \tag{12}$$

- Principle to choose market structure: Enough but not too many.

  There are many ways of arranging markets so that the equilibrium allocation is equivalent to that in ADE, as we'll see. ENOUGH: Note that if the number of markets open is too few, we cannot achieve the allocation in the ADE (incomplete market). Therefore, we need enough markets to do as well as possible. NOT TOO MANY: To the contrary, if the number of markets are too many, we can close some of the markets and still achieve the ADE allocation in this market arrangement. Also it means that there are many ways to achieve ADE allocation because some of the market instruments are redundant and can be substituted by others. If the number of markets are not TOO FEW nor TOO MANY, we call it JUST RIGHT.

With the above structure, loans market is redundant as there is only one representative agent in this economy. In equilibrium, $l_t = 0$. So, we can choose to close loans market. We will see that even though there is no trade in certain markets in equilibrium, we can solve for prices in those markets, because prices are determined even though there is no trade in equilibrium, and agents do not care if actually trade occurs or not because they just look at prices in the market (having market means agents do not care about the rest of the world but the prices in the market). Using this technique, we can determine prices of all market instruments even though they are redundant in equilibrium. This is the virtue of Lucas Tree Model and this is the fundamental for all finance literature (actually, we can price any kinds of financial instruments in this way. we will see this soon.)

2 Feb 6

2.1 From ADE to SME

- We have seen that we need enough markets to get optimal allocation with sequential market arrangement. Market structure depends on commodity space. There should be markets to trade consumption good, capital services and labor services. Using consumption good at time t as the numarie and consider relative price, we can see that only 2 markets are needed. To transfer resources, only one intertemporal market is efficient since two ways of trade are equivalent, which are to trade c_t with c_{t+1} and to trade k_t and k_{t+1}. The reason why we can normalize price of c_t to 1 at any t is that with infinite horizon, future is identical at any time, as in the way we write Bellman equation.

- As we know from last class, we can write budget constraint with loans.

$$k_t\left(1+r_t\right)+w_t+l_t=c_t+k_{t+1}+\frac{l_{t+1}}{R_{t+1}}$$

where r_t =rental price of capital and R_t = price of IOUs. But we can close the market of loans without changing the resulting allocation. This is because we need someone to lend you loans in order that you borrow loans, but there is only one agents in the economy. Then, the budget constraint becomes

$$k_t\left(1+r_t\right)+w_t=c_t+k_{t+1} \tag{13}$$

2.2 Define SME

- Long definition first:

Consumer's problem is

$$\max_{x_{1t},x_{2t},x_{3t},k_{t+1},c_t,l_t}\sum_{t=0}^{\infty}\beta^t u\left(c_t\right) \tag{14}$$

subject to

$$\frac{l_{t+1}}{R_{t+1}}+x_{1t}=r_t x_{3t}+w_t x_{2t}+l_t \quad \forall t \tag{15}$$

The producer's problem is for all $t=0,1,2,...$

$$\max_{\{y_t\}}\{y_{1t}-w_t y_{2t}-r_t y_{3t}\} \tag{16}$$

subject to

$$y_{1t}\leq F(y_{3t},y_{2t})$$

Definition 2.1 *A Sequential Market Equilibrium (SME) is* $\{x_{1t}^*,x_{2t}^*,x_{3t}^*,l_t^*\}\ \{r_t^*,R_t^*,w_t^*\}\ \{y_{1t}^*,y_{2t}^*,y_{3t}^*\}_{t=0}^{\infty}$, *and there exists* $\{c_t^*,k_{t+1}^*\}_{t=0}^{\infty}$ *such that (i) consumer maximizes: given* $\{\tilde{r}_t,\tilde{w}_t\}_{t=0}^{\infty}$, $\{x_{1t}^*,x_{2t}^*,x_{3t}^*,c_t^*,k_{t+1}^*,l_t^*\}$ *solves optimization problem. (ii) firm maximize: given* $\{\tilde{r}_t,\tilde{w}_t\}_{t=0}^{\infty}$, $\{y_{1t}^*,y_{2t}^*,y_{3t}^*\}$ *solves the producer problem. (iii) markets clear:* $x_{it}^*=y_{it}^*$, $\forall i,t$ *and* $l_{t+1}^*=0$.

- There is a short way to write SME.

  First, let's look at the properties of SME:

  1) As all the solutions are interior and k_{t+1} or l_{t+1} cannot go to infinity in equilibrium, $R_{t+1} = (1-\delta) + r_{t+1}$.

  2) In equilibrium $x_{2t}^* = 1$ and $x_{3t}^* = k_t^*$.

  Consumer's Problem in SME can be written as follows:

  $$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t, n_t) \tag{17}$$

  subject to

  $$c_t + k_{t+1} = w_t + [(1-\delta) + r_t]k_t \qquad \forall t = 0, 1, 2, ... \tag{18}$$

  $$k_0 \text{ is given} \tag{19}$$

Definition 2.2 *A Sequential Market Equilibrium (SME) is $\{c_t^*, k_{t+1}^*\}$ $\{r_t^*, w_t^*\}_{t=0}^{\infty}$, such that*

(1) consumer maximizes: given $\{r_t^*, w_t^*\}_{t=0}^{\infty}$ $\{c_t^*, k_{t+1}^*\}$ solves optimization problem.

(2)

$$\left(c_t^* + k_{t+1}^* - (1-\delta)\, k_t^*, 1, k_t^*\right) \in \arg\max y_{1t} - w_t^* y_{2t} - r_t^* y_{3t} \tag{20}$$

subject to

$$y_{1t} = F\left(y_{3t}, y_{2t}\right)$$

- – FOC to problem 20:

  $$\begin{aligned} 1 &= \lambda_t \\ w_t &= \lambda_t F_L\left(k_t^*, 1\right) \\ r_t &= \lambda_t F_k\left(k_t^*, 1\right) \end{aligned}$$

  then

  $$\begin{aligned} w_t &= F_L\left(k_t^*, 1\right) \\ r_t &= F_k\left(k_t^*, 1\right) \end{aligned}$$

  – (2) in the above definition can be substituted with (2')

  $$\begin{aligned} w_t &= F_L\left(k_t^*, 1\right) \\ r_t &= F_k\left(k_t^*, 1\right) \\ c_t^* + k_{t+1}^* - (1-\delta)\, k_t^* &= F\left(k_t^*, 1\right) \end{aligned}$$

Homework 2.3 *show (2')*$\Rightarrow$ (2)

Homework 2.4 *Explain the implication of CRS (constant return to scale) assumption on firms.*

- Another way of writing SME is:

A Sequential Market Equilibrium (SME) is $\{c_t^*, k_{t+1}^*\}$ $\{r_t^*, w_t^*\}_{t=0}^{\infty}$, such that

Definition 2.5 *(1) consumer maximizes (2') factor prices equal to marginal productivity (3) allocation is feasible.*

2.3 Compare ADE and SME

- Show the equivalence of ADE and SME:

Theorem 2.6 *If* $\{x^*, y^*, q^*\} \in AD(\mathcal{E})$*, then, there exists* $\{c_t^*, k_{t+1}^*, r_t^*, w_t^*\}_{t=0}^{\infty} \in SME(\mathcal{E})$

Proof: Pick the $\{c_t^*, k_{t+1}^*\}$ implied by consumer's problem in ADE. Define the following price:

$$
\begin{aligned}
q^*\left(\{\{0,1,0\},\{0,0,0\},...\}\right) &= w_0^* && \text{wage at time 0} \\
&\ ... \\
q^*\left(\{\{0,0,0\},...\{0,1,0\},...\}\right) &= w_t^* && \text{wage at time t} \\
&\ ...
\end{aligned}
$$

$$
\begin{aligned}
q^*\left(\{\{0,0,1\},\{0,0,0\},...\}\right) &= r_0^* && \text{rental at time 0} \\
&\ ... \\
q^*\left(\{\{0,0,0\},...\{0,0,1\},...\}\right) &= r_t^* && \text{wage at time t} \\
&\ ...
\end{aligned}
$$

Thus, we have constructed $\{r_t^*, w_t^*\}_{t=0}^{\infty}$. Next, we need to verify the condition for SME.