1 Mar 4

1.1 Review

- Stochastic RCE with and without state-contingent asset

  Consider the economy with shock to production. People are allowed to purchase state-contingent asset for next period.

Consumer's problem is

$$V(z, K, a; G, q_z) = \max_{c, a'(z')} \{u(c) + \beta \sum_{z'} \Gamma_{zz'} V(z', K', a'(z'); G, q_{z'})\} \tag{1}$$

subject to

$$\begin{aligned} c + \sum_{z'} q_{z'}(z, K)\, a'_{z'} &= [1 - \delta + r(z, K)]a + w(z, K) & (2) \\ r(z, K) &= z f_1(K, H(K)) & (3) \\ w(z, K) &= z f_2(K, H(K)) & (4) \\ K' &= G(z, K) & (5) \end{aligned}$$

Essentially, we can get Euler equation:

$$u_c(c) = \beta \sum_{z'} \Gamma_{zz'} [1 - \delta + z f_1(K, 1)] u_{c'}(c') \tag{6}$$

This condition is what we see more in macro literature. But the consumer's problem we have above is a long-hand version.

To derive it, we use

FOC:

$$\begin{aligned} \frac{\partial}{\partial c} &: \quad u_c(c(z, K, a)) = \lambda \\ \frac{\partial}{\partial a'_{z'}} &: \quad \Gamma_{zz'} V_3(z', K', a'(z')) = \lambda q_{z'}(z, K) \end{aligned}$$

Envelope condition:

$$V_3 = [1 - \delta + r(z', K')] u_c$$

Thus, we have

$$q_{z'}(z,K)\,u_c(c) = \beta\Gamma_{zz'}[1-\delta+zf_1(K,1)]u_{c'}(c') \tag{7}$$

Add over z' and use NA condition

$$\sum_{z'} q_{z'}(z,K) = 1$$

and substitute consistency condition

$$a = K$$

We will get (6).

As we see in the homework, the equilibrium of this economy with a complete market can be found in economy without complete market. The reason is that state-contingent asset price $q_{z'}(z,K)$ is adjusted in the way such that agents save the same amount independent of z'.

- Wealth distribution in economy with heterogenous agents

  Assume there are I types of agents, there are $2I$ necessary conditions for equilibrium allocation:

I budget constraint equations:

$$c^i + a^{i\prime} = w + a^i(1+r-\delta) \tag{8}$$

I FOC conditions:

$$u_c^i = \beta(1+r-\delta)\,u'_{c^{i\prime}} \tag{9}$$

And there are $2I$ unknowns $\{c^i, a^i\}$ in steady state. But in steady state, the I FOC degenerate to the same one

$$1 = \beta(1+r-\delta)$$

$$f_k\left(\sum_i a^i\right) = \frac{1}{\beta} - (1-\delta) \tag{10}$$

Therefore, the model says nothing about wealth distribution.

If the economy starts with $f_1\left(\sum_i a^i, 1\right) = \frac{1}{\beta} - (1-\delta)$, then wealth ranking stays. If not, asset holding of different types will move parallel toward steady state level.

1.2 Finance

We will study Lucas Tree Model (Lucas 1978[1]) and look at the things that Finance people talk about. Lucas tree model is a simple but powerful model.

1.2.1 The Model

Suppose there is a tree which produces random amount of fruits every period. We can think of these fruits as dividends and use d_t to denote the stochastic process of fruits production. $d_t \in \left\{d^1, ...d^{nd}\right\}$. Further, assume d_t follows Markov process. Formally:

$$d_t \sim \Gamma(d_{t+1} = d_i \mid d_t = d_j) = \Gamma_{ji} \tag{11}$$

Let h_t be the history of realization of shocks, i.e., $h_t = (d_0,\ d_1,\ ...,\ d_t)$. Probability that certain history h_t occurs is $\pi(h_t)$.

Household in the economy consumes the only good, which is fruit. We assume representative agent in the economy, and there is no storage technology. In an equilibrium, the first optimal allocation is that the representative household eats all the dividends every period. We will look at what the price has to be when agents use markets and start to trade. First, we study the Arrow-Debreu world. And then, we use sequential markets to price all kinds of derivatives, where assets are entitlement to consumption upon certain date-event.

1.2.2 Arrow-Debreu World

Consumers;s problem

$$\max_{\{c(h_t)\}_{t=0}^{\infty}} \sum_t \beta^t \sum_{h_t \in H_t} \pi(h_t) u(c_t(h_t)) \tag{12}$$

subject to

$$\sum_t \sum_{h_t \in H_t} p(h_t) c_t(h_t) = \overline{a} = \sum_t \sum_{h_t \in H_t} p(h_t) d_t(h_t) \tag{13}$$

Equilibrium allocation is autarky

$$c_t(h_t) = d_t(h_t) \tag{14}$$

Now the key thing is to find the price which can support such equilibrium allocation.

[1] Lucas, R. (1978). "Asset prices in an exchange economy." Econometrica 46: 1429-1445

Normalize

$p(h_0) = 1$

Take first order condition of the above maximization problem and also substitute (14)

FOC

$$\begin{aligned} \beta^t \pi(h_t) u_c(d_t(h_t)) &= p_t(h_t)\lambda && (15) \\ u_c(d_0) &= \lambda && (16) \end{aligned}$$

We get the expression for the price of the state contingent claim in the Arrow-Debreu market arrangement.

$$p_t(h_t) = \frac{\beta^t \pi(h_t) u_c(d_t(h_t))}{u_c(d_0)} \quad (17)$$

Note that the price $p_t(h_t)$ is in terms of time 0 consumption.

1.2.3 Sequences of Markets

In sequential market, we can think of stock market where the tree is the asset. Household can buy and sell the asset. Let s_t be share of asset and q_t be the asset price at period t. The budget constraint at every time-event is then:

$$qs' + c = s(q + d) \quad (18)$$

First, we can think of any financial instruments and use the A-D prices $p_t(h_t)$ to price them.

1. The value of the tree in terms of time 0 consumption is indeed

$$\sum_t \sum_{h_t \in H_t} p(h_t) d_t(h_t)$$

2. A contract that gives agent the tree in period 3 and get it back in period 4: This contract is worth the same as price of harvests in period 3:

$$\sum_{h_3 \in H_3} p(h_3) d_3(h_3)$$

3. Price of 3-year bond: 3 year bond gives agents 1 unit of good at period 3 with any kinds of history. The price is thus

$$\sum_{h_3 \in H_3} p(h_3)$$

1.2.4 Market Equilibrium

We will write it in a recursive form. Then, first can we get rid of h_t and write it in a recursive form? Or are prices stationary? The answer depends on whether the stochastic process is stationary.

Homework 1.1 *Show prices q are stationary (only indexed by z)*

Note: In order to be consistent with notation we have with stochastic economy, we are using z to denote the stochastic process of dividend. z process is the dividend process since this is the only random factor in the economy.

Now the consumer's optimization problem turns out to be:

$$V(z,s) = \max_{c,s'} u(c) + \beta \sum_{z'} \Gamma_{zz'} V(z',s') \tag{19}$$

subject to

$$c + s'q(z) = s[q(z) + d(z)] \tag{20}$$

To solve the problem,

FOC:

$$\begin{aligned} u_c(z) &= \lambda_z \\ \beta \sum_{z'} \Gamma_{zz'}[q(z') + d(z')]\lambda_{z'} &= \lambda_z q(z) \end{aligned}$$

So, we get $\forall z$,

$$u_c(z)\, q(z) = \beta \sum_{z'} \Gamma_{zz'}[q(z') + d(z')]u_c(z')$$

We write out the whole system of equation for all possible z,

$$\left\{ \begin{array}{c} u_c(z^1)\, q(z^1) = \beta \sum_{z'} \Gamma_{zz'}[q(z') + d(z')]u_c(z') \\ \cdots \\ u_c(z^{nz})\, q(z^{nz}) = \beta \sum_{z'} \Gamma_{zz'}[q(z') + d(z')]u_c(z') \end{array} \right. \tag{21}$$

Elements in (21) are marginal utility of consumption in different states and dividends, which are numbers, and price $q's$. Therefore, it is system of linear equations in $q's$. And there are nz linear equations and nz unknowns. We can then solve this system and obtain prices in sequential markets.

Remark: There is one equity premium puzzle in finance. This puzzle basically says that standard representative agent neoclassical growth model with CRRA utility function with "normal" parameter values fails to explain the huge difference between risky stock returns and riskless bond in US.