1 March 18

1.1 Review

- Last class, we introduced the Lucas (1978) model. There is a tree and the tree yields a random number of fruits at each period. There is a representative agent. In the previous class, we said that the equilibrium has to be such that $c_t = d_t$ since markets clear only if the consumption of the agent equals to her endowment due to the fact that we have a representative agent. Then, we were able to compute the prices that will support this allocation as the solution to the agent's maximization problem. Thus we got the prices that will induce the agent to consume all he has at each period. Then we set up a problem where the agent is able to trade at each period and let him maximize by choosing his consumption, c_t, and the amount of share of the tree to buy, s_{t+1}. From the solution to the agent's problem in the sequence of markets structure, we were able to characterize the price of a share of the tree at each node, $q(h_t)$. We derived these prices by deriving the FOC from consumer's problem and imposing the equilibrium conditions on them. Today we see more on the characterization of these prices and we go on to asset pricing using these tools.

1.2 Asset Pricing

- What is the state of the economy?

  It's the number of fruits from the tree (d). The dividend is the aggregate state variable in this economy.

- s, the share that the consumer has today, is the individual state variable.

- Consumer's Problem:

$$V(d,s) = \max_{s',c} \text{u(c)} + \beta \sum_{d'} \Gamma_{dd'} V(d',s')$$

$$s.t. \quad c + s'q(d) = s[q(d) + d]$$

In equilibrium, the solution has to be such that c=d and $s' = 1$. Impose these on the FOC and get the prices that induce the agent to choose that particular allocation.

Homework. *Show that the prices, $\{q_i\}_{i=1}^{I}$ are characterized by the following system of equations:*

$$u'(d_i) = \beta \sum_j \Gamma_{ij} u'(d_j) \frac{[q_j + d_j]}{q_i} \quad \forall i$$

Now let's write the above system of equations in the matrix form so we can write a closed form for q.

$$\begin{bmatrix} q_1 \\ .. \\ .. \\ q_I \end{bmatrix} = \begin{bmatrix} \beta\Gamma_{11}\frac{u'(d_1)}{u'(d_1)} & \beta\Gamma_{12}\frac{u'(d_2)}{u'(d_1)} & .. & \beta\Gamma_{1J}\frac{u'(d_J)}{u'(d_1)} \\ .. & \beta\Gamma_{22}\frac{u'(d_2)}{u'(d_2)} & .. & .. \\ .. & .. & .. & .. \\ \beta\Gamma_{I1}\frac{u'(d_1)}{u'(d_I)} & .. & .. & \beta\Gamma_{IJ}\frac{u'(d_J)}{u'(d_I)} \end{bmatrix} \left[\begin{bmatrix} q_1 \\ .. \\ .. \\ q_I \end{bmatrix} + \begin{bmatrix} d_1 \\ .. \\ .. \\ d_I \end{bmatrix} \right]$$

Let q= $\begin{bmatrix} q_1 \\ .. \\ .. \\ q_I \end{bmatrix}$ and d= $\begin{bmatrix} d_1 \\ .. \\ .. \\ d_I \end{bmatrix}$ and A= $\begin{bmatrix} \beta\Gamma_{11}\frac{u'(d_1)}{u'(d_1)} & \beta\Gamma_{12}\frac{u'(d_2)}{u'(d_1)} & .. & \beta\Gamma_{1J}\frac{u'(d_J)}{u'(d_1)} \\ .. & \beta\Gamma_{22}\frac{u'(d_2)}{u'(d_2)} & .. & .. \\ .. & .. & .. & .. \\ \beta\Gamma_{I1}\frac{u'(d_1)}{u'(d_I)} & .. & .. & \beta\Gamma_{IJ}\frac{u'(d_J)}{u'(d_I)} \end{bmatrix}$

and let b=Ad
we have,

$$q = Aq + b$$

so that,

$$q = (I - A)^{-1}b$$

- Now consider the same problem but now the agent can also buy bonds for the price of p(d) which entitles him to get 1 unit of the good the next period. The agent's new budget constraint is:

$$b'p(d) + c + s'q(d) = s[q(d) + d] + b$$

The equilibrium quantity of bonds is 0 because there is noone to buy those bonds from or sell them to. Now using this fact, we will find the p(d) that induces the agent to choose to buy 0 bonds.

$$p_i u'(d_i) = \beta \sum_j \Gamma_{ij} u'(d_j) \quad \forall i$$

The price of a bond, p(d), is characterized by the above set of equations.

You can see the pattern here: We can choose any kind of asset and then price it in the same way.

1.2.1 Options

Definition 1 *An option is an asset that gives you the right to buy a share at a prespecified price if you choose.*

In general, to price any kind of assets and options, we only need to know prices of consumption at each node.

The price of an option is a sum of gain under the option at each node (considering the decision of whether to exercise the option or not), multiplied by the price of consumption at each node. In order to write an expression for the price of an option at a certain node (q(h_t)), we need to first compute the price of a unit of consumption good at node h_{t+1} in terms of units of consumption goods at node h_t. One way to do this is introduce state contingent claims to the problem of the agent (y'(d')) and compute its price $p_{dd'}$.

Let's say we want to price options at node h_t. Assume that the set of the possible aggregate shock contains three elements. Start from h_t, possible nodes in the next period are h^1_{t+1} and h^2_{t+1} and h^3_{t+1}. Now denote the price of consumption at node h^1_{t+1} in terms of units of consumption at node h_t by p^y_{i1}. We want to see what p^y_{i1} is, as well as p^y_{i2}, p^y_{i3}. In order to do this we use state contingent claims because the price of a state contingent claim at node h_t that entitles the agent to get 1 unit of consumption good at node h^1_{t+1} is p^y_{i1}. So if we write down the problem of the agent with these state contingent claims and compute the prices of these state contingent claims, we'll get exactly what we need: The price of consumption at node h^1_{t+1} (or h^2_{t+1} or h^3_{t+1}) in terms of units of consumption good at t.

Once we add the state contingent claims, the agent's problem becomes the following:

$$V(d,s,b,y) = \max_{c,s',b',y'(d')} \text{u(c)} + \beta \sum_{d'} \Gamma_{dd'} V(d',s',b',y'(d'))$$

$$s.t. \sum_{d'} y'(d') p^y_{dd'} + b'p(d) + c + s'q(d) = s[q(d)+d] + b + y(d)$$

Homework *Show that the expression for the price of the state contingent claims, p^y_{ij}* is as follows:

$$p^y_{ij} = \beta \Gamma_{ij} \frac{u'(d_j)}{u'(d_i)}$$

An option that you buy at period t, entitles you to max$\{0,$q$(h^1_{t+1}) - \overline{q}\}$ at node h^1_{t+1},max$\{0,$q$(h^2_{t+1}) - \overline{q}\}$ at node h^2_{t+1}, and max$\{0,$q$(h^3_{t+1}) - \overline{q}\}$ at node h^3_{t+1}. And therefore, all we need to do to compute its price is multiply what it entitles the agent at each node by the price of consumption at that node in terms of consumption at t (which we now know because we already have an expression for p^y_{ij}) and sum over all the nodes:

$$q_i^0(\overline{q}) = \sum_j max\{0, q_j - \overline{q}\} p_{ij}^y$$

Homework *(i) Price a two period option that can be exercised any time before its maturity (i.e. if you buy the option today, it can be exercised either tomorrow or the day after)*

(ii) Price a two period option that can be exercised only at its maturity.

Homework *Come up with an asset and price it.*

Homework *Find the formula that relates the price of the bond to the price of the state contingent claims in the above problem (i.e. p(d) to* $p_{dd'}^y$*).*

Homework *Give a formula for q(d) in terms of* $p_{dd'}^y$*.*

1.3 RCE with Government

When we are considering RCE with government, there are several issues that we need to consider before we begin writing down the problem of the agent and the government's budget constraint. We need to make some choices about the economy that we are modelling:

Can the government issue debt? If no: The government is restricted by his period by period budget constraint. He cannot run a budget deficit or a surplus. Whatever he gets as revenues from taxes, he spends that and no more or no less. His budget constraint needs to hold at each period, the government cannot borrow from the public. Here the government expenditures are exactly equal to the tax revenues. If yes: The government can issue bonds at each period When it turns out that the government's expenditures are higher than its revenues (the tax revenues) he can issue debt or when it turns out that the expenditures are less than his revenues, he can retire debt that was issued before. If we model the economy so that the government is allowed to issue debt, then we need to deal with the issue of restricting it to accumulate public debt indefinitely. This is where the No-Ponzi Scheme comes into play. But we'll have more on that later.

We will do the second case today. So the economy is as follows:

- Government issues debt and raises tax revenues to pay for a constant stream of expenditures.
- Government debt is issued at face value with a strem of interest rate $\{r_{b,t}\}$
- No shocks.
- No labor/leisure choice.

Let's write down the problem of the consumer. Notice that the consumer can transfer resources across time in two ways here: He can either save in the form of capital or he can buy bonds. In equilibrium, the rates of return on both ways of saving should be the same by no arbitrage so that r$_b = r_k$. Since the rates of return on both is the same, the agent shouldn't care in what form he saves, i.e. the composition of the asset portfolio doesn't matter. So let a denote the agent's asset which consists of physical capital holding k and financial asset b. We don't need to make the distinction between the two. And let r denote the rate of return on a (which is in turn the rate of return on capital and rate of return on bonds).

Aggregate state variables are K and B where B is the government debt. Notice that G, the government expenditures, is not a state variable here since it's constant across time. The individual state variable is a, the consumer's asset holdings.

The consumer's problem:

$$V(K,B,a) = \max_{c,a'} \text{ u(c)} + \beta V(K',B',a')$$

subject to

$$\begin{aligned} c + a' &= a + [ra + w](1-\tau) \\ r &= r(K) = f_k(K,1) - \delta \\ w &= w(K) = f_n(K,1) \\ K' &= H(K,B) \\ B' &= \Psi(K,B) \\ \tau &= \tau(K,B) \end{aligned}$$

Solution is: a'=$\psi(K,B,a)$

Definition 2 *Given $\tau(K,B)$, a RCE is a set of functions $\{V(.),\psi(.),H(.),\Psi(.)\}$ such that*

1. *(Household's optimization) Given $\{H(.),\Psi(.)\}, \{V(.),\psi(.)\}$ solve the household's problem.*
2. *(Consistency) $H(K,B) + \Psi(K,B) = \psi(K,B,K+B)$*
3. *(No Arbitrage Condition)*

$$r_b(K,B) = 1 + F_K(H(K,B),1) - \delta$$

(The rate of return on bond is equal to the rate of return on capital; notice we already used this fact when we were writing down the problem of the consumer by letting r denote the rate of return on both and not distinguishing between them)

4. *(Government Budget Constraint)*

$$\Psi(K,B) + [f(K,1) - \delta K + (f_k(K,1)-\delta)B](1-\tau(K,B)) = \bar{G} + B[1 + f_k(K,1) - \delta]$$

(So that the government's resources each period are the bonds that it issues ($\Psi(K,B)$), plus its revenues from tax on rental income, wage income, and income on the interest on bonds. Its uses are the government expenditures (G), and the debt that it pays back.)

5. *(No Ponzi Scheme Condition)* $\exists \underline{B}, \overline{B}, \underline{K}, \overline{K}$ *such that* $\forall K, B \in [\underline{K}, \overline{K}]x[\underline{B}, \overline{B}]$

$$\Psi(K,B) \in [\underline{B}, \bar{B}]$$

Homework *Consider an economy with two countries indexed by i*$\in \{A, B\}$. *Each country is populated by a continuum of infintely lived identical agents that is taken to be of measure one. Each country has a production function* $f^i(K^i, N^i)$ *(different technologies across countries). Assume that output and capital can be transferred between countries at no cost. But labor cannot move across countries.Define recursive equilibria for this economy.*

2 March 20: Measure Theory

- We will use measure theory as a tool to describe a society with heterogenous agents. Most of the previous models that we dealt with, there was a continuum of identical agents, so we saw the economy as consisting of only one type of agent.But from now on in the models that we deal with there will be heterogenous agents in the economy. The agents will differ in various ways: in their preferences, in the shocks they get, etc. Therefore, the decisions they make will differ also. In order to describe such a society, we need to be able to keep track of each type of agent. We use measure theory to do that.
- What is measure?

  Measure is a way to describe society without having to keep track of names. But before we define measure, there are several definitions we need to learn.

Definition 3 *For a set A,* $\mathcal{A}$ *is a set of subsets of A.*

Definition 4 $\sigma - a\lg ebra$ $\mathcal{A}$ *is a set of subsets of A, such that,*

1. $A, \emptyset \in \mathcal{A}$
2. $B \in \mathcal{A} \Rightarrow B^c/A \in \mathcal{A}$ *(closed in complementarity)*
   where $B^c/A = \{a \in A : a \notin B\}$
3. *for* $\{B_i\}_{i=1,2...}$, $B_i \in \mathcal{A} \Rightarrow [\cap_i B_i] \in \mathcal{A}$ *(closed in countable intersections)*

- An example for the third property: Think of a σ-algebra defined on the set of people in a classroom. The property of being closed in countable intersections says that if the set of tall people is in the σ-algebra, and the set of women is in the σ-algebra, then the set of tall women should be in the σ-algebra also.

- Consider the set A={1,2,3,4} Here are some examples of σ-algebras defined on the set A:

$\mathcal{A}^1= \{\emptyset, A\}$
$\mathcal{A}^2 = \{\emptyset, A, \{1\}, \{2,3,4\}\}$
$\mathcal{A}^3 = \{\emptyset, A, \{1\}, \{2\}, \{2,3,4\}, \{1,3,4\}, \{3,4\}, \{1,2\}\}$
$\mathcal{A}^4 =$ The set of all subsets of A.

Remark 5 *A topology is a set of subsets of a set also, just like a σ-algebra. But the elements of a topology are open intervals and it does not satisfy the property of closedness in complementality (since a complement of an element is not an element of the topology). Therefore topology is not a σ-algebra.*

Remark 6 *Topologies and Borel sets are also family of sets but we use them to deal with continuity, and σ-algebra we use to deal with weight.*

- Think of the σ-algebras defined on the set A above. Which one provides us with the least amount of information? It is $\mathcal{A}^1$. Why? Because from $\mathcal{A}^1$, we only know whether an element is in the set A or not. Think of the example of the classroom. From $\mathcal{A}^1$,all we get to is whether a person is in that classroom or not, we learn nothing about the tall people, short people, males, females, etc. The more sets there are in a σ-algebra, the more information we have.This is where the Borel sets are useful. A Borel set is a σ-algebra which is generated by a family of open sets. Since Borel σ-algebra contains all the subsets generated by intervals, you can recognize any subset of set, using Borel σ-algebra. In other words, Borel σ-algebra corresponds to complete information.

- Now we are ready to define measure:

Definition 7 *A measure is a function $x : \mathcal{A} \to \mathcal{R}_+$ such that*

1. $x(\emptyset) = 0$
2. *if $B_1, B_2 \in \mathcal{A}$ and $B_1 \cap B_2 = \emptyset \Rightarrow x(B_1 \cup B_2) = x(B_1) + x(B_2)$ (finite additivity)*
3. *if $\{B_i\}_{i=1}^{\infty} \in \mathcal{A}$ and $B_i \cap B_j = \emptyset$ for all $i \neq j \Rightarrow x(\cup_i B_i) = \sum_i x(B_i)$ (countable additivity)*

Definition 8 *Probability (measure) is a measure such that $x(A) = 1$*

Homework *Show that the space of measures over the interval [0,1] is not a topological vector space.*
Homework *Show that the space of sign measures is a topological vector space.*
Homework *Show that the countable union of elements of a $\sigma - a\lg ebra$ is also element of the $\sigma - a\lg ebra$.*

- Consider the set A=[0,1] where a $\in A$ denotes wealth. So A is the set of wealth levels normalized to 1.
- We will define x: $\mathcal{A} \rightarrow \mathcal{R}_+$ as a probability measure so that the total population is normalized to one. Using measure, we can represent various statistics in a simple form:

1. The total population:

$$\int_A dx = x(A) = 1$$

2. Average wealth:

$$\int_A adx$$

   We go through each levels of wealth in the economy and multiply the wealth level by the proportion of people that have that wealth level, and because the size of the society is normalized to 1, this gives us average wealth.

3. Variance of wealth:

$$\int_A [a - \int_A adx]^2 dx$$

4. Coefficient of variation:

$$\frac{\{\int_A [a - \int_A adx]^2 dx\}^{1/2}}{\int_A adx}$$

5. Wealth level that seperates the 1% richest and the poorest 99% is $\widetilde{a}$ that solves the following equation:

$$0.99 = \int_A 1_{[a \geq \widetilde{a}]} dx$$

Homework *Write the expression for the Gini index.*

Remark 9 *Notation:*

$$\int_A 1_{[a \leq \widetilde{a}]} dx = x([0, \widetilde{a}]) = \int_A 1_{[a \leq \widetilde{a}]} x(da)$$

2.1 Introduction to the Economy with Heterogenous Agents

Imagine a Archipelago that has a continuum of islands. There is a fisherman on each island. The fishermen get an endowment s each period. s follows a Markov process with transition $\Gamma_{ss'}$ and,

s$\in \{s^1, ..., s^{n_s}\}$

The fishermen cannot swim. There is a storage technology such that, if the fishermen store q units of fish today, they get 1 unit of fish tomorrow. The problem of the fisherman is:

$$V(s,a) = \max_{c,a'} \text{u(c)} + \beta \sum_{s'} \Gamma_{ss'} V(s', a')$$

$$s.t. c + qa\prime = s + a$$

$$c, a' \succeq 0$$

Homework *Can we apply the Contraction Mapping Theorem to this problem?*

3 March 25: Economy with Heterogenous Agents

3.1 Measure Theory (continued)

- Consider the set A={1,2,3,4}and the following $\sigma - a\lg ebras$ defined on it:

$\mathcal{A}^1 = \{\emptyset, A\}$
$\mathcal{A}^2 = \{\emptyset, A, \{1,2\}, \{3,4\}\}$

- Remember from last class that the more sets there are in the $\sigma - a\lg ebra$, the more we know.

Definition 10 *f:$\mathcal{A} \to \mathcal{R}$ is measurable with respect to $\mathcal{A}$ if,*
$B_c = \{b \in A : f(b) \leq c\} \in \mathcal{A} \ \forall c \in \mathcal{A}$

Example 1 : Consider the following set A, the elements of which denote the possible pairs of today's and tomorrow's temperatures.

$$A = \{[-273, 10000] \times [-273, 10000]\}$$

And let $\mathcal{A}$ be a $\sigma - a\lg ebra$ defined on this set A such that,
$\mathcal{A} = \{[-273, -273.5]_t, [-273.5, -274]_t, [-274, -274.5]_t, ..., [-273, 10000]_{t+1}\}$
(the subscripts just denote which day the temperatures are for, today (t) or tomorrow (t+1))

Knowing which set in $\mathcal{A}$ a certain element lies provides us with no information about tomorrow's temperature. The 0.5 intervals are all for today's temperatures. Now imagine we say we will have a party only if tomorrow's temperature is above a certain level. Can we have a function defined on the $\mathcal{A}$ above and know whether there will be a party tomorrow as the outcome of this function? The answer is no. The particular $\sigma-$algebra that we defined above is not appropriate for a function the arguments of which are tomorrow's temperature. Any function whose argument is temperatures tomorrow is not measurable with respect to the $\sigma-$algebra above.

For a function to be measurable with respect to a $\sigma-$algebra, the set of points over which the function changes value should be in that $\sigma-$algebra, and this is what the above definition of measurability translates into. In other words, you should be able to tell apart the arguments over which the function changes value.

Example 2 : Consider the following $\sigma-$algebra $\mathcal{A}$ defined on set A={1,2,3,4}:

$$\mathcal{A}'= \{\emptyset, A, \{1,2\}, \{3,4\}\}$$

Now think of the function that gives \$1 for odd numbers and \$0 for even. Is this function measurable with respect to $\mathcal{A}'$? No, because the arguments where the functions changes values are not elements of $\mathcal{A}'$. This function would be measurable with respect to the following $\sigma - a\lg ebra$:

$$\mathcal{A}'' = \{\emptyset, A, \{1,3\}, \{2,4\}\}$$

One of the ways we need the notion of measurability in the context of the economies we deal with is the following: Every function that affects what people do at time t has to be "t-measurable", in other words, cannot depend on the future, t+1.

3.2 Transition Function

Definition 11 *A transition function $Q : A \times \mathcal{A} \to \mathcal{R}$ such that:*

1. $\forall \bar{B} \in \mathcal{A}$, $Q(., \bar{B}) : A \to \mathcal{R}$ *is measurable,*
2. $\forall \bar{a} \in A$, $Q(\bar{a}, .) : \mathcal{A} \to \mathcal{R}$ *is a probability measure.*

Q function is a probability that a type a agent ends up in the type which belongs to B. What the first condition above says is the following: Whatever we need to know for today has to be sufficient to specify what the probability tomorrow is.

Consider the set A the elements of which are the possiblel states of the world, say good and bad.

A={good, bad}

And let Γ be the transition matrix associated with A.

$$\Gamma_{ii'} = \begin{bmatrix} \Gamma_{gg} & \Gamma_{gb} \\ \Gamma_{bg} & \Gamma_{bb} \end{bmatrix}$$

The $\sigma-$algebra we would want to use is $\mathcal{A} = \{\emptyset, A, \{good\}, \{bad\}\}$

Notice that what we need to know to compute the probability of a certain state tomorrow is today's state and the $\sigma-$algebra lets us know that.

Homework *Verify that the* $A = \{\emptyset, A, \{good\}, \{bad\}\}$ *is a* $\sigma-$*algebra.*

Homework *Take A=[0,1] and* $A = \{Borel\ sets\ on\ A\}$ *For* $a \in A, B \in A$,

$$Q(a, B) = \int_A B dx$$

Verify that Q is a transition function.

- The measure x defined on a sigma algebra over the set of intervals of wealth is complete description of the society today.
- x'(B): Measure of people who have characteristics in B$\in \mathcal{A}$ tomorrow.
- The pair x,Q will tell us about tomorrow.

  Pick one measure that's defined over the $\sigma-$algebra, $\mathcal{A}$, on the set of wealth levels A=[0,1]

  $$x'(B) = \int_A Q((s, a), B) dx$$

  So with x and Q, we get x'.i.e. with the measure of people today and the transition function Q, we get the measure of people with wealth level in a certain interval B=[a,b] tomorrow.

  What people are now (x) + What people do (Q) $\rightarrow x'$

Example Consider a society where people are characterized as "good guys" or "bad guys". Define the sigma algebra $\mathcal{A}$ as

$$\mathcal{A} = \{\emptyset, A, \{good\}, \{bad\}\}$$

And suppose that today everybody in the society is a "good guy" so that,

$$\begin{aligned} x(\{good\}) &= 1 \\ x(\{bad\}) &= 0 \end{aligned}$$

Let Γ_{gg} denote the probability that someone will be a good guy tomorrow given he is a good guy today. So the measure of people that are good guys tomorrow is,

$$x'(\{good\}) = \Gamma_{gg}$$

and the measure of people who are bad guys tomorrow,

$$x'(\{bad\}) = \Gamma_{gb}$$

and the measure of people who are good guys two periods from now,

$$x''(\{good\}) = \Gamma_{gg}x'(\{good\}) + \Gamma_{bg}x'(\{bad\})$$

Notice that when we are writing the expressions above for the proportion of people of good guys or bad guys at a certain period, we are implicitly using the Law of Large Numbers. LLN says that the proportion of a certain characteristic in that population will converge to the probability of that characteristic with a sufficiently large sample size. In other words, we are not interested in sampling uncertainty. For example, the proportion of the good guys in the society tomorrow is just the probability of a good guy staying a good guy multiplied by the proportion of good guys today (which is 1 in the example above) PLUS the probability that a bad guy will become a good guy multiplied by the proportion of bad guys today (which is zero in the example above).

This is the same idea with the example of tossing a coin. The probability of getting heads is 1/2 each time you toss a coin. So if you toss the coin a countable number of times, the measure of realizations that are heads will be 1/2. In our society above, the probability of each good guy staying a good guy is Γ_{gg} and with sufficiently large number of people, this is equal to the measure of good guys tomorrow.

3.3 Economy with Heterogenous Agents

Now we go back to our economy with fishermen and use the tools that we introduced above for this economy.

- Every period the farmer wakes up and receives his endowment of fish, s, which follows a Markov process with transition matrix Γ. There is a storage technology and if the farmer saves q units of fish today, tomorrow he gets 1 unit of fish. His savings is denoted by a'.

(s,a) is the type of a fisherman and the set consisting of all possible such pairs is,

$$S \times A = \{s^1, s^2, ..., s^n\} \times [0, \overline{a}]$$

Let $\mathcal{A}$ be the set of Borel sets on SxA. And define a probability measure x on $\mathcal{A}$,

$$x : \mathcal{A} \to [0, 1]$$

- The fisherman's problem is:

$$V(s,a) = \max_{c,a'} \text{ u(c)} + \beta \sum_{s'} \Gamma_{ss'} V(s',a')$$

subject to

$$c + qa\prime = s + a$$

$$c \succeq 0 \text{ and a' } \in [0, \overline{a}]$$

- With the decision rule a'=g(s,a) and the transition matrix for the endowment process $\Gamma_{ss'}$, we can construct the transition matrix. The transition function Q(s,a,B) tells us the probability that a fisherman with (s,a) today ends up in some $B_s \times B_a \in \mathcal{A}$ tomorrow (where B_s and B_a are the projections of B over the spaces S and A).

The transition function is constructed as follows

$$Q(s,a,B) = 1_{[g(s,a) \in B_a]} \sum_{s' \in B_s} \Gamma_{ss'}$$

Homework *Verify that Q constructed as above is a transition function.*
Homework *Compute the stationary distribution associated with the transition matrix* $\Gamma = \begin{bmatrix} 0.85 & 0.15 \\ 0.1 & 0.9 \end{bmatrix}$
Homework *Pr {losing a job}=0.05 and Pr{finding a job}=0.5. Find the stationary distribution for states of employment..*

Example Take some Markov process with transition matrix $\Gamma = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix}$
The stationary distribution associated with this transition matrix is, $x^* = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}$

- Define the updating operator T as,

$$x\prime(B) = T(x,Q)$$

- For example if you apply the operator on x and Q twice, we get the two periods ahead measure, etc. :

$$\begin{aligned} x''(B) &= T^2(x,Q) \\ x'''(B) &= T^3(x,Q) \end{aligned}$$

x^* can be written as,

$$x^* = \lim_{n \to \infty} \text{ T}^n(x^0, Q) \quad \forall x^0$$

In other words, under some conditions on Q, no matter how society is today, if you wait long enough you'll get x^*.

But what are these sets of conditions? Here instead of formally laying it out, we briefly explain the conditions we need on Q in order to have a unique stationary distribution:

1. There should be no castes. If you are in a society with a caste system and if you're born in a certain class, you stay in that class. So clearly in such a society, initial conditions matter.

   So we need Q to satistfy the "American dream, American nightmare" condition: No matter what your initial type is, there is a positive probability of going to any different type in the sufficiently near future. So that no matter how poor you are, you can be rich; and no matter how rich you are, you can become poor.

   Consider the following transition matrix,

$$\begin{bmatrix} 0.9 & 0.1 & 0 & 0 \\ 0.3 & 0.7 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0.5 & 0.5 \end{bmatrix}$$

   This transition matrix does not satisfy what we call the "Monotone Mixing Condition". MMC basically says that the transition function should be such that it allows a sufficient mixing of all types of agents. But notice that with the above transition matrix, there is no mixing between types 1 or 2 and types 3 or 4 (for example, the probability of becoming a type 3 given you are type 1 is 0, etc.).

Remark 12 *Remember what the growth model had to say about the wealth distribution: Any initial condition we had, it stayed. So the model had nothing interesting to say about the wealth distribution. But here we have a much stronger result, now we know that no matter where we start we get a unique stationary distribution.*

Homework *For transition function Q, suppose there is a stationary distribution x^*.Write down formula for probability that people in the top 1% richest group remain in that percentile in stationary distribution x^*.*

4 March 27: Economy with Heterogenous Agents (continued)

4.1 Review

Last class we introduced transition functions and we talked about what they mean as well as how they are constructed. Then we defined a stationary distribution and briefly mentioned the properties we need for the transition function in order to have a unique stationary distribution. Today we will talk more about the stationary distributions and we will see more on what happens in the land of the fisherladies.

4.2 Stationary Distribution

Remark 13 *To aviod confusion note the equivalence between the following two kinds of notation:*

In the previous class, we denoted the sigma algebra defined on SxA by $\mathcal{A}$. So that we had,

$$\begin{aligned} x &: \mathcal{A} \to [0,1] \\ Q &: S \times A \times \mathcal{A} \to [0,1] \end{aligned}$$

But we also might use the convention where the $\sigma-$algebra defined on SxA is denoted by $\mathcal{S}x\mathcal{A}$, so that,

$$\begin{aligned} x &: \mathcal{S} \times \mathcal{A} \to [0,1] \\ Q &: S \times A \times \mathcal{S} \times \mathcal{A} \to [0,1] \end{aligned}$$

I will use the first convention here.

- Consider the transition matrix $\Gamma = \begin{bmatrix} 1 & 0 \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix}$

With this transition matrix, if you are type 1, you always stay type 1. This is not a "nice" property because remember that in order to satisfy the monotone mixing condition we need sufficient mixing of all types. No matter what type you are, there should be some positive probability to become each one of the other types. So the transition matrix above is not "nice".

- If Q satisfies a certain set of conditions, then $\exists$ a unique x* such that,

$$x^*(B) = \int_A Q((s,a), B) dx^*(B) \quad \forall B \in \mathcal{A}$$

  and it is globally stable (no matter which distribution x the economy starts at, the economy asymptotically goes to x*) :

$$x^* = \lim_{n\to\infty} \ \mathrm{T}^n(x^0, Q) \quad \forall x^0$$

Now we will see with examples, how initial conditions cease to matter after long periods:

Suppose we are state s_1 today. $\Gamma_{s_1,.}$ is the conditional distribution of tomorrow's shock given today's is s_1.

Homework *Verify this.*

Homework *Also verify that* $\Gamma^T \Gamma_{s_1,\cdot}$ *is the conditional probability distribution function of the two period ahead shock given today's shock is* s_1*.*

Now go further to 10,000 periods ahead; we can write conditional distribution of the state 10,000 periods ahead given the shock today is s_1 as $(\Gamma^T)^{10{,}000}\Gamma_{s_1,\cdot}$

You can see that the more that exponent for Γ^T grows, the less what we are multiplying that with will matter for the result. If Γ satisfies the conditions that we mentioned previously, then

$$(\Gamma^T)^{10{,}000}\Gamma_{s_1,\cdot} = x_s^*$$

In other words, if you have shock s_1 today, especially in the case of persistence, it will continue to govern the shocks you have in the periods ahead, but only for a while; after a while the affect of your initial shock will fade away.

Homework *Consider the following transition matrix of unemployment,*

$$\Gamma = \begin{bmatrix} 0.94 & 0.06 \\ 0.5 & 0.5 \end{bmatrix}$$

where $\Gamma_{ee} = 0.94$*,* $\Gamma_{eu} = 0.06$ *, etc.*

The probability that someone will be employed two periods ahead of today given they are employed today is,

$$\Pr\{s^{u} = e | s = e\} = 0.94 \times 0.94 + 0.94 \times 0.5$$

And,

$$\Pr\{s^{\infty} = e | s = e\} = x_e^*$$

Compute x_e^**.*

Suppose that the society is described by measure x today. For example, suppose that the probability distribution over states at time t is given by $\pi_t = (p_t^1, ..., p_t^N)^T$. (There are N possible states and p_t^n stands for the probability of state n today). So given that the the shocks follow a Markov process with transition matrix Γ , the probability of being in state j tomorrow is given by,

$$p_{t+1}^j = \sum_i \Gamma_{ij} p_t^i$$

and written in a compact form, this is,

$$\pi_{t+1} = \Gamma^T \pi_t$$

So a stationary distribution of a Markov chain satisfies,

$$\pi^* = \Gamma^T \pi^*$$

Homework *Solve* $\Gamma^T x_s^* = x_s^*$

Remark 14 *x^* is the unconditional probability of the individual's type far away into the future. It is also the measure of peopole with that particular type far away in the future; this is because we have a continuum of people and Law of Large Numbers works.*

4.3 Back to the fisherladies

Recall the problem of the "fisherlady":

$$V(s,a) = \max_{a' \in [0,\overline{a}]} \text{u(s+a-}qa\prime) + \beta \sum_{s'} \Gamma_{ss'} V(s', a')$$

The First Order Conditions are,

$$u_c(\text{s+a-}qa\prime) = \frac{\beta}{q} \sum_{s'} \Gamma_{ss'} u_c(\text{s'+a'-qa''})$$

This is a second order difference equation and there are many solutions that satisfy it. But only one of those solutions does not violate compactness. So if we can find some natural bounds for a', or impose conditions so that a' has bounds, the above first order conditions along with the bounds that we have for savings, will characterize the optimal solution.

You'll notice that $a' \in [0, \overline{a}]$ is already one of the constraints of the above maximization problem. But now rather than just imposing such a constraint, we will find a natural reason that savings should have a lower bound and we will consider a condition that ensures an upper bound for savings.

For the lower bound, we assume that there is no technology which allows negative amount of saving and this sounds natural since storing a negative amount of fish does not make much sense. So savings has a lower bound because Mother Nature says so.

For the time being, consider an economy where there is no uncertainty. In such an economy, the following theorem holds,

Theorem 15 *If $\beta < q$, then $\exists \overline{a}$ such that, if $a_0 < \overline{a}$, $a_t < \overline{a}$ $\forall t$.*

You can see this formally through the usual Euler equations,

$$u_c(c_t) = \frac{\beta}{q} u_c(c_{t+1}) \quad \forall t$$

From these equations, it's clear that $\beta < q \Rightarrow u_c(c_{t+1}) > u_c(c_t) \Rightarrow c_{t+1} < c_t$ $\forall t$

If you are impatient enough compared with the returns from technology, you will consume today rather than tomorrow. Gains from saving will disappear eventually and you will stop saving more.

Now think about the economy with uncertainty. Here, the fisherman has the risk of getting a very bad shock tomorrow. So the fisherman would save just in case he has this bad shock; he would want to store some fish today in order to insure himself against getting very small number of fish tomorrow so he is not hungry in case that happens. In this case we need to think more about how to put an upper bound on savings, because with uncertainty even if $\beta < q$, the fisherman is willing to save due to gains from insurance. The kind of savings to protect oneself from risk in the future in the absence of state contingent commodities, we call precautionary savings. In order to ensure an upper bound for savings, we need to bound the gains from insurance somehow. The way to do this is to impose the condition on the utility function that its negative curvature (keeping in mind that the utility function is concave) is diminishing as wealth increases. This means that wealthier agents are less risk-averse. Formally, that u' is convex. The wealthier the agent is, the smaller the variance of his endowment next period proportional to his wealth so he doesn't want to save if he is very wealthy

So in the economy with uncertainty, in order to have an upper bound on savings, we need the first derivative of the utility function to be convex so that the following Jensen's Inequality holds:

$$\frac{\beta}{q}\int \Gamma_{ss'} u_c(c') > \frac{\beta}{q} u_c(\int \Gamma_{ss'} c')$$

Theorem 16 *If $\beta < q$ and u' is convex then $\exists \overline{a}$ such that $a_0 < \overline{a}$, g(s,a)$< \overline{a}$ $\forall s$.*

4.4 Various statistics to describe x*

(Refer to last year's notes for more on these statistics)

Now we are interested in ways to summarize certain properties of x*. There are various ways to do this.

Homework *Compute the ratio of total wealth held by the top 10% to the bottom 10%.*

Homework *Compute wealth level that seperates the top 10% wealthy from the rest of the population. And also compute the wealth level that seperates the bottom 10% from the rest of the population.*

- One of the statistics we can use to measure inequality in a society is variance. The problem with using this statistic is that it is unit dependent. So using coefficient of variation or the variance of the log is more reasonable since they are unit independent (the coefficient of variation is the standard deviation divided by the mean).

- A statistic that is useful to analyze mobility is the autocorrelation of wealth.

  Another way to analyze mobility is through the persistence matrix. An example of a persistence matrix is the one the elements of which denote the probability of an agent that is in the ith quantile today is in the jth quantile tomorrow.

Homework *Verify that the largest eigenvalue of a Markov transition matrix is 1.*

- The second largest eigenvalue of a transition matrix is a measure of persistence. The bigger it is, the longer it takes for the stationary distribution to take over (so that the longer the initial conditions matter).

4.5 Economy with Heterogenous Agents (with trade)

Let the fisherladies in our previous economy now trade with each other. So they are now allowed to use borrowing and lending to transfer resources across time. There is no storage technology anymore. Also, we are making the assumption that state contingent claims cannot be traded.

An important issue here is again the compactness of the asset space. For the economy with storage technology, the lower bound was set by Mother Nature but here there is no such lower bound because agents can borrow and lend so that they can have negative amounts of savings. We need to impose a lower bound on savings so that the agents cannot run a Ponzi scheme. The condition needs to ensure that the agent is able to pay back at each state, including the worst possible state. Letting s^1 denote the worst possible shock, the lower bound on savings is characterized by the following:

$$0 + \underline{a}q = s^1 + \underline{a}$$

This means that the asset level $\underline{a}$ has to satisfy the following: if your debt position is $\underline{a}$ and you draw the worst possible earning tomorrow, you can still enjoy a nonnegative consumption, by borrowing again up to the level $\underline{a}$. Solution of this equation is:

$$\underline{a} = \frac{s^1}{q-1}$$

Since $q < 1$, $\underline{a}$ is negative.

The fisherlady's problem is:

$$V(s, a; q) = \max_{a' \in [\underline{a}, \overline{a}_q]} \text{u(s+a-}qa\prime\text{)} + \beta \sum_{s'} \Gamma_{ss'} V(s', a'; q)$$

Remark 17 *Before, q was just a technology determined number. It was the rate of return from the storage technology. But now it is the interest rate on loans.*

The prices are constant in the steady state, therefore we are only focusing on the steady state equilibria here. We are assuming that we are in a steady state and that the price in the steady state is q.

Solution: $a' = g(s, a; q)$

And we know x*(q) so we can calculate the total assets in a stationary distribution of an economy when the interest rate is q as,

$$\int a dx^*(q)$$

Equilibrium requires that the total amount of wealth that q generates should equal to 0. This is because in this economy no physical assets can be held (there is no storage technology) and the asset of each individual is the liability of the other. Therefore when you sum them up, you should get 0.

Finding the steady state will mean finding the q that solves,

$$\int a dx^*(q) = 0$$

We can show that a steady state exists by showing that there exists a q that solves the above equation as follows:

1. Let f(q)=$\int a dx^*(q)$
2. Show that f is continous in q.
3. Show that f(q)>0 for some q.
4. Show that f(q)<0 for some q.

Then by the Intermediate Value Theorem, we can conclude that $\exists q$ s.t. f(q)=0.

Roughly, the above can be shown as follows:

Consider the case when q→ β from above. In this case, people will save like crazy because there is no cost of transferring consumption across periods (the rate of return on your savings offsets your discount factor). Agents would like to save more no matter how much they own. Therefore, as q→ β aggregate savings grows without bound (Everybody is lending). Therefore, we know that f(q)>0 as q→ β.

Next consider the case where $\frac{1}{q} < 0$. This means that the rate of return on savings is negative. Clearly, nobody will save positive amounts in this case (Nobody will lend because the rate of return you get the next period on what you lent is a negative number).Therefore, aggregate savings will be negative. So we know that f(q)<0 for $\frac{1}{q} < 0$.

By the above and the continuity of f, then we can conclude that $\exists q$ s.t. f(q)=0.