1 April 1

1.1 More words on notation

We will clarify some possible confusion on the notations we have used so far. x is a generic notation for measure; $x : \mathcal{A} \to R_+$.

1. $x(B)$ is the measure of set B for $B \in \mathcal{A}$.

2. In the economy we saw, q is the price of asset that agents take as given. $x(q)$ is the distribution of agents associated with price q. In the fisher-women economy with storage technology, q is exogenously given by mother nature. So, we do not need to index measure by constant q. But in the loan economy, q is interest rate of borrowing and lending and it is endogenous. Assume $q \in [\underline{q}, \overline{q}]$. $x(q) \in X$, which is the whole set of measure associated with all possible q.

3. $x^*(q)$ is the stationary distribution of agents when the price is q. To obtain $x^*(q)$, we should

   (1) solve the household's problem and get optimal decision rule $g(s, a; q)$ and value function $V(s, a; q)$.

   (2) construct transition function $Q(s, a, B; q)$, which tells the probability of type (s, a) agents in set B tomorrow. And we need verify that Q has nice property (monotone mixing condition) such that there is a unique x^* associated with Q.

   (3) find the stationary distribution $x^*(q)$.

4. A generic measure of B in stationary distribution of the above economy is $x^*(B; q)$.

5. Total asset demand in steady state when price is q is $\int_{S\times A} a dx^*(q)$. And in equilibrium of economy without storage technology, $\int_{S\times A} a dx^*(q) = 0$. Note that the distribution of agents is over both state space of shocks and asset space.

6. Integration of a measurable function. $x : \mathcal{A} \to R_+$ is a measure. $f : A \to R$ is a measurable function with respect to x. That is changes in f can be distinguish with x. Then, we can write the integration in the following equivalent ways.

$$\int f dx = \int_A f(a) x(da) = \int_A f(a) dx$$

1.2 Definition of RCE in loan economy with incomplete markets

Definition 1 *A stationary competitive equilibrium for a loan economy with incomplete markets is* $\{q^*, g(s, a; q^*), V(s, a; q^*), Q(s, a, B; q^*), x^*(q^*)\}$ *such that*

(1) Given q^*, $g(s,a;q^*)$ and $V(s,a;q^*)$ solve household's problem.
(2) Q is a transition function constructed from $g(s,a;q^*)$ and Γ.
(3) x^* is a stationary distribution for transition function Q.

$$x^*(B) = \int_{S\times A} Q(s,a,B;q^*)\,dx^* \tag{1}$$

(4) Market clears

$$\int_{S\times A} a dx^* = 0$$

1.3 Wealth persistence and inequality

- In the economy with incomplete markets, people differ in wealth. But is wealth persistent in such economies? Simply speaking, we want to know the sign of autocorrelation of asset holding. First, the persistence in labor income depends on Γ. But income persistence is different from wealth persistence because people can save and dissave. Why do people save? Because they are risk averse and want to smooth consumption. Suppose an agent with normal wealth level receive a high endowment shock s, then she will consume some and save some for the rainy days. And if in the next period, she gets a negative shock, she will consume part of her saving. If she gets a string of bad shocks, her wealth will keep decreasing. We can also analyze the opposite situation. Therefore, wealth is persistent. The autocorrelation depends on Γ, β, σ and state space of shocks.

$$\rho_a = f(\Gamma, \beta, \sigma, s)$$

- Is wealth more equally distributed or unequally distributed? How does wealth inequality evolve? So far, we just look at stationary distribution of agents in stationary equilibrium. Therefore, wealth inequality does not change over time. Although on individual level, wealth is persistent and changes over time, the economy as a whole does not change.

1.4 Liquidity constraint

From household's optimization problem with price q taken as given, we can get Euler equation

$$u_c(c) = \frac{\beta}{q}\sum_{s'} \Gamma_{ss'} u_c(c_{s'}) \tag{2}$$

But when there is liquidity constraint, this Euler equation may not hold at some time. (**??**) is not true at the boundary. We can look at a simple two-period model. Agents have wealth a in period 1 and wage w' in period 2. Interest rate is r. The optimization problem is

$$\max_{c\geq 0, c'\geq 0, s\geq 0} u(c) + \beta u(c')$$

subject to

$$
\begin{aligned}
c+s &= a & (3) \\
c' &= (1+r)\,s+w' & (4)
\end{aligned}
$$

Constraints (3) and (4) can be written as

$$
\begin{aligned}
c+\frac{c'}{1+r} &= a+\frac{w'}{1+r} & (5) \\
s &\geq 0 & (6)
\end{aligned}
$$

People cannot borrow with the constraint of $s \geq 0$. When there is no such borrowing constraint, from FOC, we have the following Euler equation

$$u_c(c) = \beta (1+r)\, u_c(c')$$

When there is a borrowing constraint $s \geq 0$, there are two cases:

(1) constraint $s \geq 0$ does not bind, then the Euler equation holds.

(2) constraint $s \geq 0$ do bind, then $s = 0, c = a, c' = w'$. Euler equation does not hold.

$$u_c(c) \geq \beta (1+r)\, u_c(c')$$

Today's consumption is lower or equal to the first best case.

Homework: Write Kuhn-Tucker problem for this economy.

Therefore, some studies test the equality of Euler equation as the evidence of liquidity constraint. But according to our definition, if there is solvency constraint, will this constraint always be binding? It depends on how bad the constraint is. For example, if people have log utility function, $u(c) = \log c$. The bad shock of endowment is $s_b = 0$. Then, since $u_c(c) = \frac{1}{c}$, marginal utility of consuming nothing is infinity, then nobody will lead themselves to the situation of binding borrowing constraint. They will get enough saving to avoid this worst case. Therefore, when there exists borrowing constraint, Euler equation may still hold. (Analogy: every rational agent will keep herself away from the edge of a cliff so that she will not fall from the cliff.)

1.5 Growth model with incomplete markets

We add a production technology to the economy of many agents with incomplete markets. For now, we assume agent value consumption but not leisure. Every period, each agent receives s efficiency units of labor. This shock follows a Markov process with a Markov transition matrix $\Gamma_{ss'}$. You can think of it as the hours that agents can use for either leisure or labor. Since agents do not value leisure, they just use all of their efficiency units as a labor supply. (Previously, we assume $s = 1$, $\forall i$. That is everyone has one unit of efficiency labor supply).

The agent's problem is

$$V(s, a; K) = \max_{c \geq 0, a \in [0, \overline{a}]} u(c) + \beta \sum_{s'} \Gamma_{ss'} V(s', a'; K) \tag{7}$$

subject to

$$c + a' = a\left(1 + r\left(K\right)\right) + w\left(K\right)s$$

The production function is $f(K, N)$. The firm uses aggregate labor N and capital K as inputs and produce consumption goods. Note that the wage and capital rental rate which clear the market are:

$$w^* = F_N(K, N) \tag{8}$$
$$r^* = F_K(K, N\ ) - \delta \tag{9}$$

How to compute aggregate labor N? Of course we can write

$$N = \int_{S \times A} s dx \tag{10}$$

But we need know N without knowing x. This can be done because s does not depend on people's choice. An example:$S = [0.2, 1]$, $\Gamma = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix}$. The stationary distribution associated with Γ is $[0.5, 0.5]$. Thus $N = 0.5 \times 0.2 + 0.5 \times 1 = 0.6$. From this example, we know the aggregate labor is average endowment of the economy. It depends on the stationary distribution of Γ, but does not depend on people's choice.

Remark 2 *If leisure enters utility function, N is endogenous.*

The optimal solution to the agent's problem is $g\left(s, a; K\right)$ and $V\left(s, a; K\right)$. Then, we can construct transition function $Q\left(s, a, B; K\right)$ and find the stationary distribution $x^*\left(K\right)$ associated with Q.

Definition 3 *Stationary recursive competitive equilibrium in the growth model with incomplete markets is $\left\{K, g\left(s, a; K\right), V\left(s, a; K\right), Q\left(s, a, B; K\right), x^*\left(K\right)\right\}$ such that*

(1) Given K, $g\left(s, a; K\right)$ and $V\left(s, a; K\right)$ solve household's problem.

(2) Q is a transition function constructed from $g\left(s, a; K\right)$ and Γ.

(3) x^* is a stationary distribution for transition function Q.

$$x^*\left(B\right) = \int_{S \times A} Q\left(s, a, B; K\right) dx^*$$

(4) Market clears

$$F_K\left[\int a dx^*\left(K\right), N\right] - \delta = r\left(K\right) \tag{11}$$

(11) is one equation with one unknown of K. Therefore we can find the equilibrium. In writing (11), we mean that in equilibrium, aggregate capital endogenous from people's choice induces the price $r\left(K\right)$ that people take as given. Another way of writing the market clearing condition is

$$\int_{S \times A} a dx^*\left(K\right) = K$$

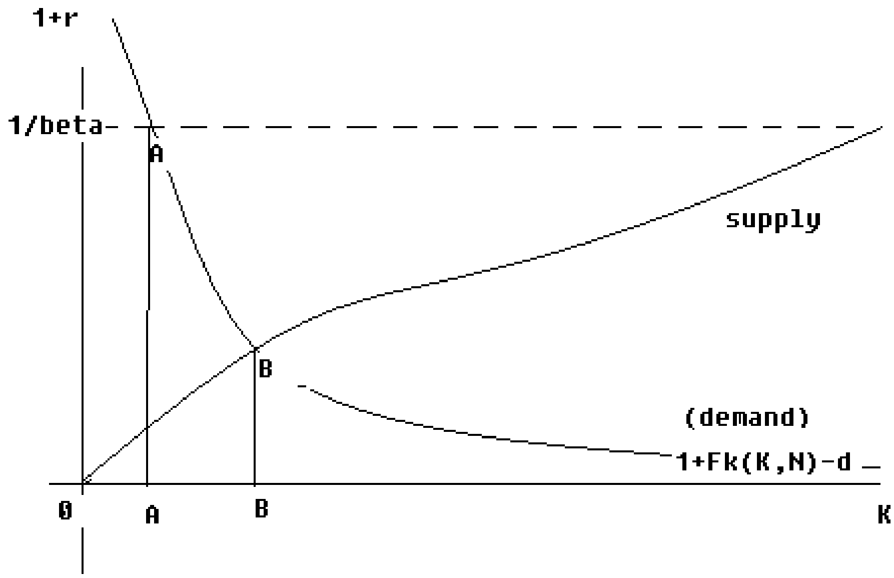


Figure 1:

1.6 Aggregate precautionary saving

In this economy, interest rate is endogenous since it is the marginal product of capital. We can analyze aggregate demand and supply of capital as a function of interest rate.

When $r \to \frac{1}{\beta} - 1$, interest rate is very high relative to time preference. People are very patient and always want to save more. Therefore, aggregate capital supply goes to infinity. But capital demand goes to zero with high rental rate.

When $r \to -1$, people will not save at all. Aggregate capital supply goes to zero, but aggregate capital demand goes to infinity.

With intermediate value theorem, there is a capital level K such that stationary equilibrium exists.

On graph, K_B is the equilibrium capital level in the economy with idiosyncratic risks. While K_A is equilibrium capital level when there is no risk in the economy. Why? Because when there is no risk, the Euler equation in steady state is

$$u_c(c^*) = \beta(1+r)u_c(c^*)$$

Thus

$$1 + r = \frac{1}{\beta}$$

We use aggregate precautionary saving to describe the additional wealth level in the society because of incomplete insurance. If we assume that the earning risks cannot be insured (i.e., the agents cannot trade state contingent securities), agents are expected to save a part of their earning in the form of capital in order to "prepare for the bad time in the future". The saving for "preparing for the bad time in the future" is what we call "precautionary saving". In the economy with complete markets there is no precautionary saving, because there is no such risk (agents end up receiving the same amount by trading Arrow securities).

Among studies in the literature, the two works are very important: one is Aiyagari (1994 Quarterly Journal of Economics). The other is Huggett (1993 Journal of Economic Dynamic and Control). Their finding is that aggregate precautionary saving is at most 3% increase of the aggregate saving rate.

1.7 Growth model with leisure and incomplete markets

$$V\left(s, a; \frac{K}{N}\right) = \max_{c \geq 0, a \in [0, \overline{a}], n \in [0,1]} u(c, n) + \beta \sum_{s'} \Gamma_{ss'} V\left(s', a'; \frac{K}{N}\right) \tag{12}$$

subject to

$$c + a' = a\left(1 + r\left(\frac{K}{N}\right)\right) + w\left(\frac{K}{N}\right) sn$$

The optimal solution is

$$\begin{aligned} a' &= g\left(s, a; \frac{K}{N}\right) \\ n &= h\left(s, a; \frac{K}{N}\right) \end{aligned}$$

In equilibrium,

$$\frac{K}{N} = \frac{\int a dx^* \left(\frac{K}{N}\right)}{\int sh\left(s, a; \frac{K}{N}\right) dx^* \left(\frac{K}{N}\right)} \tag{13}$$

2 April 3

2.1 Continuity of aggregate excess demand of capital

To show the existence of steady state equilibrium, we want to use intermediate value theorem to find equilibrium price. Thus, we need condition to guarantee that aggregate excess demand in steady state is a continuos function of price.

Theorem 4 *Stocky and Lucas (12.13)*

If (1) $S \times A$ is compact.

(2) $(s_n, a_n, q_n) \to (s_0, a_0, q_0)$, implies $Q(s_n, a_n, \cdot; q_n) \to Q(s_0, a_0, \cdot; q_0)$

(3) $x^*(q_n)$ *is unique.*
Then, for a measurable function f*,*

$$\int f(s,a)\,dx^*(q_n) \rightarrow \int f(s,a)\,dx^*(q)$$

Let's verify the above conditions in a loan economy.

(1) $S \times A$ is compact by assumption. Interest rate is bounded away from $\frac{1}{\beta}$, $q \in [\underline{q}, \overline{q}]$, then $\underline{a}$ is saving under the lowest interest rate and $\overline{a}$ is saving level with highest possible interest rate.

(2) Decision rule g is continuos from Theorem of Maximum. Therefore, the constructed transition function is continuos.

(3) Monotone mixing condition ensures a unique $x^*(q_n)$.

Therefore, the above theorem holds. Aggregate excess demand is continuos and there exists a steady state equilibrium.

2.2 Non Steady State Equilibrium

The steady state equilibrium that we have studied so far cannot be used for analyze policy change. So, let's now look at non-steady state equilibrium version of Aiyagari's economy.

First, we should know why the following problem is not well defined in equilibrium.

$$V(s,a,K;G) = \max_{c \geq 0, a \in [0,\overline{a}]} u(c) + \beta \sum_{s'} \Gamma_{ss'} V(s',a',K';G) \tag{14}$$

subject to

$$\begin{aligned} c + a' &= a(1+r) + ws \\ r &= F_K(K,N) - \delta \\ w &= F_N(K,N) \\ K' &= G(K) \end{aligned}$$

And in equilibrium,

$$K = \int a dx \tag{15}$$

In steady state equilibrium, r is a constant, indexed by K. But outside steady state, price is not a constant any more. r is marginal product of capital. Aggregate labor does not change in this economy, thus the sufficient statistic of r and w is aggregate capital $K = \int a dx$. Therefore, for individual problem, this is well defined as long as with the conceived aggregate capital, interest rate is not too low to have unbounded asset problem. That is $G(K) > K^*$. Then, agents know the sequence of prices and they make their optimal decision.

But when it comes to equilibrium, agent's conjecture does not hold. How do agents conjecture the law of motion for K? $K' = G(K)$ means that aggregate capital is sufficient statistic for aggregate capital tomorrow. Usually this does not hold. Consider two societies with same aggregate capital today. For both societies to have the same capital again tomorrow, both decision rule g has to be an affine function of asset holding.

$$g(a) = \alpha + \beta a$$

Then, from

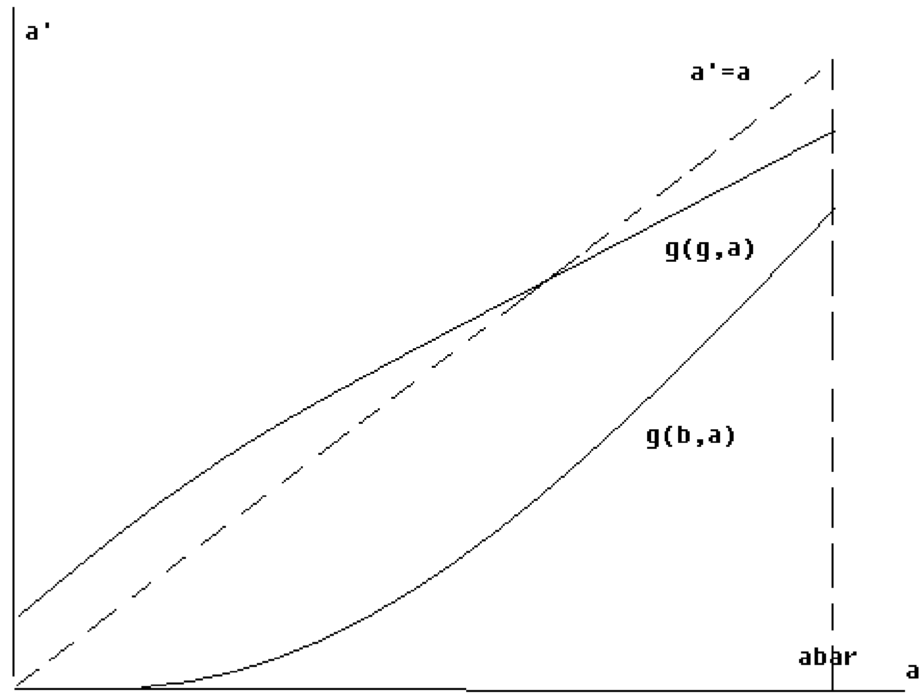
$$\begin{aligned} K &= \int a dx \\ K' &= \int g(s,a)\, dx \end{aligned}$$

it is true that for affine function of g,

$$K' = h\left(\int a dx\right) = h(K)$$

Homework: Define a linear function.

But saving function is not linear. We can look at two examples. One is economy with finite horizon. For a society populated by old men, they will dissave their wealth by holding parties, say. But for a society populated by young men with same level of wealth, they will keep saving for the future. So, the aggregate wealth of next period will not be equal for the two societies. Another example is Aiyagari economy with zero borrowing constraint. Suppose there are two shocks, s_g and s_b. Then, for asset close to zero, $g(s_b, a) = 0$ because agents want to borrow but are constrained. Therefore, $g_a(s_b, a)$ is close to zero. But for large asset holding, $g_a(s, a) \simeq 1$. Therefore, the saving rule is nonlinear.



Another way of illustration is that if wealth in the society is redistributed, then, under linear saving rule, total saving will not change. But in the above economy, when we transfer wealth from lucky guys to an unlucky one with bad shock and little asset holding, the latter will not save as the former agent does. Therefore, g is nonlinear. Aggregate capital K is not sufficient to predict K'. In equilibrium, $K' \neq G(K)$. We need know the whole distribution of wealth to forecast K'.

The well-defined problem in the nonsteady state equilibrium is:

$$V(s, a, x; G) = \max_{c \geq 0, a \in [0, \overline{a}]} u(c) + \beta \sum_{s'} \Gamma_{ss'} V(s', a', x'; G) \tag{16}$$

subject to

$$\begin{aligned} c + a' &= a(1+r) + ws \\ r &= F_K(\int adx, N\ ) - \delta \\ w &= F_N(\int adx, N\ ) \\ x' &= G(x) \end{aligned}$$

Here, $G(x)$ tells how the society distribution evolve. G maps probability measure onto probability measure.

Definition 5 *Nonsteady state equilibrium in a growth model with a continuum of agents with idiosyncratic shock and incomplete markets is* $\{V(s,a,x), g(s,a,x), G(x)\}$ *such that*

(1) $V(s,a,x), g(s,a,x)$ *solve household's problem, given* G.

(2)

$$G(x)(B) = \int_{S\times A} \sum_{s'\in B_S} \Gamma_{ss'} 1_{g(s,a,x;G)\in B_a} dx \tag{17}$$

Note that in the above definition, we have implicitly defined transition function Q.

Now, let's study rational expectation under this nonsteady state equilibrium. We will see to find equilibrium is horribly hard. First, households have expectation on evolution of wealth distribution, $x' = G^E(x)$. The optimal solution of households is $g(s,a,x;G^E)$. This evolution is true in equilibrium when given G^E, indeed households' action aggregate to generate a distribution of $x' = G^E(x)$. In math,

$$G(x)(B) = \sum_{s'\in B_s} \int_{S\times A} \Gamma_{ss'} 1_{g(s,a,x;G^E)\in B_a} dx$$

Equilibrium is a fixed point over space of functional mapping from expectation to expectation. This problem cannot be solved.

2.3 Policy analysis

"$\underbrace{\textit{Recursive Equilibrium}}_{2}$ is $\underbrace{\textit{horribly}}_{1}$ hard when done $\underbrace{\textit{properly}}_{3}$." But to know the effect of policy change, we have to study nonsteady state equilibrium. There are three things we can do.

We start with <u>*horribly*</u>. What makes horrible is that price depends on distribution. We have seen the case when price does not depend on x. In fisherwomen economy with storage technology, q is exogenous. Therefore, we can avoid the horribly hard problem by using exogenous price.

The problem of having endogenous price is that marginal rate of substitution is endogenous. In Huggett economy with borrowing and lending, price is endogenous from market clearing condition $\int a dx(q) = 0$. Aiyagari economy, r and w are marginal product of factors, thus endogenous.

2.4 Unemployment Insurance

Let's work on analysis of unemployment insurance in an economy with exogenous price. State space of shocks $S = \{e, u\}$. Transition matrix is $\Gamma_{ss'}$. τ is unemployment insurance ratio that any agent has to pay from her labor income when she has a job. b is unemployment benefit that a person can get when unemployed.

Homework: Compute average duration of having a job.

Individual optimization problem is

$$V(s,a) = \max_{c,a'} u(c) + \beta \sum_{s'} \Gamma_{ss'} V(s',a') \tag{18}$$

subject to

$$c + qa' = \begin{cases} w(1-\tau) + a & \text{if } s = e \\ b + a & \text{if } s = u \end{cases} \tag{19}$$

Optimal solution is $g(s,a;\tau,b)$. Construct transition function and find stationary distribution $x(\tau,b)$. (Note, we index distribution by policy parameters τ and b).

Equilibrium condition is

$$\int b 1_{s=u} dx = \int w\tau 1_{s=e} dx \tag{20}$$

A simpler way is the have

$$bx_{u\cdot} = w\tau x_{e\cdot} \tag{21}$$

where $x_{u\cdot}$ denotes the proportion unemployed people today.

Homework: Define steady state equilibrium of the above economy.

2.5 Unemployment Insurance Policy Analysis

Suppose we have a choice of cutting unemployment benefit by half. How do you think of this policy change? This is a question of policy analysis. To compare two policies, solving stationary equilibrium associated to each policy and compare the welfare of agents in the two stationary equilibria is wrong. Because the problem is NOT "whether you would like to be born in an economy with policy A or rather be born in an economy with policy B", but "if you are living in an economy with policy A, would you support the change of policy to policy B or rather stay with policy A." If we simply compare the economic performance of both policies, we only get direct effect of policy changes. In the transition from one policy to another, there is also effect on wage through changes in saving behavior.

To assess policy analysis, we have to

(1) construct a measure of goodness.

(2) tell what the outcome would be. That is find decision rule and distribution of agents through changes of policies.