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1.1 Economy with technology changes

In an Aiyagari economy, suppose the production is given by $A_t F(K_t, N_t)$ where $N_t = \int s dx, K_t = \int a dx$. Individual idiosyncratic shock $s\tilde{}\Gamma_{ss'}$. The agent's problem is

$$V(s, a; K) = \max u(c) + \beta \sum_{s'} \Gamma_{ss'} V(s', a'; K) \tag{1}$$

subject to

$$c + a' = a[1 + r(K)] + sw(K) \tag{2}$$

The decision rule is $g(s, a; K)$. Wealth distribution $x[g(K)]$ is derived from this $g(.)$ function. And aggregate capital in steady state equilibrium is

$$K^* = \int a dx [g(K^*)] \tag{3}$$

Now, we want to know what happens if A doubles?

1.1.1 In a Representative Agent economy

- We have learnt how to work this out in a representative agent economy. With same K, marginal product of capital doubles, r increases, people keep increasing their saving until new capital level and this new A generate interest rate equal to $\frac{1}{\beta} - 1$. So, in a RA economy, we know aggregate capital increases until it reach the new steady state capital level. To solve the equilibrium path, we can solve SPP because market is complete and equilibrium is PO.

  Euler equation is a second order difference equation

  $$\varphi(K_t, K_{t+1}, K_{t+2}) = 0$$

  with K_0 given.

  In steady state when technology is A, we can solve steady state capital level K^0 from

  $$\varphi\left(K^0, K^0, K^0\right) = 0$$

  We write out the SPP as

  $$\Omega(K) = \max u[AF(K, 1) - K'] + \beta\Omega(K')$$

  And EE is

  $$u_c = \beta AF'\left(K^0, 1\right) u'_c$$

  In steady state,

  $$1 = \beta AF'\left(K^0, 1\right)$$

  and we solve for K^0.

- When new technology takes place, $\widehat{A} = 2A$, Euler equation changes to

$$\widehat{\varphi}\left(K_t, K_{t+1}, K_{t+2}\right) = 0 \tag{4}$$

because feasibility condition changes. Social planner's problem is now

$$\Omega\left(K\right) = \max u\left[2AF\left(K,1\right) - K'\right] + \beta\Omega\left(K'\right) \tag{5}$$

Upon the technology change, capital level is still at K^0, so FOC gives

$$\frac{u_c}{u_c'} = 2\beta AF'\left(K^0, 1\right)$$

c_t and c_{t+1} have to adjust to satisfy this optimality condition. We can predict that growth rate of consumption goes up.

And we can compute the transition path for capital. Upon technology change, we have

$$\widehat{\varphi}\left(K^0, K^{01}, K^{02}\right) = 0 \tag{6}$$

But we need a second condition to get the whole path. As there is only one K^{01} with which solution to (6) satisfies feasibility. The whole path of capital level in this economy is obtained.

Problem 1 *Solve the capital level in new steady state for this economy.*

1.1.2 Technology changes with incomplete market

Now how to solve this problem with incomplete market?

In steady state equilibrium,

$$K^{0*} = \int adx\left[g\left(K^{0*}\right)\right] \tag{7}$$

When A doubles, we know what happens in new steady state, g^{1*}, K^{1*}, x^{1*}. But to get the whole transition path, we need solve optimal decision rule with K^{0*}, x^{0*} taken as given as initial condition.

To see what happens when the world changes, we assume:

1. After T periods, the economy has converged to g^{1*}, K^{1*}, x^{1*}. And price $r_t = r\left(K^{1*}\right), w_t = w\left(K^{1*}\right)$ for $t > T$.

2. Assume $\{r_t\}_{t\leq T}$ and $\{w_t\}_{t\leq T}$ are given by $\overrightarrow{r}$ and $\overrightarrow{w}$ respectively.

Individual's problem is

$$\Phi\left(s, a, \overrightarrow{r}, \overrightarrow{w}, T, K^{1*}\right) = \max_{c_t} \sum_{t=0}^{T} \beta^t u\left(c_t\right) + \beta^T E\left\{V^1\left(s^T, a^T; K^{*1}\right)\right\}$$

subject to

$$c_t + a_{t+1} = \left(1 + r_t - \delta\right)a_t + sw_t$$

Hence, we assume the transition takes place over a finite number of periods. In T periods, the economy gets to new steady state. In the meantime, households face price of $\overrightarrow{r}$ and $\overrightarrow{w}$ and choose $\{c_t\}$. This problem yields decision rule $g_t\left[s, a, \overrightarrow{r}, \overrightarrow{w}, T, K^{1*}\right]$ which is state dependent.

We can compare this problem with the one when unemployment insurance policy changes. In that case, we can use the decision rule from the new situation g^{1*} because the change in environment does not affect price. But in the current model, interest rate increases and people adjust their decision rule accordingly. So, we have to use both recursive and nonrecursive methods to solve the problem.

Problem 2 *In the Aiyagari economy with unemployment insurance, suppose unemployment insurance is paid by consumption tax τ. Describe the algorithm to access the policy changes in τ. Consider two cases when government has period by period budget constraint and when government can borrow with bond.*

1.2 Aiyagari economy with aggregate uncertainty

When we model business cycle, the economy does not converge to any steady state because there exists aggregate uncertainty. We can define an economy with a moderately stupid agents. By "moderately stupid", we mean that agents choose to ignore some relevant information in making decision. (For example, when a person wants to predict the outcome of a football game next week, she may ignore the news that one key player had a quarrel with his wife. But she forecast the outcome using the information that this key player will play in this game.)

In the Aiyagari economy with aggregate uncertainty, aggregate shock is denoted as z which follows Markov transition matrix $\Pi_{zz'}$. We allow the probability of idiosyncratic shocks to depend on aggregate production shock. Therefore, the individual's problem is

$$V(z, x, s, a; G) = \max_{c, a \geq 0} u(c) + \beta \sum_{s', z'} V(z', x', s', a'; G) \Gamma_{s'|szz'} \Pi_{z'|z} \tag{8}$$

subject to

$$c + a' = a(1 + r(z, K) - \delta) + sw(z, K) \tag{9}$$

$$x' = G(z, x) \tag{10}$$

$$K = \int a dx \tag{11}$$

Problem 3 *Talk about $\Gamma_{s'|szz'}\Pi_{z'|z}$.*

We assume agents are too stupid to solve this problem. Therefore, they exclude information embedded in distribution measure x and only use information contained in aggregate capital K. Then the agent's problem becomes to

$$\Psi(z, K, s, a; H) = \max_{c, a \geq 0} u(c) + \beta \sum_{s', z'} \Psi(z', K', s', a'; H) \Gamma_{s'|szz'} \Pi_{z'|z} \tag{12}$$

subject to

$$\begin{aligned} c + a' &= a(1 + r(z, K) - \delta) + sw(z, K) \qquad (13) \\ K' &= H(z, K) \qquad (14) \end{aligned}$$

Note, in equilibrium, the conjectured law of motion for K is not what really happens in the economy.

We will define such an economy full of stupid agents. If there is no much loss in doing so, we will use this economy in the study. There are three grounds for agents to use $K' = H(z, K)$ in optimization problem:

1. Knowing more does not mean that they can forecast better.
2. Forecasting better does not mean that they can be happier.
3. Forecasting better does not mean that they will behave differently.

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2.1 Aiyagari economy with aggregate uncertainty(continued)

In the economy with moderately stupid agents, individual problem is

$$V(z, K, s, a; H) = \max_{c, a \geq 0} u(c) + \beta \sum_{s', z'} V(z', K', s', a'; H) \Gamma_{s'|szz'} \Pi_{z'|z} \qquad (15)$$

subject to

$$\begin{aligned} c + a' &= a(1 + r(z, K) - \delta) + sw(z, K) \qquad (16) \\ K' &= H(z, K) \qquad (17) \end{aligned}$$

Since agent does not use all the information to forecast the economy,

$$\beta \sum_{s', z'} V(z', K', s', a'; H) \Gamma_{s'|szz'} \Pi_{z'|z} \neq E[V(z', K', s', a') | allinformation] \qquad (18)$$

she is forecasting K' wrongly. In equilibrium, $K' \neq H(z, K)$. So what? This problem is a quantitative dent to it. In equilibrium,

$$K' = H(z, K) + \varepsilon(z, x)$$

where $\varepsilon(z, x)$ is the forecasting error from being stupid.

When is being such stupid is not important?

1. When $\varepsilon(z, x)$ is small. Say if the period of economy is one minute. Changes in capital is quite small within a minute.

2. If $\varepsilon(z, x)$ is irrelevant.

Let m be a set of moments of x, $m \in R^n$. That is

$$\begin{aligned} m_1 &= \int a dx = K \\ m_2 &= \int a^2 dx \\ &\dots \\ &m_n \end{aligned}$$

m is information set about distribution measure x, but it only contains incomplete, finite amount of information. We want to see whether m is sufficient statistic for prices.

If agents use m in their optimization problem, we call them slightly stupid. The value function for slightly stupid agents is $V(z, m, s, a; H^n)$ such that

$$V(z, m, s, a; H^n) = \max_{c,a'} u(c) + \beta \sum_{s',z'} V(z', m', s', a'; H^n) \Gamma_{s'|szz'} \Pi_{z'|z} \tag{19}$$

subject to

$$c + a' = a(1 + r(z, m) - \delta) + sw(z, m) \tag{20}$$
$$m' = H^n(z, m) \tag{21}$$

As $n \to \infty$, $H^n(m) \to x'$, as we can know the probability distribution from moment generating function.

Now, let's construct an economy where m is sufficient statistic for prices. We will define an equilibrium where everyone is moderately stupid. Practically speaking, we are happy with "a ε approximating equilibrium" for $m \in R^n$, such that in an economy where all agents use $m \in R^n$ moments to choose what they do.

If

$$h^n[z, m, s, a, ; H^n] \simeq h^{n+1}\left[z, m, s, a; H^{n+1}\right]$$

everyone use just n moments, then being smarter (using $n + 1$ moments) does not make any difference.

Krusell and Smith (JPE 1998)[1] shows that the ε is very small. And the approximation $R^2 = 0.999992$ when agents only use the first one moment in making decision. Hence, it is fine to work with decision from n moments.

2.2 Economy with Private information

What is the most important (worst) thing we have done with the model up to the last class? We exogenously closed markets for state contingent loans and thus prevented exogenously the economy from collapsing to the representative agent economy. But the economists cannot choose what people can do and what

[1] Krusell, Per and Smith, Anthony, Jr. (1998), "Income and Wealth Heterogeneity in the Macroeconomy", Journal of Political Economy, 106-5, 867-896.

they cannot do. From now, we do not do this. Instead, we will define the fundamental environment and assume more on what information agents have and what agents can see. We will look at two big classes for models. One is the economy with private information. In other words, there is asymmetric information or incomplete information in the mod. The second class is the models with lack of commitment. In the world without commitment, the contract among agents need to be self-enforceable. Otherwise, agents will just quit the contract and walk away.

2.3 Model on unemployment insurance[2]

Consider an economy where the probability of finding a job $p(a)$ is a function of effort $a \in [0,1]$. And we assume that once the agent gets a job, she will have wage w for ever. Thus, the individual problem is

$$\max_{a_t} E \sum_t \beta^t \left[u(c_t) - a_t \right]$$

There are two cases: when the agent has got a job, she will pay no effort and enjoy w for ever. The life long utility is

$$V^E = \sum_t \beta^t u(w) = \frac{u(w)}{1-\beta} \tag{22}$$

When the agent is still unemployed, she will have nothing to consumer. Her problem is

$$V^u = \max_a \left\{ u(0) - a + \beta \left[p(a) V^E + (1 - p(a) V^{u\prime}) \right] \right\} \tag{23}$$

Problem 4 *Prove that $V^u = V^{u\prime}$ under optimal decision.*

If the optimal solution of a is interior, $a \in (0,1)$, then the first order condition gives

$$-1 + \beta p'(a) \left(V^E - V^u \right) = 0 \tag{24}$$

And since the V^u is stationary,

$$V^u = \max_a \left\{ u(0) - a + \beta \left[p(a) V^E + (1 - p(a) V^u) \right] \right\} \tag{25}$$

Solving (24)(25) gives the optimal a and V^u. Another way is to successively substitute a and obtain solution because (**??**) defines a contraction mapping operator. We can fix V_0^u, then solve (25) to get $a(V_0^u)$ and obtain V_1^u. Keeping going until $V_n^u = V_{n+1}^u$. In a word, optimal effort level a^* solves (25) with $V^u = V^{u\prime}$.

The probability of finding a job $p(a)$ is called hazard rate. If agents did not find a job with effort level a^*, next period, she will still execute the same effort

[2] The source of this part is the updated Chapter 4 of Tom Sargent's Recursive Economic Theory.

level a^*. Why? Because the duration of unemployment is not state variable in agent's problem. (If agents do not have enough realization about the difficulty of getting a job. With learning, their effort a will increase as they revise their assessment of the difficulty. But such revision of belief is not in this model.)

Now suppose resource is given to people who is unemployed to relive her suffering by a benevolent planner. This planner has to decide the minimal cost of warranting agent a utility level V: $c(V)$. To warrant utility level V, the planner tells the agent how much to consume, how much effort to exert and how much utility she will get if she stay unemployed next period. Obviously, the cost function $c(V)$ is increasing in V.

Problem 5 *Show that $c(V)$ is strictly convex.*

The cost minimization problem of the planner can be written in the following recursive problem:

$$c(V) = \min_{c,a,V^u} c + [1 - p(a)] \frac{1}{1+r} c(V^u) \tag{26}$$

subject to

$$V = u(c) - a + \beta \left[p(a) V^E + (1 - p(a)) V^u \right] \tag{27}$$

To solve the problem, construct Lagragian function

$$\mathcal{L} = -c - [1 - p(a)] \beta c(V^u) - \theta \left[V - u(c) + a - \beta \left[p(a) V^E + (1 - p(a)) V^u \right] \right]$$

FOC: (c)

$$\theta = \frac{1}{u_c} \tag{28}$$

(a)

$$c(V^u) = \theta \left[\frac{1}{\beta p'(a)} - \left(V^E - V^u \right) \right] \tag{29}$$

(V^u)

$$c'(V^u) = \theta \tag{30}$$

Envelope condition

$$c'(V) = \theta \tag{31}$$

We will work on some implication of these conditions:

1. Compare (29) and (24), we can see that the substitution between consumption and effort is different from the one in agent's problem without unemployment insurance. This is because the cost of effort is higher for work that it is from the viewpoint of planner.

2. (30) tells us that the marginal cost of warranting an extra unit of utility tomorrow is θ.,provided that tomorrow V^u is optimally chosen when today's promise is V. And (31) tells us that the marginal cost of warranting an extra unit of V today is θ.

3. Given that c is strictly convex, $V = V^u$.

4. Regardless of unemployment duration, $V = V^u$. So, effort required the the planner is the same over time. Hazard rate is still constant.

Problem 6 *Work out the model and derive the implication on your own.*

Next class, we will study the case when effort is not observable. Planner can only choose consumption and V^u. Effort level is chosen optimally by work and it is unobservable.

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3.1 Unemployment insurance(II)

Review of last class: unemployed agents can find a job with probability $p(a)$ and once they get the job, they can get w for ever. Without unemployment insurance, their problem yields

$$V^E = \sum_t \beta^t u(w) = \frac{u(w)}{1-\beta} \tag{32}$$

$$V^A = \max_a \left\{ u(0) - a + \beta \left[p(a) V^E + \left(1 - p(a) V^A\right)\right]\right\} \tag{33}$$

where we denote the utility of not finding a job as V^A, which comes from the situation when there is no unemployment insurance and people basically stay Autarky.

First order condition gives

$$-1 + \beta p'(a)\left(V^E - V^A\right) = 0 \tag{34}$$

And we know that effort level a^* does not change over time.

Now suppose there is a social planner who will warrant utility level V for unemployed agent, where V summarize all the past information. The cost minimization problem is

$$c(V) = \min_{c,a,V^u} c + [1 - p(a)]\beta c(V^u) \tag{35}$$

subject to

$$V = u(c) - a + \beta\left[p(a) V^E + (1 - p(a)) V^u\right] \tag{36}$$

FOC: (c)

$$\theta = \frac{1}{u_c} \tag{37}$$

(a)

$$c(V^u) = \theta\left[\frac{1}{\beta p'(a)} - \left(V^E - V^u\right)\right] \tag{38}$$

(V^u)

$$c'(V^u) = \theta \tag{39}$$

Envelope condition

$$c'(V) = \theta \tag{40}$$

Problem 7 *Write out envelope condition for the above problem*

The optimal promise for tomorrow is $V^u(V)$. Now, let's work out the property of $V^u(.)$.

Lemma 8 *If* $V > V^A$, *then* $c(V) > 0$, *where* V^A *is the utility for unemployed agent when they are in autarky.*

Problem 9 *Prove the above lemma.*

The intuition for the lemma is that if the planner promises the agent something more than what agent can achieve by herself, it will cost the planner something because the planner cannot do anything more than what people can do on their own.

Lemma 10 *Lagrangian multiplier* $\theta > 0$.

The second lemma tells us that if the planner promise more, she has to pay more.

In the problem without unemployment insurance, (34) implies that

$$\frac{1}{\beta p'(a)} = V^E - V^u$$

In the planner's problem, since $c(V) > 0$, (38) implies that the effort level chosen by the planner are different from agent's choice in autarky. The reason is that effort does not cost that much in planner's thought.

3.2 Unobservable effort

When a is not observable, planner can only choose c and V^u. And households choose a optimally. Now it becomes a principle-agent problem. We will solve the problem backward.

If given c and V^u, the agent will solve

$$\max_a u(c) - a + \beta \left[p(a) V^E + (1 - p(a)) V^u\right] \tag{41}$$

FOC is

$$[p'(a)\beta]^{-1} = V^E - V^u \tag{42}$$

This FOC gives an implicit function of a as a function of V^u: $a = g(V^u)$. (Because c and a are separate in the utility function, a is not a function of c).

Then, the planner solve her cost minimization problem, in which the optimality condition is also one constraint.

$$c(V) = \min_{c,a,V^u} c + [1 - p(a)]\beta c(V^u)$$

subject to

$$V = u(c) - a + \beta\left[p(a) V^E + (1-p(a)) V^u\right] \tag{43}$$

$$1 = \left[p'(a)\beta\right]\left[V^E - V^u\right] \tag{44}$$

Lagragian is

$$c + [1-p(a)]\beta c(V^u) + \theta\left[V - u(c) + a - \beta\left[p(a) V^E + (1-p(a)) V^u\right]\right]$$
$$+\eta\left[1 - \left[p'(a)\beta\right]\left[V^E - V^u\right]\right]$$

FOC: (c)

$$\theta^{-1} = u_c$$

(a)

$$c(V^u) = \theta\left[\frac{1}{\beta p'(a)} - \left(V^E - V^u\right)\right] - \eta\frac{p''(a)}{p'(a)}\left(V^E - V^u\right) \tag{45}$$

(V^u)

$$c'(V^u) = \theta - \eta\frac{p'(a)}{1-p(a)} \tag{46}$$

Envelope condition

$$c'(V) = \theta \tag{47}$$

Problem 11 *Show that in this case, the effort level that household exerts is less than that the planner wants her to exert when effort is observable.*

Again, (46) tells the marginal cost to warrant additional amount of delayed promise. (47) gives the marginal cost to increase today's utility. The Lagrangian multiplier associated with constraint (44) is positive, $\eta > 0$, which means that the constraint is binding. So,

$$\eta\frac{p'(a)}{1-p(a)} > 0$$

Therefore, we have

$$c'(V^u) < c'(V) \Rightarrow V^u < V$$

from the strict convexity of $c(.)$. The delayed promised utility decreases over time.

Let $\theta^u = \theta - \eta\frac{p'(a)}{1-p(a)}$, then $\theta^u < \theta$,which tells us about the consumption path. Consumption decreases over time because $\theta^{-1} = u_c$.

Problem 12 *Prove that c_t is a decreasing when people are unemployed.*

How about effort level?

Problem 13 *Show that a_t is increasing over time when people are unemployed.*

Overall, we get the following model implications: optimal unemployment insurance says that longer unemployment period the agent stays, the less insurance she will be insured for. In this way, the planner induces the higher effort level. Although you cannot let people do what is optimal, such behavior can be achieved by giving out less consumption and promised utility over time. This model implies that time-varying unemployment insurance plan is optimal, under which the replacement rate θ goes down over time.

Problem 14 *Show that optimal time-invariant unemployment insurance is worse. (show that it is more expensive the provide the same amount of promised utility with time-invariant scheme.)*

3.3 One side lack of commitment[3]

We will study a model with one-said lack of commitment. This is an endowment economy (no production). There is no storage technology. Consider the village of fisherladies, where young granddaughters receive $y_s \in \{y_1, y_2, ..., y_S\}$ every period. y is iid. The probability that certain y_s realizes is Π_s. h_t is a history of shocks up to period t, i.e. $h_t = \{y_0,\, y_1,\, y_2,\, ...,\, y_t\}$.

First, if the granddaughter stays autarky, she will solve the problem

$$V_{AUT} = \sum_{t=0}^{\infty} \beta^t \sum_s \Pi_s u(y_s) = \frac{\sum_s \Pi_s u(y_s)}{1-\beta}$$

Note that here V^A is the utility of the young lady before endowment shock realizes.

Now we assume that the grandmother offers a contract to the granddaughter, which transfer resources and provide insurance to her. Grandmother is subject to commitment. But the young granddaughter may leave grandmother and break her word. Thus, this model is one-sided commitment model: an agent can walk away from a contract but the other cannot. Therefore, the contract should be always in the interest of granddaughter for her to stay.

We define a contract $f_t : H_t \rightarrow c \in [0, \tau]$. We will see next class that incentives compatibility constraint requires that at each node of history H_t, the contract should guarantee a utility which is higher than that in autarky.

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Last class, we have seen that first best result is not achievable for some environment. The unemployment insurance example is a typical principle-agent problem. Principle chooses first and agents choose next. The principle has to take agents' decision rule as given, but decision. And people can affect other's behavior, but not behavior rule. As described in chapter 4, decision rule will

[3]The source of this part is the updated Chapter 15 of Tom Sargent's Recursive Economic Theory.

change over time as the intertemporal effect. We need understand the optimality condition and envelope condition to master the nature of optimal policy. From now on, we will study the key material in chapter 15, the model with lack of commitment.

4.1 One side lack of commitment(II)

The endowment shock is iid. If the granddaughter stays autarky, she will solve the problem

$$V_{AUT} = \sum_{t=0}^{\infty} \beta^t \sum_s \Pi_s u(y_s) = \frac{\sum_s \Pi_s u(y_s)}{1-\beta}$$

Notice that the problem is different from Lucas tree model because of the shock realization timing. In Lucas tree model, shock is state variable because action takes place after shock is realized. Thus, action is indexed by shock. Here action is chosen before shock realization. Therefore, shock is not a state variable and action is state contingent.

In Lucas tree model, $V(s) = \max_c u(c) + \beta \sum_{s'} \Pi_{ss'} V(s')$. Here, if we write the problem recursively, it is $V = \max_{c_s} \sum_s \Pi_s u(c_s) + \beta V$.

Remember, the grandmother will make a deal with her granddaughter. They sign a contract to specify what to do in each state. $h_t \in H_t$. Contract is thus a mapping $f_t(h_t) \to c(h_t)$. With this contract, granddaughter gives y_t to the grandmother and receives $c_t = f_t(h_{t-1}, y_t)$. But if the granddaughter decided not to observe the contract, she consumes y_t this period and cannot enter a contract in the future, i.e. she has to live in autarky in the future.

For grandmother to keep granddaughter around her, the contract has to be of interest to granddaughter because although grandmother keeps her promise, granddaughter does not. There are two possible outcome if this contract is broken. One is that granddaughter goes away with current and future endowment. The other is that they renegotiate. We ignore the second possibility as no renegotiation is allowed. But we need deal with the possibility that the granddaughter says no to the contract and steps away.

The first best outcome is to warrant a constant consumption c_t to granddaughter who is risk averse. But because of the one-side lack of commitment, the first best is not achievable. The contract should always be attractive to granddaughter, otherwise, when she gets lucky with high endowment y_s, she will feel like to leave. So, this is a dynamic contract problem which the grandmother will solve in order to induce good behavior from granddaughter. The contract is dynamic because the nature keeps moving.

We say the contract $f_t(h_t)$ is incentive compatible or satisfies participation constraint if for all h_t,

$$u(f_t(h_t)) + \sum_{\tau=1}^{\infty} \beta^{\tau} \sum_s \Pi_s u(f_{t+\tau}(h_{t+\tau})) \geq u(y_s(h_t)) + \beta V^A \tag{48}$$

The left hand side is utility guaranteed in the contract. And the right hand side is the utility that granddaughter can get by herself. The participation constraint is not binding if y_s is low. And when y_s is high, PC is binding.

4.2 Problem of the grandmother

In this model, problem of the grandmother is to find an optimal contract that maximizes the value of such a contract of warranting V to her. We define the problem using recursive formula. Firstly, let's define the value of contract to grandmother if she promised V to her granddaughter by $P(V)$. $P(V)$ can be defined recursively as the following:

$$P(V) = \max_{\{c_s, \omega_s\}_{s=1}^{S}} \sum_s \Pi_s[(y_s - c_s) + \beta P(\omega_s)] \tag{49}$$

subject to

$$u(c_s) + \beta\omega_s \geq u(y_s) + \beta V^A \quad \forall s \tag{50}$$

$$\sum_s \Pi_s[u(c_s) + \beta\omega_s] \geq V \tag{51}$$

Notice that there are $1 + S$ constraints. The choice variables c_s, ω_s are state-contingent where ω_s is the promised utility committed to granddaughter in each state. In the objective function, $\sum_s \Pi_s(y_s - c_s)$ is the expected value of net transfer.

There are two sets of constraints. (50) is PC and (51) is promise keeping constraint.

4.3 Characterization of the Optimal Contract

In order to characterize the optimal contract, construct a Lagrangian.

$$\begin{aligned} P(V) \quad = \quad & \max_{\{c_s, \omega_s, \lambda_s\}_{s=1}^{S}, \mu} \sum_s \Pi_s[(y_s - c_s) + \beta P(\omega_s)] & (52) \\ & +\mu \left[\sum_s \Pi_s[u(c_s) + \beta\omega_s] - V \right] + \sum_s \lambda_s \left[u(c_s) + \beta\omega_s - u(y_s) - \beta V^A \right] & (53) \end{aligned}$$

First order conditions are the followings:

(c_s)

$$\Pi_s = (\lambda_s + \mu\Pi_s)u'(c_s) \tag{54}$$

(ω_s)

$$-\Pi_s P'(\omega_s) = \mu\Pi_s + \lambda_s \tag{55}$$

(μ)

$$\sum_s \Pi_s[u(c_s) + \beta\omega_s] = V \tag{56}$$

(λ)

$$u(c_s) + \beta\omega_s \geq u(y_s) + \beta V_{AUT} \tag{57}$$

In addition, Envelope Theorem tells that:

$$P'(v) = -\mu \tag{58}$$

Interpret the first order conditions:

1. (54) tells that in an optimal choice of c_s, the benefit of increasing one unit of c equals the cost of doing so. The benefit comes from two parts: first is $\mu\Pi_s u'(c_s)$ as increasing consumption helps grandmother to fulfill her promise and the second part is $\lambda_s u'(c_s)$ since increase in consumption helps alleviated the participation constraint. And the cost is the probability of state s occurs.

2. (55) equates the cost of increasing one unit of promised utility and the benefit. The cost to grandmother is $-\Pi_s P'(\omega_s)$ and the benefit is $\mu\Pi_s + \lambda_s$ which helps grandmother deliver promise and alleviate participation constraint.

Problem 15 *Prove envelope condition.*

How about the contract value $P(V)$. First, $P(V)$ can be positive or negative.

Claim 16 *(1) There exits V such that $P(V) > 0$*[4]*. (2) There exits V such that $P(V) > 0$*

Problem 17 *Prove the above claim is true.*

What's the largest V we will be concerned with? When PC will be binding for sure. If PC binds for the best endowment shock y_S, then PC holds for all the shock y_s. When granddaughter gets the best shock y_S, the best autarky value is then

$$V_{AM} = u(y_S) + \beta V_A$$

And the cheapest way to guarantee V_{AM} is to give constant consumption $\overline{c_S}$, such that

$$V_{AM} = \frac{u(c_S)}{1-\beta}$$

From this case, we can see that because of lack of commitment, the grandmother will have to give more consumption in some states. While when there is no lack of commitment, strict concavity of $u(.)$ implies that constant stream of consumption beats any $\{c_t\}$ that have the same present value, as there is no PC.

Problem 18 *Show $\overline{c_S} < y_S$.*

[4] When $P(V)$ is positive, it shows that there is gain from trade.

4.4 Characterizing the Optimal Contract

We will characterize the optimal contract by considering the two cases: (i) $\lambda_s > 0$ and (ii) $\lambda_s = 0$.

Firstly, if $\lambda_s = 0$, we have the following equations from FOC and EC:

$$P'(\omega_s) = -\mu \tag{59}$$

$$P'(V) = -\mu \tag{60}$$

Therefore, for s where PC is not binding,

$$V = \omega_s$$

c_s is the same for all s. For all s such that the Participation Constraint is not binding, the grandmother offers the same consumption and promised future value.

Let's consider the second case, where $\lambda_s > 0$. In this case, the equations that characterize the optimal contract are:

$$u'(c_s) = \frac{-1}{P'(\omega_s)} \tag{61}$$

$$u(c_s) + \beta\omega_s = u(y_s) + \beta V^A \tag{62}$$

Note that this is a system of two equations with two unknowns (c_s and ω_s). So these two equations characterize the optimal contract in case $\lambda_s > 0$. In addition, we can find the following properties by carefully observing the equations:

1. The equations don't depend on V. Therefore, if a Participation Constraint is binding, promised value does not matter for the optimal contract.

2. From the first order condition with respect to ω_s, $P'(\omega_s) = P'(v) - \frac{\lambda_s}{\Pi_s}$, where $\frac{\lambda_s}{\Pi_s}$ is positive. Besides, we know that P is concave. This means that $v < \omega_s$. In words, if a Participation Constraint is binding, the moneylender promises more than before for future.

Combining all the results we have got, we can characterize the optimal contract as follows:

1. Let's fix V_0. We can find a $y_s(V_0)$, where for $\forall y_s \leq y_s(V_0)$, the participation constraint is not binding. And vice versa.

2. The optimal contract that the moneylender offers to an agent is the following:

   If $y_t \leq y_s(v_0)$, the moneylender gives $(v_0, c(v_0))$. Both of them are the same as in the previous period. In other words, the moneylender offers the agent the same insurance scheme as before.

   If $y_t > y_s(v_0)$, the moneylender gives $(v_1, c(y_s))$, where $v_1 > v_0$ and c doesn't depend on v_0. In other words, the moneylender promises larger value to the agent to keep her around.

   So the path of consumption and promised value for an agent is increasing with steps.

Problem 19 *Show heuristically that average duration of increases in wellbeing of granddaughter gets longer over time.*