1 Jan 11

- What is an equilibrium?
  An equilibrium is a statement about what the outcome of an economy is. Tells us **what happens in an economy,** and by an ecomomy we mean a well defined environment in terms of primitives such as preferences and technology.
  Then an equilibrium is a particular mapping from the environment (preference, technology, information, market structure) to allocations where,

  1. Agents maximize
  2. Agents' actions are compatible.

- One of the important questions is, given the environment what type of equilibria we should look at. The economist doesn't have the right to choose what happens, but free to define the environment.

- For the theory to be able to predict precisely what is going to happen in a well defined environment, the outcome we define as the equilibrium needs to exist and must be unique. For this reason uniqueness is property that we want the equilibrium to have. We also know with certain assumptions that will be covered we can ensure the existence and uniqueness of an equilibrium outcome.

- By now, you have learned how to solve Social planner's problem (SPP) of neoclassical growth model with representative agent (RA-NGM), using dynamic programming. Also we know that solution to SPP, if it exists, is Pareto Optimal (PO). The solution of SPP can be interpreted as the allocation to be chosen if the God exists and has control over everything (by definition!) and is benevolent (maybe by definition). In other words, the solution does not predict what is going to happen in an environment but what will prevail as the wish of the social planner. As we will see, under certain conditions we are able define the 'right' environment such that the equilibrium outcome coincides with coincides with the solution to SPP.

- Other good things for solution to SPP is that, in RA-NGM with certain assumptions, we know that (i) it exists and (ii) it's unique.

- Besides, we have two welfare theorems (FBWT, SBWT) from Dave's class. If we carefully define the environment, those two theorems guarantee (loosely) that (i) under certain conditions, Arrow-Debrew Competitive Equilibrium (ADE, or Walrasian equilibrium or valuation equilibrium) is PO, and (ii) also under certain conditions, we can construct an ADE from a PO allocation.

- Using those elements, we can argue that ADE exists and is unique, and we just need to solve SPP to derive the allocation of ADE, which is much easier task than solve a monster named ADE.

- But we have another problem: The market assumed in ADE is not palatable to us in the sense that it is far from what we see in the world. So, next, we look at an equilibrium with sequential markets (Sequential Market Equilibrium, SME). Surprisingly, we can show that, for our basic RA-NGM, the allocation in SME and the allocation of ADE turn out to be the same, which let us conclude that even the allocation of the equilibrium with sequential markets can be analyzed using the allocation of SPP.

- Lastly, we will learn that equilibrium with sequential markets with recursive form (Recursive Competitive Equilibrium, RCE) gives the same allocation as in SME, meaning we can solve the problem using our best friend = Dynamic Programming.

- (Of course, these nice properties are available for limited class of models. We need to directly solve the equilibrium, instead of solving SPP, for large class of interesting models. We will see that Dynamic Programming method is also very useful for this purpose. We will see some examples later in the course.)

- In this class, we will go over some of the 'popular' notions of equilibrium in dynamic macroeconomics that might be different from a static Walrasian equilibrium covered last semester. Neoclassical growth model (NGM), which is the workhorse of modern macroeconomics will be our departure point.

2 Jan 13

2.1 Growth Model

2.1.1 Technology

- Represented by production function:

$$f : R_+^2 \to R_+ \quad \text{such that } y_t = f(k_t, n_t) \tag{1}$$

- We assume (i) Constant Returns to Scale (CRS, or homogeneous of degree one, meaning $f(\lambda k, \lambda n) = \lambda f(k, n)$), (ii) strictly increasing in both arguments, and ((iii) INADA condition, if necessary)

2.1.2 Preference

- We assume infinitely-lived representative agent (RA).[1]
- We assume that preference of RA is (i) time-separable (with constant discount factor $\beta < 1$), (ii) strictly increasing in both consumption (iii) strictly concave
- Our assumptions let us use the utility function of the following form:

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \tag{2}$$

- Initial capital stock k_0 is given.

  With these in hand the SP problem is,

$$\max_{\{c_t, n_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{3}$$

subject to[2]

$$k_{t+1} + c_t = f(k_t, n_t) + (1-\delta)k_t \tag{4}$$

$$c_t, k_{t+1} \geq 0 \tag{5}$$

$$n_t \in [0, 1] \tag{6}$$

$$n_t + l_t = 1 \tag{7}$$

$$k_0 \text{ is given} \tag{8}$$

2.1.3 Property 1: Existence

- Need to show (i) maximand is continuous function and (ii) constraint set is compact (closedness and boundedness).
- Not go into details but be aware that commodity space is infinite dimensional space (so exactly the same argument as in 701 where commodity space is finite dimensional space is not valid here). In particular, need to define commodity space as a topological linear space with sup-norm. For those interested, the argument involves the product topology and compactness of products of compact vector spaces

[1] For now, let's treat the economy as if there were only one agent in the economy. We might interpret it as the result of normalization (so the number of population is 1) of the economy with FINITE number of identical (sharing the same technology, preference, and allocation) agents. If we proceed to the economy with mass of zero measure agents, things will be not so trivial because changing allocation of one agent does not change the aggregate amount of resources in the economy (since, by assumption, measure of an agent is zero), but let's forget it for now.

[2] We can also define f (the production function) as including depreciation of capital. In the 1st class, Victor actually took this approach, but I modified the notation to make notation consistent across classes.

2.1.4 Property 2: Uniqueness

- Need (i) convex constraint set, and (ii) strictly concave function, which through our assumptions satisfied.

2.1.5 Property 3: Pareto Optimality

- Trivial (if assume finite number of agents).
- We know how to solve SPP of RA-NGM.
- But what we want to know is equilibrium (price and allocation).
- If we can apply welfare theorems to the allocation of SPP, we can claim that "God's will realizes" and can analyze allocation of SPP instead of directly looking at an equilibrium allocation.

  Note that we want the INADA conditions everywhere so that we have interior solutions. INADA conditions make sure that the nonnegativity constraints are irrelevant. This is so that we can use the First Order Conditions and not deal with the KT conditions. Since leisure is not in the utility function, we don't have to worry about it. Agent doesn't care about it so he will work as much as he can, therefore it must be that $l_t = 1$.

Rewriting the above problem with full deprication assumption:

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(f(k_t) - k_{t+1}) \tag{9}$$

$$k_0 \;\; given \tag{10}$$

Because of the INADA conditions we know that the solution is interior. So if $\{k_{t+1}^*\}$ is a solution then it satisfies the first order conditions,

$$-\beta^t u_c(f(k_t) - k_{t+1}) + \beta^{t+1} f_k(k_t + 1) u_c(f(k_{t+1}) - k_{t+2}) = 0 \tag{11}$$

But notice that these conditions are not sufficient. The above is a second order difference equation. It has two degrees of freedom. Therefore this equation is not enough to find the solution. There can be many sequences that satisfy it. We need to more conditions to pin down the right solution: The initial condition k_0 and the transversality condition.

2.2 From SPP to Valuation Equilibrium

- In order to use the argument above, we formalize the environment of RA-NGM in the way such that we can apply welfare theorems. By using (i) existence of solution to SPP, (ii) uniqueness of solution of SPP, and (iii) welfare theorems, we can claim that ADE (i) exists, (ii) is unique, (iii) and PO. However, market arrangement of ADE is not palatable to us in the sense that set of markets that are open in the ADE is NOT close to the markets in our real world. In other words, there is notion of time in ADE: all the trades are made before the history begins and there is no more choices after the history begins. So we would like to proceed to the equilibrium concept that allows continuously open markets, which is SME and we will look at it closely later.

2.2.1 Valuation Equilibrium

- In macroeconomics, we are interested in infinite- dimensional commodity spaces. We want to look at the relationship between competitive equilibrium and Pareto optimality in models with infinite-dimensional spaces. You looked at competitive equilibrium and Pareto optimality in 701, but the proofs of the FBWT and SBWT were done in the context of finite-dimensional commodity spaces. Here we want to show that the welfare theorems hold for economies with infinite dimensional spaces. To do this, we introduce the equilibrium concept 'valuation equilibrium'.
  Before defining valuation equilibrium, we first need to define the environment, unlike the social planner problem, which is a problem of allocation, in a AD world we will have exchange among agents. This requires definition of markets in which the relevant commodities to be defined are traded.:

1. $\mathcal{L}$, Commodity space:
   $\mathcal{L}$ is a topological vector space.

   Definition 1 (Vector Space) *A vector space is a space where the operations addition and scalar multiplication are defined, and where the space is closed under these two operations. i.e. If we take two sequences* $a = \{a_i\} \in \mathcal{L}$ *and* $b = \{b_i\} \in \mathcal{L}$, *it must be that* $a + b \in \mathcal{L}$. *And if we take* $k \in \mathcal{R}^+, k > 0$, *it must be that* $a \in \mathcal{L} \Rightarrow d = ka \in \mathcal{L} \ \forall \ k > 0$.

   Definition 2 (Topological Vector Space) *A topological vector space is a vector space which is endowed with a topology such that the maps* $(x, y) \to x + y$ *and* $(\lambda, x) \to \lambda x$ *are continuous. So we have to show the continuity of the vector operations addition and scalar multiplication.*

2. $X \subset \mathcal{L}$, Consumption Possibility Set:
   Specification of the 'things' that people could do (that are feasible to them). X contains every (individually) technologically feasible consumption point.

3. $U : X \to \mathcal{R}$, Specifies the preference ordering.

4. Y, Production possibility set.

What is an allocation in this environment? An allocation is a pair (x, y). On the other hand, a <u>feasible</u> allocation is (x, y) such that x=y (agents' actions need to be compatible). What are the commodites we need to make tradable in this environment? Output, labor services, capital services. So lets define the commodity space.

$$\mathcal{L} = \{\{s_t\}^{t=0,\infty} = \{s_{it}\}_{i=1,2,3}^{t=0,\infty},\ s_{it} \in R : \sup_t |s_t| < \infty\ \}$$

so our commodity space will be the set of bounded sequences in sup norm. Interested reader can refer to Stokey and Lucas (1989) for the reasons behind the choice of this particular space. Next is the definition of consumption possibility set X with full depriciation $\delta = 1$.

$$X(k_0) = \{x \in \mathcal{L} = l_\infty^3 : \exists\{c_t, k_{t+1}\}_{t=0}^\infty \geq 0 \text{ such that}$$

$$\begin{aligned} k_{t+1} + c_t &= x_{1t} + (1-\delta)k_t \qquad \forall t & (12) \\ x_{2t} &\leq k_t \qquad \forall t & (13) \\ x_{3t} &\in [0,1] \qquad \forall t\} \end{aligned}$$

- Interpretation is that x_{1t}=received goods at period t, x_{3t}=labor supply at period t, x_{2t}=capital service at period t. $k_{t+1} + c_t = x_{1t} + (1-\delta)k_t$ comes from real accounting. Note: capital and capital service are not the same thing. Think of the difference between a house and to rent a house.

- Production possibility set Y.

  Firm's problem is relatively simple as firm do not have intertemporal decision. Firms just rent production factors and produce period by period. $Y = \Pi_{t=0}^\infty \widehat{Y}$:

$$Y = \Pi_{t=0}^\infty \widehat{Y}_t : \widehat{Y}_t = \{y_{1t} \geq 0, y_{2t}, y_{3t} \leq 0 : y_{1t} \leq f(-y_{2t}, -y_{3t})\} \qquad (14)$$

  Interpretation is that y_1=production at period t, y_{3t}=labor input at period t, y_{2t}=capital input at period t.

- Then defining preferences over this space $U : X \to R$.

$$U(x) = \sum_{t=0}^\infty \beta^t u(c_t(x)) \qquad (15)$$

  c_t is unique given x because each x implies a sequence $\{c_t, k_{t+1}\}_{t=0}^\infty$. If $x_{3t} = k_t$, $c_t = x_{1t} + (1-\delta)x_{2t} - x_{2t+1}$.

- And the budget constraint of the agent

$$v(x) \leq 0$$

where the v is the valuation function, i.e. gives the value of a particular commodity bundle. Note that this formulation is a little different than the usual dot product representation of prices that we are used to. The reason behind this is the fact that it is not always possible to represent the prices as a vector due to the nature of infinete dimensional spaces as we will see below.

2.2.2 Prices

We want the price of a good at time t in terms of good at time 0. But that's not what v^* tells us. v^* is the cost or value of a commodity point. It is an arbitrary continuous linear function that is defined on our commodity space; it may not always be possible to find a sequence of prices to represent it with. Now we will go over the conditions under which this can be done. This is the Prescott-Lucas Theorem. This theorem tells us under which conditions the valuation function, or the pricing scheme, can be represented as an inner product for the infinite time horizon case.

Theorem 3 *(based on Prescott and Lucas 1972) If, in addition to the conditions to SBWT, $\beta < 1$ and u is bounded, then $\exists \hat{p}$ such that $(x^*, y^*, \hat{p})$ is a Quasi-Equilibrium and*

$$\hat{p}(x) = \sum_{t=0}^{\infty} \sum_{i=1}^{3} \hat{p}_{it} x_{it} \tag{16}$$

i.e. price system has an inner product representation.

Theorem 4 *(based on Prescott and Lucas 1972) If, in addition to the conditions to SBWT, $\beta < 1$ and u is bounded, then $\exists \hat{p}$ such that $(x^*, y^*, \hat{p})$ is a QE and*

$$\hat{p}(x) = \sum_{t=0}^{\infty} \sum_{i=1}^{3} \hat{p}_{it} x_{it} \tag{17}$$

i.e. price system has an inner product representations.

The result above is a special case of the more general theorem proved by Prescott and Lucas (1972). Before stating the theorem, let's define some notations. Let L^n be the subspace of L such that, for $x \in L^n$, $x = ((x_{11}, x_{21,x31}),$ (x_{12}, x_{22}, x_{32}), (x_{13}, x_{23}, x_{33}), ..., $(x_{1n-1}, x_{2n-1}, x_{3n-1})$, $(0,0,0)$, $(0,0,0)$, ...), i.e. $x_{it} = 0$ for $t \geq n$. Also Let x^n denote the projection of $x \in L$ on L^n.

Now we are ready to state the theorem in a more general form.

Theorem 5 *(Prescott and Lucas 1972) If (i) X is convex, (ii) preference is convex (these two conditions are same as those in the SBWT), (iii) for every n, $x^n \in X$ and $y^n \in Y$, (iv) if $x, x' \in X$ and $U(x) > U(x')$, then there exists and integer N such that, for $\forall n \geq N$, $U(x^n) > U(x')$, then, for a QE (x^*, y^*, p^*) with non-satiation point x^*, there exists $\hat{p}$ such that (1) $\hat{p}(x) = \lim_{n\to\infty} p(x^n)$ for a $p \in Dual(L)$, and (2) $(x^*, y^*, \hat{p})$ is a QE.*

Remark 6 *The results of the theorem allows us to consider the price system of a QE as the limit of a price system of the finite commodity space and thus represent price system of a QE by inner product representations. Intuitively, the additional two conditions of the theorem ((iii) and (iv)) tell that (iii) truncated consumption or production allocation is also feasible, and (iv) truncation of the sufficiently "future" consumption does not change the preference relationship.*

With the Prescott-Lucas Theorem, from now on we can use the inner product representation of prices (assuming the conditions for this theorem hold). So the SBWT told us that for the PO allocation, we can get prices that will support it as an ADE. The Prescott-Lucas Theorem told us that these prices can be written as an inner product. Next we define the Arrow-Debreu equilibrium in its general form (i.e. with a valuation function).

3 Jan 14

3.0.3 Definition of valuation equilibrium

Definition 7 (Arrow-Debreu/Valuation Equilibrium) *Valuation equilibrium is a feasible allocation (x^*, y^*) and a valuation function v^* such that,*

1. *x^* solves the consumer's problem.*

2. *y^* solves the firm's problem.*

3. *$x^* = y^*$ markets clear.*

With this definition in hand, next go back to our original aim of constructing this environment that is using welfare theorems to show the existence and uniquenes of competitive equilibrium.

3.1 Welfare Theorems (Linking SSP and AD)

Theorem 8 (First Basic Welfare Theorem) *Suppose that for all $x \in X$ there exists a sequence $\{x_n\}_{n=0}^{\infty}$ in$X$ converging to x with $u(x_n) \geq u(x)$ for all n (local nonsatiation). If an allocation (x^*, y^*) and a continous linear functional ν constitute a competitive equilibrium, then the allocation (x^*, y^*) is Pareto optimal.*

Theorem 9 (Second Basic Welfare Theorem) *If (i) X is convex, (ii) preference is convex (for $\forall x, x' \in X$, if $x' < x$, then $x' < (1-\theta)x' + \theta x$ for any $\theta \in (0,1)$), (iii) $U(x)$ is continuous, (iv) Y is convex, (v)Y has an interior point, then with any PO allocation (x^*, y^*) such that x^* is not a saturation point, there exists a continuous linear functional ν^* such that (x^*, y^*, ν^*) is a Quasi-Equilibrium with transfers((a) for $x \in X$ which $U(x) \geq U(x^*)$ implies $\nu^*(x) \geq \nu^*(\nu^*)$ and (b) $y \in Y$ implies $\nu^*(y) \leq \nu^*(y^*)$)*

Note that an additional assumption we are making for SBWT to go through in infinitely dimensional spaces is that Y has an interior point i.e.

$$\exists \bar{y} \in Y,\ B \subset Y,\ B \ \textit{open and}\ \bar{y} \in B$$

Also that the SBWT states that under certain conditions listed above, we can find prices to support any Pareto optimal allocation as a quasi equilibrium with transfers. Transfers are not relevant in our case since we are working in an representative agent environment with identical households. Taking care of the transfers still leaves us with Quasi-Equilibrium so SBWT by itself it does not say anything about the existence of Arrow-Debreu equlibrim. The following lemma takes care of this.

Lemma 10 *If, for (x^*, y^*, ν^*) in the theorem above, the budget set has cheaper point than x^* ($\exists x \in X$ such that $\nu(x) < \nu(x^*)$), then (x^*, y^*, ν^*) is a ADE.*

With the SBWT, we established that there exists a ν that will support our PO allocation as a competitive equilibrium. What's the problem with this approach? It is that SBWT only tells us that such a ν exists, it doesn't tell us what it is. Assuming the prices has inner product representations, we will next show how to construct them using the necessary FOCs of the households and firms. Before doing that lets have a brief rewiev of where are we coming from.

Remember, our main purpose is to be able to apply the welfare theorems to the most commonly used models in macroeconomics where we have an infinite-dimensional commodity space. Until now, we set up an environment (Arrow-Debreu economy) (which consisted of the commodity space, consumption possibility set, production possibility set, and preferences) with infinite-dimensional commodity space and we stated that under certain conditions the Welfare Theorems hold in this environment. Now we will map the growth model into the environment that we talked about until here, and show that in the context of the growth model the assumptions we need for the Welfare Theorems are satisfied. Then we can conclude that any competitive equilibrium allocation is Pareto optimal and moreover we can support a PO allocation with some prices as a competitive equilibrium. This result is very important in macroeconomics. It helps us in solving for the equilibria. With the FBWT and SBWT, we can just solve for the PO allocations and then get the prices. This makes life much easier.

What's next? Constructing the prices.

3.2 The Growth Model

Houshold Problem:

$$\max_{\{c_t, l_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t(x)) \tag{18}$$

subject to

$$\sum_t (p_{1t}x_{1t} + p_{2t}x_{2t} + p_{3t}x_{3t}) = 0 \tag{19}$$

Look at the First Order Conditions with respect to x_{1t} and x_{1t+1} and using the relationships between k, c and x we can get (with full deprication),,

$$\frac{p^*_{1t}}{p^*_{1t+1}} = \frac{u_c(x^*_{1t} - k^*_{t+1})}{\beta u_c(x^*_{1t+1} - k^*_{t+2})} = F_k(k^*_t, n^*_t) \tag{20}$$

Also using the firms problem we can get,

$$\frac{p^*_{2t}}{p^*_{1t}} = F_k(k^*_t, n^*_t) \tag{21}$$

$$\frac{p^*_{3t}}{p^*_{1t}} = F_n(k^*_t, n^*_t) \tag{22}$$

where the variables with asterix are the optimal allocation from the solution to the SPP.

Note that in a AD world all the trades are made at period 0, and after the history starts, all that agents can do is to follow what was promised (full commitment is assumed). But this is an unrealistic market arrangement. To see the point this more clearly, imagine the decision of an agent who is going to be born in period t. At period 0, although the agents is not born yet, the agent also joins the market at period 0. At period 0, she trades (by solving the consumer's problem above), and she goes to limbo from period 0 (after trade) until period t-1, and she is born in period t. As we want the market arrangement of the model to be comparable to the one in the real world, this unrealistic assumption on market arrangement is not desirable. That is the motivation to consider Sequential Market Equilibrium (SME), where markets are open every period next.

4 Jan 17

4.1 Sequence of Marktes Equlibrium

Note that two things are important in the setup of this new environment: (i) there are infinitely many markets in SME (because markets are open every period), which means that there are infinitely many budget constraints to be considered, (ii) an allocation in SME has to give as much utility as in ADE to agents in order to be PO. Otherwise, agents will choose to trade in AD markets, meaning SME doesn't work. Remember that we cannot force agents to do certain things.

Also note that there are many ways of arranging markets so that the equilibrium allocation is equivalent to that in ADE. We'll see two of them. Note that if the number of markets open is TOO FEW, we cannot achieve the allocation in the ADE (incomplete market). To the contrary, if the number of markets are TOO MANY, we can close some of the markets and still achieve the ADE allocation in this market arrangement. Also it means that there are many ways to achieve ADE allocation because some of the market instruments are redundant and can be substituted by others. If the number of markets are not TOO FEW nor TOO MANY, we call it JUST RIGHT. The fact that these two arrangements have to be equal at the end of the day requires us to introduce a new tradable commodity that will utilize households to move resources in time in both directions, loans.

With sequential markets, people have capital k_t, and rent it to the firm at rental R_t. People have time 1 and rent it to firm at wage w_t. They also consume c_t and save k_{t+1}. Agents can also borrow and lending one period loan l_{t+1} at price q_t. Then the budget constraint at time t is

$$k_t R_t + w_t + l_t = c_t + k_{t+1} + q_t l_{t+1} \tag{23}$$

Next thing we notice is, we can close the market of loans without changing the resulting allocation. This is because we need someone to lend you loans in order that you borrow loans, but there is only one agents in the economy. But surprisingly, we will see that even though there is no trade in certain markets in equilibrium, we can solve for prices in those markets, because prices are determined even though there is no trade in equilibrium, and agents do not care if actually trade occurs or not because they just look at prices in the market (having market means agents do not care about the rest of the world but the prices in the market). Using this technique, we can determine prices of all market instruments even though they are redundant in equilibrium. This is the virtue of Lucas Tree Model and this is the fundamental for all finance literature (actually, we can price any kinds of financial instruments in this way. we will see this soon.). Writing down the problem of the household in SM world,

Consumer's Problem in SME can be written as follows:

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{24}$$

subject to

$$k_t R_t + w_t + l_t = c_t + k_{t+1} + q_t l_{t+1} \quad \forall t \tag{25}$$

$$k_0 \text{ is given} \tag{26}$$

and the firm solves the following static problem,

$$\max_{k_t, n_t} f(k_t, 1) - R_t k_t - w_t n_t$$

and with the well defined environments and behavioral assumptions we are ready to define equlibrium in this world.

Definition A sequence of markets equilibrium is prices and allocation $\{w_t^*, r_t^*, q_t^*, c_t^*, k_{t+1}^*, 0, n_t^*\}$ such that

- Households maximize utility,

$$\begin{aligned} \{c_t^*, k_{t+1}^*, l_t^* &= 0\}_{t=0,\infty} \in \arg \max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\ s.t.\ k_t R_t + w_t + l_t &= c_t + k_{t+1} + q_t l_{t+1} \end{aligned}$$

- Producers maximaze profits,

$$\{k_t, n_t\} \in \arg \max_{k_t, n_t} f(k_t, 1) - R_t k_t - w_t n_t$$

- Feasibility and Markets clear,

$$\begin{aligned} c_t + k_{t+1} &= f(k_t, 1) \\ l_{t+1} &= 0 \end{aligned}$$

So far we have worked with two different form of market arrangements, AD and SME, and defined equilibrium in both of these worlds. We have also seen that the problem defined in the sequential market setup is at least as hard, if not more, as the AD setup to solve. The way we went around this in the AD setup was the use of welfare theorems by making sure a unique PO solution to SPP existed, which coincided with the solution to AD problem, furthermore we were able to support this allocation with appropriate prices as the equilibrium outcome of the AD problem. If we can establish the equivalence between these two equilibrium concepts than we can figure out the euqilibrium outcome in the SME world without having to solve the sequential problem. This brings us to the following propositions.

Proposition 11 *Let the allocation* $\{x_{it}^*, y_{it}^*\}_{t=0,i=1,2,3}^{\infty}$ *and a valuation function (assuming it has an inner product representation)* $\widehat{p}(x) = \sum_{t=0}^{\infty} \sum_{i=1}^{3} p_{it}^* x_{it}$ *form an AD equilibrium. Then there exists an allocation* $\{c_t^*,\ k_{t+1}^*, n_t^*, l_{t+1}^*\}_{t=0}^{\infty}$ *, a sequence of prices* $\{R_t^*,\ w_t^*\}_{t=0}^{\infty}$ *that can be constructed such that it is an SME.*

The way to prove this proposition is first to establish an implication from AD budget set and technology constraints to SME budget set and technology constraints. Once we do so, we can show that given our constructed prices, our constructed allocations satisfy the sufficient FOCs of the SM problem and thus form an SME and this is done throuhgly in the homework solutions. The opposite implication is stated in the following proposition.

Proposition 12 *Let the allocation* $\{c_t^*,\ k_{t+1}^*, n_t^*, l_{t+1}^*\}_{t=0}^{\infty}$ *and a sequence of prices* $\{R_t^*,\ w_t^*\}_{t=0}^{\infty}$ *form an SME equilibrium. Then there exists an allocation* $\{x_{it}^*, y_{it}^*\}_{t=0,i=1,2,3}^{\infty}$*, and a valuation function (assuming it has an inner product representation)* $\widehat{p}(x) = \sum_{t=0}^{\infty} \sum_{i=1}^{3} p_{it}^* x_{it}$ *to be constructed such that it is an ADE.*

A similar line of logic can be followed to prove this proposition as well with one additional condition. We know by the definition of the AD equilibrium, the valuation function must be linear and continuous. Linearity is trivial to show in this case with our implied valuation function. The other important property continiuty seems to be harder to show but we know that in the infinite dimensional spaces, boundedness is a sufficient condition for continuity and we can exploit this fact. Once we do so, we have our AD equilibrium constructed from SME objects.

Why we did not go from SPP to SME directly? Because Welfare Theorems are available only between SPP and ADE, though what we want is to derive equivalence between SPP allocation and SME (or RCE) allocation. For some particular environments, as the equivalence result between SPP allocation and RCE allocation is available, we can exploit the result and can argue directly that some RCE allocation is indeed PO. Next we introduce uncertainty into our deterministic world. The most conventional way of doing it is introducing productivity shocks to technology.

4.2 Stochastic Growth Model

4.2.1 Markov Process

In this course, we will concentrate on Markov productivity shock. Considering shock is really a pain, so we want to use less painful one. Markov shock is a stochastic process with the following properties:

1. there are FINITE number of possible states for each time. More intuitively, no matter what happened before, tomorrow will be represented by one of a finite set.

2. what only matters for the realization tomorrow is today's state. More intuitively, no matter what kind of history we have, the only thing you need to predict realization of shock tomorrow is TODAY's realization.

More formally, for each period, suppose either z^1 or z^2 happens [3]. Denote z_t is the state of today and Z_t is a set of possible state today, i.e. $z_t \in Z_t = \{z^1, z^2\}$ for all t. Since the shock follow Markov process, the state of tomorrow will only depend on today's state. So let's write the probability that z^j will happen tomorrow, conditional on today's state being z^i as $\Gamma_{ij} = prob[z_{t+1} = z^j | z_t = z^i]$. Since Γ_{ij} is a probability, we know that

$$\sum_j \Gamma_{ij} = 1 \qquad for\ \forall i \tag{27}$$

Notice that 2-state Markov process is summarized by 6 numbers: z^1, z^2, Γ_{11}, Γ_{12}, Γ_{21}, Γ_{22}.

[3] Here we restrict our attention to the 2-state Markov process, but increasing the number of states to any finite number does not change anything fundamentally.

The great beauty of using Markov process is we can use the explicit expression of probability of future events, instead of using ambigious operator called expectation, which very often people don't know what it means when they use.

4.2.2 Representation of History

- Let's concentrate on 2-state Markov process. In each period, state of the economy is $z_t \in Z_t = \{z^1, z^2\}$.
- Denote the history of events up to t (which of $\{z^1, z^2\}$ happened from period 0 to t, respectively) by $h_t = \{z_1, z_2, ..., z_t\} \in H_t = Z_0 \times Z_1 \times ... \times Z_t$.
- In particular, $H_0 = \emptyset$, $H_1 = \{z^1, z^2\}$, $H_2 = \{(z^1, z^1), (z^1, z^2), (z^2, z^1), (z^2, z^2)\}$.
- Note that even if the state today is the same, past history might be different. By recording history of event, we can distinguish the two histories with the same realization today but different realizations in the past (think that the current situation might be "you do not have a girl friend", but we will distinguish the history where "you had a girl friend 10 years ago" and the one where you didn't (tell me if it is not an appropriate example...).)
- Let $\Pi(h_t)$ be the unconditional probability that the particular history h_t does occur. By using the Markov transition probability defined in the previous subsection, it's easy to show that (i) $\Pi(h_0) = 1$, (ii) for $h_t = (z^1, z^1)$, $\Pi(h_t) = \Gamma_{11}$ (iii) for $h_t = (z^1, z^2, z^1, z^2)$, $\Pi(h_t) = \Gamma_{12}\Gamma_{21}\Gamma_{12}$.

4.3 SPP,ADE, and SME in a Stochastic RA-NGM

4.3.1 Big Picture

- Now we have Nature, who decides the realization of productivity shock every period.
- Social Planner's Problem (the benevolent God's choice) in this world is a state-contingent plan, i,e, optimal consumption and saving (let's forget about labor-leisure choice in this section for simplicity [4]) choice for all possible nodes (imagine the nodes of a game tree. we need to solve optimal consumption and saving for each node in the tree).
- Notice that the number of nodes for which we have to solve for optimal consumption and saving is countable. This feature allows us to use the same argument as the deterministic case to deal with the problem. The only difference is that for deterministic case, the number of nodes is equal to number of periods (which is infinite but countable), but here the number of nodes is equal to the number of date-events (which is also infinite but countable).

[4] Or just assuming the consumers do not value leisure (drop leisure from utility function) is enough to let agents work as much as possible in this world.

- More mathematically, the solution of the problem is the mapping from the set of date-events (which is specified by history) to the set of feasible consumption and saving.

4.3.2 The SPP and ADE

$$\max_{\{k_{t+1}(h_t),c_t(h_t)\}} \sum_{t=0}^{\infty} \beta^t \sum_{h_t \in H_t} \Pi(h_t) u(c_t(h_t)) \tag{28}$$

subject to

$$z_t f[k_t(h_{t-1}), 1] = c_t(h_t) + k_{t+1}(h_t) \tag{29}$$

$$k_0 \qquad \text{given} \tag{30}$$

Couple of comments:

- Here capital in indexed by the time it is used. k_t is a mapping from h_{t-1} because the amount of capital used today is determined yesterday. Alternatively, you can index capital by the time when the amount is chosen, but the former notation is the tradition and more common so we use the former notation. Anyway it is just a matter of notation.
- An assumption here is leisure is not valued by consumer so time of consumer is inelastically supplied for working.
- Measurability (very loosely) means whether an object is known when agents make their choice. Choice of agents must not depend on an object which agents do not know when they make choices.

Let's denote the solution as $x^* = \{c_t^*(h_t), k_{t+1}^*(h_t)\}$. It's easy to show that (i) the utility function is strictly concave, (ii) the constraint set is convex, (iii) commodity set is same as deterministic case. Using these properties, we can show (i) existence of the solution, (ii) uniqueness of the solution, (iii) FBWT (ADE is PO), (iv) SBWT (PO allocation can be supported as an ADE), (v) price system has a nice inner product representation (Lucas and Prescott (1972))[5], (vi) some equations (derived from FOC of SPP) which characterize the ADE allocation (remember (21), (22), and (**??**)).

[5] Remember the deterministic version of Lucas-Prescott Theorem. For the stochastic model, we need two additional assumptions, corresponding (iii) and (iv) of the deterministic one. Very loosely, we need additionally, that (iii) and (iv) of the deterministic one hold for truncation with respect to certain history when probability of occurrence of the history with truncation is sufficiently small. For more details, see Lucas-Prescott paper or Harris (p62-64).