1 April 21

1.1 One sided lack of commitment

We will study a model with one-said lack of commitment. This is an endowment economy (no production). There is no storage technology. Consider the village of fisherladies, where young granddaughters receive $y_s \in \{y_1, y_2, ..., y_S\}$ every period. y is iid. The probability that a certain y_s realized is Π_s. h_t is a history of shocks up to period t, i.e. $h_t = \{y_0,\ y_1,\ y_2,\ ...,\ y_t\}$.

First, if the granddaughter stays autarky, she will enjoy total utility,

$$V_{AUT} = \sum_{t=0}^{\infty} \beta^t \sum_s \Pi_s u(y_s) = \frac{\sum_s \Pi_s u(y_s)}{1-\beta}$$

Note that here V^A is the utility of the young lady before endowment shock is realized.

Now we assume that the grandmother offers a contract to the granddaughter, which transfer resources and provide insurance to her. Grandmother can commit. But the young granddaughter may leave grandmother and break her word. Thus, this model is one-sided commitment model: an agent can walk away from a contract but the other cannot. Therefore, the contract should be always in the interest of granddaughter for her to stay.

We define a contract $f_t : H_t \to c \in [0, \tau]$. We will see next class that incentives compatibility constraint requires that at each node of history H_t, the contract should guarantee a utility which is higher than that in autarky.

Notice that the problem is different from Lucas tree model because of the shock realization timing. In Lucas tree model, shock is state variable because action takes place after shock is realized. Thus, action is indexed by shock. Here action is chosen before shock realization. Therefore, shock is not a state variable and action is state contingent.

In Lucas tree model, $V(s) = \max_c u(c) + \beta \sum_{s'} \Pi_{ss'} V(s')$. Here, if we write the problem recursively, it is $V = \max_{c_s} \sum_s \Pi_s u(c_s) + \beta V$.

Remember, the grandmother will make a deal with her granddaughter. They sign a contract to specify what to do in each state. $h_t \in H_t$. Contract is thus a mapping $f_t(h_t) \to c(h_t)$. With this contract, granddaughter gives y_t to the grandmother and receives $c_t = f_t(h_{t-1}, y_t)$. But if the granddaughter decided not to observe the contract, she consumes y_t this period and cannot enter a contract in the future, i.e. she has to live in autarky in the future.

For grandmother to keep granddaughter around her, the contract has to be of interest to granddaughter because although grandmother keeps her promise, granddaughter does not. There are two possible outcome if this contract is broken. One is that granddaughter goes away with current and future endowment. The other is that they renegotiate. We ignore the second possibility as no renegotiation is allowed. But we need deal with the possibility that the granddaughter says no to the contract and steps away.

The first best outcome is to warrant a constant consumption c_t to granddaughter who is risk averse. But because of the one-side lack of commitment, the first best is not achievable. The contract should always be attractive to granddaughter, otherwise, when she gets lucky with high endowment y_s, she will feel like to leave. So, this is a dynamic contract problem which the grandmother will solve in order to induce good behavior from granddaughter. The contract is dynamic because the nature keeps moving.

We say the contract $f_t(h_t)$ is incentive compatible or satisfies participation constraint if for all h_t,

$$u(f_t(h_t)) + \sum_{\tau=1}^{\infty} \beta^{\tau} \sum_{s} \Pi_s u(f_{t+\tau}(h_{t+\tau})) \geq u(y_s(h_t)) + \beta V^A \tag{1}$$

The left hand side is utility guaranteed in the contract. And the right hand side is the utility that granddaughter can get by herself. The participation constraint is not binding if y_s is low. And when y_s is high, PC is binding.

1.1.1 Problem of the grandmother

In this model, problem of the grandmother is to find an optimal contract that maximizes the value of such a contract of warranting V to her. We define the problem using recursive formula. Firstly, let's define the value of contract to grandmother if she promised V to her granddaughter by $\Omega(V)$. $\Omega(V)$ can be defined recursively as the following:

$$\Omega(V) = \max_{\{c_s,\omega_s\}_{s=1}^{S}} \sum_{s} \Pi_s[(y_s - c_s) + \beta\Omega(\omega_s)] \tag{2}$$

subject to

$$u(c_s) + \beta\omega_s \geq u(y_s) + \beta V^A \quad \forall s \tag{3}$$

$$\sum_{s} \Pi_s[u(c_s) + \beta\omega_s] \geq V \tag{4}$$

Notice that there are $1 + S$ constraints. The choice variables c_s, ω_s are state-contingent where ω_s is the promised utility committed to granddaughter in each state. In the objective function, $\sum_s \Pi_s(y_s - c_s)$ is the expected value of net transfer.

There are two sets of constraints. (3) is PC and (4) is promise keeping constraint.

The First Order Conditions to the grandmother's problem are:

(c_s)

$$\Pi_s = (\lambda_s + \mu\Pi_s)u'(c_s) \tag{5}$$

(ω_s)

$$-\Pi_s\Omega'(\omega_s) = \mu\Pi_s + \lambda_s \tag{6}$$

(μ)

$$\sum_{s} \Pi_s[u(c_s) + \beta\omega_s] = V \tag{7}$$

(λ)

$$u(c_s) + \beta\omega_s \geq u(y_s) + \beta V^A \tag{8}$$

In addition, Envelope Theorem tells that:

$$\Omega'(v) = -\mu \tag{9}$$

Interpreting the first order conditions:

1. (5) tells that in an optimal choice of c_s, the benefit of increasing one unit of c equals the cost of doing so. The benefit comes from two parts: first is $\mu\Pi_s u'(c_s)$ as increasing consumption helps grandmother to fulfill her promise and the second part is $\lambda_s u'(c_s)$ since increase in consumption helps alleviated the participation constraint. And the cost is the probability of state s occurs.

2. (6) equates the cost of increasing one unit of promised utility and the benefit. The cost to grandmother is $-\Pi_s\Omega'(\omega_s)$ and the benefit is $\mu\Pi_s + \lambda_s$ which helps grandmother deliver promise and alleviate participation constraint.

How about the contract value $\Omega(V)$. First, $\Omega(V)$ can be positive or negative.

Claim 1 *(1) There exits V such that $\Omega(V) > 0$*[1]*.*

What's the largest V we will be concerned with? When PC will be binding for sure. If PC binds for the best endowment shock y_S, then PC holds for all the shock y_s. When granddaughter gets the best shock y_S, the best autarky value is then

$$V_{AM} = u(y_S) + \beta V_A$$

And the cheapest way to guarantee V_{AM} is to give constant consumption $\overline{c_S}$, such that

$$V_{AM} = \frac{u(c_S)}{1-\beta}$$

From this case, we can see that because of lack of commitment, the grandmother will have to give more consumption in some states. While when there is no lack of commitment, strict concavity of $u(.)$ implies that constant stream of consumption beats any $\{c_t\}$ that have the same present value, as there is no PC.

1.1.2 Characterizing the Optimal Contract

We will characterize the optimal contract by considering the two cases: (i) $\lambda_s > 0$ and (ii) $\lambda_s = 0$.

Firstly, if $\lambda_s = 0$, we have the following equations from FOC and EC:

$$\Omega'(\omega_s) = -\mu \tag{10}$$

$$\Omega'(V) = -\mu \tag{11}$$

[1] When $P(V)$ is positive, it shows that there is gains from trade.

Therefore, for s where PC is not binding,

$$V = \omega_s$$

c_s is the same for all s. For all s such that the Participation Constraint is not binding, the grandmother offers the same consumption and promised future value.

Let's consider the second case, where $\lambda_s > 0$. In this case, the equations that characterize the optimal contract are:

$$u'(c_s) = \frac{-1}{\Omega'(\omega_s)} \tag{12}$$

$$u(c_s) + \beta\omega_s = u(y_s) + \beta V^A \tag{13}$$

Note that this is a system of two equations with two unknowns (c_s and ω_s). So these two equations characterize the optimal contract in case $\lambda_s > 0$. In addition, we can find the following properties by carefully observing the equations:

1. The equations don't depend on V. Therefore, if a Participation Constraint is binding, promised value does not matter for the optimal contract.

2. From the first order condition with respect to ω_s, $\Omega'(\omega_s) = \Omega'(v) - \frac{\lambda_s}{\Pi_s}$, where $\frac{\lambda_s}{\Pi_s}$ is positive. Besides, we know that Ω is concave. This means that $v < \omega_s$. In words, if a Participation Constraint is binding, the moneylender promises more than before for future.

Combining all the results we have got, we can characterize the optimal contract as follows:

1. Let's fix V_0. We can find a $y_s(V_0)$, where for $\forall y_s \leq y_s(V_0)$, the participation constraint is not binding. And vice versa.

2. The optimal contract that the moneylender offers to an agent is the following:

   If $y_t \leq y_s(v_0)$, the moneylender gives $(v_0, c(v_0))$. Both of them are the same as in the previous period. In other words, the moneylender offers the agent the same insurance scheme as before.

   If $y_t > y_s(v_0)$, the moneylender gives $(v_1, c(y_s))$, where $v_1 > v_0$ and c doesn't depend on v_0. In other words, the moneylender promises larger value to the agent to keep her around.

   So the path of consumption and promised value for an agent is increasing with steps.

1.2 Two sided lack of commitment

1.2.1 The Model

- Two brothers, A and B, and neither of them has access to a commitment technology. In other words, the two can sign a contract, but either of them can walk away if he does not feel like observing it.

- This is an endowment economy (no production) and there is no storage technology. Endowment is represented by $(\mathrm{y}_s^A, y_s^B) \in Y \times Y$, where y_s^i is the endowment of brother i. s=(y_s^A, y_s^B) follows a Markov process with transition matrix $\Gamma_{ss'}$.

1.2.2 First Best Allocation

We will derive the first best allocation by solving the social planner's problem:

$$\max_{\{c_i(h_t)\}_{\forall h_t, \forall i}} \lambda^A \sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t) u(c^A(h_t)) + \lambda^B \sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t) u(c^B(h_t))$$

subject to the resource constraint:

$$\sum_i c^i(h_t) - y^i(h_t) = 0 \qquad \forall h_t \quad \text{w/ multiplier } \gamma(h_t)$$

The First Order Conditions are:

$$\begin{aligned} FOC(c^A(h_t)) &: \quad \lambda^A \beta^t \Pi(h_t) u'(c^A(h_t)) - \gamma(h_t) = 0 \\ FOC(c^B(h_t)) &: \quad \lambda^B \beta^t \Pi(h_t) u'(c^B(h_t)) - \gamma(h_t) = 0 \end{aligned}$$

Combining these two yields:

$$\frac{\lambda^A}{\lambda^B} = \frac{u'(c^A(h_t))}{u'(c^B(h_t))}$$

The first best allocation will not be achieved if there is no access to a commitment technology. Therefore, the next thing we should do is look at the problem the planner is faced with in the case of lack of commitment. Due to lack of commitment, the planner needs to make sure that at each point in time and in every state of the world, h_t, both brothers prefer what they receive to autarky. Now we will construct the problem of the planner adding these participation constraints to his problem.

1.2.3 Constrained Optimal Allocation

The planner's problem is:

$$\max_{c^A(h_t), c^B(h_t)} \lambda^A \sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t) u(c^A(h_t)) + \lambda^B \sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t) u(c^B(h_t))$$

$$\sum_i c^i(h_t) - y^i(h_t) = 0 \qquad \forall h_t \quad \text{w/ multiplier } \gamma(h_t)$$

$$\sum_{r=t}^{\infty} \beta^{r-t} \sum_{h_r} \Pi(h_r|h_t) u(c^i(h_r)) \geq \Omega_i(h_t) \qquad \forall h_t, \forall i \quad \text{w/ multiplier } \mu_i(h_t)$$

where $\Omega_i(h_t) = \sum_{r=0}^{\infty} \beta^{r-t} \sum_{h_r} \Pi(h_r|h_t) u(y_i(h_t))$ (the autarky value)

- How many times does $c^A(h_{17})$ appear in this problem? Once in the objective function, once in the feasibility constraint, and it appears in the participation constraint from period 0 to period 16.
- We know that the feasibility constraint is always binding so that $\gamma(h_t) > 0 \quad \forall h_t$. On the other hand the same is not true for the participation constraint.
- Both participations cannot be binding but both can be nonbinding.
- Define $M_i(h_{-1}) = \lambda^i$

  and $M_i(h_t) = \mu_i(h_t) + M_i(h_{t-1})$

  (We will use these definitions for the recursive representation of the problem in the next class)

1.2.4 Recursive Representation of the Constrained SPP

We want to transform this problem into the recursive, because it would be easier to solve the optimal allocation with a computer. Now we will show how to transform the sequential problem with the participation constraints into its recursive representation.

Before we do this transformation, first recall the Lagrangian associated with the sequential representation of the social planner's problem:

$$\lambda^A \sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t) u(c^A(h_t)) + \lambda^B \sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t) u(c^B(h_t))$$
$$+ \sum_{t=0}^{\infty} \beta^t \sum_{h_t \in H_t} \Pi(h_t) \sum_{i=1}^{2} \mu_i(h_t) \left[\sum_{r=t}^{\infty} \beta^{r-t} \sum_{h_r} \Pi(h_r|h_t) u(c^i(h_r)) - \Omega_i(h_t) \right]$$
$$+ \sum_{t=0}^{\infty} \sum_{h_t \in H_t} \gamma(h_t) \left[\sum_{i=1}^{2} c_i(h_t) - \sum_{i=1}^{2} y_i(h_t) \right]$$

Note that here the Lagrangian multiplier associated witih the participation constraint for brother i after history h_t is $\beta^t \Pi(h_t) \mu_i(h_t)$.

Now we will use the definitions from the previous class (for $M_i(h_t)$) to rewrite the above Lagrangian in a more simple form,

Collect terms and rewrite,

$$\sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t) \sum_{i} \left\{ \lambda^i u(c^i(h_t) + \mu_i(h_t) \left[\sum_{r=t}^{\infty} \beta^{r-t} \sum_{h_r} \Pi(h_r|h_t) u(c^i(h_r)) - \Omega_i(h_t) \right] \right\}$$
$$+ \sum_{t=0}^{\infty} \sum_{h_t \in H_t} \gamma(h_t) \left[\sum_{i=1}^{2} c_i(h_t) - \sum_{i=1}^{2} y_i(h_t) \right]$$

Note that, $\sum_{r=t}^{\infty}\beta^{r-t}\sum_{h_r}\Pi(h_r|h_t)u(c^i(h_r))-\Omega_i(h_t) = u(c^i(h_t))+\sum_{r=t+1}^{\infty}\beta^{r-t}\sum_{h_r}\Pi(h_r|h_t)u(c^i(h_r))-\Omega_i(h_t)$,

and that $\Pi(h_r|h_t)\Pi(h_t) = \Pi(h_r)$ so using these, rewrite as,

$$\sum_{t=0}^{\infty}\beta^t\sum_{h_t}\Pi(h_t)\sum_i\left\{\lambda^i u(c^i(h_t) + \mu_i(h_t)u(c^i(h_t)\right\}$$
$$+\sum_{t=0}^{\infty}\sum_{h_r}\sum_i\mu_i(h_t)\left[\sum_{r=t+1}^{\infty}\beta^r\sum_{h_r}\Pi(h_r)u(c^i(h_r)) - \Omega_i(h_t)\right]$$
$$+\sum_{t=0}^{\infty}\sum_{h_t\in H_t}\gamma(h_t)\left[\sum_{i=1}^{2}c_i(h_t) - \sum_{i=1}^{2}y_i(h_t)\right]$$

Collect the terms of $u(c^i(h_r)$,

$$\sum_{t=0}^{\infty}\beta^t\sum_{h_t}\Pi(h_t)\sum_i\left\{\left[\lambda^i + \sum_{r=0}^{t-1}\mu_i(h_r)\right]u(c^i(h_t) + \mu_i(h_t)\left[u(c^i(h_t) - \Omega_i(h_t)\right]\right\}$$
$$+\sum_{t=0}^{\infty}\sum_{h_t\in H_t}\gamma(h_t)\left[\sum_{i=1}^{2}c_i(h_t) - \sum_{i=1}^{2}y_i(h_t)\right]$$

Introduce the variable $\mathrm{M}_i(h_t)$ and define it recursively as,

$$\begin{aligned} M_i(h_t) &= M_i(h_{t-1}) + \mu_i(h_t) \\ M_i(h_{-1}) &= \lambda^i \end{aligned}$$

where $\mathrm{M}_i(h_t)$ denotes the Pareto weight plus the cumulative sum of the Lagrange multipliers on the participation constraints at all periods from 1 to t.

So rewrite the Lagrangian once again as,

$$\sum_{t=0}^{\infty}\beta^t\sum_{h_t}\Pi(h_t)\sum_i\left\{M_i(h_{t-1})u(c^i(h_t) + \mu_i(h_t)\left[u(c^i(h_t) - \Omega_i(h_t)\right]\right\}$$
$$+\sum_{t=0}^{\infty}\sum_{h_t\in H_t}\gamma(h_t)\left[\sum_{i=1}^{2}c_i(h_t) - \sum_{i=1}^{2}y_i(h_t)\right]$$

Now we are ready to take the First Order Conditions:

$$\frac{u'(c^A(h_t))}{u'(c^B(h_t))} = \frac{M_A(h_{t-1}) + \mu_A(h_t)}{M_B(h_{t-1}) + \mu_B(h_t)}$$

$$\left[\sum_{r=t}^{\infty}\beta^{r-t}\sum_{h_r}\frac{\Pi(h_r)}{\Pi(h_t)}u(c^i(h_r)) - \Omega_i(h_t)\right]\mu_i(h_t) = 0$$

$$\sum_{i=1}^{2}c_i(h_t) - \sum_{i=1}^{2}y_i(h_t) = 0$$

1.2.5 Recursive Formulation

Our goal is make the problem recursive, which is very nice when we work with computer. To do this, we need to find a set of state variables which is sufficient to describe the state of the world. We are going to use x as a state variable.So the state variables are the endowment: $y = (y^A, y^B)$ and weight to brother 2: x. Define the value function as follows:

$$V = \{(V_0, V_A, V_B) \text{ such that } V_i : X \times Y \to \mathcal{R},\ i = 1, 2, V_0(x, y) = V_A(x, y) + xV_B(x, y)\}$$

What we are going find is the fixed point of the following operator (operation is defined later):

$$T(V) = \{T_0(V), T_1(V), T_2(V)\}$$

Firstly, we will ignore the participation constraints and solve the problem:

$$\max_{c_A, c_B} u(c^A(y, x)) + xu(c^B(y, x)) + \beta \sum_{y'} \Gamma_{yy'} V_0(y', x)$$

subject to

$$c^A + c^B = y^A + y^B$$

First Order Conditions yield:

$$\frac{u'(c_A)}{u'(c_B)} = x$$

Second, we will check the participation constraints. There are two possibilities here:

1. Participation constraint is not binding for either 1 or 2. Then set $x(h_t) = x(h_{t-1})$. In addition,

$$\begin{aligned} V_0^N(y, x) &= V_0(y, x) \\ V_i^N(y, x) &= u(c^i(y, x)) + \beta \sum_{y'} \Gamma_{yy'} V_i(y', x) \end{aligned}$$

2. Participation constraint is not satisfied for one of the brothers (say A).

This means that agent A is getting too little. Therefore, in order for the planner to match the outside opportunity that A has, he needs to change x so that he guarantees person A the utility from going away. We need to solve the following system of equations in this case:

$$\begin{aligned} c^A + c^B &= y^A + y^B \\ u(c^A) + \beta \sum_{y'} \Gamma_{yy'} V_A(y', x) &= u(y_A) + \beta \sum_{y'} \Gamma_{yy'} \Omega_A(y') \\ x' &= \frac{u'(c_A)}{u'(c_B)} \end{aligned}$$

This is a system of three equations and three unknowns. Denote the solution to this problem by,

$$c^A(y,x)$$
$$c^B(y,x)$$
$$x'(y,x)$$

So that,

$$\begin{aligned} V_0^N(y,x) &= V_A^N(y,x) + xV_B^N(y,x) \\ V_i^N(y,x) &= u(c^i(y,x)) + \beta \sum_{y'} \Gamma_{yy'} V_i(y', x'(y,x)) \end{aligned}$$

Thus we have obtained T(V)=V^N. And the next thing we need to do is find V* such that T(V*) $= V^*$.

Final question with this model is "how to implement this allocation?" or "Is there any equilibrium that supports this allocation?". The answer is yes. How? Think of this model as a repeated game. And define the strategy as follows: keep accepting the contract characterized here until the other guy walks away. If the other guy walks away, go to autarky forever. We can construct a Nash equilibrium by assigning this strategy to both of the brothers.

1.3 Variations of NGM

As you will see as you go along, our benchmark Neo-classical Growth model has hardtime replicating certain aspects of data and not usable for certain questions of interest such as monetary policy. In practice lots of variations and extensions of the benchmark model is utilized depending on the question being worked on. We will cover some of these extensions here.

1.3.1 Adjustment costs in investment

Behaviour of investment in our model seems to be somewhat smoother than that is observed in data. Our benchmark model produces too smooth investment (less volatile than data which seems to happen in spurts) relative to the output variability. One way to think about a cause of this observed fact is possibility of adjustment costs in investment which introduces some inertia to investment behaviour of firms. One way to introduce, suppose the law of motion for capital is,

$$k_{t+1} = (1-\delta)k_t + \varphi(x_t, k_t)$$

where x_t is investment. If we assume φ is CRTS then,

$$k_{t+1} = (1-\delta)k_t + x_t\varphi(\frac{k_t}{x_t})$$

and $\varphi' > 0$, $\varphi(\delta) = 1$ i.e. the adjustment cost is zero at the steady state level of investment. Another way would be to introduce a concave transformation technology between consumption and investment goods instead of the linear technology in the benchmark model. The problem with this approach is it is in odds with data.

1.3.2 Variable capital utilization

Benchmark model assumes in equilibrium existing capital is fully utilized. An extension is variable capital utilization would generate a smoother interest rate behaviour relative to benchmark model which seems to be generate excess volatility in the interest rate. Formally, the production function and the law of motion of capital becomes,

$$\begin{aligned} y_t &= z_t(i_t k_t)^\alpha n_t^{1-\alpha} \\ k_{t+1} &= (1-\delta(i))k_t + x_t \end{aligned}$$

where i is the intensity of capital usage.

1.3.3 Habits in consumption

Another aspect of the benchmark model is relatively more volatile consumption than data. One way to model consumption which might generate less volatility is introduction of habits in consumption. There are several ways to do this. 'Hangover effect' would be a utility function in which utility depends negatively on the previous period's consumption. A more subtle way to model is accumulation of habits over time, in which habit stock enters negatively to the utility function and current stock of habits and consumption has a positive marginal effect on habit accumulation,

$$\begin{aligned} & U(c_t, h_t) \\ h_t &= \varphi(c_{t-1}, h_{t-1}) \end{aligned}$$

One natural way to introduce consumption inertia would be through introduction of durables which enters the utility as a stock variable and provide consumption services over time.

All these variations had 'own' consumption as the driving force of habits. Another way would be 'external habits' or 'keeping up/back with the Jonesses', in which utility depends on current or past aggragate consumption level.

1.3.4 Introducing Money

The simplest way to introduce money is basically putting in the utility function in an ad hoc fashion. An indirect way of doing it is imposing a cash in advance constraint to force the households to hold money. Both of these supplies a framework which allows analysis of monetary policy on real variables. Of course any possibility of non-neutrality of money requires some nominal rigidity in some part of the model.