1 April 22

1.1 OLG

So far we have been utilizing the NGM with inifinite horizon with no demographic details. This was not because we did not have the tools to consider a finite horizon. Many interesting questions in macroeconmics should be approached within a framework where these demographic details matter. Suppose that agents can live up to period I. Agents are born with zero asset and can save. The rate of return from saving is assumed to be $[R]$. The wage rate per efficiency unit is w. Agents have a limited amount of time (normalized to one) and can allocate the time to either (i) work or (ii) enjoy leisure. Efficiency units which an agent can supply by working for a unit time changes as the agent grows older. This is captured by ε_i, where i is the age of the agent. The problem of the agent is as follows:

$$\max_{\{a_{i+1},c_t,n_i\}_{i=1}^{I}} \sum_{i=1}^{I} \beta_i(u_i(c_i) + \varphi(1-n_i)) \tag{1}$$

subject to

$$\begin{aligned} a_1 &= 0 && (2) \\ a_{i+1} + c_i &= a_i[R] + w\varepsilon_i[1-n_i] && (3) \end{aligned}$$

Notice that β_i is not β^i. Time discount factor can be different according to age. Maybe young agents discount future more (NOW is the important time for the young) and adult agents discount future less (considering the future more than kids). Different β_i can capture these.

1.1.1 Labor Earnings

What is a good theory on ε? If we look at the average wage per hour at the different age ($w\varepsilon_i$), the wage per hour increases with age, peaks at around 40, and slowly decreases until the retirement. Since w is assumed to be same for all agents, we need a theory that explains the difference in ε to replicate the hump shape of the average wage profile. What kind of theory do we have? There are two ways, in general:

1. Take $\{\varepsilon_i\}$ as exogenous; i.e., assuming that the young agents are useless because they are young.

2. Human capital theory. Assume that the difference in capital stock between the young agents and the old agents yields the difference in ε. There are three branches:

   (a) Learning-by-doing: assume that agents accumulate human capital (ε) by working. Agents learn something which enhances their human capital stock while they are working. Imagine an interns of doctor. The young doctors learn how to do operations by actually working at hospitals. This idea is represented by:

$$\varepsilon_{i+1} = \varphi_i(\varepsilon_i, n_i)$$

where n_i is hours worked of agents of age i. φ is indexed by i because learning ability can be different depending on age.

(b) Learning-by-not-doing: assume that agents accumulate human capital by actually learning (which is different from working or enjoying leisure). This idea is represented by:

$$\varepsilon_{i+1} = \varphi_i(\varepsilon_i, l_i)$$

where l_i is the time spent on learning, which is different from working or enjoying leisure. Agents allocate their time in learning to accumulate human capital.

(c) Education: the difference from learning models above is that most of education is acquired in the early stage of life. Keane and Ken Wolpin (REStat1994)[1] showed that 90% of people's fate is determined before age 16, by using structurally estimated model of the career choice.

1.1.2 Constructing Recursive Problem (1): Stationary Equilibrium

Let's define a stationary equilibrium in the recursive way for a model of learning by not doing. Stationary equilibrium means that the prices: r and w do not change over time. Firstly, let's define the problem of an agent of age i,conditional on $\mathcal{K}$, which is a capital labor ratio. Since r and w are functions of $\mathcal{K}$, we only need to record $\mathcal{K}$ instead of keeping track of prices. Individual agent's problem is:

$$V_i(a;\mathcal{K}) = \max_{c,n,a'} \{u(c, 1-n) + \beta V_{i+1}(a';\mathcal{K})\} \tag{4}$$

subject to

$$\begin{aligned}
c + a' &= a[R(\mathcal{K})] + w(\mathcal{K})n\varepsilon_i & (5)\\
\varepsilon_{i+1} &= \varphi_i(\varepsilon_i, l_i) & (6)\\
n &\in [0,1] & (7)\\
V_{I+1} &= 0 & (8)\\
a_1 &= 0 & (9)
\end{aligned}$$

Solution of the problem is sequences $\{a_{i+1}, c_i, n_i\}_{i=1}^{I}$ Now we are ready to define a stationary equilibrium.

[1] Keane , M., and Wolpin, K. (1994), "The Solution and Estimation of Discrete Choice Dynamic Programming Models by Simulation: Monte Carlo Evidence,' Review of Economics and Statistics, 76-4, 648-672.

Definition 1 *A stationary equilibrium is a set of allocations* $\{a_{i+1}^*, c_i^*, n_i^*\}_{i=1}^I$*, a pair of prices* r^* *and* w^**, and* $\mathcal{K}$ *such that:*

1. *(Agents' optimization) Given prices* r^* *and* w^**,* $\{a_{i+1}^*, c_i^*, n_i^*\}_{i=1}^I$*, solves the optimization problem of agents.*

2. *(Firm's optimization) Prices* r^* *and* w^* *are determined competitively.*

3. *(Consistency)*
$$\frac{\sum_{i=1}^I a_i^*}{\sum_{i=1}^I \varepsilon_i n_i^*} = \mathcal{K}$$

1.1.3 Constructing Recursive Problem (2): Non-Stationary Equilibrium

Now consider a non-stationary version of this economy where there is a constant population growth and technology shocks. Note that the constant growth assumption is convinient since it ensures the demographic structure of the economy is stable over time.

$$\mu_i = (\frac{1}{1+g_p})^i$$

Let's define an equilibrium which is not restricted to stationary one. Prices can change over time. First define the vector of assets, indexed by age as, $\overrightarrow{A} = \{A_i\}_{i=2}^I$, then we know,

$$K = \sum_i \mu_i A_i$$

The recursive formulation of the agent's problem is as follows:

$$V_i(z, a, \overrightarrow{A}; G, H) = \max_{c,n,a'} \Big\{ u(c, 1-n) + \beta E\{V_{i+1}(z, a', \overrightarrow{A}'; G, H) \mid z\} \Big\} \quad (10)$$

subject to

$$\begin{aligned} c + a' &= a[R(z, \overrightarrow{A})] + w(z, \overrightarrow{A})n && (11) \\ n &\in [0, 1] && (12) \\ V_{I+1} &= 0 && (13) \\ a_1 &= 0 && (14) \\ K_0 &= 0 && (15) \\ \overrightarrow{A}' &= G(z, \overrightarrow{A}) && (16) \\ N &= H(z, \overrightarrow{A}) && (17) \end{aligned}$$

The solution of this problem is

$$a_i' = g_i(a, z, \overrightarrow{A}; G, H) \tag{18}$$
$$c_i = c_i(a, z, \overrightarrow{A}; G, H) \tag{19}$$
$$n_i = h_i(a, z, \overrightarrow{A}; G, H) \tag{20}$$

Now we are ready to define a nonstationary recursive competitive equilibrium.

Definition 2 *A nonstationary equilibrium is a set of functions* $\{V_i^*(.),\ g_i^*(.),\ c_i^*(.),\ n_i^*(.)\}_{i=1}^I,\ G^*(z, \overrightarrow{A}),\ H^*(z, \overrightarrow{A}),\ r^*(z, \overrightarrow{A}),\ w^*(z, \overrightarrow{A})$ *such that:*

1. *(Agent's optimization) Given* $G^*(z, \overrightarrow{A}), H^*(z, \overrightarrow{A}), r^*(\overrightarrow{A}, z)\ w^*(\overrightarrow{A}, z)$, $\{V_i^*(.),\ g_i^*(.),\ c_i^*(.),\ n_i^*(.)\}_{i=1}^I$ *solves the agents' problem.*

2. *(Firm's optimization)* $r^*(\overrightarrow{A}, z)$ *and* $w^*(\overrightarrow{A}, z)$ *are determined competitively.*[2]

3. *(Consistency)*

$$\begin{aligned} H^*(z, \overrightarrow{A}) &= \sum_{i=1}^{I} h_i^*(z, A_i, \overrightarrow{A}; G^*, H^*)\mu_i \\ G_i^*(z, \overrightarrow{A}) &= g_i^*(z, A_i, \overrightarrow{A}; G^*, H^*) \qquad \forall i \end{aligned}$$

[2] In case of Cobb Douglas production function, $r^*(K, N) = \alpha \left(\frac{\sum_{i=1}^I K_i}{N}\right)^{\alpha-1}$ and $w^*(K, N) = (1-\alpha)\left(\frac{\sum_{i=1}^I K_i}{N}\right)^{\alpha}$.