1 Feb 8

1.1 Economy with Government Debt

Now assume that the government taxes labor income and issues debt to pay for a constant stream of government expenditures $\bar{G}$. This economy is more complicated and tricky than the previous economy without debt (where the amount of the government expenditure is equal to the tax income).

- When the government issues debt, government have the option to delay taxation.
- Government budget constraint will not be satisfied automatically in defining equilibrium.
- Tax policy, that is represented by a function $\tau(.)$, should depend on state of the economy. In particular, since the government always spends a constant expenditure, (i) the government will retire the debt that was issued before when it has a higher revenue, and , (ii) the government will issue more debt when it has a lower revenue.

The tricky part of the problem is to ensure that the government budget constraint is satisfied in the sense of present value. In other words, we want to rule out the insufficient taxation when debt keeps growing. We call such situation as "snowball effect" or "Ponzi scheme".

1.2 Defining RCE

1.2.1 State variables

- Aggregate state variable: K, B. K is the aggregate capital in the economy. B is the government debt stock. Government debt here is one period debt in the form of discount bond. Government sell bond today at price q and promise to repay one unit of good tomorrow.
- Individual state variables: a. a is a total asset holding of the agent.

Representative agent only cares about the value of her asset holding, not the composition of her asset portfolio. So, in defining RCE, we only need one state variable for the asset, not both physical capital holding k and financial asset b.In doing so, one equilibrium condition is implied: physical capital holding k and financial asset b bear the same rate of return. This condition holds because they are perfect substitutes, and by No Arbitrage argument.

1.2.2 Household's problem

$$V(K, B, a) = \max_{c,a'} u(c) + V(K', B', a') \tag{1}$$

subject to

$$
\begin{aligned}
c + a' &= a + [r(K,B)a + w(K,B)](1 - \tau(K,B)) \qquad (2)\\
K' &= H(K,B)\\
B' &= G(K,B)
\end{aligned}
$$

And the solutions are:

$$a' = g(K,B,a) \tag{3}$$

There are different ways of writing an equilibrium. Some are long and tedious, but here we are using short cut in the following sense. The functional form of w and r are given explicitly by marginal product of labor and marginal product of labor capital minus depreciation. So in the definition of equilibrium, we do not need to write out firm's problem.

Household needs to know B because B will affect future prices. In our problem, law of motion for K, B and future taxes τ depend on B, so future prices are affected by B. Why in this problem household expects K' and B' to evolve according to Φ and Ψ? We set it so and this is true in RCE.

There is no government expenditure in household's problem, because household does not care G, rather G will affect individual problem indirectly through B and τ.

1.2.3 Definition of RCE

Definition 1 *Given a feasible policy $\tau(K,B)$, a RCE is a set of functions $\{V^*, G^*, H^*, g^*, r^*,\}$ such that*

1. *(Household's optimization) Given $\{H^*, G^*\}, \{g^*, V^*\}$ solve the household problem.*
2. *(Consistency)*
$$H^*(K,B) + G^*(K,B) = h^*(K,B,K+B) \tag{4}$$
3. *(No Arbitrage Condition)*
$$r(K,B) = F_K(\Phi^*(K,B),1) - \delta \tag{5}$$
4. *(Government Budget Constraint)*
$$H^*(K,B) = \bar{G} + B(1+r(K,B))(1-\tau(K,B)) - \tau(K,B)[F(K,1) - \delta K] \tag{6}$$
5. *(No Ponzi Scheme Condition) $\exists \underline{B}$ and $\overline{B}$, such that $\forall K \in [0, \overline{\overline{K}}]$, $B \in [\underline{B}, \overline{B}]$*
$$H^*(K,B) \in [\underline{B}, \bar{B}],\ G^*(K,B) \in [\underline{K}, \overline{\overline{K}}] \tag{7}$$

Note that the market clerearing condition is implicitly there through Walras Law.

1.3 Extensions to our standard economy

1.3.1 Economy with two type of agents

Assume that in the economy there are two types of agents, called type A and type B (B denotes rich A poor in wealth terms). Measure of the agents of type A and type B are the same . Without loss of generality, we can think of the economy as the one with two agents, both of whom are price takers.

Agents can be different in many ways, including in terms of wealth, preference, ability, etc. We will first look at an economy where agents are different in wealth and efficiency in terms of their labor services. We also assume poor types care . There are measure one population of rich people and measure one population of poor people. For simplicity, we assume there are no shocks and agents do not value leisure.

The state variables are aggregate wealth of both types, K^A and K^B. Why? We know wage and rental only depends on total capital stock $K = K^A + K^B$. But K is not sufficient as aggregate state variables because agents need know tomorrow's price which depends on tomorrow's aggregate capital. To not to carry superscripts on K we can also define a new state varible λ which denotes the share of total wealth held by the agents of type A. The problem of type A agent is,

$$V^A(K, \lambda, a) = \max_{c,a'} \left\{u(c, C^A) + \beta V^A(G\left(K, \lambda\right), H(K, \lambda), a')\right\} \tag{8}$$

subject to

$$c + a' = R\left(K\right) a + e^A w\left(K\right) \tag{9}$$

Given,

$$C^A = C^A(K, \lambda)$$

Solutions are:

$$a' = g\left(K_A, K_B, a\right)$$

The problem of type B agent is similar with different functional forms. Next we define the RCE

Definition 2 *RCE is a set of functions* $\{V^i(.), g^i(.)\}$ *and* $\{G(.), H(.), C^A(.)\}$, $i \in \{A, B\}$*, such that;*

1. Given $\{G(.), H(.), C^A(.)\}, \{V^i(.), g^i(.)\}$ solves the households problem.

   (RA condition)

$$\begin{aligned}
G\left(K, \lambda\right) &= \frac{g^A(K, \lambda, 2K\lambda) + g^B(K, \lambda, 2K(1-\lambda))}{2} \\
H(K, \lambda) &= \frac{g^A(K, \lambda, 2K\lambda)}{g^A(K, \lambda, 2K\lambda) + g^B(K, \lambda, 2K(1-\lambda))} \\
C^A(K, \lambda) &= 2\lambda K R(K, \lambda) + e^A w(K, \lambda) - g^A(K, \lambda, 2K\lambda)
\end{aligned}$$

So with these powerful tools in our hand, we are able to define richer and more interesting environments than the RA framework. The important thing when defining equilibrium in these environments is to be consistent.

1.3.2 Neo-classical firm with a dynamic problem

Our analysis so far have left the firm's static problem lingering in the background and primarily focused on the HH behaviour. This is merely a matter of choice and as we will show below firm's problem can be formulated in a dynamic manner without resulting any substansive changes in our main results. The firm is defined as an entity with a unit of land. The land is not used in the production process. The firm makes the dynamic investment decision and owns the capital and households owns the shares of the firm. Then the problem of our household and firm is,

$$\begin{aligned} V(K,a) &= \max_{a',c} \{U(c) + \beta V[G(K), a']\} && (10) \\ s.t. \quad c + a' &= R(K)a \end{aligned}$$

$$\Omega(K,k) = \max_{k'} \{F(k,1) - k' + q(G(K))\Omega(G(K), k')\} \tag{11}$$

with solutions;

$$\begin{aligned} a' &= g(K,a) \\ k' &= h(K,k) \end{aligned}$$

The way to proceed in defining the recursive equilibrium and characterizing it is similar only a bit more tedious.

First lets write down the functional equations that implicitly defines the RCE.

The FOC and the EC for the firm is as follows,

$$\begin{aligned} FOC(k') &: -1 + q(K')\Omega_2(K',k') = 0 && (12) \\ EC &: \Omega_2(K',k') = F_k(k',1) && (13) \end{aligned}$$

and the Euler Equation for the firm becomes;

$$1 = q(K')F_k(k',1) \tag{EE FIRM}$$

and using the RA condition;

$$G(K) = h(K,K) = k' \tag{14}$$

we get ;

$$1 = q(G(K))F_k(G(K),1) \tag{15}$$

For the HH, the FOC and EC are;

$$\begin{aligned} FOC(a') &: -U_c(R(K)a - a') + \beta V_2[G(K), a'] = 0 && (16) \\ EC &: V_2[G(K), a'] = R(K')U_c(R(K')a' - a'') && (17) \end{aligned}$$

and HH Euler Equation is;

$$U_c(R(K)a - a') = \beta R(K')U_c(R(K')a' - a'') \tag{EE HH}$$

with the relevant RA condition;

$$\begin{aligned} a &= \Omega(K, K) && (18) \\ a' &= h[K, q(K)\Omega(K, K)] = \Omega(G(K), G(K)) && (19) \end{aligned}$$

Definition 3 *A RCE is a list of functions* $\{V, g, \Omega, h, q, G\}$ *such that,*

1. *Given* $\{q, G\}$*,* $\{V, g, \Omega, h\}$ *solves the HH and firm problems*
2. *Representative agent conditions*

$$\begin{aligned} G(K) &= h(K, K) \\ \Omega(G(K), G(K)) &= g(G(K), \Omega(K, K)) \end{aligned}$$

Next we look at the simplest possible model that can explain great deal in asset pricing.

1.4 Lucas Tree Model (Lucas 1978)

1.4.1 The Model

Suppose there is a tree which produces random amount of fruits every period. We can think of these fruits as dividends and use d_t to denote the stochastic process of fruits production. Further, assume d_t follows Markov process. Formally:

$$d_t \sim \Gamma(d_{t+1} = d_i \mid d_t = d_j) = \Gamma_{ji} \tag{20}$$

Let h_t be the history of realization of shocks, i.e., $h_t = (d_0,\ d_1,\ ...,\ d_t)$. Probability that certain history h_t occurs is $\pi(h_t)$.

Household in the economy consumes the only good, which is fruit. With usual assumption on preference retained, consumers maximize:

$$\sum_t \beta^t \sum_{h_t \in H_t} \pi(h_t) u(c_t) \tag{21}$$

Since we assume representative agent in the economy, and there is no storage technology, in an equilibrium, the representative household eats all the dividends every period. So the lifetime utility of the household will be:

$$\sum_t \beta^t \sum_{h_t \in H_t} \pi(h_t) u(d_t) \tag{22}$$

Now suppose that the household is given some STUFF at period 0 and there exists a market to trade fruits. It's trivial to guess that the equilibrium allocation will be an autarky (almost by definition), but the key thing is to find the price which can support the equilibrium allocation of autarky.

Define the household's problem.

$$\max_{\{c(h_t)\}_{t=0}^{\infty}} \sum_t \beta^t \sum_{h_t \in H_t} \pi(h_t) u(c_t(h_t)) \tag{23}$$

subject to

$$\sum_t \beta^t \sum_{h_t \in H_t} p(h_t) c(h_t) = STUFF \tag{24}$$

and

$$p_0 = 1 \tag{25}$$

Note that we are considering the Arrow-Debreu market arrangement, with consumption goods in period 0 as a numeraire.

1.4.2 First Order Condition

Take first order condition of the above maximization problem:

$$FOC \quad c(h_t) \qquad \frac{p(h_t)}{p_0} = p_t(h_t) = \frac{\beta^t \pi(h_t) u'(c(h_t))}{u'(c(h_0))} \tag{26}$$

By combining this FOC with the following equilibrium condition:

$$c(h_t) = d_t \ \forall t, h_t \tag{27}$$

We get the expression for the price of the state contingent claim in the Arrow-Debreu market arrangement.

$$p_t(h_t) = \frac{\beta^t \pi(h_t) u'(d(h_t))}{u'(d(h_0))} \tag{28}$$

1.4.3 Price the tree

Now we can compute the mysterious STUFF which satisfies the budget constraint.
What is the STUFF? STUFF is the sufficient amount to buy fruits in every period in every contingency from time 0 on, measured in period 0 consumption good. We can Imagine that the STUFF is a TREE, which bears fruits.
Tree in this model is a package of a stream of good. In asset pricing,

the price of an asset = value of all the things that the asset entitles you to get.

Therefore, the formula to compute q_t =the price of tree at period 0 is:

$$q_0 = \sum_t \sum_{h_t \in H_t} p_t d_t = \sum_t \sum_{h_t \in H_t} \frac{\beta^t \pi(h_t) u'(d(h_t))}{u'(d(h_0))} d(h_t) \tag{29}$$

1.4.4 Sequential Market

In sequential market, the household can buy and sell fruits in every period, and the tree (the asset). To consider the trade of the asset, let s_t be share of asset and q_t be the asset price at period t. The budget constraint at every time-event is then:

$$q_t s_{t+1} + c_t = s_t(q_t + d_t) \tag{30}$$

Thus, the consumer's optimization problem turns out to be:

$$\max_{\{c_t(h_t),s_{t+1}(h_t)\}_{t=0}^{\infty}} \sum_t \beta^t \sum_{h_t \in H_t} \pi(h_t)u(c_t(h_t)) \tag{31}$$

subject to

$$q_t(h_t)s_{t+1}(h_t) + c_t(h_t) = s_t(h_{t-1})[q_t(h_t) + d_t] \tag{32}$$

Again, from first order condition, we can derive q_t, which is the price of one tree after history h_t in terms of consumption goods at node h_t. To solve the problem, construct Lagrangian as follows:

$$L: \quad \sum_t \beta^t \sum_{h_t \in H_t} \pi(h_t)[u(c_t(h_t)) - \lambda_t(h_t)\{s_t(h_{t-1})[q_t(h_t)+d_t] - q_t(h_t)s_{t+1}(h_t) + c_t(h_t)\}] \tag{33}$$

Note that there are many ways to write equivalent Lagrangians. In the case above, the sequence of Lagrange multipliers is $\{\beta^t \pi(h_t)\lambda_t\}$. We write it in this way to simplify expressions of the first order conditions. First order conditions are:

$$FOC w.r.t. c_t(h_t) \qquad u'(c_t(h_t)) = \lambda_t(h_t) \tag{34}$$

$$FOC\ w.r.t.\ s_{t+1}(h_t) \qquad \pi(h_t)\lambda_t(h_t)q_t(h_t) = \beta \sum_{h_{t+1}|h_t} \pi(h_t)\lambda_{t+1}(h_{t+1})[q_{t+1}(h_{t+1}) + d_{t+1}(h_{t+1})] \tag{35}$$

Recall, d_t follows a Markov process,

$$\pi(h_{t+1}) = \pi(h_t)\Gamma_{ij} \text{ where } d_t(h_t) = d_i,\ d_{t+1} = d_j \tag{36}$$

so, combine (34) and (35), we get:

$$u'(c_t(h_t))q_t(h_t) = \beta \sum_j \Gamma_{ij} u'(c_{t+1}(h_{t+1}))[q_j + d_j] \tag{37}$$

In equilibrium, $c_t(h_t) = d_t(h_t)$. Let's pick $d_t(h_t) = d_i$, then,

$$u'(d_i)q_i = \beta \sum_j \Gamma_{ij} u'(d_j)[q_j + d_j] \tag{38}$$

From this equation, we can see that (i) the price of asset is also Markovian, and (ii) the marginal utility today is equal to marginal utility tomorrow weighted by prices at each node. Looking at the recursive version of the same problem with denoting discrete state variable as subscripts,

$$V_i(s) = \max_{s',c} u(c) + \beta \sum_{d'} \Gamma_{ij} V_j(s')$$

$$s.t. \quad c + s\prime q_i = s[q_i + d_i]$$

In equilibrium, the solution has to be such that c=d and s$^{'}$ $= 1$. Impose these on the FOC and get the prices that induce the agent to choose that particular allocation. Then the FOC for a particular state i would imply,

$$q_i = \beta \sum_j \Gamma_{ij} \frac{u'(d_j)}{u'(d_i)} [q_j + d_j] \tag{39}$$

A closer look to these conditions reveals they form a system of linear equations in prices. In order to solve for the prices of q_i, we need to solve the system of equations that consists of (**??**) for each i.

$$\begin{bmatrix} q_1 \\ .. \\ .. \\ q_I \end{bmatrix} = \begin{bmatrix} \beta\Gamma_{11}\frac{u'(d_1)}{u'(d_1)} & \beta\Gamma_{12}\frac{u'(d_2)}{u'(d_1)} & .. & \beta\Gamma_{1J}\frac{u'(d_J)}{u'(d_1)} \\ .. & \beta\Gamma_{22}\frac{u'(d_2)}{u'(d_2)} & .. & .. \\ .. & .. & .. & .. \\ \beta\Gamma_{I1}\frac{u'(d_1)}{u'(d_I)} & .. & .. & \beta\Gamma_{IJ}\frac{u'(d_J)}{u'(d_I)} \end{bmatrix} \left[\begin{bmatrix} q_1 \\ .. \\ .. \\ q_I \end{bmatrix} + \begin{bmatrix} d_1 \\ .. \\ .. \\ d_I \end{bmatrix} \right]$$

Let q=$\begin{bmatrix} q_1 \\ .. \\ .. \\ q_I \end{bmatrix}$ and d=$\begin{bmatrix} d_1 \\ .. \\ .. \\ d_I \end{bmatrix}$ and A=$\begin{bmatrix} \beta\Gamma_{11}\frac{u'(d_1)}{u'(d_1)} & \beta\Gamma_{12}\frac{u'(d_2)}{u'(d_1)} & .. & \beta\Gamma_{1J}\frac{u'(d_J)}{u'(d_1)} \\ .. & \beta\Gamma_{22}\frac{u'(d_2)}{u'(d_2)} & .. & .. \\ .. & .. & .. & .. \\ \beta\Gamma_{I1}\frac{u'(d_1)}{u'(d_I)} & .. & .. & \beta\Gamma_{IJ}\frac{u'(d_J)}{u'(d_I)} \end{bmatrix}$

and let b=Ad
we have,

$$q = Aq + b$$

so that,

$$q = (I - A)^{-1} b$$

Now, suppose that the dividend process is not Markovian. We can still get price of tree in terms of h_t good as follows:

$$q_t(h_t) = \frac{\sum_{\tau=t+1}^{\infty} \sum_{h_\tau | h_t} p(h_\tau) d_t(h_\tau)}{p(h_t)} \tag{40}$$

or

$$p(h_t) q_t(h_t) = \sum_{\tau=t+1}^{\infty} \sum_{h_\tau | h_t} p(h_\tau) d_t(h_\tau) = \sum_{h_{t+1} | h_t} p(h_{t+1}) [d_{t+1}(h_{t+1}) + q_{t+1}(h_{t+1})] \tag{41}$$

Now that we are able to price any asset we would like, we can utilize it to look into one of the most popular puzzles in economic literature.

1.4.5 Equity premium puzzle

This puzzle basically says that standard representative agent neoclassical growth model with CRRA utility function with "normal" parameter values fails to explain the huge difference between risky stock returns and riskless bond in US. For example, Dimson, Marsh, and Staunton (2002) reported that the average annual real returns of equity (over 1900-2000) is 6.7%, while the average annual returns of risk-free[1] T-bill over the same period is 0.9%. So the risk premium is around 6% annually. Of course, equity premium puzzle depends on many assumptions, as I listed above, so there are many other assumptions which might cause the problem. But if we change only σ to match this high equity premium, it is known that we need $\sigma = 20 - 50$. In other words, people have to be very very risk averse to hold T-bills regardless of the huge difference in average return. What is the average rate of return in our model economy? To be able to pin that down, we need to know the stationary distribution of the shocks to our economy (more to be discussed about stationary distributions later in the course but for now it basically gives the relative frequency of each state occuring in long run). Given the stationary distribution μ the average rate of return on bonds is given by

$$r^b = \sum_j \mu_j r_j^b$$

and to pin down r_j^b we can use our simple Lucas tree model to price the bond. We know a bond is a asset that gives a unit of consumption good for sure regardless of the state of the economy where as the return on shares is state dependent and usually pays out good when the times are good and marginal utility consumption is low and vice versa. Using the following budget constraint, with state contingent assets b, with the usual problem,

$$c + s\prime q_i + \sum_i p_{ij} b_i' = s[q_i + d_i]$$

one can show the price of the asset that pays a unit of consumption for sure next period if state j is realized,

$$p_{ij} = \beta \Gamma_{ij} \frac{U_c(j)}{U_c(i)}$$

and the price of bond and the return on it is,

$$\begin{aligned} p_i^b &= \sum_j p_{ij} \\ r_j^b &= \frac{1}{q_j^b} - 1 \end{aligned}$$

[1] We ignore the inflation risk here. If we consider the inflation risk, T-bill is also risky unless it is inflation adjusted (and it is the case).

The average rate of return on shares in our model is,

$$\sum_j \mu_j \sum_i \Gamma_{ji} \frac{q_i + d_i}{q_j} - 1 = r^{sm}$$

and it is the discrepancy between differences of these rates in the model and in real world that is in the core of this puzzle

1.4.6 Pricing an Arbitrary Asset

Because in a complete market any asset can be reproduced by buying and selling contingent claims at every node, we can use this model as a powerful asset pricing formula. For example, discount bond is a promise to pay one unit of good tomorrow no matter what happens. To reproduce bond, it suffices to buy one unit of state contingent claim at every node in the next period. Therefore, at h_t, the price of bond is:

$$p^b(h_t) = \frac{\sum_{h_{t+1}|h_t} p(h_{t+1})}{p(h_t)} \tag{42}$$

Consols is a promise to pay one unit of good forever from now on. Thus its price is:

$$p^{consol}(h_t) = \frac{\sum_{\tau=t}^{\infty} \sum_{h_\tau|h_t} p(h_\tau)}{p(h_t)} \tag{43}$$

Consider a one period call option, which is a right to buy one share of a tree at the fixed price (exercise price) $\bar{q}$. The price of this option is:

$$p^{o,\bar{q}}(h_t) = \frac{\sum_{h_{t+1}|h_t} p(h_{t+1})[q(h_{t+1}) - \bar{q}]1_{[q(h_{t+1})-\bar{q}]>0}}{p(h_t)} \tag{44}$$

where 1 is an indicator function (see the note of the next class).

1.4.7 Two Period Option

To see that we can price any kinds of assets or options using this principle, let's price two periods option. Option here is the RIGHT to buy a consumption goods at a negotiated price. When we talk about multiple period options, we have to be aware the difference between American and European option. American option can be exercised AT ANY TIME before its maturity. On the contrary, European option can be exercised ONLY AT ITS MATURITY. But the principle to price them is same. By the way, notice that American option is always more expensive than its European counterpart, because American option contains more options to its holders.

Here let's price two period American and European options at a node h_t. As a set up, assume that the set of the possible aggregate shock contains two elements. Start from h_t, possible nodes in the next periods are h^1_{t+1} and h^2_{t+1}.

In the two period ahead, there are four possible nodes, h^1_{t+2}, h^2_{t+2}, h^3_{t+2}, h^4_{t+2}, where h^1_{t+2}, and h^2_{t+2} can be reached only from h^1_{t+1}.

Firstly, remember the price of an one period option at the node $h_{t+1} : p^o(h_t)$ with negotiated price $\bar{q}$. This is:

$$p^{o1}(h_t) = \sum_{h_{t+1}|h_t} [q(h_{t+1}) - \bar{q}] 1_{[q(h_{t+1})-\bar{q}]>0} \frac{p(h_{t+1})}{p(h_t)} \tag{45}$$

where $1_{[\text{expression}]}$ is an indicator function that takes value of 1 if the [expression] is true and 0 if false, and $p(h_t)$ is the price of consumption goods at node h_t. You can also use χ for an indicator function.

Price of an European option (option which can be exercised ONLY in the two period ahead), which is just the natural extension of this one period option, is as follows:

$$p^{o2}(h_t) = \sum_{h_{t+2}|h_t} [q(h_{t+2}) - \bar{q}] 1_{[q(h_{t+2})-\bar{q}]>0} \frac{p(h_{t+2})}{p(h_t)} \tag{46}$$

Price of an American option is a little bit more tricky:

$$p^{oa2}(h_t) = \sum_{h_{t+1}|h_t} \max\left\{p^{o1}(h_{t+1}), [q(h_{t+1}) - \bar{q}]\right\} \frac{p(h_{t+1})}{p(h_t)} \tag{47}$$

In the period t+1, a holder can either (i) exercise the option (and then the option expires), or (ii) keep the option to the next period (in this case, the option is exactly the same as the one period option bought in the period t+1).

1.4.8 Final Remark

In this fashion, we can price any kinds of assets or options. For example, you can easily price future transaction[2]. This is basically finance guys are doing during while their life. They are just solving the price, without solving the allocation (because of RA assumption, we do not need to solve the asset portfolio of agents, which are the same in equilibrium).

[2] Future transaction is a contract to buy or sell a goods in a negotiated period at a negotiated price. The difference from option is that you MUST perform the transaction, no matter whether you want to do or not. Naturally, option contract is more expensive, as you are given an option not to exercise.