1 Feb 17

2 Economy with Heterogeneous Agents

2.1 Introduction

So far, in environments we anaylyzed, the type of agents does not change over time. In this case, especially, if the number of type of agents is small, as the example we did with only two different types of agents, it's easy to keep track of all the types, and so is to define an equilibrium .From now on, we will consider the economies with (i) many agents who are very different among themselves at a given time (crosssection), and (ii) change their types over time.

Since agents might trade each other, we need to keep track of the aggregate state of the world . There are two ways to do it. One is "Spanish Interior Minister way". People in the economy are given identification number and you record the types of agents according to the number. But this way is not efficient, because the id number does not tell the properties of agents: we use the id numbers just to keep track of individuals. So we take the second way. We are not going to keep track of agents by id numbers given to each agent but we use MEASURE. To further proceed, we need some knowledge on thew measure theory, so let's study it briefly, and after that we will see how measure theory is useful for our purpose.

2.2 Introduction to Measure Theory

2.2.1 Intuition

Measure theory can be understood nicely by comparing to weight. Measure is useful in literally measuring a mass in a mathematically consistent way, which is similar to the way of weighting a mass. Therefore, intuitively the following properties are expected to be satisfied by measure:

1. $measure(nothing) = 0$
2. if $A \cap B = \emptyset$, $measure(A + B) = measure(A) + measure(B)$

These properties are intuitive with weight. The weight of nothing is zero. If a body is 200 pounds, and you chop off a hand from the body and put the hand and the rest of the body together on the scales, they must weight 200 pounds. Now consider an economy with many agents. The measure of nobody in the economy is zero. If a measure of the total population is normalized to one, and you take away the rich people from the population and measure the sum of rich people and the rest of the population, they must have measure one.

2.2.2 Definitions

Definition 1 *For a set A, $\mathcal{A}$ is a set of subsets of A.*

Definition 2 *σ-algebra $\mathcal{A}$ is a set of subsets of A, with the following properties:*

1. *$A, \emptyset \in \mathcal{A}$*
2. *$B \in \mathcal{A} \Rightarrow B^c \in \mathcal{A}$ (closed in complementarity)*
3. *for $\{B_i\}_{i=1,2...}$, $B_i \in \mathcal{A} \Rightarrow [\cap_i B_i] \in \mathcal{A}$ (closed in countable intersections)*

The intuition of the property 2 of σ-algebra is as follows. If we chop off a hand from a body, and if the hand is an element of $\mathcal{A}$, the rest of the body is also an element of $\mathcal{A}$. Soon we will define measure as a function from σ-algebra to a real number Then the property of σ-algebra implies that if we can measure the chopped hand, we can measure also the rest of the body.

Examples of σ-algebra are the follows:

1. Everything (all the possible subsets of a set A)
2. $\{\emptyset, A\}$
3. $\{\emptyset, A, A_{1/2}, A_{2/2}\}$ where $A_{1/2}$ means the lower half of A (imagine A as an closed interval on $\mathcal{R}$).
4. $\{\emptyset, A, A_{1/4}, A_{2/4}, A_{3/4}, A_{4/4}, A^c_{1/4}, A^c_{2/4}, A^c_{3/4}, A^c_{4/4}, A_{1,2/4}, A_{1,2/4}, A_{1,3/4}, A_{1,4/4}, A^c_{1,2/4}, A^c_{1,3/4}, A^c_{1,4/4}\}$

Remark 3 *A convention is (i) use small letters for elements, (ii) use capital letters for sets, (iii) use "fancy" letters for set of subsets.*

Look at the examples of 3 and 4. Imagine you are given $a \in A$. If the only information we can get with respect to a is whether a is included in an element of $\mathcal{A}$ or not, it is true that we have richer information on a with σ-algebra 4 than 3 because, with 4, we can know a is included in which of $A_{1/4}, A_{2/4}, A_{3/4}, A_{4/4}$, where with 3, we only know a is included in which of $A_{1/2}, A_{2/2}$. In this sense, σ-algebra is similar to the notion of information.

Definition 4 *A measure is a function $x : \mathcal{A} \to \mathcal{R}_+$ such that*

1. *$x(\emptyset) = 0$*
2. *if $B_1, B_2 \in \mathcal{A}$ and $B_1 \cap B_2 = \emptyset \Rightarrow x(B_1 \cup B_2) = x(B_1) + x(B_2)$ (finite additivity)*
3. *if $\{B_i\}_{i=1}^{\infty} \in \mathcal{A}$ and $B_i \cap B_j = \emptyset$ for all $i \neq j \Rightarrow x(\cup_i B_i) = \sum_i x(B_i)$ (countable additivity)*

In English, countable additivity means that measure of the union of countable disjoint sets is the sum of the measure of these sets.

Definition 5 *Borel-σ-algebra is (roughly) a σ-algebra which is generated by a family of open sets.*

Remember the discussion on the information. Since Borel-σ-algebra contains all the subsets generated by intervals, you can recognize any subset of a set, using Borel-σ-algebra. In other words, Borel-σ-algebra corresponds to the complete information.

You might find that σ-algebra is similar to topology. Topology is also a set of subsets, but its elements are open intervals and it does not satisfy closedness in complementarity (complement of an element is not an element of topology). Very roughly, the difference implies that topologies are useful in dealing with continuity and σ-algebra is useful in dealing with measure.

Definition 6 *Probability (measure) is a measure such that* $x(A) = 1$