1 March 15

1.1 Problem of Fishermen (Once again)

Imagine a Archipelago that has a continuum of islands (instead of the pig farmers we used in class). There is a fisherman on each island. The fishermen get an endowment e each period which follows a Markov process with transition $\Gamma_{ee'}$ and,

s$\in \{e^1, ..., e^{n_e}\}$

The fishermen cannot swim. There is a storage technology such that, if the fishermen store q units of fish today, they get 1 unit of fish tomorrow. (e,a) is the type of a fisherman and the set consisting of all possible such pairs is,

$$E \times A = \{e^1, e^2, ..., e^n\} \times [0, \overline{a}]$$

Let $\mathcal{A}$ be the set of Borel sets on SxA. And define a probability measure x on $\mathcal{A}$,

$$x : \mathcal{A} \rightarrow [0, 1]$$

The fisherman's problem is:

$$V(e, a) = \max_{c, a' \geq 0} u(c) + \beta \sum_{e'} \Gamma_{ee'} V(e', a')$$

subject to

$$c + qa\prime = e + a$$

$$c \succeq 0 \text{ and a' } \in [0, \overline{a}]$$

With the decision rule $a' = g(e, a)$ and the transition matrix for the endowment process $\Gamma_{ee'}$

The First Order Conditions are,

$$u_c(e + a - qa\prime) = \frac{\beta}{q} \sum_{s'} \Gamma_{ee'} u_c(e\prime + a\prime - q'a'')$$

You'll notice that a$' \in [0, \overline{a}]$ is already one of the constraints of the above maximization problem. But now rather than just imposing such a constraint, we will find a natural reason that savings should have a lower bound and we will consider a condition that ensures an upper bound for savings.

For the lower bound, we assume that there is no technology which allows negative amount of saving and this sounds natural since storing a negative amount of fish does not make much sense. So savings has a lower bound because Mother Nature says so.

Here, the fisherman has the risk of getting a very bad shock tomorrow. So the fisherman would save just in case he has this bad shock; he would want to store some fish today in order to insure himself against getting very small number of fish tomorrow so he is not hungry in case that happens. In this case we need to think more about how to put an upper bound on savings, because with uncertainty even if $\beta < q$, the fisherman is willing to save due to gains from insurance. The kind of savings to protect oneself from risk in the future in the absence of state contingent commodity markets which can be used to insure against any contingency to make sure consumption is constant across states, which is usually called precautionary savings. In order to ensure an upper bound for savings, we need to bound the gains from insurance somehow. The way to do this is to impose the condition on the utility function that its negative curvature (keeping in mind that the utility function is concave) is diminishing as wealth increases. This means that wealthier agents are less risk-averse. Formally, that u' is convex. The wealthier the agent is, the smaller the variance of his endowment next period proportional to his wealth so he doesn't want to save if he is very wealthy. This is simply because of the fact that the wealth is not subject to any uncertainty but income is thus as the income wealth ratio rise, the overall uncertainty the agent faces diminishes.

So in the economy with uncertainty, in order to have an upper bound on savings, we need the first derivative of the utility function to be convex so that the following Jensen's Inequality holds:

$$\frac{\beta}{q}\int \Gamma_{ss'} u_c(c') > \frac{\beta}{q} u_c(\int \Gamma_{ss'} c')$$

Theorem 1 *If $\beta < q$ and u' is convex then $\exists \overline{a}$ such that $a_0 < \overline{a}$, $g(s,a) < \overline{a}$ $\forall s$.*

Now consider the case of lower bound. Suppose we let the fisherman borrow and lend to each other but not store any fish, how can we make sure that our agents always has the capability to pay back what they owe. What would be the endogenous lower bound to ensure this? Such a condition would make sure that in the worst case scenario our agent should be able pay the interest rate on its debt and roll over the same amount (the lowest possible amount). Thus, letting the lower bound be $\underline{a}$ and the lowest possible shock be $\underline{e}$ then,

$$\begin{aligned} 0 + q\underline{a} &= \underline{a} + \underline{e} \\ \underline{a} &= \frac{\underline{e}}{q-1} \end{aligned}$$

This is called the solvency constraint rather than a borrowing constraint. Note that when we let the fishermen to get into lending contracts with each other, we need a consistency condition to make sure agents actions are compatible with each other. Here the price will be endogenously determined will ensure the agents hold just the right amount of assets.

The stationary equilibrium of such an economy is defined as,

Definition 2 *A stationary equilibrium for an Huggett(1993) economy is a set $\{q^*, x^*(q^*), Q(s,a,B;q^*), g(s,a;q^*)\}$ such that*

1. *(Agent Optimization) Given q^*, $g(s,a;q^*)$ solves the agent's problem.*

2. *(Consistency)* $Q(s,a,B;q^*)$ *is a transition matrix associated with* $\Gamma_{ss'}$ *and* $g(s,a;q^*)$.

3. *(Stationarity)* x^* *is the unique stationary distribution associated with* $Q(s,a,B;q^*)$.

4. *(Market clear)*
$$\int adx^*(q^*) = 0$$

1.2 Transition analysis in an Heteregenous Agent Economy

Now suppose instead of a natural technology (q) we have a production function of the Cobb-Douglas form that is CRTS. What is sufficient variable to pin down the prices now? We know $\frac{K}{L}$ is sufficent, so there is a particular capital labor ratio that pins down the stationary equilibrium prices. The problem we have is,

$$V\left(e,a;\frac{K}{L}\right) = \max_{c\geq 0, a\in[0,\overline{a}]} u\left(c\right) + \beta\sum_{s'}\Gamma_{ss'}V\left(s',a';\frac{K}{L}\right) \tag{1}$$

subject to

$$c + a' = a\left(1 + r\left(\frac{K}{L}\right)\right) + w\left(\frac{K}{L}\right)e$$

The optimal solution is

$$a' = g\left(e,a;\frac{K}{L}\right)$$

In equilibrium,

$$(\frac{K}{L}) = \frac{\int adx^*\left(\frac{K}{L}\right)}{\int edx^*\left(\frac{K}{L}\right)} \tag{2}$$

where,

$$L = \int ed\gamma^*$$

and γ^* is the stationary distribution of efficiency labor units e. Now suppose we would like to do some policy analysis in this economy (such as a change in tax on capital income), which would require us to do analysis off the stationary equilibrium. Once we leave the stationarity, we know the distribution of x becomes a state variable. This is because of the fact that the law of motion for K/L depends on the wealth distribution and so does the prices. We know that dealing with an object like distribution as a state variable is extremely demanding. The way to circumvent that problem is to make prices independent of distribution by either with an assumption on technology (linear) or directly assuming a small country framework where the prices are given exogenously.

2 March 18

2.1 Growth

In our analysis so far, we have used Neo-classical Growth Model as our benchmark model and built on it for the analysis of more interesting economic questions. One peculiar characteristic of our benchmark model, unlike its name suggested, was lack of growth. Many interesting questions in economics are related to the cross-country differences of growth rates and we will cover some models that will allow for growth so that we will be able to attempt to answer such questions.

2.1.1 Exogenous growth

What does it take for an economy to grow? Before answering that question, we know in our standard NGM there is basically two ways of growth, one in which everything grows, which is not necessarily a per-capita growth, and the other is per-capita growth. We will be focusing on per-capita growth. The title exogenous growth refers to the structure of models in which growth rate is determined exogenously, and is not an outcome of the model. First and the simplest one of these is one in which the determinant of the growth rate is population growth. (Note that these notes are written to compliment the notes of Per Krusell which can be downloaded from the course website)

Growth with population Suppose the population of our economy grows at rate γ and we have the classical CRTS technology in capital and labor inputs.

$$\begin{aligned} Y_t &= AF(K_t, N_t) \qquad (3) \\ N_t &= N_0 * \gamma^t \end{aligned}$$

Note that our economy is no longer stationary but as we will see, within the exogenous growth framework we can make these economies look like stationary ones by re-normalizing the variables. Thus, at the end of the day it will only be a mathematical twist on our standard growth model. Once we do that, we will be looking for the counterpart of a steady state that we have in our stationary economies, the Balanced Growth Path, in which all the variables will be growing at constant rates but not necessarily equal. Back to our population growth model, we know

$$AF(K, N) = A[KF_k(K, N) + NF_N(K, N)] \qquad (4)$$

Question is, if N is growing at rate γ, can this economy have a balanced growth path. Can we construct one? We know by CRTS property F_K and F_N are homogenous of degree zero. If we assume capital stock grows at rate γ as well, then prices stay constant and per-capita variables are constant and output grow at the same rate. So we get growth on a balanced growth path without per-capita growth. One question is how do we model population growth in our representative agent model. One way is to assume there is a constant proportion of immigration to our economy from outside but this has to assume the immigrants are identical to our existing agents in our economy, which is a bid problematic. The other way could be to assume growing dynasties which preserves the representative agent nature of our economy. If we do so, the problem of the social planner becomes,

$$\max \sum_{t=0}^{\infty} \beta^t N_t U(\frac{C_t}{N_t}) \tag{5}$$
$$st \quad C_t + K_{t+1} = AF(K_t, N_t) + (1-\delta)K_t$$

To transform the budget set to per capita terms, divide all terms by N_t and to make the environment stationary by dividing all the variables by γ^t and assume $\mathrm{N}_0 = 1$, we get,

$$\max \sum_{t=0}^{\infty} (\beta\gamma)^t U(\widehat{c}_t) \tag{6}$$
$$st \quad \widehat{c}_t + \gamma\widehat{k}_{t+1} = AF(\widehat{k}_t, 1) + (1-\delta)\widehat{k}_t$$

So how is this transformed model any different than our NGM? By the discount factor, the agents in this economy with growth discounts the future less but everything else is identical to NGM of course with the exception of this economy growing at a constant rate.

Now suppose we have a 'labor augmenting' productivity growth with constant population normalized to one, i.e. have the following CRTS technology,

$$Y_t = AF(K_t, \gamma^t N_t) \tag{7}$$
$$AF(K_t, \gamma^t N_t) = A[K_t F_k(K_t, \gamma^t N_0) + \gamma^t N_0 F_N(K_t, \gamma^t N_0)] \tag{8}$$

Can we have an BGP? The problem is,

$$\max \sum_{t=0}^{\infty} \beta^t U(C_t) \tag{9}$$
$$st \quad C_t + K_{t+1} = AF(K_t, \gamma^t N_0) + (1-\delta)K_t$$

and since we have a population of one, these variables are already per-capita terms. For stationarity, we have to normalize the variables to 'per productivity' units, by dividing all by γ^t. Then the problem becomes,

$$\max \sum_{t=0}^{\infty} \beta^t U(\gamma^t \widehat{c}_t) \tag{10}$$
$$st \quad \widehat{c}_t + \gamma\widehat{k}_{t+1} = AF(\widehat{k}_t, 1) + (1-\delta)\widehat{k}_t$$

Suppose we have a CRRA preferences, then the question is how can we represent the preferences as a function of $\widehat{c}_t$ only. Writing the CRRA,

$$\sum_{t=0}^{\infty} \beta^t \frac{(\gamma^t \widehat{c}_t)^{1-\sigma} - 1}{1-\sigma} = \sum_{t=0}^{\infty} (\beta(\gamma^{1-\sigma}))^t \frac{\widehat{c}_t{}^{1-\sigma} - 1}{1-\sigma} \tag{11}$$

and the problem becomes,

$$\max \sum_{t=0}^{\infty} (\beta(\gamma^{1-\sigma}))^t \frac{\widehat{c}_t^{\,1-\sigma} - 1}{1-\sigma} \tag{12}$$

$$st \quad \widehat{c}_t + \gamma \widehat{k}_{t+1} = AF(\widehat{k}_t, 1) + (1-\delta)\widehat{k}_t$$

and once again it is exact same problem as the NGM with a different discount factor. Note that the existence of a solution to this problem depends on $\beta(\gamma^{1-\sigma})$.In this set-up we get per-capita growth at rate γ. Also note that CRRA is the only functional form for preferences that is compatible with BGP. This is because as per-capita output grows, for consumption to grow at a constant rate, our agent has to face the same trade-off at each period.

Now suppose we have the TFP growing at rate γ with a CRTS Cobb-Douglas technology

$$\begin{aligned} Y_t &= A_t F(k_t, 1) \\ \frac{A_{t+1}}{A_t} &= \gamma \end{aligned}$$

What would be the growth rate of this economy? We can show that like the previous cases the growth rate of the economy is the growth rate for the productivity of labor, which is $\gamma^{\frac{1}{1-\alpha}}$ in this case.