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1.0.1 Endogenous Growth

So far in the models we covered growth rate has been determined exogenously. Next we will look to models in which the growth rate is chosen by the model itself. We do know for a fixed amount of labor, the curvature of our technology limits the growth due to diminishing marginal return on capital and with depreciation there is an upper limit on capital accumulation. So if our economy is to experience sustainable growth for a long period of time, we either give up the curvature assumption on our technology or we have to be able to shift our production function up. Given a fixed amount of labor, this shift is possible either by an increasing TFP parameter or increasing labor productivity, . The simplest of such models where we can see t is the AK model, where the technology is linear in capital stock so that diminishing marginal return on capital does not set in.

AK Model We have the usual social planner's problem with linear technology and full depreciation,

$$\begin{aligned} & \max \sum_{t=0}^{\infty} \beta^t U(C_t) \\ st \;\; C_t + K_{t+1} &= AK_t \end{aligned} \tag{1}$$

and the FOCs

$$(c_t) \;\; : \;\; \beta^t U_c(.) = \lambda_t \tag{2}$$

$$(k_{t+1}) \;\; : \;\; \lambda_t = \lambda_{t+1} \tag{3}$$

together implies the Euler equation,

$$U_c(c_t) = A\beta U_c(c_{t+1}) \tag{4}$$

and on the BGP with consumption growing at rate γ with CRRA utility we get,

$$\gamma = (A\beta)^{1/\sigma} \tag{5}$$

and the growth rate is determined by the model parameters endogenously. Note that capital also grows at rate γ and the fate of the economy is determined by pre-determined parameters of the model. The capital stock will diverge to infinity if $(A\beta)^{1/\sigma} > 1$ or the economy is destined to vanish if $(A\beta)^{1/\sigma} < 1$. Also note that there is no transitional dynamics in this model (we loose the state variables in the euler equation after substituting for the balanced growth rate relation) and conditional on γ,asymptotically all economies are same regardless of the initial capital level. If we de-centralize this economy we know wages will be zero since labor has no use and gross rental rate of capital will be fixed at the A. This is at odds with what we observe in real world. We would rather like to have a model that allows for both transitional dynamics, labor and growth at the same time. Allowing for labor implies that we need a variable that proxies the increasing productivity of labor endogenously and be reproducible in terms of output, such that we are able to shift our production function continually in the output-capital space without hitting a natural bound.

Human Capital and Growth One way of doing this is, introducing the variable 'Human Capital' as an input of production, to proxy continuously and endogenously increasing labor efficiency. We have two ways of modelling the human capital, one way is to see it very much like physical capital, in the sense output has to be invested to increase the existing stock of human capital. That is the Lucas' approach, in which you can think of investing in education by building more schools as a way to increase the existing human capital stock. The alternative way would be to reserve a part of the leisure time for increasing the human capital stock, which can be thought of studying harder to get better in a fraction of the leisure time. Unfortunately, the second approach puts limit on the rate human capital can grow and might fail to generate sustainable endogenous growth. Next, we look at the Lucas' human capital model.

Lucas' Human Capital Model We have an Cobb-Douglas technology with CRTS and human capital (H) as an input of production instead of labor and the laws of motion for the inputs,

$$\begin{aligned} F(H,K) &= AK^{\alpha}H^{1-\alpha} && (6) \\ K' &= i_k + (1-\delta_k)K && (7) \\ H' &= i_h + (1-\delta_h)H && (8) \end{aligned}$$

Now that there is no limit to the accumulation of human capital and sustainable growth on a BGP is feasible. Furthermore, an analysis of the characterization of the balanced growth path will indicate that this model indeed has transitional dynamics, so unlike the AK model if economy starts out of this optimal growth path economy can adjust and converge to it by responding to prices in a de-centralized setting. If we model the law of motion for human capital as,

$$H' = (1-N) + (1-\delta_h)H \tag{9}$$

where $(1-N)$ is the time devoted to accumulating human capital, say by studying harder, we see there is a natural limit to the growth of human capital and such an economy might not have a BGP. The key ingredient of endogenous growth with labor is then the reproducibility of the human capital without such a limit.

Growth through Externalities (Romer) We have seen in the AK model the growth rate is determined solely by model primitives and endogenized but still it is not a directly or indirectly determined by the agents' choices in our model. In Lucas' human capital model, the growth rate is determined by the choice of agents, specifically by the optimal ratio of human and physical capital. The source of growth in Lucas' model is reproducibility of human capital. In the next model, Romer introduces the notion of externality generated by the aggregate capital stock to go through the problem of diminishing marginal returns to aggregate capital. In this model, the source of growth will be the aggregate capital accumulation, which is possible with a linear aggregate technology in capital as we saw in the AK model. The firms in our model will not be aware of this externality and will have the usual CRTS technology and observe the source of growth coming from the TFP parameter. As usual with externalities, the equilibrium outcome will not be optimal. Each firm has the following technology,

$$y_t = AK_t^{1-\alpha} k_t^{\alpha} n_t^{1-\alpha} \tag{10}$$

but since the firms are not aware of the positive externality they are facing they are solving the problem with the following technology.

$$\begin{aligned} y_t &= \overline{A}_t k_t^{\alpha} n_t^{1-\alpha} && (11) \\ \textit{where } \overline{A}_t &= A_t K_t^{1-\alpha} && (12) \end{aligned}$$

We can see the social planner in fact is solving an AK model in per-capita terms. So does the de-centralized version of this economy have a BGP and if it does how would it look like? Assuming CRRA preferences without leisure and , we can derive the BGP condition and pin down the growth rate from the euler equation of a typical household,

$$1 = \beta\gamma^{-\sigma}(1+r) \tag{13}$$

where $\gamma = \dfrac{c_{t+1}}{c_t}$is the growth rate at the balanced path as usual and r=MP$_k$. So to find out the marginal product of capital for the firm we differentiate the technology w.r.t. k_t,

$$1 + r_t = \alpha AK_t^{1-\alpha} k_t^{\alpha-1} n_t^{1-\alpha} + (1-\delta) \tag{14}$$

and since the prices are determined by aggregate state variables $K_t = k_t$ gives,

$$A\alpha - \delta = r \tag{15}$$

and substituting this into the euler equation we get the growth rate of consumption.

$$[(A\alpha - \delta + 1)\beta]^{\frac{1}{\sigma}} = \gamma \tag{16}$$

Solving the AK problem the SP faces we can verify the optimal growth rate for consumption is,

$$[(A - \delta + 1)\beta]^{\frac{1}{\sigma}} = \gamma^{sp}. \tag{17}$$

The important properties of the decentralized model are,

1. It is sub-optimal due to firms' unawareness of the externality they are facing and thus have lower growth rate.
2. Once again, the rental rate do not depend on the capital stock (due to the linear technology in aggregate the state variable capital stock drops out of the euler equation) and there is no transitional dynamics generated by the model.

To sum up what we have done so far, we have started with models that had exogenous growth and saw that we can make these models look an behave like our NGM after appropriate transformation. Then we went on to look at the models that generate growth endogenously and saw that a prerequisite for growth in these models is linearity of the technology in reproducible factors. We looked at the simple AK model, where the technology is linear in capital stock and analyzed the BGP of such an economy. Then we looked at Lucas' human capital model, in which we had two forms of capital, human and physical, both of which are reproducible in terms of output. Then we analyzed the model by Romer, which again has linearity in the reproducible factor at the aggregate level (capital stock), but firms were facing the CRTS technology with diminishing marginal return on capital and not aware of the positive externality they face. Next we will see another model by Romer with monopolistic competition and a R&D sector which can generate endogenous growth.

Monopolistic Competition, Endogenous Growth and R&D Romer's monopolistic competition model has three production sectors, the final goods production, intermediate goods production and R&D i.e. variety production. Our usual TFP parameter in production function will represent the 'variety' in production inputs and as we will see the growth of varieties through research and development firms will make sure a balanced growth path is sustainable. The production function in this economy is,

$$Y_t = L_{1t}^{\alpha} \int_0^{A_t} x_t(i)^{1-\alpha} di \tag{18}$$

where $x_t(i)$ is the type i intermediate good and there is a measure A_t of different intermediate goods and L_{1t} is the amount of labor allocated to the final good production. The production function exhibits CRTS. The intermediate goods are produced with the following linear technology,

$$\int_0^{A_t} \eta x_t(i) di = K_t \tag{19}$$

Now suppose the variety of goods grows at rate γ, $A_{t+1} = \gamma A_t$, is long run sustainable growth possible? The answer to this question will depend whether our final goods production technology is linear in growing terms. We do know by the curvature of the technology, optimality implies equal amount of each variety will be used in production, $x_t(i) = x_t$,then we have,

$$A_t \eta x_t = K_t \tag{20}$$

and our output at this equal variety becomes,

$$Y_t = L_{1t}^{\alpha} A_t {x_t}^{1-\alpha} \tag{21}$$

then substituting for x_t we have,

$$Y_t = \frac{L_{1t}^{\alpha}}{\eta^{1-\alpha}} A_t^{\alpha} K_t^{1-\alpha} \tag{22}$$

thus if both A_t and K_t are growing at rate γ, then production function is linear in growing terms and long run balanced growth is feasible. Note that this model becomes very similar to our previous exogenous labor productivity growth under these assumptions. The purpose of this model is to determine γ endogenously. What will be the source of growth, where does γ come from? As we will see, there will be incentives for R&D firms to produce new 'varieties' because there will be a demand for it. These new varieties will be patented to intermediate good production firms, where a patent will mean exclusive rights to produce that intermediate good. So we will have monopolistic competition in the intermediate goods production. Now suppose the law of motion for 'varieties', which is the technology in R&D sector is given by,

$$A_{t+1} = (1 + L_{2t}\zeta) A_t \tag{23}$$

where L_{2t} is the labor employed in R&D sector. Note that this is not a regular law of motion in the sense every new variety produced helps the production of further new varieties.such that there is a positive externality to variety production. Also assume leisure is not valued and we have aggragate feasibility condition for labor as,

$$L_{2t} + L_{1t} = 1 \tag{24}$$

As a homework, we have calculated the BG rate of SP version of this economy, now we will de-centralize this economy and characterize the equlibrium growth rate.and see that it is sub-optimal. The period t problem of a firm in the competitive final good production sector is,

$$\max_{x_t(i), L_{1t}} \{ L_{1t}^{\alpha} \int_0^{A_t} x_t(i)^{1-\alpha} di - w_t L_{1t} - \int_0^{A_t} q_t(i) x_t(i) di \} \tag{25}$$

and since we have CRTS with perfect competition we have zero profit with following FOCs,

$$\begin{aligned} w_t &= \alpha L_{1t}^{\alpha-1} \int_0^{A_t} x_t(i)^{1-\alpha} di & (26) \\ q_t(i) &= (1-\alpha) L_{1t}^{\alpha} x_t(i)^{-\alpha} & (27) \end{aligned}$$

notice that the inverse demand function for good of variety i is,

$$\left(\frac{q_t(i)}{(1-\alpha) L_{1t}^{\alpha}} \right)^{\frac{-1}{\alpha}} = x_t(i) \tag{28}$$

The intermediate goods industry will be monopolistic competition, in which there is only one firm, that is one patent holder, producing each variety. Each firm takes the demand of its variety and prices as given, and solves the following problem each period,

$$\begin{aligned} \Pi_t(i) &= \max_{x_t(i),K_t(i)} \{q_t(i)x_t(i) - R_t K_t(i)\} \\ s.t. \quad x_t(i) &= \frac{K_t(i)}{\eta} \end{aligned} \tag{29}$$

plugging in the inverse demand function and the technology constraint, the FOC is,

$$(1-\alpha)^2 x_t(i)^{-\alpha} L_{1t} = R_t \eta \tag{30}$$

and because of the symmetry we mentioned $x_t(i) = x_t = \frac{K_t}{\eta A_t}$ we can write this FOC as,

$$(1-\alpha)^2 (\frac{K_t}{\eta A_t})^{-\alpha} L_{1t} = R_t \eta \tag{31}$$

i.e. the rental price of capital is not equal to it's marginal product and there is opportunuties for positive profit. But also remember there is a fixed cost of entering to this industry, namely the price paid for the patent. Then as we will see the relation between the two will be one of our equilibrium conditions. Now lets look at the problem of R&D firms,

$$\max_{A_{t+1},L_{2t}} \{p_t^P (A_{t+1} - A_t) - w_t L_{2t}\} \tag{32}$$

$$s.t. A_{t+1} = (1 + L_{2t}\zeta)A_t$$

where p_t^P is the patent of the price. Free entry is assumed thus there will be zero profit in equilibrium. Notice also the R&D firm is solving a static problem without realizing the positive externality this period's decision creates on next periods production. As we will see, this and the monopoly power of the patent owners will be the sources of sub-optimality in decentralized solution.The FOC is,

$$p_t^P = \frac{w_t}{\zeta A_t} \tag{33}$$

where wage is determined in the final goods market and given this price equilibrium quantity will come from the deman function. As we mentioned before, one equilibrium condition will be that at any point in time, total profit a patent generates will be equal to price of it such that there will also be zero profit in the intermediate goods market.

$$p_t^P = \sum_{\tau=t}^{\infty} \frac{\Pi_t(i)}{(1+r)^{\tau-t}} \tag{34}$$

These conditions with constant growht equations for the growing variables is sufficient to characterize the equilibrium growth rate of this economy.