Lecture Notes for Econ 702 (Prof.Rios-Rull)

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1 Jan 28: Overview and Review of Equilibrium

1.1 Introduction

- What is an equilibrium (EQM)?
  - Loosely speaking, an equilibrium is a mapping from environments (preference, technology, information, market structure) to allocations.
  - Equilibrium allows us to characterizes what happens in a given environment, that is, given what people like, know, have...
- Two requirements of an equilibrium is (i) agents optimize. They do as best as possible. and (ii) actions of agents in the economy are compatible to each others. Examples of equilibrium concepts: Walrasian equilibrium.
- Properties of equilibrium concepts: existence and uniqueness.
  - We can prove existence of equilibrium by constructing one. We need uniqueness because otherwise, we do not have sharp prediction on what is likely to happen in a given environment.
  - Pareto optimality is not necessary property of equilibrium. Neither is tractability.

1.2 Arrow-Debreu Competitive Equilibrium

- What is Arrow-Debreu Competitive Equilibrium (ADE)?
  - All trades happen at time 0.
  - Perfect commitment
  - Everything is tradable and things are traded conditional on date and event.

[1] Makoto Nakajima's Econ 702 Lecture Notes for Spring 2002 were taken as basis when writing these notes. We are thankful to him.

– Agents are price takers.

- Agents' Problem

$$\max_{x \in X} U(x) \tag{1}$$

subject to

$$p(x) \leq 0 \tag{2}$$

$p : S \to R$ is a continuous, linear function. $S \supset X$ may include infinity dimensional subjects. Thus, p is a linear function, but not necessary a vector.

This problem can be applied to infinite horizon and/or stochastic environment.

In this problem, the single objective function is $U(x)$, and restriction is $x \in X$ and $p(x) \leq 0$. As commitment is perfect in X and trades happen at time 0, people do whatever they choose at time 0.

- Properties of ADE
  - Existence. $CE(\mathcal{E}) \neq \phi$.Existence is relatively easy to get. For uniqueness, we need more to get sufficient condition.
  - Pareto optimality. $CE(\mathcal{E}) \subset PO(\mathcal{E})$ when there is no externalities and non-satiation for utility function. But we need to be careful here. Given agents, endowment, preferences, technology and information structure, elements of competitive equilibrium are (i) price system and (ii) allocation. But $CE(\mathcal{E}) \subset PO(\mathcal{E})$ is only for allocation, which means allocation from such competitive equilibrium is Pareto optimal.
  - From $CE(\mathcal{E})$ to $PO(\mathcal{E})$, three things are required: (1) find price to get CE. (ii) right redistribution (transfer t) to give agents enough resources for the allocation. (iii) free disposal to ensure the quasi equilibrium is a true equilibrium.
  - \Second Basic Welfare Theorem. Proof uses separation theorem, for which the sufficient condition is (i) nonempty convex set.(ii) interior point.

    For any allocation $x \in PO(\mathcal{E})$, $\exists p$, such that (p, t, x) is quasi equilibrium with transfers (QET).

1.3 The Road Map

- In the first two weeks with Randy, we learned how to solve Social planner's problem (SPP) of neoclassical growth model with representative agent (RA-NGM), using dynamic programming. Also we know that solution to SPP is Pareto Optimal (PO) in our model. Other good things for solution to SPP is that, in RA-NGM, we know that (i) it exists and (ii) it's unique.

- Besides, we have two welfare theorems (FBWT, SBWT) from Dave's class. If we carefully define the environment, those two theorems guarantee (loosely) that (i) under certain conditions, Arrow-Debreu Competitive Equilibrium (ADE, or Walrasian equilibrium or valuation equilibrium) is PO, and (ii) also under certain conditions, we can construct an ADE from a PO allocation.

- Using those elements, we can argue that ADE exists and is unique, and we just need to solve SPP to derive the allocation of ADE, which is much easier task than solve a monster named ADE.

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- But we have another problem: The market assumed in ADE is not palatable to us in the sense that it is far from what we see in the world. So, next, we look at an equilibrium with sequential markets (Sequential Market Equilibrium, SME). Surprisingly, we can show that, for our basic RA-NGM, the allocation in SME and the allocation of ADE turn out to be the same, which let us conclude that even the allocation of the equilibrium with sequential markets can be analyzed using the allocation of SPP.

- Lastly, we will learn that equilibrium with sequential markets with recursive form (Recursive Competitive Equilibrium, RCE) gives the same allocation as in SME, meaning we can solve the problem using our best friend = Dynamic Programming.

- (Of course, these nice properties are available for limited class of models. We need to directly solve the equilibrium, instead of solving SPP, for large class of interesting models. We will see that Dynamic Programming method is also very useful for this purpose. We will see some examples later in the course.)

1.4 Review of Ingredients of RA-NGM

Technology

- Representative agent's problem.

$$\max_{\{c_t,k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{3}$$

subject to

$$k_{t+1} + c_t = f(k_t) \tag{4}$$

$$c_t, k_{t+1} \geq 0 \tag{5}$$

$$k_0 \text{ is given} \tag{6}$$

There are many variations of this problem, including models with distortion, stochastic environment. In writing such optimization problem, you should always specify control variable, initial condition.

Solutions is a sequence $\{c_t, k_{t+1}\}_{t=0}^{\infty} \in l_\infty$.

- Existence of a solution: We use Maximum theorem to prove existence. Sufficient condition to use Maximum theorem: (i) maximand is continuous function and (ii) constraint set is compact (closedness and boundedness). We assume u is continuous, f is bounded.
- Uniqueness: Sufficient condition includes: (i) convex constraint set, and (ii) strictly concave function. We assume u is strictly concave and f is concave
- Characterization of the solution: If u and f are differentiable and $\{c_t^*, k_{t+1}^*\}_{t=0}^{\infty}$ is the unique solution, the following condition has to be satisfied.

  $$u'(c_t) = \beta f'(k_{t+1})\, u'(c_{t+1}) \tag{7}$$

  And to rule out corner solution, Inada condition is assumed.

Homework 1.1 *Derive* (98)

(98) can be rewritten as

$$u'(f(k_t) - k_{t+1}) = \beta f'(k_{t+1})\, u'(f(k_{t+1}) - k_{t+2}) \tag{8}$$

This is a second order difference equation. We need two initial conditions to pin down the entire sequence. We have got one initial capital, we have to look for k_1 that does not go out of track. Therefore, to solve the problem as an infinite sequence is difficult. Now let's look at another way of solving it as you seen in Randy's class.

1.5 Dynamic Programming

Define $V(k)$ to be the highest utility of the agent by doing right things in life. The Bellman equation is

$$V(k) = \max_{\{c,k'\}} u(c) + \beta V(k') \tag{9}$$

subject to

$$c + k' = f(k) \tag{10}$$

The solution is

$$k' = g(k) = \arg\max u(c) + \beta V(k') \tag{11}$$

Using $g(.)$, we can construct the sequence of $\{k_{t+1}\}$.

$$\begin{aligned} k_0 &= g(k_{-1}) \\ k_1 &= g(k_0) = g^2(k_{-1}) \\ &\ldots \end{aligned}$$

We can show that the sequence constructed this way using Bellman equation satisfies the first order condition (98)

- F.O.C. of Bellman equation

$$V(k) = \max_{\{c,k'\}} u(f(k) - k') + \beta V(k') \tag{12}$$

is

$$-u'(f(k) - k') + \beta V_{k'}(k') = 0 \tag{13}$$

We know $V(.)$ is differentiable.

$$\begin{aligned} V_k(k) &= \frac{\partial}{\partial k}\{u[f(k) - g(k)] + \beta V(g(k))\} \\ &= [f_k(k) - g_k(k)]\, u_c[f(k) - g(k)] + \beta g'(k) V'(g(k)) \\ &= u_c f_k + \underbrace{g_k[-u_c + \beta V_c(g(k))]}_{=0 \text{ from FOC above}} \\ &= u_c f_k \end{aligned}$$

So,

$$-u_c + \beta f_k(k')\, u_c(c') = 0 \tag{14}$$

Therefore, we got the same FOC as what we have before.

$$-u_c + \beta f_k(k')\, u_c(c') = 0 \tag{15}$$

2 Jan 30: Neoclassical Growth Model

2.1 Review of growth model

- The model we studied last class is

$$\max_{\{c_t,k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{16}$$

subject to

$$k_{t+1} + c_t = f(k_t, 1) \tag{17}$$

$$c_t, k_{t+1} \geq 0 \tag{18}$$

$$k_0 \text{ is given} \tag{19}$$

We usually assume u and f are strictly concave, f is bounded. Then, there exists a unique solution which can be characterized by first order condition.

  - First Order Condition is a necessary but not sufficient condition for the optimal solution. Remember, the FOC is actually a second order difference equation. With only 1 initial condition k_0, there can be many sequences indexed by k_1. And the sequence may end up to be negative or infinity, which is not even feasible for this problem.
  - We can also get the optimal solution$\{c_t^*, k_{t+1}^*\}_{t=0}^{\infty}$ from Bellman equation as we see in Randy's class.

- Bellman equation for the growth model

$$V(k) = \max_{\{c,k'\}} u(c) + \beta V(k') \tag{20}$$

subject to

$$c + k' = f(k, 1) \tag{21}$$

The solution is

$$k' = g(k) \tag{22}$$

The solution is a fixed point of an functional operator, which is a contraction. Using $g(.)$, we can construct the sequence of $\{k_{t+1}\}\,.k_0, k_1 = g(k_0)\,...$

We can show that the sequence constructed this way using Bellman equation satisfies the first order condition (98)

-   - We can go back and forth between these two forms of problem. One way is to construct $\{k_{t+1}\}$ by $g(.)$. The other direction is to see the sequence $\{k_{t+1}\}$ satisfies $k_{t+1}^* = g(k_t)$.

2.2 Social Planner's Problem

- We can write the growth model as a social planner's problem (SPP). We assume the economy is populated by a huge number of identical agents. The social planner is like the God who tells people what they should do.

  - Properties of the solution to SPP:

    (1) Pareto optimal. It is a one-line proof: if the solution to SPP is not optimal, there is better allocation to make everyone happier.

    (2) Uniqueness. The social planner treats everyone the same. So the SPP is symmetric and we get unique solution. Since all the Pareto optimal allocation are solution to the SPP, PO($\mathcal{E}$) is unique.

- Road map: What we want to know is equilibrium (price and allocation). If we can apply welfare theorems to the allocation of SPP, we can claim that "God's will realizes" and can analyze allocation of SPP instead of directly looking at an equilibrium allocation. In order to use the argument above, we formalize the environment of RA-NGM in the way such that we can apply welfare theorems. By using (i) existence of solution to SPP, (ii) uniqueness of solution of SPP, and (iii) welfare theorems, we can claim that ADE (i) exists, (ii) is unique, (iii) and PO. However, market arrangement of ADE is not palatable to us in the sense that set of markets that are open in the ADE is NOT close to the markets in our real world. In other words, there is notion of time in ADE: all the trades are made before the history begins and there is no more choices after the history begins. So we would like to proceed to the equilibrium concept that allows continuously open markets, which is SME and we will look at it closely next week.

2.3 Arrow-Debreu Equilibrium (ADE)

- Elements of ADE are commodity space, consumption possibility space, production possibility space and preference set.

- Commodity space:

  Commodity space is a topological vector space S which is space of bounded real sequences with sup-norm.S includes everything people trade, which are sequences. Agents have time and rent it to firm (labor services), owe capital and rent it to firm (capital services) and buy stuff to consume some and save some for the future. Hence, $S = l_\infty^3$. $s_t = \{s_{1t}, s_{2t}, s_{3t}\}_{t=0}^{\infty}$, which are goods, labor services and capital services, respectively.

- Consumption possibility set X.

$X \subset S$ and

$$X = \{x \in S = l^3_\infty : \exists \{c_t, k_{t+1}\}_{t=0}^\infty \geq 0 \text{ such that}$$

$$\begin{aligned} k_{t+1} + c_t &= x_{1t} + (1-\delta)k_t \qquad \forall t \\ x_{2t} &\in [0,1] \qquad \forall t \\ x_{3t} &\leq k_t \qquad \forall t \\ k_0 &= \text{given}\} \end{aligned} \tag{23}$$

Interpretation is that x_{1t}=received goods at period t, x_{2t}=labor supply at period t, x_{3t}=capital service at period t. $k_{t+1} + c_t = x_{1t} + (1-\delta)k_t$ comes from real accounting. Note: capital and capital service are not the same thing. Think of the difference between a house and to rent a house.

- Preference $U : X \to R$.

$$U(x) = \sum_{t=0}^{\infty} \beta^t u(c_t(x)) \tag{24}$$

c_t is unique given x because each x implies a sequence $\{c_t, k_{t+1}\}_{t=0}^\infty$. If $x_{3t} = k_t$, $c_t = x_{1t} + (1-\delta)x_{3t} - x_{3t+1}$.

- Production possibility set Y.

  Firm's problem is relatively simple as firm do not have intertemporal decision. Firms just rent production factors and produce period by period.

  $Y = \Pi_{t=0}^\infty \widehat{Y}$:

$$Y = \Pi_{t=0}^\infty \widehat{Y}_t : \widehat{Y}_t = \{y_{1t}, y_{2t}, y_{3t} \geq 0 : y_{1t} \leq f(y_{3t}, y_{2t})\} \tag{25}$$

  Interpretation is that y_1=production at period t, y_{2t}=labor input at period t, y_{3t}=capital input at period t.

  - Note: We did not use the convention in general equilibrium that input is negative and output is positive.

    Implicitly, we assume firm is constant return to scale. So, we do not need to worry about industrial organization.

- Price.

  A price is a continuos and linear function $q : S \to R$. $q \neq 0$ and $q \in S^*$. S^* is a separating hyperplane in separation hyperplane theorem.

  - Continuity: for $s^n \to s$, $\Rightarrow q(s^n) \to q(s)$

– Linearity: $q(s^1+s^2) = q(s^1) + q(s^2)$

– $q(x)$ may not be represented

– Inner product representation: $S = l_\infty^3$, one candidate space for q is l_1. l_1 is space of sup-norm bounded sequences. If $\{z_t\} \in l_1, \sum_t^\infty |z_t| < \infty$. Then, for $s \in S$, $z \in l_1$, $\sum_t z_t s_t < \infty$. We use p to denote such price function. Not all $q(x)$ may not be represented in this inner product form. But we will see one theorem about inner product representation of price.

– If $p \in l_{1,3}$, we can write price as $p(s) = \sum_{t=0}^\infty p_{1t}s_{1t} + p_{2t}s_{2t} + p_{3t}s_{3t} < \infty$. Note here, the prices of labor service and capital service are negative, as we make input factor to be positive.

Homework 2.1 *$s \in S$, assume $s_{1t} = s_1, s_{2t} = s_2, s_{3t} = s_3, \forall t$ (that is we consider steady state, for simplicity), show that for any $r > 0$, the discounting $\left\{\frac{1}{(1+r)^t}\right\}_{t=0}^\infty$ is a price vector in $l_{1,3}$.*

- Define an ADE:

  An Arrow-Debreu Competitive Equilibrium is a triad (p^*, x^*, y^*) such that

  1. x^* solves the consumer's problem.

  $$x^* \in \arg\max_{x\in X} U(x) \tag{26}$$

  subject to

  $$q(x) \le 0 \tag{27}$$

  2. y^* solves the firm's problem.

  $$y^* \in \arg\max_{y\in Y} p(y)$$

  3. markets clear, i.e. $x^* = y^*$.

  – Note that the price system (or valuation function) p^* is an element of $Dual(L)$ and not necessarily represented as a familiar "price vector".

  – Note there are many implicit assumptions like (i) all the markets are competitive (agents are price taker), (ii) absolute commitment (economy with a lack of commitment is also a topic of macroeconomics, maybe from your 2nd year on), (iii) all the future events are known, with the probability of each events when trade occurs (before the history begins).

2.4 Welfare Theorems

Theorem 2.2 *(FBWT) If the preferences of consumers U are locally nonsatiated ($\exists \{x_n\} \in X$ that converges to $x \in X$ such that $U(x_n) > U(x)$), then allocation (x^*, y^*) of an ADE is PO.*

Homework 2.3 *Show U is locally nonsatiated.*

Theorem 2.4 *(SBWT) If (i) X is convex, (ii) preference is convex (for $\forall x, x' \in X$, if $x' < x$, then $x' < (1-\theta)x' + \theta x$ for any $\theta \in (0,1)$), (iii) $U(x)$ is continuous, (iv) Y is convex, (v) Y has an interior point, then with any PO allocation (x^*, y^*) such that x^* is not a satiation point, there exists a continuous linear functional q^* such that (x^*, y^*, p^*) is a Quasi-Equilibrium with transfer.*

- We can get rid of transfers in this economy. Everyone is the same, so, given $q(x_i) \leq t_i$, $\sum_i t_i = 0$, $\Rightarrow t_i = 0$ for all i.
- Quasi equilibrium and true equilibrium.

  Quasi equilibrium is allocation from cost minimization problem. That is, (a) for $x \in X$ which $U(x) \geq U(x^*)$ implies $q^*(x) \geq q^*(x^*)$ and (b) $y \in Y$ implies $q^*(y) \leq q^*(y^*)$.

  If, for (x^*, y^*, q^*) in the theorem above, the budget set has cheaper point than x^*, that is, $\exists x \in X$ such that $q(x) < q(x^*)$,

  then (x^*, y^*, p^*) is a ADE.

Homework 2.5 *Show that conditions for SBWT are satisfied in the PO allocation of RA-NGM.*

Now we established that the ADE of the RA-NGM exists, is unique, and is PO. The next thing we would like to establish is that the price system $q^*(x)$ takes the familiar form of inner product of price vector and allocation vector, which we will establish next.

2.5 Inner Product Representations of Prices

Theorem 2.6 *(based on Prescott and Lucas 1972) If, in addition to the conditions to SBWT, $\beta < 1$ (or some analog stochastic version about state) and u is bounded, then $\exists p^* \in l_{1,3}$ such that (x^*, y^*, p^*) is a QE.*

That is, price system has an inner product representations.

Remark 2.7 *For OLG (overlapping generation model), there may be no enough discounting. We will see how it works in that case.*

Remark 2.8 *Actually, for now, the condition $\beta < 1$ is what we need to know as you can see on class. For bounded utility function, remember that most of the familiar period utility functions (CRRA (including log utility function), CARA) in macroeconomics do not satisfy the conditions, as the utility function is not bounded. There is a way to get away with it, but we you not need to go into details (for those interested, see Stokey, Lucas, and Prescott, Section 16.3, for example).*

- Now the agent's problem can be written as

$$x^* \in \arg\max_{x \in X} U(x) \tag{28}$$

subject to

$$\sum_{t=0}^{\infty} p_{1t}s_{1t} + p_{2t}s_{2t} + p_{3t}s_{3t} = 0 \tag{29}$$

- In ADE: all the trades are made before the history begins and there is no more choices after the history begins. However, market arrangement of ADE is not palatable to us in the sense that set of markets that are open in the ADE is NOT close to the markets in our real world. We will allow people to trade every period and use sequence of budget constraints in agent's problem. Next week we will proceed to the equilibrium concept that allows continuously open markets, which is sequential market equilibrium.

3 Feb 2

3.1 Review

- We established the equivalence between ADE and SPP. Unlike the SPP allocation, ADE can kind of tell us what happened on the world as people act optimally and compactibly.

- ADE exists. And ADE allocation is optimal by FBWT. SBWT tells us the any SPP allocation can be got from a QET. And there are three key points about QET.

    - With identical agents, transfer is zero

- If there is a cheaper point, quasi equilibrium is a true equilibrium
- Price may have an inner product representation given the condition in Prescott and Lucas (1972) is satisfied

- ADE definition: $\exists\ p^*$ such that

$$x^* = \arg\max_{x \in X} U(x) \tag{30}$$

subject to

$$p^*(x) \leq 0 \tag{31}$$

or

$$\sum_{t=0}^{\infty} p_{1t}^* x_{1t} + p_{2t}^* x_{2t} + p_{3t}^* x_{3t} = 0 \tag{32}$$

3.2 Prices in Arrow-Debreu Competitive Equilibrium

- How to get p^*?

The basic intuition is from college economics: Price is equal to marginal rate of substitution. Remember, p_{1t}^* is the price of consumption good at period t in terms of consumption good at time 0.

- Denote λ as Lagrangian multiplier associated with (32) .Normalize price of time 0 consumption to be 1. From first order condition, we can get

$$\frac{\partial U(x^*)}{\partial x_{1t}} = \lambda p_{1t}^* \text{ for } t \geq 1 \tag{33}$$

$$\frac{\partial U(x^*)}{\partial x_{10}} = \lambda \tag{34}$$

Therefore Price of time t consumption good is equal to marginal rate of substitution between consumption at time t and consumption at time 0. ADE allocation x^* can be solved from SPP. As the functional form $U(.)$ is known, we can construct price series p_{1t}^*.

- To get price of labor service p_{2t}^* (related to wage), we have to look at the firm's problem under current setting. Consumer's problem says nothing about wage because leisure is not valued in utility function, although usually the marginal rate of substitution between consumption and leisure is a natural candidate.

– Firm's problem

$$y_t^* \in \arg\max_{y_t} p_{1t}^* y_{1t} + p_{2t}^* y_{2t} + p_{3t}^* y_{3t} \tag{35}$$

subject to

$$y_{1t} = f\left(y_{3t}, y_{2t}\right) \tag{36}$$

Denote λ_t as Lagrangian multiplier associated with (36)First order condition:

$$\begin{aligned} p_{1t}^* &= \lambda_t \\ p_{2t}^* &= -\lambda_t f_L\left(y_{3t}^*, y_{2t}^*\right) \end{aligned}$$

Thus, we can construct price p_{2t}^*. We have

$$\frac{p_{2t}^*}{p_{1t}^*} = -f_L\left(y_{3t}^*, y_{2t}^*\right) \tag{37}$$

And the wage rate at period t is price of labor service at time t in terms of time t consumption good.

– Capital service price and arbitrage.

$$\frac{p_{3t}^*}{p_{1t}^*} = -f_k(k_t^*, n_t^*) \tag{38}$$

Homework 3.1 *show (38) (note: there is no δ in this condition. And please relate it to (97))*

No Arbitrage: One freely tradable good can only have one market price. Two identical ways of transferring resource have to be priced at same level.

How people can move one unit of resources from t to t+1? There are two ways. one is to sell one unit x_{1t} at time t and get p_{1t}^*, then at time t+1, agents can get $\frac{p_{1t}^*}{p_{1t+1}^*}$ unit of consumption good x_{t+1}. The other way is to save one more unit of capital k_{1t+1}, at time t+1, agents can get rental of the additional unit of capital service and also non-depreciation part of k_{1t+1}. The relative price of doing so is $(1-\delta) + \frac{p_{3t+1}^*}{p_{1t+1}^*}$.

Hint 3.2 *The reason why we can see $(1-\delta)$ is the following: In ADE foc, all the arguments are $x's$ and $y's$. To relate to k_{1t+1}, we make use of consumption possibility space definition, $k_{t+1} + c_t = x_{1t} + (1-\delta) k_t$ and impose $x_{3t} = k_t$. Therefore, the benefit to save one more unit of capital is $(1-\delta) + \frac{p_{3t+1}^*}{p_{1t+1}^*}$. For details, see homework solution.*

From no arbitrage argument, we know

$$\frac{p_{1t}^*}{p_{1t+1}^*} = (1-\delta) + \frac{p_{3t+1}^*}{p_{1t+1}^*} \tag{39}$$

- In sum, we can solve SPP to get allocation of ADE, and then construct price using FOC of household and firm's problem.

3.3 Sequential Market Arrangements

- So far, rational expectation does not really apply. Agents do not need perfect forecasting as all the trades are decided at time 0. In agent's optimization problem, there is only one static constraint. At time t comes, people just execute their decision for this date.

- ADE allocation can be decentralized in different trade arrangement.

  - We will look at SME. Two things are important here: (i) Allocation in equilibrium with sequential market arrangements cannot Pareto-optimally dominate ADE allocation. (ii) But with various market arrangement, the allocation may be worse than ADE allocation. Say, labor market can be shut down or other arrangement to make people buy and trade as bad as it happens. (This topic is about endogenous theory of market institution).

- Sequence of Markets.

  With sequential markets, people have capital k_t, and rent it to the firm at rental $(1+r_t)$. People have time 1 and rent it to firm at wage w_t. They also consume c_t and save k_{t+1}. Agents can also borrow and lending one period loan l_{t+1} at price q_t. Then the budget constraint at time t is

  $$k_t(1+r_t) + w_t + l_t = c_t + k_{t+1} + q_t l_{t+1} \tag{40}$$

  - Principle to choose market structure: Enough but not too many.

    There are many ways of arranging markets so that the equilibrium allocation is equivalent to that in ADE, as we'll see. ENOUGH: Note that if the number of markets open is too few, we cannot achieve the allocation in the ADE (incomplete market). Therefore, we need enough markets to do as well as possible. NOT TOO MANY: To the contrary, if the number of markets are too many, we can close some of the markets and still achieve the ADE allocation in this market arrangement. Also it means that there are many ways to achieve ADE allocation because some of the market instruments are redundant and can be substituted by others. If the number of markets are not TOO FEW nor TOO MANY, we call it JUST RIGHT.

With the above structure, loans market is redundant as there is only one representative agent in this economy. In equilibrium, $l_t = 0$. So, we can choose to close loans market. We will see that even though there is no trade in certain markets in equilibrium, we can solve for prices in those markets, because prices are determined even though there is no trade in equilibrium, and agents do not care if actually trade occurs or not because they just look at prices in the market (having market means agents do not care about the rest of the world but the prices in the market). Using this technique, we can determine prices of all market instruments even though they are redundant in equilibrium. This is the virtue of Lucas Tree Model and this is the fundamental for all finance literature (actually, we can price any kinds of financial instruments in this way. we will see this soon.)

4 Feb 6

4.1 From ADE to SME

- We have seen that we need enough markets to get optimal allocation with sequential market arrangement. Market structure depends on commodity space. There should be markets to trade consumption good, capital services and labor services. Using consumption good at time t as the numarie and consider relative price, we can see that only 2 markets are needed. To transfer resources, only one intertemporal market is efficient since two ways of trade are equivalent, which are to trade c_t with c_{t+1} and to trade k_t and k_{t+1}. The reason why we can normalize price of c_t to 1 at any t is that with infinite horizon, future is identical at any time, as in the way we write Bellman equation.

- As we know from last class, we can write budget constraint with loans.

$$k_t (1 + r_t) + w_t + l_t = c_t + k_{t+1} + \frac{l_{t+1}}{R_{t+1}}$$

where r_t =rental price of capital and R_t = price of IOUs. But we can close the market of loans without changing the resulting allocation. This is because we need someone to lend you loans in order that you borrow loans, but there is only one agents in the economy. Then, the budget constraint becomes

$$k_t (1 + r_t) + w_t = c_t + k_{t+1} \tag{41}$$

4.2 Define SME

- Long definition first:

Consumer's problem is

$$\max_{x_{1t},x_{2t},x_{3t},k_{t+1},c_t,l_t} \sum_{t=0}^{\infty} \beta^t u\left(c_t\right) \tag{42}$$

subject to

$$\frac{l_{t+1}}{R_{t+1}} + x_{1t} = r_t x_{3t} + w_t x_{2t} + l_t \quad \forall t \tag{43}$$

The producer's problem is for all $t = 0, 1, 2, ...$

$$\max_{\{y_t\}} \{y_{1t} - w_t y_{2t} - r_t y_{3t}\} \tag{44}$$

subject to

$$y_{1t} \leq F(y_{3t}, y_{2t})$$

Definition 4.1 *A Sequential Market Equilibrium (SME) is* $\{x_{1t}^*, x_{2t}^*, x_{3t}^*, l_t^*\}$ $\{r_t^*, R_t^*, w_t^*\}$ $\{y_{1t}^*, y_{2t}^*, y_{3t}^*\}_{t=0}^{\infty}$, *and there exists* $\{c_t^*, k_{t+1}^*\}_{t=0}^{\infty}$ *such that (i) consumer maximizes: given* $\{\tilde{r}_t, \tilde{w}_t\}_{t=0}^{\infty}$, $\{x_{1t}^*, x_{2t}^*, x_{3t}^*, c_t^*, k_{t+1}^*, l_t^*\}$ *solves optimization problem. (ii) firm maximize: given* $\{\tilde{r}_t, \tilde{w}_t\}_{t=0}^{\infty}$, $\{y_{1t}^*, y_{2t}^*, y_{3t}^*\}$ *solves the producer problem. (iii) markets clear:* $x_{it}^* = y_{it}^*$, $\forall i, t$ *and* $l_{t+1}^* = 0$.

- There is a short way to write SME.

  First, let's look at the properties of SME:

  1) As all the solutions are interior and k_{t+1} or l_{t+1} cannot go to infinity in equilibrium, $R_{t+1} = (1-\delta) + r_{t+1}$.

  2) In equilibrium $x_{2t}^* = 1$ and $x_{3t}^* = k_t^*$.

  Consumer's Problem in SME can be written as follows:

$$\max_{\{c_t,k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t, n_t) \tag{45}$$

  subject to

$$c_t + k_{t+1} = w_t + [(1-\delta) + r_t]k_t \qquad \forall t = 0, 1, 2, ... \tag{46}$$

$$k_0 \text{ is given} \tag{47}$$

Definition 4.2 *A Sequential Market Equilibrium (SME) is* $\{c_t^*, k_{t+1}^*\}$ $\{r_t^*, w_t^*\}_{t=0}^{\infty}$*, such that*

(1) consumer maximizes: given $\{r_t^*, w_t^*\}_{t=0}^{\infty}$ $\{c_t^*, k_{t+1}^*\}$ *solves optimization problem.*

(2)

$$\left(c_t^* + k_{t+1}^* - (1-\delta)\, k_t^*, 1, k_t^*\right) \in \arg\max y_{1t} - w_t^* y_{2t} - r_t^* y_{3t} \tag{48}$$

subject to

$$y_{1t} = F\left(y_{3t}, y_{2t}\right)$$

- – FOC to problem 48:

$$\begin{aligned} 1 &= \lambda_t \\ w_t &= \lambda_t F_L\left(k_t^*, 1\right) \\ r_t &= \lambda_t F_k\left(k_t^*, 1\right) \end{aligned}$$

  then

$$\begin{aligned} w_t &= F_L\left(k_t^*, 1\right) \\ r_t &= F_k\left(k_t^*, 1\right) \end{aligned}$$

  – (2) in the above definition can be substituted with
  (2')

$$\begin{aligned} w_t &= F_L\left(k_t^*, 1\right) \\ r_t &= F_k\left(k_t^*, 1\right) \\ c_t^* + k_{t+1}^* - (1-\delta)\, k_t^* &= F\left(k_t^*, 1\right) \end{aligned}$$

Homework 4.3 *show (2')*$\Rightarrow$ (2)

Homework 4.4 *Explain the implication of CRS (constant return to scale) assumption on firms.*

- Another way of writing SME is:

A Sequential Market Equilibrium (SME) is $\{c_t^*, k_{t+1}^*\}$ $\{r_t^*, w_t^*\}_{t=0}^{\infty}$, such that

Definition 4.5 *(1) consumer maximizes (2') factor prices equal to marginal productivity (3) allocation is feasible.*

4.3 Compare ADE and SME

- Show the equivalence of ADE and SME:

Theorem 4.6 *If* $\{x^*, y^*, q^*\} \in AD(\mathcal{E})$*, then, there exists* $\{c_t^*, k_{t+1}^*, r_t^*, w_t^*\}_{t=0}^{\infty} \in SME(\mathcal{E})$

Proof: Pick the $\{c_t^*, k_{t+1}^*\}$ implied by consumer's problem in ADE. Define the following price:

$$\begin{aligned}
q^*(\{\{0,1,0\},\{0,0,0\},...\}) &= w_0^* && \text{wage at time 0} \\
&\ \ ... \\
q^*(\{\{0,0,0\},...\{0,1,0\},...\}) &= w_t^* && \text{wage at time t} \\
&\ \ ...
\end{aligned}$$

$$\begin{aligned}
q^*(\{\{0,0,1\},\{0,0,0\},...\}) &= r_0^* && \text{rental at time 0} \\
&\ \ ... \\
q^*(\{\{0,0,0\},...\{0,0,1\},...\}) &= r_t^* && \text{wage at time t} \\
&\ \ ...
\end{aligned}$$

Thus, we have constructed $\{r_t^*, w_t^*\}_{t=0}^{\infty}$. Next, we need to verify the condition for SME.

5 FEBRUARY 11

5.1 Review

- Last class, our purpose was to construct a new market arrangement, sequence of markets, because it is much closer to what we think markets are like in the real world. The fact that all trade takes place at time 0 in the Arrow-Debreu world is not very realistic so we wanted to allow the agents to trade at each period.

- We used certain properties of equilibrium to write a shorter version of SME that did not bother to distinguish between the choice of the firm and the household (for convenience).

- Now we will show that the allocations of the Arrow-Debreu equilibrium and the sequence of markets equilibrium are the same. Namely we will outline the proof of the following theorem:

(i) If (x^*, y^*, p^*) is an Arrow-Debreu equilibrium, we can construct the sequence of markets equilibrium with (x^*, y^*).

(ii) If $(\widetilde{x}, \widetilde{y}, \widetilde{r}, \widetilde{w})$ is an sequence of markets equilibrium, we can construct the Arrow-Debreu equilibrium with $(\widetilde{x}, \widetilde{y})$.

Proof. (Outline) ■

Remark 5.1 *Refer to the solution key of Hw 3 for the complete proof.*

First showing ADE $\Rightarrow$ SME

$(\mathrm{q}^*, x^*, y^*) \Rightarrow \exists \left\{c_t^*, k_{t+1}^*, r_t^*, w_t^*\right\}_{t=0}^{\infty} \in SME$

The first thing we need to do is construct the sequence of markets equilibrium prices from q^*. Remember that $\mathrm{q}^*(x)$ is a function that assigns a value to each commodity bundle in terms of consumption goods AT TIME 0. The prices in the AD world DO NOT correspond to the usual price for consumption goods, wage and rent. In order to get r_t^*, w_t^* we need to transform these prices in terms of units of consumption at time 0, to prices in terms of units of consumption goods at time t. The question is: How much does one unit of time 0 consumption exchange for unit of consumption at time t?

1 unit of time 0 consumption $\rightarrow \frac{1}{q^*(\{0,0,0\},\{1,0,0\},..........)}$

i.e. 1 unit of time 0 consumption can get you $\frac{1}{q^*(\{0,0,0\},\{1,0,0\},..........)}$ units of time 1 consumption.

Thus, we can write the following,

$w_0^* = q^*(\{0,1,0\}, \{0,0,0\}, ...)$

$r_0^* = q^*(\{0,0,1\}, \{0,0,0\}, ...)$

$w_1^* = \frac{q^*(\{0,0,0\},\{0,1,0\},..........)}{q^*(\{0,0,0\},\{1,0,0\},..........)}$

....

Homework *In the same way, write down the expressions for* r_t^*, w_t^*.

Now the following will be our strategy to show that from ADE we can get to SME:

First construct a candidate $\{\widetilde{r}_t, \widetilde{w}_t\}_{t=0}^{\infty}$ and $\left\{\widetilde{c}_t, \widetilde{k}_{t+1}, \widetilde{n}_t\right\}_{t=0}^{\infty}$ from (q^*, x^*, y^*).

- For $\widetilde{c}_t, \widetilde{k}_{t+1}, \widetilde{n}_t$, pick c_t^*, k_{t+1}^*, n_t^* so that

$$
\begin{aligned}
\widetilde{c}_t &= x_{1t}^* + (1-\delta)x_{3t}^* - x_{3t+1}^* && \forall t \\
\widetilde{n}_t &= x_{2t}^* && \forall t \\
\widetilde{k}_t &= x_{3t}^* && \forall t
\end{aligned}
$$

- For $\widetilde{r}_t, \widetilde{w}_t$, pick

$$
\begin{aligned}
\widetilde{r}_t &= \frac{p_{3t}^*}{p_{1t}^*} = F_k(k_t^*, n_t^*) && \forall t \\
\widetilde{w}_t &= \frac{p_{2t}^*}{p_{1t}^*} = F_n(k_t^*, n_t^*) && \forall t
\end{aligned}
$$

Now verify that these candidates solve the firm's and the consumer's maximization problem. For firms, this is obvious from the condition that marginal productivities equal to the prices of factors of production.

But for consumer, we need to show that,

$$\left\{\widetilde{c}_t, \widetilde{k}_{t+1}, \widetilde{n}_t\right\}_{t=0}^{\infty} \in \underset{\{c_t, k_{t+1}, n_t\}_{t=0}^{\infty}}{\operatorname{argmax}} \sum_{t=0}^{\infty} \beta^t u(c_t, 1-n_t)$$

$$\text{s.t.} \quad c_t + k_{t+1} = \widetilde{w}_t n_t + (1 + \widetilde{r}_t - \delta)k_t$$

We know that the objective function is strictly concave. The next thing we need is that the constraint set is convex.

Homework *Define F as,*

F=$\{c_t, k_{t+1}, n_t\}_{t=0}^{\infty} \mid c_t + k_{t+1} = \widetilde{w}_t + (\widetilde{r}_t + 1 - \delta)k_t \quad \forall t$

Show that F is convex

Once we know that above (i.e. the strict concavity of the objective function and convexity of the constraint set) we can say that the solution to the consumer's problem exists, is unique and the First Order Conditions characterize it (together with the Transversality Condition).

Then showing that if c_t^*, k_{t+1}^*, n_t^* satisfies the FOC in the AD world given q^* , it also satisfies the FOC from the consumer's problem above will be enough to complete the proof.

Question: Can we prove it another way, for example through contradiction? Yes, but that will not make our life any easier. Because even when you suppose that there is another allocations other than c_t^*, k_{t+1}^*, n_t^* that solves the consumer's problem in the sequence of markets, you will still need the properties that the solution satisfies as we derived to get the contradiction.

Now showing SME $\Rightarrow$ ADE

We need to build the AD objects (x's and y's) from the SME allocation.

$$\begin{aligned}
x_{1t}^* &= \widetilde{c}_t + \widetilde{k}_{t+1} - (1-\delta)\widetilde{k}_t \quad && \forall t \\
x_{2t}^* &= \widetilde{n}_t && \forall t \\
x_{3t}^* &= \widetilde{k}_t && \forall t
\end{aligned}$$

And the candidate for q^* will be,

$$q^*(x) = \frac{\sum\limits_{t=0}^{\infty} -x_{1t} + x_{2t}\widetilde{w}_t + x_{3t}\widetilde{r}_t}{\prod\limits_{s=0}^{t} (1+r_s^*)}$$

Note that this is a function on a whole sequence. We have to define q not just one a point but everywhere. The other way (ADE to SME) was easy because the wage and the rental prices were just numbers.

Homework *Show that this candidate for $q^*(x)$ is indeed a price (Hint: Show that it is continous and linear).*

Remark 5.2 *What does continous mean in infinite dimensional space? Bounded. In this context, it implies that the value of the bundle of commodities has to be finite and for that we need prices to go to zero sufficiently fast.*

$\frac{1}{\prod_{s=0}^{t}(1+r_s^*)} \rightarrow 0$ *A sufficient condition for this is that $r_s > 0$ for some s. i.e. that the interest rates are not negative too often.*

Thus we can say that a sufficient condition for the prices that we contructed to be bounded and thus continous is that the interest rates are positive. ***(BAK)***

Now from market clearing we know that the following has to hold,

$$x^* = y^*$$

Homework *Show that x^* and y^* solves the problem of the consumer and the firm.*

And that's the end of the second part of the proof.

5.2 ROAD MAP

What have we done so far?

- We know that the social planner's problem can be solved recursively (you learned this in Randy's class). So with dynamic programming methods, we get a good approximation of the optimal policy (g(k)) and get $\left\{c_t^*, k_{t+1}^*\right\}_{t=0}^{\infty}$ .Then we learned that this allocation is Pareto Optimal and that it can be supported as a quasi-equilibrium with transfers.
- We also learned that this allocation $\left\{c_t^*, k_{t+1}^*\right\}_{t=0}^{\infty}$ is also the sequence of markets equilibrium allocation and it is the ONLY one.

- So now we know that the dynamic programming problem gives us not only what is good but also what wiil happen in the sequence of markets.

  What next?

- The question that we now want to address is: What happens if there are heterogenous agents in the economy (versus the representative agent model that we have been dealing with so far) and if the solution is not Pareto Optimal?

- What can we do when we do not have the luxury of having an economy that does not satisfy the Welfare Theorems or when there are different agents?

- Can we still use dynamic programming to deal with problems like this?

- We will define equilibria recursively so that we can write the problem of the households as a dynamic programming problem and we will use the same methods Randy used to find the optimal policy rule g(k). But now the objects that the agents are choosing over are not sequences. They choose what they will do for today and tomorrow and prices are not a sequence anymore but a function of the states.

- We will do the construction of such equilibria after a short digression on shocks.

5.3 SHOCK AND HISTORY

We will now look at the stochastic RA-NGM.

What is a shock? Unanticipated change? Not really:

In a stochastic environment, we don't know exactly what will happen but we know where it's coming from (we know something about the stochastic process, i.e. the process that the shocks are following)

5.3.1 Markov Chains

In this course, we will concentrate on Markov productivity shock. Markov shock is a stochastic process with the following properties.

1. There are finite number of possible states for each time. More intuitively, no matter what happened before, tomorrow will be represented by one of a finite set.

2. The only thing that matters for the realization tomorrow is today's state. More intuitively, no matter what kind of history we have, the only thing you need to predict realization of shock tomorrow is today's realization.

   More formally, for each period, suppose either z^1 or z^2 happens[2] [3]. Denote z_t is the state of today and Z_t is a set of possible state today, i.e. $z_t \in Z_t = \{z^1, z^2\}$ for all t. Since the shock follow Markov process, the state of tomorrow will only depend on today's state. So let's write the probability that z^j will happen tomorrow, conditional on today's state being z^i as $\Gamma_{ij} = prob[z_{t+1} = z^j | z_t = z^i]$. Since Γ_{ij} is a probability, we know that

$$\sum_j \Gamma_{ij} = 1 \qquad for\ \forall i \tag{49}$$

   Notice that 2-state Markov process is summarized by 6 numbers: z^1, z^2, Γ_{11}, Γ_{12}, Γ_{21}, Γ_{22}.

The great beauty of using Markov process is we can use the explicit expression of probability of future events, instead of using weird operator called expectation, which very often people don't know what it means when they use.

5.3.2 Representation of History

- Let's concentrate on 2-state Markov process. In each period, state of the economy is $z_t \in Z_t = \{z^1, z^2\}$.
- Denote the history of events up to t (which of $\{z^1, z^2\}$ happened from period 0 to t, respectively) by

  $h_t = \{z_0, z_1, z_2, ..., z_t\} \in H_t = Z_0 \times Z_1 \times ... \times Z_t$.
- In particular, $H_0 = \emptyset$, $H_1 = \{z^1, z^2\}$, $H_2 = \{(z^1, z^1), (z^1, z^2), (z^2, z^1), (z^2, z^2)\}$.
- Note that even if the state today is the same, past history might be different. By recording history of event, we can distinguish the two histories with the same realization today but different realizations in the past (think that the current situation might be "you do not have a girl friend", but we will distinguish the history where "you had a girl friend 10 years ago" and the one where you didn't (tell me if it is not an appropriate example...).)

[2] In this class, superscript denotes the state, and subscript denotes the time.

[3] Here we restrict our attention to the 2-state Markov process, but increasing the number of states to any finite number does not change anything fundamentally.

- Let $\Pi(h_t)$ be the unconditional probability that the particular history h_t does occur. By using the Markov transition probability defined in the previous subsection, it's easy to show that (i) $\Pi(h_0) = 1$, (ii) for $h_t = (z^1, z^1)$, $\Pi(h_t) = \Gamma_{11}$ (iii) for $h_t = (z^1, z^2, z^1, z^2)$, $\Pi(h_t) = \Gamma_{12}\Gamma_{21}\Gamma_{12}$.

- $\Pr\{z_{t+1} = z^i \mid z_t = z^j, z_{t-1}, z_{t-2,\ldots\ldots\ldots\ldots}\} = \Gamma_{ji}$

- Having finite support of the distribution is very convenient.

  Homework *Show that a Markov chain of memory 2 can be represented as a Markov chain of memory 1.*

5.3.3 Social Planner's Problem with Shocks

- Social Planner's Problem (the benevolent God's choice) in this world is a state-contingent plan, i.e. optimal consumption and saving (let's forget about labor-leisure choice in this section for simplicity) choice for all possible nodes (imagine the nodes of a game tree. we need to solve optimal consumption and saving for each node in the tree).

- Notice that the number of nodes for which we have to solve for optimal consumption and saving is countable. This feature allows us to use the same argument as the deterministic case to deal with the problem. The only difference is that for deterministic case, the number of nodes is equal to number of periods (which is infinite but countable), but here the number of nodes is equal to the number of date-events (which is also infinite but countable).

- More mathematically, the solution of the problem is the mapping from the set of date-events (which is specified by history) to the set of feasible consumption and saving.

$$\max_{\{c_t(h_t), k_{t+1}(h_t)\}} \sum_{t=0}^{\infty} \beta^t \sum_{h_t \in H_t} \Pi(h_t) u(c_t(h_t))$$

subject to

$$c_t(h_t) + k_{t+1}(h_t) = (1-\delta)k_t(h_{t-1}) + z_t F[k_t(h_{t-1}), 1] \quad \forall t \ \forall h_t$$

$$k_0, z_0 \text{ given}$$

What is the dynamic programming version of this problem?

When we are writing the dynamic programming version, we need to carefully specify what the states are. States should be things that matter and change and that are predetermined. We will have more on this later.

$$V(z,k) = \max_{c,k\prime} u(c) + \beta \sum_{z\prime \in Z} \Gamma_{zz\prime} V(z\prime, k\prime)$$

subject to

$$c + k\prime = (1-\delta)k + zF(k,1)$$

$$k_0, z_0 \text{ given}$$

6 FEB 13: ADE and SME in a stochastic RA-NGM

6.1 Review

- Recall $\Gamma_{ij} = \Pr\{z_{t+1} = z^j \mid z_t = z^i\}$
- $\Gamma_{i.} = 1$ i.e. the probability of going SOMEWHERE given today's state is z^i is 1.
- $\Pi_t(h_t; \Gamma, z_0)$ is a function from the set of histories up to t.
- A Markov matrix Γ is a square matrix such that
    1. $\Gamma_{i.} = 1$
    2. $\Gamma_{ij} \geq 0$
- $\Pi_t(h_t; \Gamma, z_0)$: Here Γ denotes possible Markov matrices and z_0 denotes possible initial shocks. Why do we have ; ? This is because Γ and z_0 are given in the problem. They will not be changing while we do the analysis, they are like the parameters of the problem.

6.2 ADE

We will now go over Arrow-Debreu with uncertainty with the inner product representation of prices (rather than using a general continous linear function)

We first need to define the commodity space, the consumption possibility set and the production possibility set.

As in the deterministic environment, define commodity space as space of bounded real sequences with sup-norm $L = l^3_\infty$.

But before in the deterministic case, we only had 3 commodities for each period. Now we have 3 commodities for each date-event (h_t).

Define the consumption possibility set X as:

$X = \{x \in L = l^3_\infty : \exists \{c_t(h_t), k_{t+1}(h_t)\}_{t=0}^\infty \geq 0$ such that

$$\begin{aligned} k_{t+1}(h_t) + c_t(h_t) &= x_{1t}(h_t) + (1-\delta)k_t(h_t) \qquad \forall t \; \forall h_t \\ x_{2t}(h_t) &\in [0,1] \qquad \forall t \; \forall h_t \\ x_{3t}(h_t) &\leq k_t(h_t) \qquad \forall t \;\; \forall h_t \\ & k_0, z_0 \text{ given}\} \end{aligned}$$

- Notice that the only difference from before is that now all the constraints has to hold for all periods AND all histories.

  Define the production possibility set Y as:

  $$Y = \{y \in L : y_{1t}(h_t) \leq F(y_{3t}(h_t), y_{2t}(h_t)) \quad \forall t \; \forall h_t\}$$

  The consumer's problem in ADE is:

  $$\max_{x \in X} \sum_{t=0}^{\infty} \beta^t \sum_{h_t \in H_t} \Pi(h_t) u(c_t(h_t))$$

  subject to

  $$\sum_{t=0}^{\infty} \sum_{h_t \in H_t} \sum_{i=1}^{3} \hat{p}_{it}(h_t) x_{it}(h_t) \leq 0$$

  We know that the solution to this problem is Pareto Optimal.

  Recall the dynamic programming version of the social planner's problem:

$$V(z,k)=\max_{c,k\prime} u(c)+\beta\sum_{z\prime\in Z}\Gamma_{zz\prime}V(z\prime,k\prime)$$

subject to

$$c+k\prime=(1-\delta)k+zF(k,1)$$

k_0, z_0 given

Remember that the state needs to be changing and predetermined. For example, Γ is not a state.

Solution to the above problem is a policy rule k'=g(k) and from this policy rule we can draw the whole path for capital. Also Second Welfare Theorem tells us that the solution can be supported as a quasi-equilibrium with transfers.

6.3 SME

$\mathrm{p}_1(h_{17})$: Price of one unit of the consumption good in period 17 at history h.

$\mathrm{p}_1(\widetilde{h}_{17})$:Price of one unit of the consumption good in period 17 at history $\widetilde{h}$.

We want to have sequence of markets that are complete. We want the agents to be able to transfer resources not just across time but also across different states of the world.

For this, we need state contingent assets.

The budget constraint for the representative agent in SME world is:

$$c_t(h_t)+k_{t+1}(h_t)+\sum_{z_{t+1}\in Z} q_t(h_t,z_{t+1})l_{t+1}(h_{t+1})=k_t(h_{t-1})[1+r_t(h_t)]+w(h_t)+l_t(h_t)$$

Here $l_{t+1}(h_{t+1})$ is the state contingent claim. By deciding how much $l_{t+1}(h_{t+1})$ to get for each possible h_{t+1}, the agent decides how much of the good he is buying for each possible realization of tomorrow.

Homework *What should the expression below be equal to?*

$$\sum_{z\prime \in Z} q_t(h_t, z\prime) = ?$$

Note that this is the price of an asset that pays one unit of the good to the agent the next period at each state of the world.

A sequence of markets equilibrium is $\{c_t(h_t), k_{t+1}(h_t), l_{t+1}(h_t, z_{t+1})\}$, $\{w(h_t), r(h_t), q_t(h_t, z_{t+1})\}$ such that,

1. Given $\{w(h_t), r(h_t), q_t(h_t, z_{t+1})\}$, $\{c_t(h_t), k_{t+1}(h_t), l_{t+1}(h_t, z_{t+1})\}$ solves the consumer's problem.
2. $w(h_t) = z_t F_2(k_t(h_{t-1}), 1)$
   r(h$_t$) $= z_t F_1(k_t(h_{t-1}), 1)$
3. $l_{t+1}(h_t, z_{t+1}) = 0 \qquad \forall h_t, z_{t+1}$

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What is it that people buy and sell in the sequence of markets?

Consider an economy with two periods. At t=0, the agent's endowment of the good is 2 units. At t=1,, two things can happen: The good state or the bad state. The bad state happens with probability π, and the bad state with probability (1-π). In the good state, the agent's endowment is 3 units of the good and in the bad state the agent's endowment is 1 unit of the good.

How many date events are there? 3 date events. Because in addition to the first period, we also have the two possible "events" that can take place at t=1.

The consumer's problem in this economy is:

$$\max \; u(c_0) + \pi u(c_b) + (1 - \pi) u(c_g)$$

$$s.t.\;\; c_0 + p_g c_g + p_b c_b = 2 + 3p_g + p_b$$

Suppose the solution to the consumer's problem is $\{2, 2, 4\}$. What does this mean?

He signs a contract in period 0, then he consumes c_0 (regardless of anything). After period 0, nature determines whether the good state or the bad state happens. NO TRADE happens in period 1. All trade already took place at t=0. All that takes place at t=1, is the fullfilment of whatever promises for deliveries were made at t=0. For example, the given allocation above tells us the following: The guy signs a contract at t=0 promising that he will give up his endowment of 3 units of the good in the good state for delivery of 2 units AND he will give up his endowment of 1 unit of the good in the bad state for the delivery of 4 units. And it also tells us that he will consumer 2 units of the good at period 0, no matter what happens.

Remember not to think of this concept as just insurance. Because insurance is only a subset of possible state contingent claims. We are talking about any kind of state contingent claims here, not just the ones which are only geared towards insuring you agains the bad state.

Now let's extend this to three periods. We will now have 7 commodities.

The agent's objective function is:

$$u(c_0) + \pi u(c_b) + (1-\pi)u(c_g) + \pi^2 u(c_{bb}) + \pi(1-\pi)\left[u(c_{bg}) + u(c_{gb})\right] + (1-\pi)^2 u(c_{gg})$$

In the Arrow-Debreu world, in complete markets, how many commodities are traded? 7 commodities. It is 7 commodities because the agent need to decide what he wants for each date-event. For t=2, we have four date events, for t=1 we have two date-events, and for t=0 we have one. These date-events are the nodes.

Recall that in the AD world, after period 0, all people do is honour their commitment and deliver promises. No trade takes place after period 0.

How about in the sequence of markets? Trades can occur at more than one node. We want to implement the same type of allocation as in AD with a market arrangement that is simpler and recurrent. Think of the same world that we described above, with two periods and two states. And take note of the fact that at each one of those nodes, trade CAN take place now, unlike in the AD arrangement.

In the sequence of markets, how many things are traded at period 0? Only 2. This is because in the sequence of marketts, the agent does not trade for two periods ahead. Also once we go on to t=1, at one of the nodes, say the good state, the agent again only trades for two commodities, he does not do anything about the other state anymore, because the bad state has not happened.

We will characterize what happens in this world through backwards induction. We will first go to the last period (t=1 in this case) and work backwards.

So at t=1, the agent is either at the good state or the bad state. Let's first consider the node associated with the good state.At this node, the agent consumes c_g and he chooses what he will consume if tomorrow's period is good again (c_{gg}) and he chooses what he will consumer if tomorrow's period is bad (c_{gb}). His objective function consists of the utility that he gets from consuming c_g and the expected value of his utility in the next period.

$$V_g(x_g; p) = \max \; u(c_g) + \pi u(c_{gb}) + (1-\pi)u(c_{gg})$$

$$s.t. \;\; c_g + \frac{p_{gb}}{p_g}c_{gb} + \frac{p_{gg}}{p_g}c_{gg} = x_g + 1\frac{p_{gb}}{p_g} + 3\frac{p_{gg}}{p_g}$$

x_g : The agent's past choice on what to get at the node associated with the good state at t=1.

Now consider the node associated with the bad state:

$$V_b(x_b; p) = \max \; u(c_b) + \pi u(c_{bb}) + (1-\pi)u(c_{bg})$$

$$s.t. \;\; c_b + \frac{p_{bb}}{p_b}c_{bb} + \frac{p_{bg}}{p_b}c_{bg} = x_b + 1\frac{p_{bb}}{p_b} + 3\frac{p_{bg}}{p_b}$$

We have basically collapsed what the agent cares for after period 1 to the V functions.

Now go to time 0. The consumer's problem is:

$$\max \quad u(c_0) + \pi V_b(x_b; p) + (1-\pi) V_g(x_g; p)$$

$$s.t. \quad c_0 + x_g p_g + x_b p_b = 2 + 3p_g + p_b$$

Constructing ADE from SME and vice versa in this environment:

This is trivial because this time we don't even need to bother with constructing the prices from one world to the other. Notice that in the formulations of the consumer's problem in the sequence of markets, we already have been implicitly using the AD prices given that we know the allocations will be the same. The p's are the AD prices and the SM prices are, for example, $\frac{p_{bg}}{p_b}$, etc.

However, one thing you should be aware of is that ADE gives us certain prices and allocations; whereas in SME we need to determine the prices, allocations AND x_g and x_b.

From ADE to SME:

1. Construct the SM prices:

   $p_b = 0$

   $p_g = p_g$

2. Use the same allocation:

   c_0, c_g, c_b, c_{gg}

3. Construct the missing items (x_g and x_b)

   Using the budget constraint, get x_g from the prices and allocations in state g; and get x_b from the prices and allocations in state b.

4. Verify that the following SME conditions are satisfied:

   -Markets clear at last period (This is trivial from ADE)

   -x's add up to 0 across consumers.

   -c and x solve the consumer's problem.

From SME to ADE:

1. Get rid of the x's.

2. Verify conditions of ADE.

With two periods:

In the sequence of markets how many things are traded? 9 because we have 3 commodities at each of the 3 nodes.

In the Arrow-Debreu it was 7.

Now suppose we have 100 goods instead. In Arrow-Debreu, we will have 700 things to trade. On the other hand, in SM, we will have 102 goods to trade per node and thus we will have only 306 things to trade.

In the sequence of markets, we have minimal number of trades to get the best allocation. Arrow-Debreu has nice properties but it's messy to deal with.

7.1 Back to the Growth Model

$$\max_{\{c_t(h_t),k_{t+1}(h_t)\}} \sum_{t=0}^{\infty} \beta^t \sum_{h_t \in H_t} \Pi(h_t) u(c_t(h_t))$$

$$\text{s.t. } c_t(h_t)+k_{t+1}(h_t)+\sum_{z_{t+1}} q_t(h_t, z_{t+1}) x(h_t, z_{t+1}) = k_t(h_{t-1})[1+r_t(h_t)-\delta]+w(h_t)+x(h_{t-1}, z_t(h_t))$$

Note: The notation $z_t(h_t)$ just refers to the z that is consistent with history h_t.

In the representative agent model, market clearing requires that $x(h_t, z_{t+1}) = 0 \quad \forall h_t \; \forall z_{t+1}$

Homework *Consider an economy with 2 periods. There are two states of nature: The good state and the bad state.Both states have equal probabilities. There is only one agent in the economy and he has an endowment of 1 coconut and 2 scallops. In the good state, he will have an endowment of 3 units of the goods and in the bad state, he will have an endowment of 1 unit of each good. The agent's utility function takes the following form:*

$$u(s,c) = \log s + \log c$$

Compute the equilibrium for this economy.

As before when we write down the equilibrium, we do a shortcut and we ignore the x and q. This does not mean that markets are not complete. If all agents are identical then state contingent claims have to be 0 for all nodes.

7.2 General Overview

So far we have shown the following:

SPP $\Leftrightarrow$ AD (From the Welfare Theorems)

SPP$\Leftrightarrow$ Dynamic Programming Problem(What Randy did)

AD$\Longleftrightarrow$ SME

SME $\leftarrow$ sc $\rightarrow$ RCE

RCE $\leftarrow$ sc $\rightarrow$ Dynamic Programming Problem

-sc denotes "something in common"

Notice that RCE and DP are not necessarily equivalent.

Also, SME and RCE are not necessarily equivalent.

Why would it be that SPP $\not\Leftrightarrow$ AD

-Markets may not be complete.

-Externalities

-Heterogenous Agents

So for most equilibria, we need to compute the equilibria directly. We don't have the luxury of solving the social planner's problem to get the equilibrium allocation. Solving the problem from AD and SME, it's very messy. So we will use the RCE notion to characterize what happens in the economy.

8 Feb 20: Defining RCE

8.1 Review

Consider the following two period economy:

The goods A and B at time 0 are denoted by x_0^A ,x_0^B and the goods at time 1 are denoted by x_1^A ,x_1^B.

In Arrow-Debreu, the consumer's problem is:

$$\max \quad u(x_0, x_1)$$

$$s.t. \sum_{i=0}^{1} \sum_{l=A,B} p_i^l x_i^l \quad \leq 0$$

In SME,

$$\Omega(b_1^A; q) = \max_{x_1^A, x_1^B} \quad u(x_1^A, x_1^B)$$

$$s.t. \; x_1^A + q_{1B} x_1^B = b_1^A$$

where b_1^A is what the consumer chose to bring from the past. Assume that loans are in the form of good A (don't need to transfer resources in the form of all goods. Saving in the form of only one good is enough.)

Now go to period 0,

$$\max_{\mathrm{x}_0^A, x_0^B, b_1^A} \quad \mathrm{u}(\mathrm{x}_0^A, x_0^B) + \beta\Omega(b_1^A; q)$$

$$s.t.\ x_0^A + q_0^{bond} b_1^A + q_{0B} x_0^B = endowment$$

Homework *Take this simple economy and show the equivalence between SME and ADE*

Homework *Given an ADE, write two sequence of markets equilibria. In one of them, take good A as the good used to transfer resources into the future.In the other, take it as good B. Show that the two allocations are equivalent.*

9 Feb 20

9.1 Road map

- From now on, we will look at Recursive Competitive Equilibrium (RCE).
  - In Randy's class, we learned that a Sequential Problem of SPP can be solved using Dynamic Programming. Now we will see that we can use the same Dynamic Programming technique to solve an equilibrium, RCE.
  - First, we know the equivalence between an allocation of SPP and an allocation of ADE, using Welfare Theorems. And we showed that ADE can be represented as SME, where the market arrangements are more palatable. From today, we will see that SME is equivalent to RCE.
  - When Welfare Theorems holds, we do not need to directly solve the equilibrium, because we know that allocation of SPP can be supported as an equilibrium and it is unique, meaning the SPP allocation is the only equilibrium. But if (i) assumptions of Welfare Theorems do not hold or (ii) we have more than one agent, thus we have many equilibrium depending on the choice of the Pareto weight in the Social Planner's Problem, we can solve the equilibrium directly, both in theory and empirically using computer. Since (i) solving ADE is "almost impossible", (ii) solving SME is "very hard", but (iii) solving RCE is "possible", RCE is important for analyzing this class of economies, where Welfare Theorems fail to hold.

- In ADE and SME, sequences of allocations and prices characterize the equilibrium, but in RCE, what characterize the equilibrium are functions from state space to space of controls and values.

9.2 Recursive representation in equilibrium

Remember that the consumer's problem in SME is as follows:

$$\max_{\{k_{t+1},c_t\}_{t=0}^{\infty}} \sum_t \beta^t u(c_t) \tag{50}$$

$$c_t + k_{t+1} = w_t + [1 + r_t]k_t \tag{51}$$

How to translate the problem using recursive formulation? First we need to define the state variables. state variables need to satisfy the following criteria:

1. PREDETERMINED: when decisions are made, the state variables are taken as given.
2. It must MATTER for decisions of agents: there is no sense of adding irrelevant variables as state variable.
3. It VARIES across time and state: otherwise, we can just take it as a parameter.

Be careful about the difference between aggregate state and individual state. Aggregate state is not affected by individual choice. But aggregate state should be consistent with the individual choice (we will consider the meaning of "consistency" more formally later), because aggregate state represents the aggregated state of individuals. In particular, in our RA-NGM, as we have only one agent, aggregate capital turns out to be the same as individual state in equilibrium, but this does not mean that the agent decide the aggregate state or the agent is forced to follow the average behavior, but rather the behavior of the agent turns out to be the aggregate behavior, in equilibrium.

Also note that prices (wages, and rental rates of capital) is determined by aggregate capital, rather than individual capital, and since individual takes aggregate state as given, she also takes prices as given (because they are determined by aggregate state). Again, the aggregate capital turns out to coincide with the individual choice, but it is not because of the agent's choice, rather it is the result of consistency requirement.

One notational note. Victor is going to use a for individual capital and K for aggregate capital, in order to avoid the confusion between K and k. But the problem with aggregate and individual capital is often called as "big-K, small-k" problem, because the difference of aggregate capital and individual capital is crucial. So for our case, the counterpart is "big-K, small-a" problem.

Having said that we guess that candidates for state variables are $\{K, a, w, r\}$. But we do not need $\{r, w\}$. Why? Because they are redundant: K is the sufficient statistics to calculate $\{r, w\}$ and K is a state variable, we do not need $\{r, w\}$ as state variables.

Now let's write the representative consumer's problem in the recursive way. At this point, the time subscript has not be got rid of. People care about today's period utility and the continuation utility from tomorrow t+1:

$$V_t(K, a; G) = \max_{c,a'}[u(c) + \beta V_{t+1}(K', a'; G)] \tag{52}$$

subject to

$$c + a' = w + [1 + r - \delta]a \tag{53}$$

$$w = w(K) \tag{54}$$

$$r = r(K) \tag{55}$$

$$K' = G(K) \tag{56}$$

Fundamental rules to write a well-defined problem:

- All the variables in the problem above: $([u(c) + \beta V(K', a'; G^e)])$ have to be either (i) a parameter or an argument of the value function V(.) (state variable), (ii) a choice variable (so appear below max operator, c and a' here), (iii) or defined by a constraint, in order for the problem to be well defined. In the case above, note (i) c and a' is a choice variable, (ii) K' is defined by (56) (which we will discuss below), (iii) the variables in (53) (especially r and w) are also defined by constraints, which only contains state variables (K), thus we know that the problem is well defined.

- Agents need to make expectations about tomorrow's price to make consumption - saving choice. Because prices $\{r, w\}$ are given by marginal product of production functions. Agents have to make "forecast" or "expectations" about the future aggregate state of the world.

- We index the value function with G because the solution of the problem above depends on the choice of G. But what is "appropriate" G? This is revealed when we see the definition of an equilibrium below.

Homework 9.1 *Show the mapping defined by (52) is a contraction mapping. And prove the existence of FP and give the solution's properties.*

9.3 Recursive Competitive Equilibrium:

Now, let's define the Recursive Competitive Equilibrium:

Definition 9.2 *A Representative Agent Recursive Competitive Equilibrium with arbitrary expectation G^E is $\{V(.), g(.), G(.)\}$ such that*

1. $\{V(.), g(.)\}$ *solves consumer's problem:*

$$V(K, a, G^E) = \max_{c,a'}[u(c) + \beta V(K', a'; G^E)]$$

subject to

$$c + a' = w(K) + [1 + r(K) - \delta]a \tag{57}$$

$$K' = G^E(K) \tag{58}$$

Solution is $g\left(K, a; G^E\right)$.

2. *Aggregation of individual choice:*

$$K' = G(K; G^E) = g(K, K; G^E) \tag{59}$$

Some comments on the second condition. The second condition means that if a consumer turns out to be average this period (her individual capital stock is K, which is aggregate capital stock), the consumer will choose to be average in the next period (she chooses $G(K)$, which is a belief on the aggregate capital stock in the next period if today's aggregate capital stock is K). You can interpret this condition as "consistency" condition, because this condition guarantees that in an equilibrium, individual choice turns out to be consistent with the aggregate law of motion.

Agents have rational expectation when $G = G^E$. To compute this equilibrium, we can define G^E first, then get g and $G(.)$. The whole sequence of equilibrium choice is obtained by iteration.

Now, let's define A Representative Agent Recursive Competitive Equilibrium with rational expectation.

Definition 9.3 *A Representative Agent Recursive Competitive Equilibrium with rational expectation is $\{V(.), g(.), G(.)\}$ such that*

1. $\{V(.), g(.)\}$ *solves consumer's problem:*

$$V(K,a,G) = \max_{c,a'}[u(c) + \beta V(K',a';G^E)]$$

subject to

$$c + a' = w(K) + [1 + r(K) - \delta]a \tag{60}$$

$$K' = G(K) \tag{61}$$

Solution is $g(K,a;G)$.

2. *Aggregation of individual choice:*

$$K' = G(K) = g(K,K;G) \tag{62}$$

In other words, a RA RCE with rational expectation is a RA RCE with expectation G^E while with additional condition imposed:

$$G\left(K, G^E\right) = G^E(K)$$

9.4 Solve SPP and RCE

When we look at SPP in recursive form, we find a contraction mapping. The SPP is solved as the fixed point of contraction mapping. In math, we define

$$T(V_0)(K) = \max_{K' \in X} R(K,K') + \beta V_1(K')$$

where T maps a continuos, concave function to a continuos and concave function. And we can show T is a contraction mapping. To find the fixed point of this contraction V^*, we can use iteration: for any continuos and concave function V_0,

$$V^* = \lim_{n \to \infty} T^n(V_0)$$

such that

$$V^* = \max_{K' \in X} R(K,K') + \beta V^*(K')$$

But to solve a RE RA RCE, we cannot use such fixed point theorem because we need find (V, G, g) jointly. Similarly, we can define the following mapping $\widehat{T}$ which has three parts corresponding to (V, G).

$$\begin{aligned} V_1(K,a) &= \widehat{T}_1(V_0, G_0) = \max_{c,a'} u(c) + \beta V_0(G_0(K), a') \\ s.j. \quad c + a' &= w(K) + [1 - \delta + r(K)]a \end{aligned}$$

and the decision rule is

$$a' = g(K, a; G_0)$$

$$G_1(K) = \widehat{T}_2(V_0, G_0) = g(K, K; G_0)$$

We can see the first component of $\widehat{T}$ mapping gives V, and the second part gives G. Fixed point of this mapping $\widehat{T}$ is RE RA RCE.

But, $\widehat{T}$ is not a contraction. It is more difficult to find RE RA RCE in theory, but we will see how we can solve the problem on computer later.

- Another comment about RCE: If there are multiple equilibria in the economy, it is problematic to define RCE. The reason is that RCE solution is functions. Given today's state variable, tomorrow's state is unique. When we construct SME out of RCE $\{..., K_i, K_j, ...\}$, given K_i, there is only one unique K_j.

10 Feb 25

10.1 From RCE to SME

Homework 10.1 *Prove that a RCE with RE is a SME.*

Hint 10.2 *You can show by construction. Suppose we have a RCE. Using a_0 (given) and $G(K)$, we can derive a whole sequence of $\{k_t,\ c_t\}_{t=0}^{\infty}$. Using the constructed sequences of allocation, we can construct sequence of prices $\{r_t,\ w_t\}_{t=0}^{\infty}$. Remember that we have necessary and sufficient conditions for SME. we just need to show that the necessary and sufficient conditions are satisfied by the constructed sequences.*

10.2 RCE for the Economy with Endogenous Labor-Leisure Choice

Let's try to write down the problem of consumer. The first try:

$$V(K, a; G) = \max_{c,n,a'}\{u(c, n) + \beta V(K', a'; G)\} \tag{63}$$

subject to

$$c + a' = [1 - \delta + r(K)]a + w(K)n \tag{64}$$

$$K' = G(K) \tag{65}$$

This is an ill-defined problem. Why? Something is missing! $r(K)$ and $w(K)$ are wrong function of price because now K is not sufficient determinant of w and r.From firm's problem, we know

$$w = f_2(K, N)$$

Now we have two options to add the missing piece.

Option 1: write $V(K, a; G, w(.), r(.))$. And the equilibrium condition would be

$$\begin{aligned} w(K). &= f_2(K, N) \\ r(K) &= f_1(K, N) \end{aligned}$$

Option 2: write $V(K, a; G, H)$ where H function is agent's expectation about aggregate labor as function of aggregate capital.

$$N = H(K)$$

then, the price function is

$$\begin{aligned} w(K) &= f_2(K, H(K)) \\ r(K) &= f_1(K, H(K)) \end{aligned}$$

We will use option 2 to write RCE with RE.

Homework 10.3 *Define RCE using option 1*

From now on, we will only look at RCE with rational expectation. Now the consumer's problem is

$$V(K, a; G, H) = \max_{c,n,a'}\{u(c, n) + \beta V(K', a'; G, H)\} \tag{66}$$

subject to

$$c + a' = [1 - \delta + f_1(K, H(K))]a + f_2(K, H(K))n \tag{67}$$

$$K' = G(K) \tag{68}$$

And the solutions are:

$$a' = g(K, a; G, H) \tag{69}$$

$$n = h(K, a; G, H) \tag{70}$$

Definition 10.4 *A RCE is a set of functions* $\{V(.),\ G(.)\,, H(.),\ g(.)\,,\ h(.)\}$ *such that*

1. *Given* $\{G(.)\,, H(.)\}$, $\{V(.),\ g(.),\ h(.)\}$ *solves the consumer's problem.*
2. $$G(K) = g(K, K; G, H) \tag{71}$$
$$H(K) = h(K, K; G, H) \tag{72}$$

10.3 More on solving RCE

We have known that we can define mapping for

$$V^0(K,a) = T\left(V^1(.)\right) = \max_{a' \in X} u(a, a') + \beta V^1(K', a')$$

Note that we cannot solve a mapping since that's a mechanic thing. Mapping we have here is from a functional space to a functional space. We can only solve equation. For example, the Bellman equation is a functional equation which we can solve.

$$V(K,a) = \max_{a' \in X} u(a, a') + \beta V(K', a')$$

When the mapping we defined above is a contraction mapping (sufficient condition is monotonicity and discounting), then there is a unique fixed point. This fixed point can be obtained by iteration. For RCE, if we fix G and H, we can construct the contraction mapping and get fixed point by iteration. The reason why the value function is fixed point is that in infinite horizon economy, today's view of future is the same as that of tomorrow. For finite horizon economy, we have to solve problem backward, starting from $V_{T-1}(.) = \max u(.) + \beta V_T(.)$

To solve RCE, there are two steps. First, given G and H, we can solve the problem by some approximation methods (you will see this in late May). Second, we have to verify that G and H are consistent in equilibrium. That is agent's expectation is actually correct as what happens in life. Since there is no contraction mapping for

$$\begin{aligned} G'(K) &= g\left(K, K; G^0, H^0\right) \\ H'(K) &= h\left(K, K; G^0, H^0\right) \end{aligned}$$

it is hard to prove existence directly. But we can construct one and verify the equilibrium condition. This is the way to solve RCE.

Although compared with SPP, RCE is hard to solve, it can be used to characterize more kinds of economies, including those environments when welfare theorem does not hold.

10.4 RCE for non-PO economies

What we did with RCE so far can be claimed to be irrelevant. Why? Because, since the Welfare Theorems hold for these economies, equilibrium allocation, which we would like to investigate, can be solved by just solving SPP allocation. But RCE can be useful for analyzing much broader class of economies, many of them is not PO (where Welfare Theorems do not hold). That's what we are going to do from now. Let's define economies whose equilibria are not PO, because of distortions to prices, heterogeneity of agents, etc.

10.5 Economy with Externality

Suppose agents in this economy care about other's leisure. We would like to have beer with friends and share time with them. So other people's leisure enters my utility function. That is, the preference is given by

$$u(c, n, N)$$

where L=1-N is the aggregate leisure.

One example may be

$\log c + \log(1-n) + (1-n)(1-N)$[17]

With externality in the economy, competitive equilibrium cannot be solved from SPP.

The problem of consumer is as follows:

$$V(K, a) = \max_{c,n,a'}\{u(c, n, N) + \beta V(K', a')\} \tag{73}$$

subject to

$$\begin{aligned} c + a' &= [1-\delta+r]a + wn & (74)\\ r &= F_k(K, N) & (75)\\ w &= F_N(K, N) & (76)\\ K' &= G(K) & (77)\\ N &= H(K) & (78) \end{aligned}$$

And the solutions are:

$a' = g(K, a)$

$n = h(K, a)$

We can define RCE in this economy.

Homework 10.5 *Please define a RCE for this economy. Compare the equilibrium with social planner's solution and explain the difference.*

Comments:

1. We will not write G and H in value function since this is the way we see in literature. But you should feel it.
2. What if you only wanna hang out with some friends? Write $\frac{N}{12}$ in the utility function. This is the way we can work with RA framework. We can see how far we can get from RA model. To think how to write a problem with unemployment in a RA model, for example. But if you only wanna hang out with rich guys, RA is not enough. We will see how to model economy with certain wealth distribution later.

10.6 Economy with tax (1)

What is the government? It is an economic entity which takes away part of our income and uses it. The traditional (or right-wing) way of thinking of the role of the government is to assume that the government is taking away part of our disposable income and throw away into ocean. If you are left-wing person, you might think that the government return tax income to household as transfer or they do something we value.

Let's first look at the first version where income tax is thrown into ocean. For now, we assume that the government is restricted by period-by-period budget constraint (so the government cannot run deficit nor surplus).

The consumer's problem is as follows:

$$V(K, a) = \max_{c,n,a'}[u(c, n) + \beta V(G(K), a\prime)] \tag{79}$$

subject to

$$c + a' = a + \{nf_2(K, H(K)) + [f_1(K) - \delta]a\}(1 - \tau) \tag{80}$$

Income tax is proportional tax and only levied on income not on wealth. Depreciation is exempt from tax too.

The government period by period constraint is trivial in this case:

government expenditure $= \tau[f(K, H(K)) - \delta K]$

Remark 10.6 *Notice that the economy does not achieve Pareto Optimality, thus solved by SPP. Because in SPP, marginal rate of substitution equals to marginal rate of transformation. But in this economy, income tax affected equilibrium allocation in the following way: (i) the distortion is in favor of leisure against consumption. Why? Tax is only on income which is needed to get consumption not on leisure, but agent can simply work less to get higher utility. (ii) the distortion is in favor of today against tomorrow. The reason is the return of saving is less due to tax.*

10.7 Economy with tax (2)

Now let's look at an economy where the tax income is returned to household in the form of lump sum transfers.

Consumer's problem is

$$V(K,a)=\max_{c,n,a'}[u(c,n)+\beta V(G(K),a\prime)] \tag{81}$$

subject to

$$c+a'=a+\{nf_2(K,H(K))+[f_1(K)-\delta]a\}(1-\tau)+T \tag{82}$$

Where T is lump sum transfer. From government period by period constraint, we know

$$T=\tau[f(K,H(K))-\delta K]$$

The equilibrium in this economy is not Pareto optimal. The reason is that agents tend to work less in order to pay less tax. And they do not realize the lump sum tax they will get from government is affected by their action. But we can not blame them because agent only have power to control what she does, not other's action. Only in a RA world, her action happens to be the aggregate state. We have to separate agent's problem from equilibrium condition.

10.8 Economy with shocks to production

When there is shocks to production, should it be included in state variables?Yes, because shocks matters in two ways: (1) it changes rate of return. (2) it affects the way that economy evolves. Therefore, the state variables are: z,K,a. Consumer's problem is

$$V(z,K,a;G,q_z)=\max_{c,a'(z')}\{u(c)+\beta\sum_{z'}\Gamma_{zz'}V(z',K',a'(z');G,q_{z'})\} \tag{83}$$

subject to

$$c+\sum_{z'} q_{z'}(z,K)\,a'_{z'} = [1-\delta+r(z,K)]a+w(z,K) \tag{84}$$

$$r(z,K) = zf_1(K,H(K)) \tag{85}$$

$$w(z,K) = zf_2(K,H(K)) \tag{86}$$

$$K' = G(z,K) \tag{87}$$

- There is a complete set of markets for all possible contingences. So people can sign contract to trade state-contingent goods. What we have in the question above is state-contingent asset. $q(z,z')$ has a fancy name of pricing kernel and it has to induce equilibrium in this economy. Since there is only one RA, in equilibrium, there is no trade.

The decision rule is:

$$a'_{z'} = g_{z'}(z,K,a;G,q_z) \tag{88}$$

Agent is free to choose any asset holding conditional on any z'. That's why there are n_z decision rules. But in equilibrium, there is only one K' get realized which cannot depend on z'.

First, we can get n_z market clearing condition for equilibrium:

$$G(z,K) = g_{z'}(z,K,K) \tag{89}$$

But there are n_z+1 functions to solve in equilibrium: $g_{z'}$ and G. So there is one missing condition. We will see in next class that the missing condition is No Arbitrage condition: If one is free to store capital rather than trade state-contingent claim, the result is the same.

11 Feb 27

We have talked about stochastic RCE from last class. The consumer's problem is

$$V(z,K,a) = \max_{c,a'(z')}\{u(c)+\beta\sum_{z'}\Gamma_{zz'}V(z',K',a'(z'))\} \tag{90}$$

subject to

$$c+\sum_{z'} q_{z'}(z,K)\,a'_{z'} = [1-\delta+r(z,K)]a+w(z,K) \tag{91}$$

$$r(z,K) = zf_1(K,H(K)) \tag{92}$$

$$w(z,K) = zf_2(K,H(K)) \tag{93}$$

$$K' = G(z,K) \tag{94}$$

The decision rule is:

$$a'_{z'} = g_{z'}(z, K, a; G, q_z) \tag{95}$$

In RCE,

$$G(z, K) = g_{z'}(z, K, K) \tag{96}$$

which gives us n_z conditions. But we need $n_z + 1$ conditions. The missing condition is NA.

11.1 No Arbitrage condition in stochastic RCE

If the agent wanna have one unit of capital good for tomorrow, there are two ways to achieve this. One is the give up one unit of consumption today and store it for tomorrow's one unit of capital good. The cost is 1. The other way is to purchase state-contingent asset to get one unit of capital good for tomorrow. How to do this? Buy one unit of state-contingent asset for all the possible z'. That is $a'(z') = 1, \forall z'$.The total cost is

$$\sum_{z'} q_{z'}(z, K)$$

No Arbitrage condition is

$$\sum_{z'} q_{z'}(z, K) = 1 \tag{97}$$

11.2 Steady State Equilibrium

In a sequential market environment, steady state equilibrium is an equilibrium where $k_t = k$, $\forall t$. In a deterministic economy without leisure nor distortion, we can first look at the steady state of SPP. To find steady state, we use Euler equation and equate all the $k's$. Note: Euler equation is a second order difference equation, so there are k's at three different time involved, $k_{t,}k_{t+1}, k_{t+2}$.

In a RCE, steady state equilibrium is when

$$K = G(K)$$

When there is shock in economy, strictly speaking steady state does not exist in the sense of $K = G(K)$. Because now z is evolving stochastically and $K' = G(z, K)$. But we will see the probability measure of (K, z) can be found as a stationary once the capital is set at right range. And of course, the shock has to be stationary somehow itself.

Comments:

1. RCE is stationary automatically in the sense that there is no time subscript in value function and decision rule.

2. For some growing economy, we can always transform it into a non-growing economy, as you may see with Randy.

3. The way that econometricians and macroeconomists look at data are different. Econometricians believe there is a true data generating process underlying the data. Macroeconomists think that real data are generated by people's choice. They test models by comparing the properties of data generated by model to the real data. We will see how to use model to look at data later in the class.

11.3 FOC in stochastic RCE

$$u'\left(c\left(z,K,K\right)\right)=\beta\sum_{z'}\Gamma_{zz'}u'\left[c\left(z',K',K'\right)\right]\left[1-\delta+r\left(z',K'\right)\right] \tag{98}$$

where from budget constraint,

$$c\left(z,K,a\right)=\left[1-\delta+r\left(z,K\right)\right]a+w\left(z,K\right)-\sum_{z'}q_{z'}\left(z,K\right)a'_{z'}$$

Comment: in (98), the RA condition $a=K$ is used. It is allowed because the substitution is done after we derive first order condition. Agent only optimizes with respect to a', not K'. So, we get correct FOC first. Then, we can apply equilibrium condition that $a'=K'$.

To derive FOC, envelope condition is used.

FOC (a'):

$$-u'\left(c\left(z,K,K\right)\right)q_{z'}\left(z,K\right)+\beta\Gamma'_{zz'}V_3\left(z',K',a'_{z'}\right)=0$$

By envelop condition

$$V_3\left(z',K',a'_{z'}\right)=\left[1-\delta+r\left(z',K'\right)\right]u'\left[c\left(z',K',K'\right)\right] \tag{99}$$

Homework 11.1 *Derive Envelope condition for this problem.*

Therefore,

$$u'\left(c\left(z,K,K\right)\right)=\beta\frac{\Gamma'_{zz'}}{q_{z'}\left(z,K\right)}\left[1-\delta+r\left(z',K'\right)\right]u'\left[c\left(z',K',K'\right)\right] \tag{100}$$

If we can get $c(z, K, K)$ from SPP, (100) is an equation of $q_{z'}(z, K)$.

$$q_{z'}(z, K) = \beta \frac{\Gamma'_{zz'} u' [c(z', K', K')]}{u'(c(z, K, K))} [1 - \delta + r(z', K')] \tag{101}$$

(101) gives the price that induce household to choose the same allocation c and a' as from SPP. And such price ensure that agent's decision $g_{z'}(z, K, a)$ does not depend on z' in equilibrium:

$$G(z, K) = g_{z'}(z, K, K)$$

Remark 11.2 *Price q are related to but not the same as probability $\Gamma'_{zz'}$. It is also weighted by intertemporal rate of substitution to measure people's evaluation on consumption at some event. One simple example: in two period economy where good state and bad state happen with equal probability, to induce people to choose endowment of 2 and 1 at time 1, price for bad state must be higher since consumption at bad state is more valuable to people.*

Remark 11.3 *In this version of stochastic RCE, agent chooses state-contingent asset a' for next period before shocks are realized. When next period comes, z' realizes and production takes place using the saving a'. There are other different timings. Say, consumer chooses consumption and saving after shocks for next period get revealed.*

(98) and (101) are equilibrium condition for RCE. We can also get (98) in the following way:

(101) holds for all z'. If we sum (101) over z' and use the No Arbitrage condition (97), we can get

$$\sum_{z'} \beta \frac{\Gamma'_{zz'} u' [c(z', K', K')]}{u'(c(z, K, K))} [1 - \delta + r(z', K')] = \sum_{z'} q_{z'}(z, K) = 1$$

Therefore,

$$u'(c(z, K, K)) = \beta \sum_{z'} \Gamma_{zz'} u' [c(z', K', K')] [1 - \delta + r(z', K')]$$

Up to this point, we know that people will save the same amount regardless of tomorrow's state, because the price of state-contingent asset will induce them to do so. Therefore, an equivalent way to write RCE is to let agent choose tomorrow's capital without trade of state-contingent asset. And we can define RCE without $q's$.

Homework 11.4 *Show that if there is a law saying that people have no right to buy state-contingent commodities. Then in equilibrium, the law is not binding. In other word, consumer's problem is equivalent to*

$$V(z, K, a) = \max_{c,a'} u(c) + \sum_{z'} \Gamma_{zz'} V(z', K', a')$$

subject to

$$\begin{aligned} c + a' &= [1 - \delta + r(z, K)]a + w(z, K) \\ K' &= G(z, K) \end{aligned}$$

11.4 Economy with Two Types of Agents

Assume that in the economy there are two types of agents, called type A and type B. Measure of the agents of type A and type B are the same. Without loss of generality, we can think of the economy as the one with two agents, both of whom are price takers.

Agents can be different in many ways, including in terms of wealth, preference, ability, etc. We will first look at an economy where agents are different in wealth. There are 1/2 population of rich people and 1/2 population of poor people. For simplicity, we assume there are no shocks and agents do not value leisure.

The state variables are aggregate wealth of both types, K^A and K^B. Why? We know wage and rental only depends on total capital stock $K = K^A + K^B$. But K is not sufficient as aggregate state variables because agents need know tomorrow's price which depends on tomorrow's aggregate capital.

Agents' preference is the same, so the problem for both types are:

$$V(K_A, K_B, a) = \max_{c,a'} \{u(c) + \beta V(G_A(K_A, K_B), G_B(K_A, K_B))\} \tag{102}$$

subject to

$$c + a' = [r(K) + 1 - \delta] a + w(K) \tag{103}$$

Solutions are:

$$a' = g(K_A, K_B, a)$$

In RCE, the equilibrium condition is:

$$\begin{aligned} G_A(K_A, K_B) &= g(K_A, K_B, K_A) \\ G_B(K_A, K_B) &= g(K_A, K_B, K_B) \end{aligned}$$

Homework 11.5 *Show that necessary condition for K to be sufficient state variable is that agents' decision rules are linear.*

Homework 11.6 *Show*

$$G_A(K_A, K_B) = G_B(K_B, K_A)$$

Homework 11.7 *What does the theory say about the wealth distribution in steady state equilibrium for 2-type-agent economy above? Compare it with steady state wealth distribution in island economy where markets do not exist.*

12 Mar 4

12.1 Review

- Stochastic RCE with and without state-contingent asset

  Consider the economy with shock to production. People are allowed to purchase state-contingent asset for next period.

Consumer's problem is

$$V(z, K, a; G, q_z) = \max_{c, a'(z')} \{u(c) + \beta \sum_{z'} \Gamma_{zz'} V(z', K', a'(z'); G, q_{z'})\} \tag{104}$$

subject to

$$\begin{aligned} c + \sum_{z'} q_{z'}(z, K)\, a'_{z'} &= [1 - \delta + r(z, K)]a + w(z, K) & (105) \\ r(z, K) &= z f_1(K, H(K)) & (106) \\ w(z, K) &= z f_2(K, H(K)) & (107) \\ K' &= G(z, K) & (108) \end{aligned}$$

Essentially, we can get Euler equation:

$$u_c(c) = \beta \sum_{z'} \Gamma_{zz'}[1 - \delta + z f_1(K, 1)] u_{c'}(c') \tag{109}$$

This condition is what we see more in macro literature. But the consumer's problem we have above is a long-hand version.

To derive it, we use

FOC:

$$
\begin{aligned}
\frac{\partial}{\partial c} &: \qquad u_c\left(c\left(z, K, a\right)\right) = \lambda \\
\frac{\partial}{\partial a'_{z'}} &: \qquad \Gamma_{zz'} V_3\left(z', K', a'\left(z'\right)\right) = \lambda q_{z'}\left(z, K\right)
\end{aligned}
$$

Envelope condition:

$$V_3 = [1 - \delta + r\left(z', K'\right)]u_c$$

Thus, we have

$$q_{z'}\left(z, K\right) u_c\left(c\right) = \beta \Gamma_{zz'} [1 - \delta + z f_1\left(K, 1\right)] u_{c'}\left(c'\right) \tag{110}$$

Add over z' and use NA condition

$$\sum_{z'} q_{z'}\left(z, K\right) = 1$$

and substitute consistency condition

$$a = K$$

We will get (109).

As we see in the homework, the equilibrium of this economy with a complete market can be found in economy without complete market. The reason is that state-contingent asset price $q_{z'}\left(z, K\right)$ is adjusted in the way such that agents save the same amount independent of z'.

- Wealth distribution in economy with heteregenous agents

  Assume there are I types of agents, there are $2I$ necessary conditions for equilibrium allocation:

I budget constraint equations:

$$c^i + a^{i\prime} = w + a^i\left(1 + r - \delta\right) \tag{111}$$

I FOC conditions:

$$u_c^i = \beta\left(1 + r - \delta\right) u'_{c^{i\prime}} \tag{112}$$

And there are $2I$ unknowns $\{c^i, a^i\}$ in steady state. But in steady state, the I FOC degenerate to the same one

$$1 = \beta\left(1 + r - \delta\right)$$

$$f_k\left(\sum_i a^i\right) = \frac{1}{\beta} - (1 - \delta) \tag{113}$$

Therefore, the model says nothing about wealth distribution.

If the economy starts with $f_1\left(\sum_i a^i, 1\right) = \frac{1}{\beta} - (1 - \delta)$, then wealth ranking stays. If not, asset holding of different types will move parallel toward steady state level.

12.2 Finance

We will study Lucas Tree Model (Lucas 1978[4]) and look at the things that Finance people talk about. Lucas tree model is a simple but powerful model.

12.2.1 The Model

Suppose there is a tree which produces random amount of fruits every period. We can think of these fruits as dividends and use d_t to denote the stochastic process of fruits production. $d_t \in \left\{d^1, ...d^{nd}\right\}$. Further, assume d_t follows Markov process. Formally:

$$d_t \sim \Gamma(d_{t+1} = d_i \mid d_t = d_j) = \Gamma_{ji} \tag{114}$$

Let h_t be the history of realization of shocks, i.e., $h_t = (d_0,\ d_1,\ ...,\ d_t)$. Probability that certain history h_t occurs is $\pi(h_t)$.

Household in the economy consumes the only good, which is fruit. We assume representative agent in the economy, and there is no storage technology. In an equilibrium, the first optimal allocation is that the representative household eats all the dividends every period. We will look at what the price has to be when agents use markets and start to trade. First, we study the Arrow-Debreu world. And then, we use sequential markets to price all kinds of derivatives, where assets are entitlement to consumption upon certain date-event.

[4]Lucas, R. (1978). "Asset prices in an exchange economy." Econometrica 46: 1429-1445

12.2.2 Arrow-Debreu World

Consumers;s problem

$$\max_{\{c(h_t)\}_{t=0}^{\infty}} \sum_t \beta^t \sum_{h_t \in H_t} \pi(h_t) u(c_t(h_t)) \tag{115}$$

subject to

$$\sum_t \sum_{h_t \in H_t} p(h_t) c_t(h_t) = \overline{a} = \sum_t \sum_{h_t \in H_t} p(h_t) d_t(h_t) \tag{116}$$

Equilibrium allocation is autarky

$$c_t(h_t) = d_t(h_t) \tag{117}$$

Now the key thing is to find the price which can support such equilibrium allocation.

Normalize

$$p\left(h_0\right) = 1$$

Take first order condition of the above maximization problem and also substitute (117)

FOC

$$\begin{aligned} \beta^t \pi(h_t) u_c(d_t(h_t)) &= p_t(h_t)\lambda && (118) \\ u_c\left(d_0\right) &= \lambda && (119) \end{aligned}$$

We get the expression for the price of the state contingent claim in the Arrow-Debreu market arrangement.

$$p_t(h_t) = \frac{\beta^t \pi(h_t) u_c(d_t(h_t))}{u_c(d_0)} \tag{120}$$

Note that the price $p_t(h_t)$ is in terms of time 0 consumption.

12.2.3 Sequences of Markets

In sequential market, we can think of stock market where the tree is the asset. Household can buy and sell the asset. Let s_t be share of asset and q_t be the asset price at period t. The budget constraint at every time-event is then:

$$qs' + c = s(q + d) \tag{121}$$

First, we can think of any financial instruments and use the A-D prices $p_t(h_t)$ to price them.

1. The value of the tree in terms of time 0 consumption is indeed

$$\sum_t \sum_{h_t \in H_t} p(h_t) d_t(h_t)$$

2. A contract that gives agent the tree in period 3 and get it back in period 4: This contract is worth the same as price of harvests in period 3:

$$\sum_{h_3 \in H_3} p(h_3) d_3(h_3)$$

3. Price of 3-year bond: 3 year bond gives agents 1 unit of good at period 3 with any kinds of history. The price is thus

$$\sum_{h_3 \in H_3} p(h_3)$$

12.2.4 Market Equilibrium

We will write it in a resursive form. Then, first can we get rid of h_t and write it in a recursive form? Or are prices stationary? The answer depends on whether the stochastic process is stationary.

Homework 12.1 *Show prices q are stationay (only indexed by z which is essentially the same as dividend).*

Now the consumer's optimization problem turns out to be:

$$V(z,s) = \max_{c,s'} u(c) + \beta \sum_{z'} \Gamma_{zz'} V(z',s') \tag{122}$$

subject to

$$c + s' q(z) = s[q(z) + d(z)] \tag{123}$$

To solve the problem,

FOC:

$$\begin{aligned} u_c(z) &= \lambda_z \\ \beta \sum_{z'} \Gamma_{zz'} [q(z') + d(z')] \lambda_{z'} &= \lambda_z q(z) \end{aligned}$$

So, we get $\forall z$,

$$u_c(z)\,q(z) = \beta \sum_{z'} \Gamma_{zz'}[q(z') + d(z')]u_c(z')$$

We write out the whole system of equation for all possible z,

$$\begin{cases} u_c(z^1)\,q(z^1) = \beta \sum_{z'} \Gamma_{zz'}[q(z') + d(z')]u_c(z') \\ \quad\quad\quad\quad\quad\quad\quad\quad \cdots \\ u_c(z^{nz})\,q(z^{nz}) = \beta \sum_{z'} \Gamma_{zz'}[q(z') + d(z')]u_c(z') \end{cases} \tag{124}$$

Elements in (124) are marginal utility of consumption in different states and dividends, which are numbers, and price $q's$. Therefore, it is systerm of linear equations in $q's$. And there are nz linear equations and nz unknowns. We can then solve this system and obtain prices in sequential markets.

13 March 18

13.1 Review

- Last class, we introduced the Lucas (1978) model. There is a tree and the tree yields a random number of fruits at each period. There is a representative agent. In the previous class, we said that the equilibrium has to be such that $c_t = d_t$ since markets clear only if the consumption of the agent equals to her endowment due to the fact that we have a representative agent. Then, we were able to compute the prices that will support this allocation as the solution to the agent's maximization problem. Thus we got the prices that will induce the agent to consume all he has at each period. Then we set up a problem where the agent is able to trade at each period and let him maximize by choosing his consumption, c_t, and the amount of share of the tree to buy, s_{t+1}. From the solution to the agent's problem in the sequence of markets structure, we were able to characterize the price of a share of the tree at each node, $q(h_t)$. We derived these prices by deriving the FOC from consumer's problem and imposing the equilibrium conditions on them. Today we see more on the characterization of these prices and we go on to asset pricing using these tools.

13.2 Asset Pricing

- What is the state of the economy?

It's the number of fruits from the tree (d). The dividend is the aggregate state variable in this economy.

- s, the share that the consumer has today, is the individual state variable.
- Consumer's Problem:

$$V(d,s) = \max_{s',c} \text{u(c)} + \beta \sum_{d'} \Gamma_{dd'} V(d',s')$$

$$s.t. \;\; c + s\prime q(d) = s[q(d) + d]$$

In equilibrium, the solution has to be such that c=d and $s^{'} = 1$. Impose these on the FOC and get the prices that induce the agent to choose that particular allocation.

Homework. *Show that the prices, $\{q_i\}_{i=1}^{I}$ are characterized by the following system of equations:*

$$u'(d_i) = \beta \sum_j \Gamma_{ij} u'(d_j) \frac{[q_j + d_j]}{q_i} \quad \forall i$$

Now let's write the above system of equations in the matrix form so we can write a closed form for q.

$$\begin{bmatrix} q_1 \\ .. \\ .. \\ q_I \end{bmatrix} = \begin{bmatrix} \beta\Gamma_{11}\frac{u'(d_1)}{u'(d_1)} & \beta\Gamma_{12}\frac{u'(d_2)}{u'(d_1)} & .. & \beta\Gamma_{1J}\frac{u'(d_J)}{u'(d_1)} \\ .. & \beta\Gamma_{22}\frac{u'(d_2)}{u'(d_2)} & .. & .. \\ .. & .. & .. & .. \\ \beta\Gamma_{I1}\frac{u'(d_1)}{u'(d_I)} & .. & .. & \beta\Gamma_{IJ}\frac{u'(d_J)}{u'(d_I)} \end{bmatrix} \left[\begin{bmatrix} q_1 \\ .. \\ .. \\ q_I \end{bmatrix} + \begin{bmatrix} d_1 \\ .. \\ .. \\ d_I \end{bmatrix} \right]$$

Let q= $\begin{bmatrix} q_1 \\ .. \\ .. \\ q_I \end{bmatrix}$ and d= $\begin{bmatrix} d_1 \\ .. \\ .. \\ d_I \end{bmatrix}$ and A= $\begin{bmatrix} \beta\Gamma_{11}\frac{u'(d_1)}{u'(d_1)} & \beta\Gamma_{12}\frac{u'(d_2)}{u'(d_1)} & .. & \beta\Gamma_{1J}\frac{u'(d_J)}{u'(d_1)} \\ .. & \beta\Gamma_{22}\frac{u'(d_2)}{u'(d_2)} & .. & .. \\ .. & .. & .. & .. \\ \beta\Gamma_{I1}\frac{u'(d_1)}{u'(d_I)} & .. & .. & \beta\Gamma_{IJ}\frac{u'(d_J)}{u'(d_I)} \end{bmatrix}$

and let b=Ad

we have,

$$q = Aq + b$$

so that,

$$q = (I - A)^{-1} b$$

- Now consider the same problem but now the agent can also buy bonds for the price of p(d) which entitles him to get 1 unit of the good the next period. The agent's new budget constraint is:

$$b' p(d) + c + s' q(d) = s[q(d) + d] + b$$

The equilibrium quantity of bonds is 0 because there is noone to buy those bonds from or sell them to. Now using this fact, we will find the p(d) that induces the agent to choose to buy 0 bonds.

$$p_i u'(d_i) = \beta \sum_j \Gamma_{ij} u'(d_j) \quad \forall i$$

The price of a bond, p(d), is characterized by the above set of equations.

You can see the pattern here: We can choose any kind of asset and then price it in the same way.

13.2.1 Options

Definition 13.1 *An option is an asset that gives you the right to buy a share at a prespecified price if you choose.*

In general, to price any kind of assets and options, we only need to know prices of consumption at each node.

The price of an option is a sum of gain under the option at each node (considering the decision of whether to exercise the option or not), multiplied by the price of consumption at each node. In order to write an expression for the price of an option at a certain node (q(h_t)), we need to first compute the price of a unit of consumption good at node h_{t+1} in terms of units of consumption goods at node h_t. One way to do this is introduce state contingent claims to the problem of the agent (y'(d')) and compute its price $p_{dd'}$.

Let's say we want to price options at node h_t. Assume that the set of the possible aggregate shock contains three elements. Start from h_t, possible nodes in the next period are h^1_{t+1} and h^2_{t+1} and h^3_{t+1}. Now denote the price of consumption at node h^1_{t+1} in terms of units of consumption at node h_t by p^y_{i1}. We want to see what p^y_{i1} is, as well as p^y_{i2}, p^y_{i3}. In order to do this we use state contingent claims because the price of a state contingent claim at node h_t that entitles the agent to get 1 unit of consumption good at node h^1_{t+1} is p^y_{i1}. So if we write down the problem of the agent with these state contingent claims and compute the prices of these state contingent claims, we'll get exactly what we need: The price of consumption at node h^1_{t+1} (or h^2_{t+1} or h^3_{t+1}) in terms of units of consumption good at t.

Once we add the state contingent claims, the agent's problem becomes the following:

$$V(d,s,b,y) = \max_{c,s',b',y'(d')} \text{u(c)} + \beta \sum_{d'} \Gamma_{dd'} V(d',s',b',y'(d'))$$

$$s.t. \sum_{d'} y'(d') p^y_{dd'} + b'p(d) + c + s'q(d) = s[q(d)+d] + b + y(d)$$

Homework *Show that the expression for the price of the state contingent claims,* p^y_{ij} is as follows:

$$p^y_{ij} = \beta \Gamma_{ij} \frac{u'(d_j)}{u'(d_i)}$$

An option that you buy at period t, entitles you to max{0,q(h^1_{t+1})$-\overline{q}$} at node h^1_{t+1},max{0,q(h^2_{t+1})$-\overline{q}$} at node h^2_{t+1}, and max{0,q(h^3_{t+1}) $-\overline{q}$} at node h^3_{t+1}. And therefore, all we need to do to compute its price is multiply what it entitles the agent at each node by the price of consumption at that node in terms of consumption at t (which we now know because we already have an expression for p^y_{ij}) and sum over all the nodes:

$$q^0_i(\overline{q}) = \sum_j max\{0, q_j - \overline{q}\} p^y_{ij}$$

Homework *(i) Price a two period option that can be exercised any time before its maturity (i.e. if you buy the option today, it can be exercised either tomorrow or the day after)* *(ii) Price a two period option that can be exercised only at its maturity.*

Homework *Come up with an asset and price it.*

Homework *Find the formula that relates the price of the bond to the price of the state contingent claims in the above problem (i.e. $p(d)$ to $p^y_{dd'}$).*

Homework *Give a formula for $q(d)$ in terms of $p^y_{dd'}$.*

13.3 RCE with Government

When we are considering RCE with government, there are several issues that we need to consider before we begin writing down the problem of the agent and the government's budget constraint. We need to make some choices about the economy that we are modelling:

Can the government issue debt? If no: The government is restricted by his period by period budget constraint. He cannot run a budget deficit or a surplus. Whatever he gets as revenues from taxes, he spends that and no more or no less. His budget constraint needs to hold at each period, the government cannot borrow from the public. Here the government expenditures are exactly equal to the tax revenues. If yes: The government can issue bonds at each period When it turns out that the government's expenditures are higher than its revenues (the tax revenues) he can issue debt or when it turns out that the expenditures are less than his revenues, he can retire debt that was issued before. If we model the economy so that the government is allowed to issue debt, then we need to deal with the issue of restricting it to accumulate public debt indefinitely. This is where the No-Ponzi Scheme comes into play. But we'll have more on that later.

We will do the second case today. So the economy is as follows:

- Government issues debt and raises tax revenues to pay for a constant stream of expenditures.
- Government debt is issued at face value with a strem of interest rate $\{\mathrm{r}_{b,t}\}$
- No shocks.
- No labor/leisure choice.

Let's write down the problem of the consumer. Notice that the consumer can transfer resources across time in two ways here: He can either save in the form of capital or he can buy bonds. In equilibrium, the rates of return on both ways of saving should be the same by no arbitrage so that $\mathrm{r}_b = r_k$. Since the rates of return on both is the same, the agent shouldn't care in what form he saves, i.e. the composition of the asset portfolio doesn't matter. So let a denote the agent's asset which consists of physical capital holding k and financial asset b. We don't need to make the distinction between the two. And let r denote the rate of return on a (which is in turn the rate of return on capital and rate of return on bonds).

Aggregate state variables are K and B where B is the government debt. Notice that G, the government expenditures, is not a state variable here since it's constant across time. The individual state variable is a, the consumer's asset holdings.

The consumer's problem:

$$V(K, B, a) = \max_{c,a'} \text{ u(c) } + \beta V(K', B', a')$$

subject to

$$\begin{aligned} c + a' &= a + [ra + w](1 - \tau) \\ r &= r(K) = f_k(K, 1) - \delta \\ w &= w(K) = f_n(K, 1) \\ K' &= H(K, B) \\ B' &= \Psi(K, B) \\ \tau &= \tau(K, B) \end{aligned}$$

Solution is: a'=$\psi(K, B, a)$

Definition 13.2 *Given $\tau(K, B)$, a RCE is a set of functions $\{V(.), \psi(.), H(.), \Psi(.)\}$ such that*

1. *(Household's optimization) Given $\{H(.), \Psi(.)\}, \{V(.), \psi(.)\}$ solve the household's problem.*
2. *(Consistency) $H(K, B) + \Psi(K, B) = \psi(K, B, K + B)$*
3. *(No Arbitrage Condition)*

   $$r_b(K, B) = 1 + F_K(H(K, B), 1) - \delta$$

   (The rate of return on bond is equal to the rate of return on capital; notice we already used this fact when we were writing down the problem of the consumer by letting r denote the rate of return on both and not distinguishing between them)

4. *(Government Budget Constraint)*

$$\Psi(K,B)+[f(K,1)-\delta K+(f_k(K,1)-\delta)B](1-\tau(K,B))=\bar{G}+B[1+f_k(K,1)-\delta]$$

(So that the government's resources each period are the bonds that it issues ($\Psi(K,B)$), plus its revenues from tax on rental income, wage income, and income on the interest on bonds. Its uses are the government expenditures (G), and the debt that it pays back.)

5. *(No Ponzi Scheme Condition)* $\exists \underline{B}$, $\overline{B}, \underline{K}$, $\overline{K}$ *such that* $\forall K, B \in [\underline{K}, \overline{K}]x[\underline{B}, \overline{B}]$

$$\Psi(K,B)\in[\underline{B},\bar{B}]$$

Homework *Consider an economy with two countries indexed by* $i \in \{A, B\}$*. Each country is populated by a continuum of infintely lived identical agents that is taken to be of measure one. Each country has a production function* $f^i(K^i, N^i)$ *(different technologies across countries). Assume that output and capital can be transferred between countries at no cost. But labor cannot move across countries.Define recursive equilibria for this economy.*

14 March 20: Measure Theory

- We will use measure theory as a tool to describe a society with heterogenous agents. Most of the previous models that we dealt with, there was a continuum of identical agents, so we saw the economy as consisting of only one type of agent.But from now on in the models that we deal with there will be heterogenous agents in the economy. The agents will differ in various ways: in their preferences, in the shocks they get, etc. Therefore, the decisions they make will differ also. In order to describe such a society, we need to be able to keep track of each type of agent. We use measure theory to do that.

- What is measure?

  Measure is a way to describe society without having to keep track of names. But before we define measure, there are several definitions we need to learn.

Definition 14.1 *For a set A,* $\mathcal{A}$ *is a set of subsets of A.*

Definition 14.2 $\sigma - a\lg ebra$ $\mathcal{A}$ *is a set of subsets of A, such that,*

1. A, $\emptyset \in \mathcal{A}$
2. $B \in \mathcal{A} \Rightarrow B^c/A \in \mathcal{A}$ *(closed in complementarity)*
   where $B^c/A = \{a \in A : a \notin B\}$

3. for $\{B_i\}_{i=1,2...}$*,* $B_i \in \mathcal{A} \Rightarrow [\cap_i B_i] \in \mathcal{A}$ *(closed in countable intersections)*

- An example for the third property: Think of a σ-algebra defined on the set of people in a classroom. The property of being closed in countable intersections says that if the set of tall people is in the σ-algebra, and the set of women is in the σ-algebra, then the set of tall women should be in the σ-algebra also.
- Consider the set A={1,2,3,4} Here are some examples of σ-algebras defined on the set A:

$\mathcal{A}^1 = \{\emptyset, A\}$

$\mathcal{A}^2 = \{\emptyset, A, \{1\}, \{2,3,4\}\}$

$\mathcal{A}^3 = \{\emptyset, A, \{1\}, \{2\}, \{2,3,4\}, \{1,3,4\}, \{3,4\}, \{1,2\}\}$

$\mathcal{A}^4 =$ The set of all subsets of A.

Remark 14.3 *A topology is a set of subsets of a set also, just like a σ-algebra. But the elements of a topology are open intervals and it does not satisfy the property of closedness in complementality (since a complement of an element is not an element of the topology). Therefore topology is not a σ-algebra.*

Remark 14.4 *Topologies and Borel sets are also family of sets but we use them to deal with continuity, and σ-algebra we use to deal with weight.*

- Think of the σ-algebras defined on the set A above. Which one provides us with the least amount of information? It is $\mathcal{A}^1$. Why? Because from $\mathcal{A}^1$, we only know whether an element is in the set A or not. Think of the example of the classroom. From $\mathcal{A}^1$,all we get to is whether a person is in that classroom or not, we learn nothing about the tall people, short people, males, females, etc. The more sets there are in a σ-algebra, the more information we have.This is where the Borel sets are useful. A Borel set is a σ-algebra which is generated by a family of open sets. Since Borel σ-algebra contains all the subsets generated by intervals, you can recognize any subset of set, using Borel σ-algebra. In other words, Borel σ-algebra corresponds to complete information.
- Now we are ready to define measure:

Definition 14.5 *A measure is a function* $x : \mathcal{A} \to \mathcal{R}_+$ *such that*

1. $x(\emptyset) = 0$

2. *if* $B_1, B_2 \in \mathcal{A}$ *and* $B_1 \cap B_2 = \emptyset \Rightarrow x(B_1 \cup B_2) = x(B_1) + x(B_2)$ *(finite additivity)*

3. *if* $\{B_i\}_{i=1}^{\infty} \in \mathcal{A}$ *and* $B_i \cap B_j = \emptyset$ *for all* $i \neq j \Rightarrow x(\cup_i B_i) = \sum_i x(B_i)$ *(countable additivity)*

Definition 14.6 *Probability (measure) is a measure such that* $x(A) = 1$

Homework *Show that the space of measures over the interval [0,1] is not a topological vector space.*

Homework *Show that the space of sign measures is a topological vector space.*

Homework *Show that the countable union of elements of a* $\sigma - a\lg ebra$ *is also element of the* $\sigma - a\lg ebra$.

- Consider the set A=[0,1] where a $\in A$ denotes wealth. So A is the set of wealth levels normalized to 1.
- We will define x: $\mathcal{A} \to \mathcal{R}_+$ as a probability measure so that the total population is normalized to one. Using measure, we can represent various statistics in a simple form:

1. The total population:

$$\int_A dx = x(A) = 1$$

2. Average wealth:

$$\int_A a dx$$

   We go through each levels of wealth in the economy and multiply the wealth level by the proportion of people that have that wealth level, and because the size of the society is normalized to 1, this gives us average wealth.

3. Variance of wealth:

$$\int_A [a - \int_A a dx]^2 dx$$

4. Coefficient of variation:

$$\frac{\{\int_A [a - \int_A adx]^2 dx\}^{1/2}}{\int_A adx}$$

5. Wealth level that seperates the 1% richest and the poorest 99% is $\widetilde{a}$ that solves the following equation:

$$0.99 = \int_A 1_{[a \geq \widetilde{a}]} dx$$

Homework *Write the expression for the Gini index.*

Remark 14.7 *Notation:*

$$\int_A 1_{[a \leq \widetilde{a}]} dx = x([0, \widetilde{a}]) = \int_A 1_{[a \leq \widetilde{a}]} x(da)$$

14.1 Introduction to the Economy with Heterogenous Agents

Imagine a Archipelago that has a continuum of islands. There is a fisherman on each island. The fishermen get an endowment s each period. s follows a Markov process with transition $\Gamma_{ss'}$ and,

s$\in \{s^1, ..., s^{n_s}\}$

The fishermen cannot swim. There is a storage technology such that, if the fishermen store q units of fish today, they get 1 unit of fish tomorrow. The problem of the fisherman is:

$$V(s,a) = \max_{c,a'} \text{u(c)} + \beta \sum_{s'} \Gamma_{ss'} V(s', a')$$

$$s.t. c + qa\prime = s + a$$

$$c, a' \succeq 0$$

Homework *Can we apply the Contraction Mapping Theorem to this problem?*

15 March 25: Economy with Heterogenous Agents

15.1 Measure Theory (continued)

- Consider the set A={1,2,3,4}and the following $\sigma - a\lg ebras$ defined on it:

$\mathcal{A}^1 = \{\emptyset, A\}$

$\mathcal{A}^2 = \{\emptyset, A, \{1,2\}, \{3,4\}\}$

- Remember from last class that the more sets there are in the $\sigma - a\lg ebra$, the more we know.

Definition 15.1 *f:*$\mathcal{A} \to \mathcal{R}$ *is measurable with respect to* $\mathcal{A}$ *if,*

$B_c = \{b \in A : f(b) \leq c\} \in \mathcal{A}\ \forall c \in \mathcal{A}$

Example 1 : Consider the following set A, the elements of which denote the possible pairs of today's and tomorrow's temperatures.

$$A = \{[-273, 10000] \times [-273, 10000]\}$$

And let $\mathcal{A}$ be a $\sigma - a\lg ebra$ defined on this set A such that,

$\mathcal{A} = \{[-273, -273.5]_t, [-273.5, -274]_t, [-274, -274.5]_t, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, [-273, 10000]_{t+1}\}$

(the subscripts just denote which day the temperatures are for, today (t) or tomorrow (t+1))

Knowing which set in $\mathcal{A}$ a certain element lies provides us with no information about tomorrow's temperature. The 0.5 intervals are all for today's temperatures. Now imagine we say we will have a party only if tomorrow's temperature is above a certain level. Can we have a function defined on the $\mathcal{A}$ above and know whether there will be a party tomorrow as the outcome of this function? The answer is no. The particular $\sigma-$algebra that we defined above is not appropriate for a function the arguments of which are tomorrow's temperature. Any function whose argument is temperatures tomorrow is not measurable with respect to the $\sigma-$algebra above.

For a function to be measurable with respect to a $\sigma-$algebra, the set of points over which the function changes value should be in that $\sigma-$algebra, and this is what the above definition of measurability translates into. In other words, you should be able to tell apart the arguments over which the function changes value.

Example 2 : Consider the following $\sigma-$algebra $\mathcal{A}$ defined on set A={1,2,3,4}:

$\mathcal{A}'= \{\emptyset, A, \{1,2\}, \{3,4\}\}$

Now think of the function that gives \$1 for odd numbers and \$0 for even. Is this function measurable with respect to $\mathcal{A}'$? No, because the arguments where the functions changes values are not elements of $\mathcal{A}'$. This function would be measurable with respect to the following $\sigma - a\lg ebra$:

$\mathcal{A}'' = \{\emptyset, A, \{1,3\}, \{2,4\}\}$

One of the ways we need the notion of measurability in the context of the economies we deal with is the following: Every function that affects what people do at time t has to be "t-measurable", in other words, cannot depend on the future, t+1.

15.2 Transition Function

Definition 15.2 *A transition function $Q : A \times \mathcal{A} \rightarrow \mathcal{R}$ such that:*

1. $\forall \bar{B} \in \mathcal{A}$, $Q(., \bar{B}) : A \rightarrow \mathcal{R}$ *is measurable,*
2. $\forall \bar{a} \in A$, $Q(\bar{a}, .) : \mathcal{A} \rightarrow \mathcal{R}$ *is a probability measure.*

Q function is a probability that a type a agent ends up in the type which belongs to B. What the first condition above says is the following: Whatever we need to know for today has to be sufficient to specify what the probability tomorrow is.

Consider the set A the elements of which are the possiblel states of the world, say good and bad.

A={good, bad}

And let Γ be the transition matrix associated with A.

$$\Gamma_{ii'} = \begin{bmatrix} \Gamma_{gg} & \Gamma_{gb} \\ \Gamma_{bg} & \Gamma_{bb} \end{bmatrix}$$

The $\sigma-$algebra we would want to use is $\mathcal{A} = \{\emptyset, A, \{good\}, \{bad\}\}$

Notice that what we need to know to compute the probability of a certain state tomorrow is today's state and the $\sigma-$algebra lets us know that.

Homework *Verify that the* $A = \{\emptyset, A, \{good\}, \{bad\}\}$ *is a* $\sigma-$*algebra.*

Homework *Take A=[0,1] and* $A = \{Borel\ sets\ on\ A\}$ *For* $a \in A, B \in A$,

$$Q(a, B) = \int_A B dx$$

Verify that Q is a transition function.

- The measure x defined on a sigma algebra over the set of intervals of wealth is complete description of the society today.
- x'(B): Measure of people who have characteristics in B$\in \mathcal{A}$ tomorrow.
- The pair x,Q will tell us about tomorrow.

  Pick one measure that's defined over the $\sigma-$algebra, $\mathcal{A}$, on the set of wealth levels A=[0,1]

  $$x'(B) = \int_A Q((s, a), B) dx$$

  So with x and Q, we get x'.i.e. with the measure of people today and the transition function Q, we get the measure of people with wealth level in a certain interval B=[a,b] tomorrow.

  What people are now (x) + What people do (Q) $\rightarrow x'$

Example Consider a society where people are characterized as "good guys" or "bad guys". Define the sigma algebra $\mathcal{A}$ as

$$\mathcal{A} = \{\emptyset, A, \{good\}, \{bad\}\}$$

And suppose that today everybody in the society is a "good guy" so that,

$$\begin{aligned} x(\{good\}) &= 1 \\ x(\{bad\}) &= 0 \end{aligned}$$

Let Γ_{gg} denote the probability that someone will be a good guy tomorrow given he is a good guy today. So the measure of people that are good guys tomorrow is,

$$x'(\{good\}) = \Gamma_{gg}$$

and the measure of people who are bad guys tomorrow,

$$x'(\{bad\}) = \Gamma_{gb}$$

and the measure of people who are good guys two periods from now,

$$x''(\{good\}) = \Gamma_{gg}x'(\{good\}) + \Gamma_{bg}x'(\{bad\})$$

Notice that when we are writing the expressions above for the proportion of people of good guys or bad guys at a certain period, we are implicitly using the Law of Large Numbers. LLN says that the proportion of a certain characteristic in that population will converge to the probability of that characteristic with a sufficiently large sample size. In other words, we are not interested in sampling uncertainty. For example, the proportion of the good guys in the society tomorrow is just the probability of a good guy staying a good guy multiplied by the proportion of good guys today (which is 1 in the example above) PLUS the probability that a bad guy will become a good guy multiplied by the proportion of bad guys today (which is zero in the example above).

This is the same idea with the example of tossing a coin. The probability of getting heads is 1/2 each time you toss a coin. So if you toss the coin a countable number of times, the measure of realizations that are heads will be 1/2. In our society above, the probability of each good guy staying a good guy is Γ_{gg} and with sufficiently large number of people, this is equal to the measure of good guys tomorrow.

15.3 Economy with Heterogenous Agents

Now we go back to our economy with fishermen and use the tools that we introduced above for this economy.

- Every period the farmer wakes up and receives his endowment of fish, s, which follows a Markov process with transition matrix Γ. There is a storage technology and if the farmer saves q units of fish today, tomorrow he gets 1 unit of fish. His savings is denoted by a'.

(s,a) is the type of a fisherman and the set consisting of all possible such pairs is,

$$S \times A = \{s^1, s^2, ..., s^n\} \times [0, \overline{a}]$$

Let $\mathcal{A}$ be the set of Borel sets on SxA. And define a probability measure x on $\mathcal{A}$,

$x : \mathcal{A} \rightarrow [0, 1]$

- The fisherman's problem is:

$$V(s, a) = \max_{c,a'} \text{u(c)} + \beta \sum_{s'} \Gamma_{ss'} V(s', a')$$

subject to

$$c + qa\prime = s + a$$

$$c \succeq 0 \text{ and a' } \in [0, \overline{a}]$$

- With the decision rule a'=g(s,a) and the transition matrix for the endowment process $\Gamma_{ss'}$, we can construct the transition matrix. The transition function Q(s,a,B) tells us the probability that a fisherman with (s,a) today ends up in some $\text{B}_s \times B_a \in \mathcal{A}$ tomorrow (where B_s and B_a are the projections of B over the spaces S and A).

The transition function is constructed as follows

$$Q(s, a, B) = 1_{[g(s,a) \in B_a]} \sum_{s' \in B_s} \Gamma_{ss'}$$

Homework *Verify that Q constructed as above is a transition function.*

Homework *Compute the stationary distribution associated with the transition matrix* $\Gamma = \begin{bmatrix} 0.85 & 0.15 \\ 0.1 & 0.9 \end{bmatrix}$

Homework *Pr {losing a job}=0.05 and Pr{finding a job}=0.5. Find the stationary distribution for states of employment..*

Example Take some Markov process with transition matrix $\Gamma = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix}$

The stationary distribution associated with this transition matrix is, $\text{x}^* = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}$

- Define the updating operator T as,

$$x\prime(B) = T(x, Q)$$

- For example if you apply the operator on x and Q twice, we get the two periods ahead measure, etc. :

$$\begin{aligned} x''(B) &= T^2(x, Q) \\ x'''(B) &= T^3(x, Q) \end{aligned}$$

x* can be written as,

$$x^* = \lim_{n \to \infty} \ \mathrm{T}^n(x^0, Q) \quad \forall x^0$$

In other words, under some conditions on Q, no matter how society is today, if you wait long enough you'll get x*.

But what are these sets of conditions? Here instead of formally laying it out, we briefly explain the conditions we need on Q in order to have a unique stationary distribution:

1. There should be no castes. If you are in a society with a caste system and if you're born in a certain class, you stay in that class. So clearly in such a society, initial conditions matter.

   So we need Q to satistfy the "American dream, American nightmare" condition: No matter what your initial type is, there is a positive probability of going to any different type in the sufficiently near future. So that no matter how poor you are, you can be rich; and no matter how rich you are, you can become poor.

   Consider the following transition matrix,

$$\begin{bmatrix} 0.9 & 0.1 & 0 & 0 \\ 0.3 & 0.7 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0.5 & 0.5 \end{bmatrix}$$

   This transition matrix does not satisfy what we call the "Monotone Mixing Condition". MMC basically says that the transition function should be such that it allows a sufficient mixing of all types of agents. But notice that with the above transition matrix, there is no mixing between types 1 or 2 and types 3 or 4 (for example, the probability of becoming a type 3 given you are type 1 is 0, etc.).

Remark 15.3 *Remember what the growth model had to say about the wealth distribution: Any initial condition we had, it stayed. So the model had nothing interesting to say about the wealth distribution. But here we have a much stronger result, now we know that no matter where we start we get a unique stationary distribution.*

Homework *For transition function Q, suppose there is a stationary distribution x^*. Write down formula for probability that people in the top 1% richest group remain in that percentile in stationary distribution x^*.*

16 March 27: Economy with Heterogenous Agents (continued)

16.1 Review

Last class we introduced transition functions and we talked about what they mean as well as how they are constructed. Then we defined a stationary distribution and briefly mentioned the properties we need for the transition function in order to have a unique stationary distribution. Today we will talk more about the stationary distributions and we will see more on what happens in the land of the fisherladies.

16.2 Stationary Distribution

Remark 16.1 *To aviod confusion note the equivalence between the following two kinds of notation:*

In the previous class, we denoted the sigma algebra defined on SxA by $\mathcal{A}$. So that we had,

$$\begin{aligned} x &: \mathcal{A} \to [0,1] \\ Q &: S \times A \times \mathcal{A} \to [0,1] \end{aligned}$$

But we also might use the convention where the $\sigma-$algebra defined on SxA is denoted by $\mathcal{S}x\mathcal{A}$, so that,

$$\begin{aligned} x &: \mathcal{S} \times \mathcal{A} \to [0,1] \\ Q &: S \times A \times \mathcal{S} \times \mathcal{A} \to [0,1] \end{aligned}$$

I will use the first convention here.

- Consider the transition matrix $\Gamma = \begin{bmatrix} 1 & 0 \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix}$

With this transition matrix, if you are type 1, you always stay type 1. This is not a "nice" property because remember that in order to satisfy the monotone mixing condition we need sufficient mixing of all types. No matter what type you are, there should be some positive probability to become each one of the other types. So the transition matrix above is not "nice".

- If Q satisfies a certain set of conditions, then ∃ a unique x* such that,

$$x^*(B) = \int_A Q((s,a),B)dx^*(B) \quad \forall B \in \mathcal{A}$$

  and it is globally stable (no matter which distribution x the economy starts at, the economy asymptotically goes to x*) :

$$x^* = \lim_{n\to\infty} \ \mathrm{T}^n(x^0, Q) \quad \forall x^0$$

Now we will see with examples, how initial conditions cease to matter after long periods:

Suppose we are state s_1 today. $\Gamma_{s_1,\cdot}$ is the conditional distribution of tomorrow's shock given today's is s_1.

Homework *Verify this.*

Homework *Also verify that $\Gamma^T\Gamma_{s_1,\cdot}$ is the conditional probability distribution function of the two period ahead shock given today's shock is s_1.*

Now go further to 10,000 periods ahead; we can write conditional distribution of the state 10,000 periods ahead given the shock today is s_1 as $(\Gamma^T)^{10{,}000}\Gamma_{s_1,\cdot}$

You can see that the more that exponent for Γ^T grows, the less what we are multiplying that with will matter for the result. If Γ satisfies the conditions that we mentioned previously, then

$$(\Gamma^T)^{10{,}000}\Gamma_{t_1,\cdot} = x_s^*$$

In other words, if you have shock s_1 today, especially in the case of persistence, it will continue to govern the shocks you have in the periods ahead, but only for a while; after a while the affect of your initial shock will fade away.

Homework *Consider the following transition matrix of unemployment,*

$$\Gamma = \begin{bmatrix} 0.94 & 0.06 \\ 0.5 & 0.5 \end{bmatrix}$$

where $\Gamma_{ee} = 0.94$*,* $\Gamma_{eu} = 0.06$ *, etc.*

The probability that someone will be employed two periods ahead of today given they are employed today is,

$$\Pr\{s^{u} = e | s = e\} = 0.94 \times 0.94 + 0.94 \times 0.5$$

And,

$$\Pr\{s^{\infty} = e | s = e\} = x_e^*$$

Compute x_e^**.*

Suppose that the society is described by measure x today. For example, suppose that the probability distribution over states at time t is given by $\pi_t = (p_t^1, ..., p_t^N)^T$. (There are N possible states and p_t^n stands for the probability of state n today). So given that the the shocks follow a Markov process with transition matrix Γ , the probability of being in state j tomorrow is given by,

$$p_{t+1}^j = \sum_i \Gamma_{ij} p_t^i$$

and written in a compact form, this is,

$$\pi_{t+1} = \Gamma^T \pi_t$$

So a stationary distribution of a Markov chain satisfies,

$$\pi^* = \Gamma^T \pi^*$$

Homework *Solve* $\Gamma^T x_s^* = x_s^*$

Remark 16.2 x^* *is the unconditional probability of the individual's type far away into the future. It is also the measure of peopole with that particular type far away in the future; this is because we have a continuum of people and Law of Large Numbers works.*

16.3 Back to the fisherladies

Recall the problem of the "fisherlady":

$$V(s,a) = \max_{a' \in [0,\overline{a}]} \text{u(s+a-}qa\prime) + \beta \sum_{s'} \Gamma_{ss'} V(s', a')$$

The First Order Conditions are,

$$u_c(\text{s+a-}qa\prime) = \frac{\beta}{q} \sum_{s'} \Gamma_{ss'} u_c(\text{s'+a'-qa''})$$

This is a second order difference equation and there are many solutions that satisfy it. But only one of those solutions does not violate compactness. So if we can find some natural bounds for a', or impose conditions so that a' has bounds, the above first order conditions along with the bounds that we have for savings, will characterize the optimal solution.

You'll notice that a$' \in [0, \overline{a}]$ is already one of the constraints of the above maximization problem. But now rather than just imposing such a constraint, we will find a natural reason that savings should have a lower bound and we will consider a condition that ensures an upper bound for savings.

For the lower bound, we assume that there is no technology which allows negative amount of saving and this sounds natural since storing a negative amount of fish does not make much sense. So savings has a lower bound because Mother Nature says so.

For the time being, consider an economy where there is no uncertainty. In such an economy, the following theorem holds,

Theorem 16.3 *If $\beta < q$, then $\exists \overline{a}$ such that, if $a_0 < \overline{a}$, $a_t < \overline{a}$ $\forall t$.*

You can see this formally through the usual Euler equations,

$$u_c(c_t) = \frac{\beta}{q} u_c(c_{t+1}) \quad \forall t$$

From these equations, it's clear that $\beta < q \Rightarrow u_c(c_{t+1}) > u_c(c_t) \Rightarrow c_{t+1} < c_t \quad \forall t$

If you are impatient enough compared with the returns from technology, you will consume today rather than tomorrow. Gains from saving will disappear eventually and you will stop saving more.

Now think about the economy with uncertainty. Here, the fisherman has the risk of getting a very bad shock tomorrow. So the fisherman would save just in case he has this bad shock; he would want to store some fish today in order to insure himself against getting very small number of fish tomorrow so he is not hungry in case that happens. In this case we need to think more about how to put an upper bound on savings, because with uncertainty even if $\beta < q$, the fisherman is willing to save due to gains from insurance. The kind of savings to protect oneself from risk in the future in the absence of state contingent commodities, we call precautionary savings. In order to ensure an upper bound for savings, we need to bound the gains from insurance somehow. The way to do this is to impose the condition on the utility function that its negative curvature (keeping in mind that the utility function is concave) is diminishing as wealth increases. This means that wealthier agents are less risk-averse. Formally, that u' is convex. The wealthier the agent is, the smaller the variance of his endowment next period proportional to his wealth so he doesn't want to save if he is very wealthy

So in the economy with uncertainty, in order to have an upper bound on savings, we need the first derivative of the utility function to be convex so that the following Jensen's Inequality holds:

$$\frac{\beta}{q}\int \Gamma_{ss'}u_c(c') > \frac{\beta}{q}u_c(\int \Gamma_{ss'}c')$$

Theorem 16.4 *If $\beta < q$ and u' is convex then $\exists \overline{a}$ such that $a_0 < \overline{a}$, g(s,a)$< \overline{a}$ $\forall s$.*

16.4 Various statistics to describe x*

(Refer to last year's notes for more on these statistics)

Now we are interested in ways to summarize certain properties of x*. There are various ways to do this.

Homework *Compute the ratio of total wealth held by the top 10% to the bottom 10%.*

Homework *Compute wealth level that seperates the top 10% wealthy from the rest of the population. And also compute the wealth level that seperates the bottom 10% from the rest of the population.*

- One of the statistics we can use to measure inequality in a society is variance. The problem with using this statistic is that it is unit dependent. So using coefficient of variation or the variance of the log is more reasonable since they are unit independent (the coefficient of variation is the standard deviation divided by the mean).

- A statistic that is useful to analyze mobility is the autocorrelation of wealth.

  Another way to analyze mobility is through the persistence matrix. An example of a persistence matrix is the one the elements of which denote the probability of an agent that is in the ith quantile today is in the jth quantile tomorrow.

Homework *Verify that the largest eigenvalue of a Markov transition matrix is 1.*

- The second largest eigenvalue of a transition matrix is a measure of persistence. The bigger it is, the longer it takes for the stationary distribution to take over (so that the longer the initial conditions matter).

16.5 Economy with Heterogenous Agents (with trade)

Let the fisherladies in our previous economy now trade with each other. So they are now allowed to use borrowing and lending to transfer resources across time. There is no storage technology anymore. Also, we are making the assumption that state contingent claims cannot be traded.

An important issue here is again the compactness of the asset space. For the economy with storage technology, the lower bound was set by Mother Nature but here there is no such lower bound because agents can borrow and lend so that they can have negative amounts of savings. We need to impose a lower bound on savings so that the agents cannot run a Ponzi scheme. The condition needs to ensure that the agent is able to pay back at each state, including the worst possible state. Letting s^1 denote the worst possible shock, the lower bound on savings is characterized by the following:

$$0 + \underline{a}q = s^1 + \underline{a}$$

This means that the asset level $\underline{a}$ has to satisfy the following: if your debt position is $\underline{a}$ and you draw the worst possible earning tomorrow, you can still enjoy a nonnegative consumption, by borrowing again up to the level $\underline{a}$. Solution of this equation is:

$$\underline{a} = \frac{s^1}{q-1}$$

Since $q < 1$, $\underline{a}$ is negative.

The fisherlady's problem is:

$$V(s,a;q) = \max_{a' \in [\underline{a}, \overline{a_q}]} \text{u(s+a-}qa'\text{)} + \beta \sum_{s'} \Gamma_{ss'} V(s', a'; q)$$

Remark 16.5 *Before, q was just a technology determined number. It was the rate of return from the storage technology. But now it is the interest rate on loans.*

The prices are constant in the steady state, therefore we are only focusing on the steady state equilibria here. We are assuming that we are in a steady state and that the price in the steady state is q.

Solution: $a' = g(s, a; q)$

And we know x*(q) so we can calculate the total assets in a stationary distribution of an economy when the interest rate is q as,

$$\int adx^*(q)$$

Equilibrium requires that the total amount of wealth that q generates should equal to 0. This is because in this economy no physical assets can be held (there is no storage technology) and the asset of each individual is the liability of the other. Therefore when you sum them up, you should get 0.

Finding the steady state will mean finding the q that solves,

$$\int adx^*(q) = 0$$

We can show that a steady state exists by showing that there exists a q that solves the above equation as follows:

1. Let f(q)=$\int adx^*(q)$
2. Show that f is continous in q.
3. Show that f(q)>0 for some q.
4. Show that f(q)<0 for some q.

Then by the Intermediate Value Theorem, we can conclude that $\exists q$ s.t. f(q)=0.

Roughly, the above can be shown as follows:

Consider the case when q$\rightarrow$ β from above. In this case, people will save like crazy because there is no cost of transferring consumption across periods (the rate of return on your savings offsets your discount factor). Agents would like to save more no matter how much they own. Therefore, as q$\rightarrow$ β aggregate savings grows without bound (Everybody is lending). Therefore, we know that f(q)>0 as q$\rightarrow$ β.

Next consider the case where $\frac{1}{q} < 0$. This means that the rate of return on savings is negative. Clearly, nobody will save positive amounts in this case (Nobody will lend because the rate of return you get the next period on what you lent is a negative number).Therefore, aggregate savings will be negative. So we know that f(q)<0 for q<0.

By the above and the continuity of f, then we can conclude that $\exists q$ s.t. f(q)=0.

17 April 1

17.1 More words on notation

We will clarify some possible confusion on the notations we have used so far. x is a generic notation for measure; $x : \mathcal{A} \to R_+$.

1. $x(B)$ is the measure of set B for $B \in \mathcal{A}$.

2. In the economy we saw, q is the price of asset that agents take as given. $x(q)$ is the distribution of agents associated with price q. In the fisherwomen economy with storage technology, q is exogenously given by mother nature. So, we do not need to index measure by constant q. But in the loan economy, q is interest rate of borrowing and lending and it is endogenous. Assume $q \in \left[\underline{q}, \overline{q}\right]$. $x(q) \in X$, which is the whole set of measure associated with all possible q.

3. $x^*(q)$ is the stationary distribution of agents when the price is q. To obtain $x^*(q)$, we should

   (1) solve the household's problem and get optimal decision rule $g(s, a; q)$ and value function $V(s, a; q)$.

   (2) construct transition function $Q(s, a, B; q)$, which tells the probability of type (s, a) agents in set B tomorrow. And we need verify that Q has nice property (monotone mixing condition) such that there is a unique x^* associated with Q.

   (3) find the stationary distribution $x^*(q)$.

4. A generic measure of B in stationary distribution of the above economy is $x^*(B; q)$.

5. Total asset demand in steady state when price is q is $\int_{S\times A} a dx^*(q)$. And in equilibrium of economy without storage technology, $\int_{S\times A} a dx^*(q) = 0$. Note that the distribution of agents is over both state space of shocks and asset space.

6. Integration of a measurable function. $x : \mathcal{A} \to R_+$ is a measure. $f : A \to R$ is a measurable function with respect to x. That is changes in f can be distinguish with x. Then, we can write the integration in the following equivalent ways.

$$\int f dx = \int_A f(a)\, x(da) = \int_A f(a)\, dx$$

17.2 Definition of RCE in loan economy with incomplete markets

Definition 17.1 *A stationary competitive equilibrium for a loan economy with incomplete markets is* $\{q^*, g(s,a;q^*), V(s,a;q^*), Q(s,a,B;q^*), x^*(q^*)\}$ *such that*

(1) Given q^*, $g(s,a;q^*)$ *and* $V(s,a;q^*)$ *solve household's problem.*

(2) Q *is a transition function constructed from* $g(s,a;q^*)$ *and* Γ.

(3) x^* *is a stationary distribution for transition function* Q.

$$x^*(B) = \int_{S\times A} Q(s,a,B;q^*)\, dx^* \tag{125}$$

(4) Market clears

$$\int_{S\times A} a dx^* = 0$$

17.3 Wealth persistence and inequality

- In the economy with incomplete markets, people differ in wealth. But is wealth persistent in such economies? Simply speaking, we want to know the sign of autocorrelation of asset holding. First, the persistence in labor income depends on Γ. But income persistence is different from wealth persistence because people can save and dissave. Why do people save? Because they are risk averse and want to smooth consumption. Suppose an agent with normal wealth level receive a high endowment shock s, then she will consume some and save some for the rainy days. And if in the next period, she gets a negative shock, she will consume part of her saving. If she gets a string of bad shocks, her wealth will keep decreasing. We can also analyze the opposite situation. Therefore, wealth is persistent. The autocorrelation depends on Γ, β, σ and state space of shocks.

$$\rho_a = f(\Gamma, \beta, \sigma, s)$$

- Is wealth more equally distributed or unequally distributed? How does wealth inequality evolve? So far, we just look at stationary distribution of agents in stationary equilibrium. Therefore, wealth inequality does not change over time. Although on individual level, wealth is persistent and changes over time, the economy as a whole does not change.

17.4 Liquidity constraint

From household's optimization problem with price q taken as given, we can get Euler equation

$$u_c(c) = \frac{\beta}{q} \sum_{s'} \Gamma_{ss'} u_c(c_{s'}) \tag{126}$$

But when there is liquidity constraint, this Euler equation may not hold at some time. (168) is not true at the boundary. We can look at a simple two-period model. Agents have wealth a in period 1 and wage w' in period 2. Interest rate is r. The optimization problem is

$$\max_{c \geq 0, c' \geq 0, s \geq 0} u(c) + \beta u(c')$$

subject to

$$\begin{aligned} c + s &= a & (127) \\ c' &= (1+r)s + w' & (128) \end{aligned}$$

Constraints (186) and (187) can be written as

$$\begin{aligned} c + \frac{c'}{1+r} &= a + \frac{w'}{1+r} & (129) \\ s &\geq 0 & (130) \end{aligned}$$

People cannot borrow with the constraint of $s \geq 0$. When there is no such borrowing constraint, from FOC, we have the following Euler equation

$$u_c(c) = \beta(1+r) u_c(c')$$

When there is a borrowing constraint $s \geq 0$, there are two cases:

(1) constraint $s \geq 0$ does not bind, then the Euler equation holds.

(2) constraint $s \geq 0$ do bind, then $s = 0, c = a, c' = w'$. Euler equation does not hold.

$$u_c(c) \geq \beta(1+r) u_c(c')$$

Today's consumption is lower or equal to the first best case.

Homework: Write Kuhn-Tucker problem for this economy.

Therefore, some studies test the equality of Euler equation as the evidence of liquidity constraint. But according to our definition, if there is solvency constraint, will this constraint always be binding? It depends on how bad the constraint is. For example, if people have log utility function, $u(c) = \log c$. The bad shock of endowment is $s_b = 0$. Then, since $u_c(c) = \frac{1}{c}$, marginal utility of consuming nothing is infinity, then nobody will lead themselves to the situation of binding borrowing constraint. They will get enough saving to avoid this worst case. Therefore, when there exists borrowing constraint, Euler equation may still hold. (Analogy: every rational agent will keep herself away from the edge of a cliff so that she will not fall from the cliff.)

17.5 Growth model with incomplete markets

We add a production technology to the economy of many agents with incomplete markets. For now, we assume agent value consumption but not leisure. Every period, each agent receives s efficiency units of labor. This shock follows a Markov process with a Markov transition matrix $\Gamma_{ss'}$. You can think of it as the hours that agents can use for either leisure or labor. Since agents do not value leisure, they just use all of their efficiency units as a labor supply. (Previously, we assume $s = 1$, $\forall i$. That is everyone has one unit of efficiency labor supply).

The agent's problem is

$$V(s, a; K) = \max_{c \geq 0, a \in [0, \overline{a}]} u(c) + \beta \sum_{s'} \Gamma_{ss'} V(s', a'; K) \tag{131}$$

subject to

$$c + a' = a(1 + r(K)) + w(K)s$$

The production function is $f(K, N)$. The firm uses aggregate labor N and capital K as inputs and produce consumption goods. Note that the wage and capital rental rate which clear the market are:

$$\begin{aligned} w^* &= F_N(K, N) && (132) \\ r^* &= F_K(K, N) - \delta && (133) \end{aligned}$$

How to compute aggregate labor N? Of course we can write

$$N = \int_{S \times A} s dx \tag{134}$$

But we need know N without knowing x. This can be done because s does not depend on people's choice. An example:$S = [0.2, 1]$, $\Gamma = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix}$. The stationary distribution associated with Γ is $[0.5, 0.5]$. Thus $N = 0.5 \times 0.2 + 0.5 \times 1 = 0.6$. From this example, we know the aggregate labor is average endowment of the economy. It depends on the stationary distribution of Γ, but does not depend on people's choice.

Remark 17.2 *If leisure enters utility function, N is endogenous.*

The optimal solution to the agent's problem is $g(s, a; K)$ and $V(s, a; K)$. Then, we can construct transition function $Q(s, a, B; K)$ and find the stationary distribution $x^*(K)$ associated with Q.

Definition 17.3 *Stationary recursive competitive equilibrium in the growth model with incomplete markets is $\{K, g(s, a; K), V(s, a; K), Q(s, a, B; K), x^*(K)\}$ such that*

(1) Given K, $g(s, a; K)$ and $V(s, a; K)$ solve household's problem.

(2) Q is a transition function constructed from $g(s, a; K)$ and Γ.

(3) x^* is a stationary distribution for transition function Q.

$$x^*(B) = \int_{S\times A} Q(s, a, B; K)\, dx^*$$

(4) Market clears

$$F_K\left[\int a dx^*(K), N\right] - \delta = r(K) \tag{135}$$

(135) is one equation with one unknown of K. Therefore we can find the equilibrium. In writing (135), we mean that in equilibrium, aggregate capital endogenous from people's choice induces the price $r(K)$ that people take as given. Another way of writing the market clearing condition is

$$\int_{S\times A} a dx^*(K) = K$$

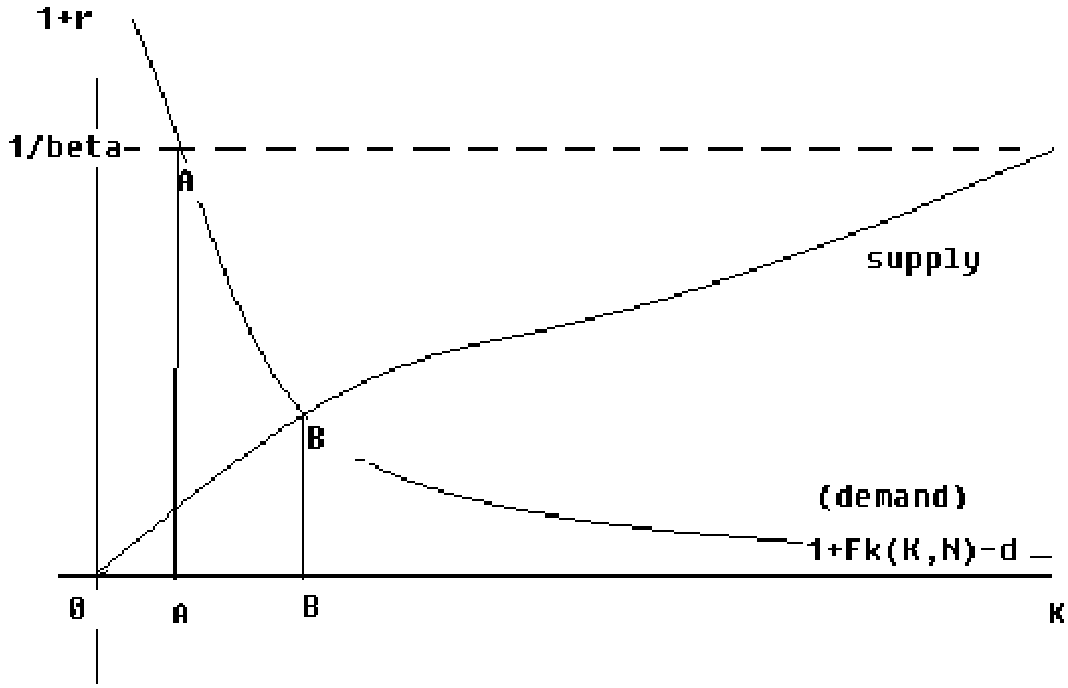

Figure 1:

17.6 Aggregate precautionary saving

In this economy, interest rate is endogenous since it is the marginal product of capital. We can analyze aggregate demand and supply of capital as a function of interest rate.

When $r \to \frac{1}{\beta} - 1$, interest rate is very high relative to time preference. People are very patient and always want to save more. Therefore, aggregate capital supply goes to infinity. But capital demand goes to zero with high rental rate.

When $r \to -1$, people will not save at all. Aggregate capital supply goes to zero, but aggregate capital demand goes to infinity.

With intermediate value theorem, there is a capital level K such that stationary equilibrium exists.

On graph, K_B is the equilibrium capital level in the economy with idiosyncratic risks. While K_A is equilibrium capital level when there is no risk in the economy. Why? Because when there is no risk, the Euler equation in steady state is

$$u_c(c^*) = \beta(1+r)\,u_c(c^*)$$

Thus

$$1 + r = \frac{1}{\beta}$$

We use aggregate precautionary saving to describe the additional wealth level in the society because of incomplete insurance. If we assume that the earning risks cannot be insured (i.e., the agents cannot trade state contingent securities), agents are expected to save a part of their earning in the form of capital in order to "prepare for the bad time in the future". The saving for "preparing for the bad time in the future" is what we call "precautionary saving". In the economy with complete markets there is no precautionary saving, because there is no such risk (agents end up receiving the same amount by trading Arrow securities).

Among studies in the literature, the two works are very important: one is Aiyagari (1994 Quarterly Journal of Economics). The other is Huggett (1993 Journal of Economic Dynamic and Control). Their finding is that aggregate precautionary saving is at most 3% increase of the aggregate saving rate.

17.7 Growth model with leisure and incomplete markets

$$V\left(s, a; \frac{K}{N}\right) = \max_{c \geq 0, a \in [0, \overline{a}], n \in [0,1]} u(c, n) + \beta \sum_{s'} \Gamma_{ss'} V\left(s', a'; \frac{K}{N}\right) \tag{136}$$

subject to

$$c + a' = a\left(1 + r\left(\frac{K}{N}\right)\right) + w\left(\frac{K}{N}\right) sn$$

The optimal solution is

$$\begin{aligned} a' &= g\left(s, a; \frac{K}{N}\right) \\ n &= h\left(s, a; \frac{K}{N}\right) \end{aligned}$$

In equilibrium,

$$\frac{K}{N} = \frac{\int a dx^* \left(\frac{K}{N}\right)}{\int sh\left(s, a; \frac{K}{N}\right) dx^* \left(\frac{K}{N}\right)} \tag{137}$$

18 April 3

18.1 Continuity of aggregate excess demand of capital

To show the existence of steady state equilibrium, we want to use intermediate value theorem to find equilibrium price. Thus, we need condition to guarantee that aggregate excess demand in steady state is a continuos function of price.

Theorem 18.1 *Stocky and Lucas (12.13)*

If (1) $S \times A$ is compact.

(2) $(s_n, a_n, q_n) \to (s_0, a_0, q_0)$, implies $Q(s_n, a_n, \cdot; q_n) \to Q(s_0, a_0, \cdot; q_0)$

(3) $x^*(q_n)$ is unique.

Then, for a measurable function f,

$$\int f(s,a)\, dx^*(q_n) \to \int f(s,a)\, dx^*(q)$$

Let's verify the above conditions in a loan economy.

(1) $S \times A$ is compact by assumption. Interest rate is bounded away from $\frac{1}{\beta}$, $q \in [\underline{q}, \overline{q}]$, then $\underline{a}$ is saving under the lowest interest rate and $\overline{a}$ is saving level with highest possible interest rate.

(2) Decision rule g is continuos from Theorem of Maximum. Therefore, the constructed transition function is continuos.

(3) Monotone mixing condition ensures a unique $x^*(q_n)$.

Therefore, the above theorem holds. Aggregate excess demand is continuos and there exists a steady state equilibrium.

18.2 Non Steady State Equilibrium

The steady state equilibrium that we have studied so far cannot be used for analyze policy change. So, let's now look at non-steady state equilibrium version of Aiyagari's economy.

First, we should know why the following problem is not well defined in equilibrium.

$$V(s,a,K;G) = \max_{c \geq 0, a \in [0,\overline{a}]} u(c) + \beta \sum_{s'} \Gamma_{ss'} V(s', a', K'; G) \tag{138}$$

subject to

$$\begin{aligned} c + a' &= a(1+r) + ws \\ r &= F_K(K, N) - \delta \\ w &= F_N(K, N) \\ K' &= G(K) \end{aligned}$$

And in equilibrium,

$$K = \int a dx \tag{139}$$

In steady state equilibrium, r is a constant, indexed by K. But outside steady state, price is not a constant any more. r is marginal product of capital. Aggregate labor does not change in this economy, thus the sufficient statistic of r and w is aggregate capital $K = \int adx$. Therefore, for individual problem, this is well defined as long as with the conceived aggregate capital, interest rate is not too low to have unbounded asset problem. That is $G(K) > K^*$. Then, agents know the sequence of prices and they make their optimal decision.

But when it comes to equilibrium, agent's conjecture does not hold. How do agents conjecture the law of motion for K? $K' = G(K)$ means that aggregate capital is sufficient statistic for aggregate capital tomorrow. Usually this does not hold. Consider two societies with same aggregate capital today. For both societies to have the same capital again tomorrow, both decision rule g has to be an affine function of asset holding.

$$g(a) = \alpha + \beta a$$

Then, from

$$\begin{aligned} K &= \int adx \\ K' &= \int g(s, a) dx \end{aligned}$$

it is true that for affine function of g,

$$K' = h\left(\int adx\right) = h(K)$$

Homework: Define a linear function.

But saving function is not linear. We can look at two examples. One is economy with finite horizon. For a society populated by old men, they will dissave their wealth by holding

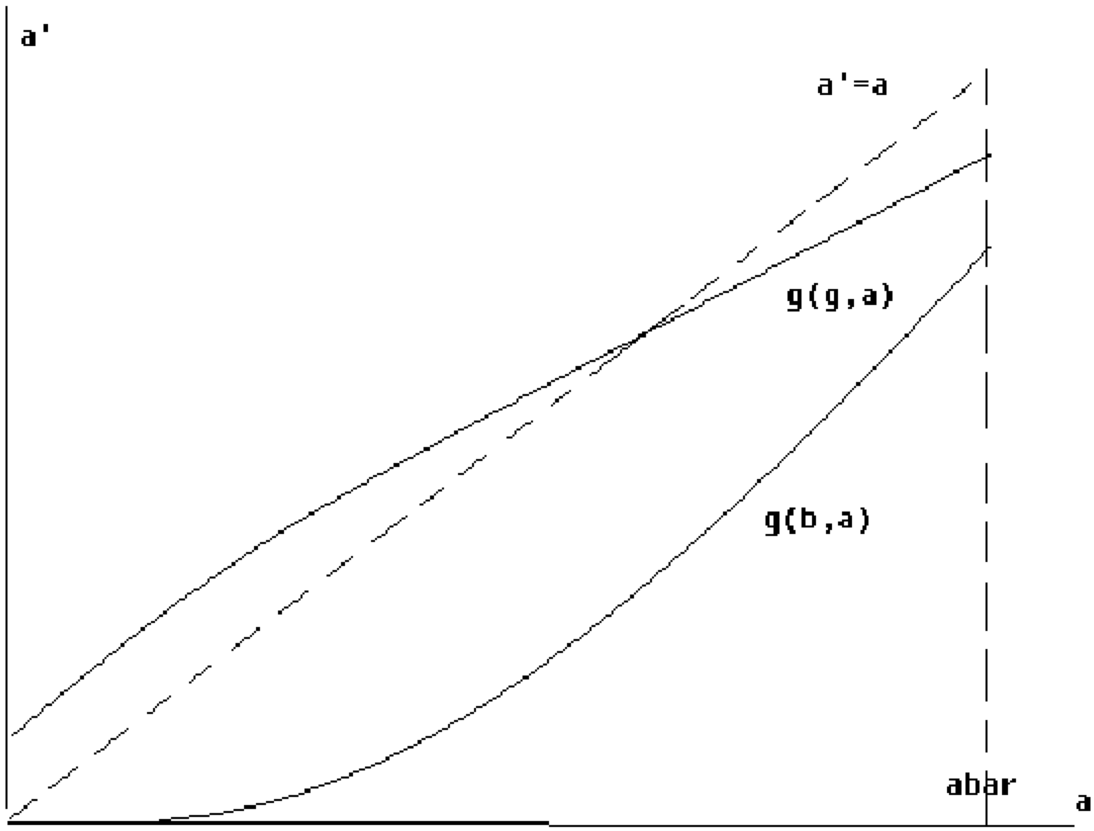

Figure 2:

parties, say. But for a society populated by young men with same level of wealth, they will keep saving for the future. So, the aggregate wealth of next period will not be equal for the two societies. Another example is Aiyagari economy with zero borrowing constraint. Suppose there are two shocks, s_g and s_b. Then, for asset close to zero, $g(s_b, a) = 0$ because agents want to borrow but are constrained. Therefore, $g_a(s_b, a)$ is close to zero. But for large asset holding, $g_a(s, a) \simeq 1$. Therefore, the saving rule is nonlinear.

Another way of illustration is that if wealth in the society is redistributed, then, under linear saving rule, total saving will not change. But in the above economy, when we transfer wealth from lucky guys to an unlucky one with bad shock and little asset holding, the latter will not save as the former agent does. Therefore, g is nonlinear. Aggregate capital K is not sufficient to predict K'. In equilibrium, $K' \neq G(K)$. We need know the whole distribution of wealth to forecast K'.

The well-defined problem in the nonsteady state equilibrium is:

$$V(s,a,x;G) = \max_{c\geq 0, a\in[0,\overline{a}]} u(c) + \beta \sum_{s'} \Gamma_{ss'} V(s',a',x';G) \tag{140}$$

subject to

$$\begin{aligned} c + a' &= a(1+r) + ws \\ r &= F_K(\int adx, N\ ) - \delta \\ w &= F_N(\int adx, N\ ) \\ x' &= G(x) \end{aligned}$$

Here, $G(x)$ tells how the society distribution evolve. G maps probability measure onto probability measure.

Definition 18.2 *Nonsteady state equilibrium in a growth model with a continuum of agents with idiosyncratic shock and incomplete markets is* $\{V(s,a,x), g(s,a,x), G(x)\}$ *such that*

(1) $V(s,a,x), g(s,a,x)$ *solve household's problem, given* G*.*

(2)

$$G(x)(B) = \int_{S\times A} \sum_{s'\in B_S} \Gamma_{ss'} 1_{g(s,a,x;G)\in B_a} dx \tag{141}$$

Note that in the above definition, we have implicitly defined transition function Q.

Now, let's study rational expectation under this nonsteady state equilibrium. We will see to find equilibrium is horribly hard. First, households have expectation on evolution of wealth distribution, $x' = G^E(x)$. The optimal solution of households is $g\left(s,a,x;G^E\right)$. This evolution is true in equilibrium when given G^E, indeed households' action aggregate to generate a distribution of $x' = G^E(x)$. In math,

$$G(x)(B) = \sum_{s'\in B_s} \int_{S\times A} \Gamma_{ss'} 1_{g(s,a,x;G^E)\in B_a} dx$$

Equilibrium is a fixed point over space of functional mapping from expectation to expectation. This problem cannot be solved.

18.3 Policy analysis

"$\underset{2}{\underline{\textit{Recursive Equilibrium}}}$ is $\underset{1}{\underline{\textit{horribly}}}$ hard when done $\underset{3}{\underline{\textit{properly}}}$." But to know the effect of policy change, we have to study nonsteady state equilibrium. There are three things we can do.

We start with $\underline{\textit{horribly}}$. What makes horrible is that price depends on distribution. We have seen the case when price does not depend on x. In fisherwomen economy with storage technology, q is exogenous. Therefore, we can avoid the horribly hard problem by using exogenous price.

The problem of having endogenous price is that marginal rate of substitution is endogenous. In Huggett economy with borrowing and lending, price is endogenous from market clearing condition $\int adx\,(q) = 0$. Aiyagari economy, r and w are marginal product of factors, thus endogenous.

18.4 Unemployment Insurance

Let's work on analysis of unemployment insurance in an economy with exogenous price. State space of shocks $S = \{e, u\}$. Transition matrix is $\Gamma_{ss'}$. τ is unemployment insurance ratio that any agent has to pay from her labor income when she has a job. b is unemployment benefit that a person can get when unemployed.

Homework: Compute average duration of having a job.

Individual optimization problem is

$$V(s,a) = \max_{c,a'} u(c) + \beta \sum_{s'} \Gamma_{ss'} V(s',a') \tag{142}$$

subject to

$$c + qa' = \begin{cases} w(1-\tau) + a & \text{if } s = e \\ b + a & \text{if } s = u \end{cases} \tag{143}$$

Optimal solution is $g(s,a;\tau,b)$. Construct transition function and find stationary distribution $x(\tau,b)$. (Note, we index distribution by policy parameters τ and b).

Equilibrium condition is

$$\int b 1_{s=u} dx = \int w\tau 1_{s=e} dx \tag{144}$$

A simpler way is the have

$$bx_{u\cdot} = w\tau x_{e\cdot} \tag{145}$$

where $x_{u\cdot}$ denotes the proportion unemployed people today.

Homework: Define steady state equilibrium of the above economy.

18.5 Unemployment Insurance Policy Analysis

Suppose we have a choice of cutting unemployment benefit by half. How do you think of this policy change? This is a question of policy analysis. To compare two policies, solving stationary equilibrium associated to each policy and compare the welfare of agents in the two stationary equilibria is wrong. Because the problem is NOT "whether you would like to be born in an economy with policy A or rather be born in an economy with policy B", but "if you are living in an economy with policy A, would you support the change of policy to policy B or rather stay with policy A." If we simply compare the economic performance of both policies, we only get direct effect of policy changes. In the transition from one policy to another, there is also effect on wage through changes in saving behavior.

To assess policy analysis, we have to

(1) construct a measure of goodness.

(2) tell what the outcome would be. That is find decision rule and distribution of agents through changes of policies.

19 April 8

19.1 Unemployment Insurance Policy Analysis (Continued)

Given policy parameter (τ, θ), agent's problem is:

$$V(s,a) = \max_{c\geq 0, a'\geq 0} u(c) + \beta \sum_{s'} \Gamma_{ss'} V(s', a') \tag{146}$$

subject to

$$c + qa' = w(1-\tau) 1_{s=e} + \theta 1_{s=u} + a \tag{147}$$

Optimal solution is $g(s, a; \tau, \theta)$. We can find stationary distribution $x(\tau, \theta)$.

We assume that government has to balance its period by period budget constraint.

$$\int b 1_{s=u} dx = \int w\tau 1_{s=e} dx \tag{148}$$

Since the fraction of people who are unemployed/employed are exogenous, we use $x_{e\cdot}$ denote the proportion of employed people and $x_{u\cdot}$ for unemployed people. Then the government budget constraint is

$$\theta x_{u\cdot} = w\tau x_{e\cdot} \tag{149}$$

From (149), the unemployment benefit, or called replacement rate, θ is totally determined given τ since $x_{e\cdot}$, $x_{u\cdot}$ and w are all exogenous. The unemployment insurance policy is trivially computed given τ.

Remark 19.1 *Markets are incomplete in this economy. People want to trade state contingent claims or borrow but constrained from doing so. But we can assume "chicken government" in the sense that government has power which is beyond people's ability. (People like chicken. People do not know how to make chicken. Government knows how to make chicken. Government makes chicken). In this economy, government provide unemployment insurance. In the next model, we will see that government can also borrow.*

Now, suppose the current policy is $\widehat{\tau}$, will it be a better policy if τ is set to zero? In other words, should we get ride of unemployment insurance? First thing we should know is that the goodness of policy is measured in social welfare. Under current policy $\widehat{\tau}$, social welfare is

$$\int u\left[c\left(s,a;\widehat{\tau}\right)\right] dx\left(\widehat{\tau}\right)$$

where the optimal decision of individual is $c\left(s,a;\widehat{\tau}\right)$ and $x\left(\widehat{\tau}\right)$ is wealth distribution in steady state indexed by policy parameter $\widehat{\tau}$. To investigate the effect of changing policies, can we compare the following social welfare, $\int u\left[c\left(s,a;\widehat{\tau}\right)\right] dx\left(\widehat{\tau}\right)$ and $\int u\left[c\left(s,a;0\right)\right] dx\left(0\right)$? The two terms are both in steady state, which means that people have managed to adjust their behavior to the prevailing policies. But it does not make sense to do such comparison in welfare analysis of policies. To compare welfare among policies, we have to put the economy in the same initial conditions (steady state obtained under $\widehat{\tau}$) and then impose different policies (in our example, the choices are to continue with $\widehat{\tau}$ or to have $\tau = 0$). Another illustrative example: suppose there is a full coverage unemployment policy. People will get the same endowment whether they are unemployed or not. In this case, people will have not incentive to save against risk of s. Now if all the benefit is abolished, people will want to dissave when they are hit by bad shock, but they do not have much assets. The whole adjustment to steady state is a long-run thing. Therefore, we get nothing from direct welfare comparison of two steady states. Another example is that suppose Nigeria is now imposing a perfect set of policy for economy. But will you choose to live in Nigeria or in US which has less perfect policy today? Although in 500 years, Nigeria may be a much better place to live than US, but for now, you will not choose to move there. So, we have to compare policies under same initial conditions.

$$\text{Initial Condition}\left\langle \begin{array}{ll} \pi^A & \text{welfare } u\left[\pi^A, IC\right] \\ \pi^B & \text{welfare } u\left[\pi^B, IC\right] \end{array} \right.$$

In this case, because price does not depend on the whole wealth distribution, policy analysis is easy. We can analyze in the following steps:

1. Solve agent's problem when policy parameter is $\widehat{\tau}$ and 0 respectively. Decision rules are $g\left(s,a;\widehat{\tau}\right)$ and $g\left(s,a;0\right)$.

2. The current society wealth distribution is $x\left(\widehat{\tau}\right)$. When the policy continue to be $\widehat{\tau}$, social welfare is

$$\begin{aligned} W\left(x\left(\widehat{\tau}\right),\widehat{\tau}\right) &= \int u\left[c\left(s,c;\widehat{\tau}\right)\right] dx\left(\widehat{\tau}\right) + \beta \int u\left[c\left(s,c;\widehat{\tau}\right)\right] dx\left(\widehat{\tau}\right) \\ &\quad +\ldots + \beta^t \int u\left[c\left(s,c;\widehat{\tau}\right)\right] dx\left(\widehat{\tau}\right) + \ldots \qquad (150) \\ &= \frac{1}{1-\beta} \int u\left[c\left(s,c;\widehat{\tau}\right)\right] dx\left(\widehat{\tau}\right) \qquad (151) \end{aligned}$$

   where $W\left(x\left(\widehat{\tau}\right),\widehat{\tau}\right)$ denotes the welfare for economy with distribution $x\left(\widehat{\tau}\right)$ under policy $\tau = \widehat{\tau}$.

3. If the policy parameter change and stay at 0 now, the evolution of wealth distribution can be obtained in the following way. Construct transition function $Q\left(s,a,B;0\right)$ from agent's decision rule $g\left(s,a;0\right)$ and transition matrix Γ. The sequence of wealth distribution is

$$\begin{aligned} x_0 &= x\left(\widehat{\tau}\right) \\ x_1\left(B\right) &= \int_{S\times A} Q\left(s,a,B;0\right) dx_0,\ \forall B \in \mathcal{A} \\ &\cdots \\ x_t\left(B\right) &= \int_{S\times A} Q\left(s,a,B;0\right) dx_{t-1},\ \forall B \in \mathcal{A} \qquad (152) \\ &\cdots \\ &\downarrow \\ &\quad x\left(0\right) \end{aligned}$$

since $x(0)$ is the unique stationary distribution associated with $\tau = 0$. Note here, equivalently, we can also define a mapping operator

$$T(Q,x) = \int_{S\times A} Q(s,a,B;0)\,dx$$

and find the sequence of distribution.

Then, the social welfare under the new policy is

$$\begin{aligned} W(x(\widehat{\tau}),0) &= \int u[c(s,c;\widehat{\tau})]\,dx_0 + \beta \int u[c(s,c;\widehat{\tau})]\,dx_1 \\ &\quad +\ldots+\beta^t \int u[c(s,c;\widehat{\tau})]\,dx_t + \ldots \end{aligned} \tag{153}$$

where $W(x(\widehat{\tau}),0)$ denotes the welfare for economy with distribution $x(\widehat{\tau})$ under policy $\tau = 0$.

4. Compare $W(x(\widehat{\tau}),\widehat{\tau})$ and $W(x(\widehat{\tau}),0)$. The one with higher social welfare is better.

But in most times, price is not exgonenous. And also, the government's budget constraint may depend on the whole wealth distribution.

19.2 Second example with unemployment insurance policy

Assume that under this unemployment insurance policy, people who have jobs have to pay a proportion of their whole income as unemployment premium. The interest rate on storage is r. That is, with policy (τ,θ), the agent solves the following problem:

$$V(s,a) = \max_{c\geq 0, a'\geq 0} u(c) + \beta \sum_{s'} \Gamma_{ss'} V(s',a') \tag{154}$$

subject to

$$c + a' = [w1_{s=e} + ra](1-\tau) + \theta 1_{s=u} + a \tag{155}$$

With this economy, government's revenue depends on total value of wealth, including with labor income and interest from storage. Now, We want to find the implication of a new policy in which τ is cut by one half.

Case 1:

Assume the government is facing period by period budget constraint

$$\int (1_{s=e}w + ra)\,\tau dx = \theta x_{u\cdot} \tag{156}$$

We rewrite it as

$$w\tau x_{e\cdot} + \int ra\tau dx = \theta x_{u\cdot}$$

Because $\int ra\tau dx$ is capital revenue and depends on the whole wealth distribution $x(\tau,\theta)$, we cannot infer one policy parameter θ from the other one τ. What we can do is that we guess a sequence of θ_t and see whether it satisfies government period by period budget constraint. Steps:

1. Current policy is $\left(\widehat{\tau}, \widehat{\theta}\right)$, $\tau = \frac{\widehat{\tau}}{2}$, guess a sequence $\{\theta_t\}_{t=0}^{\infty}$ which agents take as given in their optimization problem.

2. Solve

$$\max_{\{c_t(h_t), a_{t+1}(h_t)\}} \sum_t \beta^t \sum_{h_t} \Pi(h_t) u(c_t(h_t)) \tag{157}$$

subject to

$$\begin{aligned} c_t(h_t) + a_{t+1}(h_t) &= \left[a_t(h_{t-1}) + w1_{s(h_t)=e}\right](1-\tau) \\ &\quad +1_{s(h_t)=u}\theta_t + a(h_{t-1}) \;\; \forall t, h_t \\ &\quad a_0,\ s_0\ given \end{aligned} \tag{158}$$

where $h_t = \{s_0, s_1, ...\}$ is a history of an agent. The solution of the problem is:

$$g_t\left(s, a; \tau, \{\theta_t\}\right)$$

Find distribution x_t accordingly.

3. The government budget constraint is satisfied for all t.

$$\begin{aligned} w\frac{\widehat{\tau}}{2}x_{e\cdot} + \int ra\tau dx_0 &= \theta_0 x_{u\cdot} \\ w\frac{\widehat{\tau}}{2}x_{e\cdot} + \int ra\tau dx_1 &= \theta_1 x_{u\cdot} \\ &\ \ ... \\ w\frac{\widehat{\tau}}{2}x_{e\cdot} + \int ra\tau dx_t &= \theta_t x_{u\cdot} \\ &\ \ ... \end{aligned} \tag{159}$$

In practise, we can assume after 100 years, say, economy converges to new steady state. Therefore, (159) is a system of 100 equations with 100 unknowns. We can solve for $\{\theta_t\}_{t=0}^{99}$

Case 2:

We can see from last example that the economy does not converge to new steady state immediately. Because if so, government period by period budget constraint does not hold. But if we assume that government can borrow and lending, then there is little constraint on policy parameter and θ can be any constant. We will work on the implication of change policy $\left(\widehat{\tau}, \widehat{\theta}\right)$ to $\left(\overline{\tau}, \overline{\theta}\right)$.

Now, given initial wealth distribution x, a policy (τ, θ) is a feasible policy if

1. Agents solve

$$V(s,a) = \max_{c\geq 0, a'\geq 0} u(c) + \beta \sum_{s'} \Gamma_{ss'} V(s', a') \tag{160}$$

subject to

$$c + a' = [w 1_{s=e} + ra](1-\tau) + \theta 1_{s=u} + a \tag{161}$$

Decision rule is

$$g(s, a; \tau, \theta)$$

Note that since government can issue domestic debt, individual wealth can take the form of storage or debt. But both kinds of assets have the same rate of return which is exogenously given by storage technology.

Transition function is $Q(s, a, B; \tau, \theta)$ and the distribution mapping operator on distribution is $T(Q, x)$. And the wealth distribution evolves

$$\begin{aligned} x_0 &= x \\ x_1 &= T(Q, x_0) \\ &\dots \\ x_t &= T(Q, x_{t-1}) \\ &\dots \end{aligned}$$

2. Government budget constraint is satisfied

Present value of government expenditure = Present value of government revenue

That is

$$\sum_{t=0}^{\infty}\frac{\theta x_{u\cdot}}{(1+r)^t}=\sum_{t=0}^{\infty}\frac{\int\tau\left(1_{s=e}w+ra\right)dx_t}{(1+r)^t}$$

3. Government cannot issue ridiculous amount of debt. Let D_t denote the total government debt at period t. Law of motion for D_t is

$$D_{t+1}=(1+r)\,D_t-\int\tau\left[w1_{s=e}+ra\right]dx_t+\theta x_{u\cdot}$$

The total government debt and households asset cannot be negative

$$D_{t+1}+\int adx_t\geq 0 \tag{162}$$

Therefore, we make sure that the society does not store negative amount.

Remark If we assume the government can borrow from aboard, then there is no (18) constraint.

Remark If (18) is violated, storage becomes negative which cannot be true. Therefore, price will not be exogenously given by storage. It is endogenous to clear the asset market demand and supply.

20 April 14

20.1 Economy with technology changes

In an Aiyagari economy, suppose the production is given by $A_tF(K_t,N_t)$ where $N_t=\int sdx, K_t=\int adx$. Individual idiosyncratic shock $s\tilde{}\Gamma_{ss'}$. The agent's problem is

$$V(s,a;K)=\max u(c)+\beta\sum_{s'}\Gamma_{ss'}V(s',a';K) \tag{163}$$

subject to

$$c+a'=a\left[1+r(K)\right]+sw(K) \tag{164}$$

The decision rule is $g(s,a;K)$. Wealth distribution $x\left[g(K)\right]$ is derived from this $g(.)$ function. And aggregate capital in steady state equilibrium is

$$K^*=\int adx\left[g(K^*)\right] \tag{165}$$

Now, we want to know what happens if A doubles?

20.1.1 In a Representative Agent economy

- We have learnt how to work this out in a representative agent economy. With same K, marginal product of capital doubles, r increases, people keep increasing their saving until new capital level and this new A generate interest rate equal to $\frac{1}{\beta}-1$. So, in a RA economy, we know aggregate capital increases until it reach the new steady state capital level. To solve the equilibrium path, we can solve SPP because market is complete and equilibrium is PO.

  Euler equation is a second order difference equation

  $$\varphi\left(K_t, K_{t+1}, K_{t+2}\right)=0$$

  with K_0 given.

  In steady state when technology is A, we can solve steady state capital level K^0 from

  $$\varphi\left(K^0, K^0, K^0\right)=0$$

  We write out the SPP as

  $$\Omega(K)=\max u\left[AF(K,1)-K'\right]+\beta\Omega\left(K'\right)$$

  And EE is

  $$u_c=\beta AF'\left(K^0,1\right)u_c'$$

  In steady state,

  $$1=\beta AF'\left(K^0,1\right)$$

  and we solve for K^0.

- When new technology takes place, $\widehat{A}=2A$, Euler equation changes to

  $$\widehat{\varphi}\left(K_t, K_{t+1}, K_{t+2}\right)=0 \tag{166}$$

  because feasibility condition changes. Social planner's problem is now

  $$\Omega(K)=\max u\left[2AF(K,1)-K'\right]+\beta\Omega\left(K'\right) \tag{167}$$

  Upon the technology change, capital level is still at K^0, so FOC gives

  $$\frac{u_c}{u_c'}=2\beta AF'\left(K^0,1\right)$$

  c_t and c_{t+1} have to adjust to satisfy this optimality condition. We can predict that growth rate of consumption goes up.

And we can compute the transition path for capital. Upon technology change, we have

$$\widehat{\varphi}\left(K^{0},K^{01},K^{02}\right)=0 \tag{168}$$

But we need a second condition to get the whole path. As there is only one K^{01} with which solution to (168) satisfies feasibility. The whole path of capital level in this economy is obtained.

Problem Solve for the capital level in the new steady state for this economy.

20.1.2 Technology changes with incomplete market

Now how to solve this problem with incomplete market?

In steady state equilibrium,

$$K^{0*}=\int adx\left[g\left(K^{0*}\right)\right] \tag{169}$$

When A doubles, we know what happens in new steady state, g^{1*},K^{1*},x^{1*}. But to get the whole transition path, we need solve optimal decision rule with K^{0*},x^{0*} taken as given as initial condition.

To see what happens when the world changes, we assume:

1. After T periods, the economy has converged to g^{1*},K^{1*},x^{1*}. And price $r_t=r\left(K^{1*}\right),w_t=w\left(K^{1*}\right)$ for $t>T$.

2. Assume $\{r_t\}_{t\leq T}$ and $\{w_t\}_{t\leq T}$ are given by $\overrightarrow{r}$ and $\overrightarrow{w}$ respectively.

Individual's problem is

$$\Phi\left(s,a,\overrightarrow{r},\overrightarrow{w},T,K^{1*}\right)=\max_{c_t}\sum_{t=0}^{T}\beta^t u\left(c_t\right)+\beta^T E\left\{V^1\left(s^T,a^T;K^{*1}\right)\right\}$$

subject to

$$c_t+a_{t+1}=\left(1+r_t-\delta\right)a_t+sw_t$$

Hence, we assume the transition takes place over a finite number of periods. In T periods, the economy gets to new steady state. In the meantime, households face price of $\overrightarrow{r}$ and

$\overrightarrow{w}$ and choose $\{c_t\}$. This problem yields decision rule $g_t\left[s, a, \overrightarrow{r}, \overrightarrow{w}, T, K^{1*}\right]$ which is state dependent.

We can compare this problem with the one when unemployment insurance policy changes. In that case, we can use the decision rule from the new situation g^{1*} because the change in environment does not affect price. But in the current model, interest rate increases and people adjust their decision rule accordingly. So, we have to use both recursive and nonrecursive methods to solve the problem.

Problem In the Aiyagari economy with unemployment insurance, suppose unemployment insurance is paid by consumption tax τ. Describe the algorithm to access the policy changes in τ . Consider two cases when government has period by period budget constraint and when government can borrow with bond.

20.2 Aiyagari economy with aggregate uncertainty

When we model business cycle, the economy does not converge to any steady state because there exists aggregate uncertainty. We can define an economy with a moderately stupid agents. By "moderately stupid", we mean that agents choose to ignore some relevant information in making decision. (For example, when a person wants to predict the outcome of a football game next week, she may ignore the news that one key player had a quarrel with his wife. But she forecast the outcome using the information that this key player will play in this game.)

In the Aiyagari economy with aggregate uncertainty, aggregate shock is denoted as z which follows Markov transition matrix $\Pi_{zz'}$. We allow the probability of idiosyncratic shocks to depend on aggregate production shock. Therefore, the individual's problem is

$$V\left(z, x, s, a; G\right) = \max_{c, a \geq 0} u\left(c\right) + \beta \sum_{s', z'} V\left(z', x', s', a'; G\right) \Gamma_{s'|szz'} \Pi_{z'|z} \tag{170}$$

subject to

$$\begin{aligned} c + a' &= a(1 + r\left(z, K\right) - \delta) + sw\left(z, K\right) && (171) \\ x' &= G\left(z, x\right) && (172) \\ K &= \int a dx && (173) \end{aligned}$$

Problem Talk about $\Gamma_{s'|szz'}\Pi_{z'|z}$

We assume agents are too stupid to solve this problem. Therefore, they exclude information embedded in distribution measure x and only use information contained in aggregate capital K. Then the agent's problem becomes to

$$\Psi(z, K, s, a; H) = \max_{c, a \geq 0} u(c) + \beta \sum_{s', z'} \Psi(z', K', s', a'; H) \Gamma_{s'|szz'} \Pi_{z'|z} \tag{174}$$

subject to

$$\begin{aligned} c + a' &= a(1 + r(z, K) - \delta) + sw(z, K) && (175) \\ K' &= H(z, K) && (176) \end{aligned}$$

Note, in equilibrium, the conjectured law of motion for K is not what really happens in the economy.

We will define such an economy full of stupid agents. If there is no much loss in doing so, we will use this economy in the study. There are three grounds for agents to use $K' = H(z, K)$ in optimization problem:

1. Knowing more does not mean that they can forecast better.
2. Forecasting better does not mean that they can be happier.
3. Forecasting better does not mean that they will behave differently.

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21.1 Aiyagari economy with aggregate uncertainty(continued)

In the economy with moderately stupid agents, individual problem is

$$V(z, K, s, a; H) = \max_{c, a \geq 0} u(c) + \beta \sum_{s', z'} V(z', K', s', a'; H) \Gamma_{s'|szz'} \Pi_{z'|z} \tag{177}$$

subject to

$$\begin{aligned} c + a' &= a(1 + r(z, K) - \delta) + sw(z, K) && (178) \\ K' &= H(z, K) && (179) \end{aligned}$$

Since agent does not use all the information to forecast the economy,

$$\beta \sum_{s', z'} V(z', K', s', a'; H) \Gamma_{s'|szz'} \Pi_{z'|z} \neq E[V(z', K', s', a') | allinformation] \tag{180}$$

she is forecasting K' wrongly. In equilibrium, $K' \neq H(z, K)$. So what? This problem is a quantitative dent to it. In equilibrium,

$$K' = H(z, K) + \varepsilon(z, x)$$

where $\varepsilon(z, x)$ is the forecasting error from being stupid.

When is being such stupid is not important?

1. When $\varepsilon(z, x)$ is small. Say if the period of economy is one minute. Changes in capital is quite small within a minute.

2. If $\varepsilon(z, x)$ is irrelevant.

Let m be a set of moments of x, $m \in R^n$. That is

$$\begin{aligned} m_1 &= \int a dx = K \\ m_2 &= \int a^2 dx \\ &\cdots \\ &m_n \end{aligned}$$

m is information set about distribution measure x, but it only contains incomplete, finite amount of information. We want to see whether m is sufficient statistic for prices.

If agents use m in their optimization problem, we call them slightly stupid. The value function for slightly stupid agents is $V(z, m, s, a; H^n)$ such that

$$V(z, m, s, a; H^n) = \max_{c,a'} u(c) + \beta \sum_{s',z'} V(z', m', s', a'; H^n) \Gamma_{s'|szz'} \Pi_{z'|z} \tag{181}$$

subject to

$$\begin{aligned} c + a' &= a(1 + r(z, m) - \delta) + sw(z, m) & (182) \\ m' &= H^n(z, m) & (183) \end{aligned}$$

As $n \to \infty$, $H^n(m) \to x'$, as we can know the probability distribution from moment generating function.

Now, let's construct an economy where m is sufficient statistic for prices. We will define an equilibrium where everyone is moderately stupid. Practically speaking, we are happy with "a ε approximating equilibrium" for $m \in R^n$, such that in an economy where all agents use $m \in R^n$ moments to choose what they do.

If

$$h^n[z, m, s, a, ; H^n] \simeq h^{n+1}\left[z, m, s, a; H^{n+1}\right]$$

everyone use just n moments, then being smarter (using $n+1$ moments) does not make any difference.

Krusell and Smith (JPE 1998)[5] shows that the ε is very small. And the approximation $R^2 = 0.999992$ when agents only use the first one moment in making decision. Hence, it is fine to work with decision from n moments.

21.2 Economy with Private information

What is the most important (worst) thing we have done with the model up to the last class? We exogenously closed markets for state contingent loans and thus prevented exogenously the economy from collapsing to the representative agent economy. But the economists cannot choose what people can do and what they cannot do. From now, we do not do this. Instead, we will define the fundamental environment and assume more on what information agents have and what agents can see. We will look at two big classes for models. One is the economy with private information. In other words, there is asymmetric information or incomplete information in the mod. The second class is the models with lack of commitment. In the world without commitment, the contract among agents need to be self-enforceable. Otherwise, agents will just quit the contract and walk away.

21.3 Model on unemployment insurance[6]

Consider an economy where the probability of finding a job $p(a)$ is a function of effort $a \in [0,1]$. And we assume that once the agent gets a job, she will have wage w for ever. Thus, the individual problem is

$$\max_{a_t} E \sum_t \beta^t \left[u(c_t) - a_t\right]$$

There are two cases: when the agent has got a job, she will pay no effort and enjoy w for ever. The life long utility is

$$V^E = \sum_t \beta^t u(w) = \frac{u(w)}{1-\beta} \tag{184}$$

When the agent is still unemployed, she will have nothing to consumer. Her problem is

$$V^u = \max_a \left\{ u(0) - a + \beta \left[p(a) V^E + (1 - p(a) V^{u\prime}) \right] \right\} \tag{185}$$

[5]Krusell, Per and Smith, Anthony, Jr. (1998), "Income and Wealth Heterogeneity in the Macroeconomy", Journal of Political Economy, 106-5, 867-896.

[6]The source of this part is the updated Chapter 4 of Tom Sargent's Recursive Economic Theory.

Problem Prove that $V^u = V^{u\prime}$ under optimal decision.

If the optimal solution of a is interior, $a \in (0,1)$, then the first order condition gives

$$-1 + \beta p'(a)\left(V^E - V^u\right) = 0 \tag{186}$$

And since the V^u is stationary,

$$V^u = \max_a \left\{u(0) - a + \beta\left[p(a)V^E + (1 - p(a)V^u)\right]\right\} \tag{187}$$

Solving (186)(187) gives the optimal a and V^u. Another way is to successively substitute a and obtain solution because (**??**) defines a contraction mapping operator. We can fix V_0^u, then solve (187) to get $a(V_0^u)$ and obtain V_1^u. Keeping going until $V_n^u = V_{n+1}^u$. In a word, optimal effort level a^* solves (187) with $V^u = V^{u\prime}$.

The probability of finding a job $p(a)$ is called hazard rate. If agents did not find a job with effort level a^*, next period, she will still execute the same effort level a^*. Why? Because the duration of unemployment is not state variable in agent's problem. (If agents do not have enough realization about the difficulty of getting a job. With learning, their effort a will increase as they revise their assessment of the difficulty. But such revision of belief is not in this model.)

Now suppose resource is given to people who is unemployed to relive her suffering by a benevolent planner. This planner has to decide the minimal cost of warranting agent a utility level V: $c(V)$. To warrant utility level V, the planner tells the agent how much to consume, how much effort to exert and how much utility she will get if she stay unemployed next period. Obviously, the cost function $c(V)$ is increasing in V.

Problem Show that c(V) is strictly convex.

The cost minimization problem of the planner can be written in the following recursive problem:

$$c(V) = \min_{c,a,V^u} c + [1 - p(a)]\frac{1}{1+r}c(V^u) \tag{188}$$

subject to

$$V = u(c) - a + \beta\left[p(a)V^E + (1 - p(a))V^u\right] \tag{189}$$

To solve the problem, construct Lagragian function

$$\mathcal{L} = -c - [1 - p(a)]\beta c(V^u) - \theta\left[V - u(c) + a - \beta\left[p(a)V^E + (1 - p(a))V^u\right]\right]$$

FOC: (c)

$$\theta = \frac{1}{u_c} \tag{190}$$

(a)

$$c\left(V^{u}\right)=\theta\left[\frac{1}{\beta p'(a)}-\left(V^{E}-V^{u}\right)\right] \tag{191}$$

(V^{u})

$$c'\left(V^{u}\right)=\theta \tag{192}$$

Envelope condition

$$c'(V)=\theta \tag{193}$$

We will work on some implication of these conditions:

1. Compare (191) and (186), we can see that the substitution between consumption and effort is different from the one in agent's problem without unemployment insurance. This is because the cost of effort is higher for work that it is from the viewpoint of planner.

2. (192) tells us that the marginal cost of warranting an extra unit of utility tomorrow is θ.,provided that tomorrow V^{u} is optimally chosen when today's promise is V. And (193) tells us that the marginal cost of warranting an extra unit of V today is θ.

3. Given that c is strictly concave, $V=V^{u}$.

4. Regardless of unemployment duration, $V=V^{u}$. So, effort required the the planner is the same over time. Hazard rate is still constant.

Problem Work out the model and derive the implication on your own.

Next class, we will study the case when effort is not observable. Planner can only choose consumption and V^{u}. Effort level is chosen optimally by work and it is unobservable.

22 April 16

22.1 Unemployment insurance(II)

Review of last class: unemployed agents can find a job with probability $p(a)$ and once they get the job, they can get w for ever. Without unemployment insurance, their problem yields

$$V^{E}=\sum_{t}\beta^{t}u(w)=\frac{u(w)}{1-\beta} \tag{194}$$

$$V^A = \max_a \left\{ u(0) - a + \beta \left[p(a) V^E + \left(1 - p(a) V^A \right) \right] \right\} \tag{195}$$

where we denote the utility of not finding a job as V^A, which comes from the situation when there is no unemployment insurance and people basically stay Autarky.

First order condition gives

$$-1 + \beta p'(a) \left(V^E - V^A \right) = 0 \tag{196}$$

And we know that effort level a^* does not change over time.

Now suppose there is a social planner who will warrant utility level V for unemployed agent, where V summarize all the past information. The cost minimization problem is

$$c(V) = \min_{c,a,V^u} c + [1 - p(a)] \beta c(V^u) \tag{197}$$

subject to

$$V = u(c) - a + \beta \left[p(a) V^E + (1 - p(a)) V^u \right] \tag{198}$$

FOC: (c)

$$\theta = \frac{1}{u_c} \tag{199}$$

(a)

$$c(V^u) = \theta \left[\frac{1}{\beta p'(a)} - \left(V^E - V^u \right) \right] \tag{200}$$

(V^u)

$$c'(V^u) = \theta \tag{201}$$

Envelope condition

$$c'(V) = \theta \tag{202}$$

The optimal promise for tomorrow is $V^u(V)$. Now, let's work out the property of $V^u(.)$.

Lemma 22.1 *If $V > V^A$, then $c(V) > 0$, where V^A is the utility for unemployed agent when they are in autarky.*

Problem Prove the above lemma.

The intuition for the lemma is that if the planner promises the agent something more than what agent can achieve by herself, it will cost the planner something because the planner cannot do anything more than what people can do on their own.

Lemma 22.2 *Lagrangian multiplier $\theta > 0$.*

The second lemma tells us that if the planner promise more, she has to pay more.

In the problem without unemployment insurance, (196) implies that

$$\frac{1}{\beta p'(a)} = V^E - V^u$$

In the planner's problem, since $c(V) > 0$, (200) implies that the effort level chosen by the planner are different from agent's choice in autarky. The reason is that effort does not cost that much in planner's thought.

22.2 Unobservable effort

When a is not observable, planner can only choose c and V^u. And households choose a optimally. Now it becomes a principle-agent problem. We will solve the problem backward.

If given c and V^u, the agent will solve

$$\max_a u(c) - a + \beta\left[p(a)V^E + (1 - p(a))V^u\right] \tag{203}$$

FOC is

$$[p'(a)\beta]^{-1} = V^E - V^u \tag{204}$$

This FOC gives an implicit function of a as a function of V^u: $a = g(V^u)$. (Because c and a are separate in the utility function, a is not a function of c).

Then, the planner solve her cost minimization problem, in which the optimality condition is also one constraint.

$$c(V) = \min_{c,a,V^u} c + [1 - p(a)]\beta c(V^u)$$

subject to

$$\begin{aligned} V &= u(c) - a + \beta \left[p(a) V^E + (1 - p(a)) V^u \right] && (205) \\ 1 &= \left[p'(a) \beta \right] \left[V^E - V^u \right] && (206) \end{aligned}$$

Lagragian is

$$c + [1 - p(a)] \beta c(V^u) + \theta \left[V - u(c) + a - \beta \left[p(a) V^E + (1 - p(a)) V^u \right] \right]$$
$$+ \eta \left[1 - \left[p'(a) \beta \right] \left[V^E - V^u \right] \right]$$

FOC: (c)

$$\theta^{-1} = u_c$$

(a)

$$c(V^u) = \theta \left[\frac{1}{\beta p'(a)} - \left(V^E - V^u \right) \right] - \eta \frac{p''(a)}{p'(a)} \left(V^E - V^u \right) \tag{207}$$

(V^u)

$$c'(V^u) = \theta - \eta \frac{p'(a)}{1 - p(a)} \tag{208}$$

Envelope condition

$$c'(V) = \theta \tag{209}$$

Problem Show that in this case, the effort level that household exerts is less than that the planner wants her to exert when effort is observable.

Again, (208) tells the marginal cost to warrant additional amount of delayed promise. (209) gives the marginal cost to increase today's utility. The Lagrangian multiplier associated with constraint (206) is positive, $\eta > 0$, which means that the constraint is binding. So,

$$\eta \frac{p'(a)}{1 - p(a)} > 0$$

Therefore, we have

$$c'(V^u) < c'(V) \Rightarrow V^u < V$$

from the strict concavity of $c(.)$. The delayed promised utility decreases over time.

Let $\theta^u = \theta - \eta \frac{p'(a)}{1-p(a)}$, then $\theta^u < \theta$,which tells us about the consumption path. Consumption decreases over time because $\theta^{-1} = u_c$.

Problem Prove that c_t is a decreasing when people are unemployed.

How about effort level?

Problem Show that a_t is increasing over time when people are unemployed.

Overall, we get the following model implications: optimal unemployment insurance says that longer unemployment period the agent stays, the less insurance she will be insured for. In this way, the planner induces the higher effort level. Although you cannot let people do what is optimal, such behavior can be achieved by giving out less consumption and promised utility over time. This model implies that time-varying unemployment insurance plan is optimal, under which the replacement rate θ goes down over time.

Problem Show that optimal time-invariant unemployment insurance is worse. (show that it is more expensive the provide the same amount of promised utility with time-invariant scheme.)

22.3 One side lack of commitment[7]

We will study a model with one-said lack of commitment. This is an endowment economy (no production). There is no storage technology. Consider the village of fisherladies, where young granddaughters receive $y_s \in \{y_1, y_2, ..., y_S\}$ every period. y is iid. The probability that certain y_s realizes is Π_s. h_t is a history of shocks up to period t, i.e. $h_t = \{y_0,\ y_1,\ y_2,\ ...,\ y_t\}$.

First, if the granddaughter stays autarky, she will solve the problem

$$V_{AUT} = \sum_{t=0}^{\infty} \beta^t \sum_s \Pi_s u(y_s) = \frac{\sum_s \Pi_s u(y_s)}{1-\beta}$$

Note that here V^A is the utility of the young lady before endowment shock realizes.

Now we assume that the grandmother offers a contract to the granddaughter, which transfer resources and provide insurance to her. Grandmother is subject to commitment. But the young granddaughter may leave grandmother and break her word. Thus, this model is one-sided commitment model: an agent can walk away from a contract but the other

[7]The source of this part is the updated Chapter 15 of Tom Sargent's Recursive Economic Theory.

cannot. Therefore, the contract should be always in the interest of granddaughter for her to stay.

We define a contract $f_t : H_t \to c \in [0, \tau]$. We will see next class that incentives compatibility constraint requires that at each node of history H_t, the contract should guarantee a utility which is higher than that in autarky.

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Last class, we have seen that first best result is not achievable for some environment. The unemployment insurance example is a typical principle-agent problem. Principle chooses first and agents choose next. The principle has to take agents' decision rule as given, but decision. And people can affect other's behavior, but not behavior rule. As described in chapter 4, decision rule will change over time as the intertemporal effect. We need understand the optimality condition and envelope condition to master the nature of optimal policy. From now on, we will study the key material in chapter 15, the model with lack of commitment.

23.1 One side lack of commitment(II)

The endowment shock is iid. If the granddaughter stays autarky, she will solve the problem

$$V_{AUT} = \sum_{t=0}^{\infty} \beta^t \sum_s \Pi_s u(y_s) = \frac{\sum_s \Pi_s u(y_s)}{1-\beta}$$

Notice that the problem is different from Lucas tree model because of the shock realization timing. In Lucas tree model, shock is state variable because action takes place after shock is realized. Thus, action is indexed by shock. Here action is chosen before shock realization. Therefore, shock is not a state variable and action is state contingent.

In Lucas tree model, $V(s) = \max_c u(c) + \beta \sum_{s'} \Pi_{ss'} V(s')$. Here, if we write the problem recursively, it is $V = \max_{c_s} \sum_s \Pi_s u(c_s) + \beta V$.

Remember, the grandmother will make a deal with her granddaughter. They sign a contract to specify what to do in each state. $h_t \in H_t$. Contract is thus a mapping $f_t(h_t) \to c(h_t)$. With this contract, granddaughter gives y_t to the grandmother and receives $c_t = f_t(h_{t-1}, y_t)$. But if the granddaughter decided not to observe the contract, she consumes y_t this period and cannot enter a contract in the future, i.e. she has to live in autarky in the future.

For grandmother to keep granddaughter around her, the contract has to be of interest to granddaughter because although grandmother keeps her promise, granddaughter does not.

There are two possible outcome if this contract is broken. One is that granddaughter goes away with current and future endowment. The other is that they renegotiate. We ignore the second possibility as no renegotiation is allowed. But we need deal with the possibility that the granddaughter says no to the contract and steps away.

The first best outcome is to warrant a constant consumption c_t to granddaughter who is risk averse. But because of the one-side lack of commitment, the first best is not achievable. The contract should always be attractive to granddaughter, otherwise, when she gets lucky with high endowment y_s, she will feel like to leave. So, this is a dynamic contract problem which the grandmother will solve in order to induce good behavior from granddaughter. The contract is dynamic because the nature keeps moving.

We say the contract $f_t(h_t)$ is incentive compatible or satisfies participation constraint if for all h_t,

$$u(f_t(h_t)) + \sum_{\tau=1}^{\infty} \beta^{\tau} \sum_s \Pi_s u(f_{t+\tau}(h_{t+\tau})) \geq u(y_s(h_t)) + \beta V^A \tag{210}$$

The left hand side is utility guaranteed in the contract. And the right hand side is the utility that granddaughter can get by herself. The participation constraint is not binding if y_s is low. And when y_s is high, PC is binding.

23.2 Problem of the grandmother

In this model, problem of the grandmother is to find an optimal contract that maximizes the value of such a contract of warranting V to her. We define the problem using recursive formula. Firstly, let's define the value of contract to grandmother if she promised V to her granddaughter by $P(V)$. $P(V)$ can be defined recursively as the following:

$$P(V) = \max_{\{c_s,\omega_s\}_{s=1}^{S}} \sum_s \Pi_s[(y_s - c_s) + \beta P(\omega_s)] \tag{211}$$

subject to

$$u(c_s) + \beta\omega_s \geq u(y_s) + \beta V^A \quad \forall s \tag{212}$$

$$\sum_s \Pi_s[u(c_s) + \beta\omega_s] \geq V \tag{213}$$

Notice that there are $1 + S$ constraints. The choice variables c_s, ω_s are state-contingent where ω_s is the promised utility committed to granddaughter in each state. In the objective function, $\sum_s \Pi_s(y_s - c_s)$ is the expected value of net transfer.

There are two sets of constraints. (212) is PC and (213) is promise keeping constraint.

23.3 Characterization of the Optimal Contract

In order to characterize the optimal contract, construct a Lagrangian.

$$P(V) = \max_{\{c_s,\omega_s,\lambda_s\}_{s=1}^S,\mu} \sum_s \Pi_s[(y_s - c_s) + \beta P(\omega_s)] \tag{214}$$

$$+\mu\left[\sum_s \Pi_s[u(c_s) + \beta\omega_s] - V\right] + \sum_s \lambda_s \left[u(c_s) + \beta\omega_s - u(y_s) - \beta V^A\right] \tag{215}$$

First order conditions are the followings:

(c_s)

$$\Pi_s = (\lambda_s + \mu\Pi_s)u'(c_s) \tag{216}$$

(ω_s)

$$-\Pi_s P'(\omega_s) = \mu\Pi_s + \lambda_s \tag{217}$$

(μ)

$$\sum_s \Pi_s[u(c_s) + \beta\omega_s] = V \tag{218}$$

(λ)

$$u(c_s) + \beta\omega_s \geq u(y_s) + \beta V_{AUT} \tag{219}$$

In addition, Envelope Theorem tells that:

$$P'(v) = -\mu \tag{220}$$

Interpret the first order conditions:

1. (216) tells that in an optimal choice of c_s, the benefit of increasing one unit of c equals the cost of doing so. The benefit comes from two parts: first is $\mu\Pi_s u'(c_s)$ as increasing consumption helps grandmother to fulfill her promise and the second part is $\lambda_s u'(c_s)$ since increase in consumption helps alleviated the participation constraint. And the cost is the probability of state s occurs.

2. (217) equates the cost of increasing one unit of promised utility and the benefit. The cost to grandmother is $-\Pi_s P'(\omega_s)$ and the benefit is $\mu\Pi_s + \lambda_s$ which helps grandmother deliver promise and alleviate participation constraint.

Problem Prove the Envelope Condition.

How about the contract value $P(V)$. First, $P(V)$ can be positive or negative.

Claim (1) There exits V such that P(V) > 04 . (2) There exits V such that P(V) > 0

Problem Prove the above claim is true.

What's the largest V we will be concerned with? When PC will be binding for sure. If PC binds for the best endowment shock y_S, then PC holds for all the shock y_s. When granddaughter gets the best shock y_S, the best autarky value is then

$$V_{AM} = u(y_S) + \beta V_A$$

And the cheapest way to guarantee V_{AM} is to give constant consumption $\overline{c_S}$, such that

$$V_{AM} = \frac{u(c_S)}{1-\beta}$$

From this case, we can see that because of lack of commitment, the grandmother will have to give more consumption in some states. While when there is no lack of commitment, strict concavity of $u(.)$ implies that constant stream of consumption beats any $\{c_t\}$ that have the same present value, as there is no PC.

Problem Show $\overline{c_S} < y_s$

23.4 Characterizing the Optimal Contract

We will characterize the optimal contract by considering the two cases: (i) $\lambda_s > 0$ and (ii) $\lambda_s = 0$.

Firstly, if $\lambda_s = 0$, we have the following equations from FOC and EC:

$$P'(\omega_s) = -\mu \tag{221}$$
$$P'(V) = -\mu \tag{222}$$

Therefore, for s where PC is not binding,

$$V = \omega_s$$

c_s is the same for all s. For all s such that the Participation Constraint is not binding, the grandmother offers the same consumption and promised future value.

Let's consider the second case, where $\lambda_s > 0$. In this case, the equations that characterize the optimal contract are:

$$u'(c_s) = \frac{-1}{P'(\omega_s)} \tag{223}$$
$$u(c_s) + \beta\omega_s = u(y_s) + \beta V^A \tag{224}$$

Note that this is a system of two equations with two unknowns (c_s and ω_s). So these two equations characterize the optimal contract in case $\lambda_s > 0$. In addition, we can find the following properties by carefully observing the equations:

1. The equations don't depend on V. Therefore, if a Participation Constraint is binding, promised value does not matter for the optimal contract.

2. From the first order condition with respect to ω_s, $P'(\omega_s) = P'(v) - \frac{\lambda_s}{\Pi_s}$, where $\frac{\lambda_s}{\Pi_s}$ is positive. Besides, we know that P is concave. This means that $v < \omega_s$. In words, if a Participation Constraint is binding, the moneylender promises more than before for future.

Combining all the results we have got, we can characterize the optimal contract as follows:

1. Let's fix V_0. We can find a $y_s(V_0)$, where for $\forall y_s \leq y_s(V_0)$, the participation constraint is not binding. And vice versa.

2. The optimal contract that the moneylender offers to an agent is the following:

   If $y_t \leq y_s(v_0)$, the moneylender gives $(v_0, c(v_0))$. Both of them are the same as in the previous period. In other words, the moneylender offers the agent the same insurance scheme as before.

   If $y_t > y_s(v_0)$, the moneylender gives $(v_1, c(y_s))$, where $v_1 > v_0$ and c doesn't depend on v_0. In other words, the moneylender promises larger value to the agent to keep her around.

   So the path of consumption and promised value for an agent is increasing with steps.

Problem Show heuristically that average duration of increases in wellbeing of granddaughter gets longer over time.

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24.1 One Sided Lack of Commitment (II) (continued)

Last class we showed that the promise keeping constraint is always binding but that the participation constraint may or may not be binding. For shocks that are above a certain value, the participation constraint will be binding and for shocks that are less, it will not be binding.

Homework. *Show that the following holds:*

Participation constraint is not binding for $\widetilde{y}_s \Rightarrow$ *Participation constraint is not binding* $\forall y_s$ *such that* $y_s \leq \widetilde{y}_s$

Participation constraint is binding for $\widetilde{y}_s \Rightarrow$ *Participation constraint is binding* $\forall y_s$ *such that* $y_s \geq \widetilde{y}_s$

Cross sectional analysis with lots of granddaugthers:

First Period:

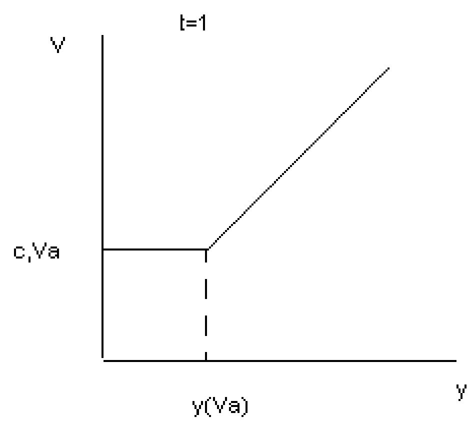


Figure 3:

The grandmother starts out by guaranteeing all of them the autarky value, V^A. But then once the shocks are realized, things will change. The following is what happens to the granddaughters according to the shocks they get:

- The granddaugthers who get bad shocks (those who get shocks less than the critical value y$^*(V^A)$):

  The grandmother will give them what they were promised, so that the unlucky ones will remain with promised utility V^A. For the unlucky granddaugthers, the grandmother does not need to increase their promised utility because their outside opportunity is not better than the deal she's already offering them.

- The granddaughters who get good shocks (those whoe get shocks above the critical value y$^*(V^A)$) :

  The grandmother will have to increase their promised utility. Otherwise, the granddaughters will not be willing to stay since their outside opportunity (u(y$_s$) + βV^A)

is better than the deal she's offering them. Therefore, the grandmother will have to give them just enough promised utility ω_s and consumption c_s such that it gives the granddaughters what they would get if they left. This promised utility should therefore satisfy:

$$u(y_s) + \beta V^A = u(c_s) + \beta\omega_s$$

Note that $y^*(V^A)$ is the lowest endowment value such that the granddaughters are willing to stay with c_s and V^A. Any shocks to endowment that are higher than this critical value will make the granddaughthers willing to go so that their promised utility needs to be increased to keep them around.

Second period:

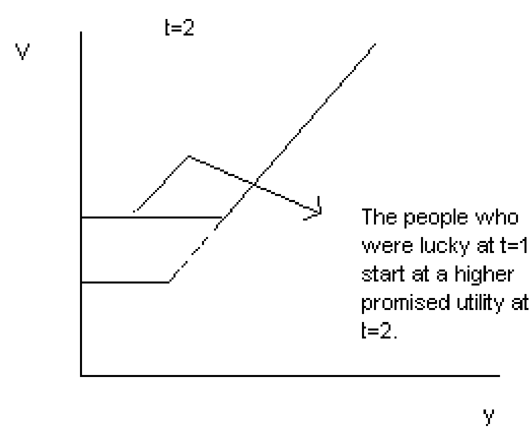


Figure 4:

- The granddaughters who were lucky in the previous period:

  They will all start at a higher promised utility. A fraction of them who are lucky again in the second period will get higher promised utilities and the remaining unlucky ones will have the same promised utility as in the previous period.

- The granddaugthers who were unlucky in the previous period:

  They will all start at V^A again. A fraction of them who are unlucky again this period will get V^A again and the remaining who are lucky this period will get a higher promised utility.

You can see the pattern here: The average consumption over these two periods will go up because the grandmother gives more when the granddaughters get lucky. So as time passes, things are getting worse for the grandmother because she keeps having to promise more. On the other hand, over time, the granddaughters are doing better (As long as they had one good shock in the past, their promised utility is higher)

The grandmother is willing to sign anything with P(V)$\geq$ 0. When she offers only the autarky value, with P(V^A) she gets all the gains from trade.

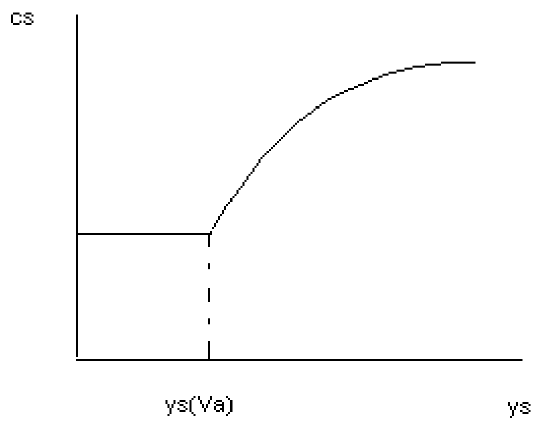


Figure 5:

In the above graph, you can see that the consumption stays the same until the critical value, y$_s^*(V^A)$. However, once the granddaughter gets a shock higher than that the grandmother has to increase the granddaughter's consumption. But an important thing to notice here is that the consumption is growing at a slower rate than endowment. To see why this is true, recall that u is concave so that the marginal utility of the consumer is decreasing. Therefore, for a certain increase in y$_s$, c$_s$ will increase by less because the promised utility ω_s is also increasing (recall the equation $u(y_s) + \beta V^A = u(c_s) + \beta \omega_s$). A "stupid" thing to do for the grandmother would be to set c$_s = y_s$. It would be "stupid", because she can give less now and more in the future; so she can manage to keep the granddaughter around by offering some consumption less than y$_s$.

In the beginning the grandmother gets a lot by guaranteeing the granddaughter a steady flow of utility, but then later she starts giving. This is because as time passes by, what she needs to promise increases because the granddaughter has good shocks (each shock increases the level of promised utility for good). In other words, the grandmother is at first a net receiver and then a net giver.

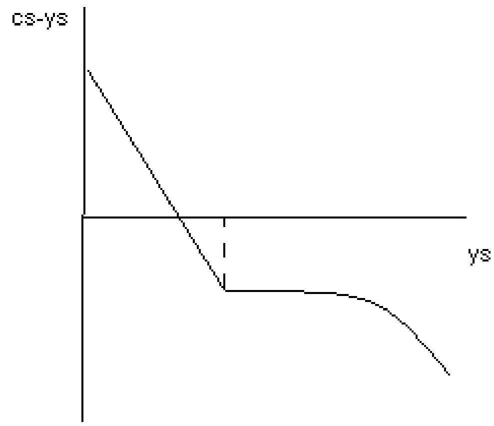


Figure 6:

24.2 Economy with Two Sided Lack of Commitment

24.2.1 The Model

- Two brothers, A and B, and neither of them has access to a commitment technology. In other words, the two can sign a contract, but either of them can walk away if he does not feel like observing it.
- This is an endowment economy (no production) and there is no storage technology. Endowment is represented by $(y_s^A, y_s^B) \in Y \times Y$, where y_s^i is the endowment of brother i. s=(y_s^A, y_s^B) follows a Markov process with transition matrix $\Gamma_{ss'}$.

24.2.2 First Best Allocation

We will derive the first best allocation by solving the social planner's problem:

$$\max_{\{c_i(h_t)\}_{\forall h_t, \forall i}} \lambda^A \sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t) u(c^A(h_t)) + \lambda^B \sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t) u(c^B(h_t))$$

subject to the resource constraint:

$$\sum_i c^i(h_t) - y^i(h_t) = 0 \qquad \forall h_t \quad \text{w/ multiplier } \gamma(h_t)$$

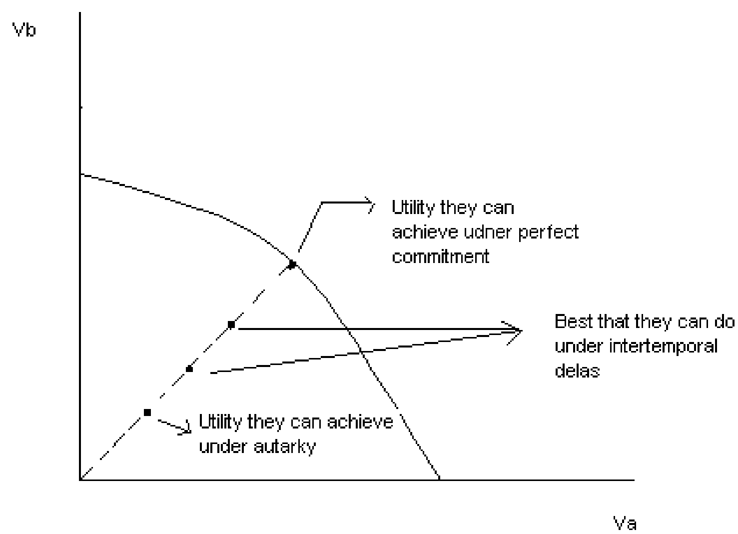
The First Order Conditions are:

$$\begin{aligned} FOC(c^A(h_t)) &: \lambda^A \beta^t \Pi(h_t) u'(c^A(h_t)) - \gamma(h_t) = 0 \\ FOC(c^B(h_t)) &: \lambda^B \beta^t \Pi(h_t) u'(c^B(h_t)) - \gamma(h_t) = 0 \end{aligned}$$

Combining these two yields:

$$\frac{\lambda^A}{\lambda^B} = \frac{u'(c^A(h_t))}{u'(c^B(h_t))}$$

Homework. *Show an implication of the above First Order Conditions under CRRA.*



The first best allocation will not be achieved if there is no access to a commitment technology. Therefore, the next thing we should do is look at the problem the planner is faced with in the case of lack of commitment. Due to lack of commitment, the planner needs to make sure that at each point in time and in every state of the world, h$_t$, both brothers prefer what they receive to autarky. Now we will construct the problem of the planner adding these participation constraints to his problem.

24.2.3 Constrained Optimal Allocation

The planner's problem is:

$$\max_{c^A(h_t),c^B(h_t)} \lambda^A \sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t)u(c^A(h_t)) + \lambda^B \sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t)u(c^B(h_t))$$

$$\sum_i c^i(h_t) - y^i(h_t) = 0 \qquad \forall h_t \quad \text{w/ multiplier } \gamma(h_t)$$

$$\sum_{r=t}^{\infty} \beta^{r-t} \sum_{h_r} \Pi(h_r|h_t)u(c^i(h_r)) \geq \Omega_i(h_t) \qquad \forall h_t, \forall i \quad \text{w/ multiplier } \mu_i(h_t)$$

where $\Omega_i(h_t) = \sum_{r=0}^{\infty} \beta^{r-t} \sum_{h_r} \Pi(h_r|h_t)u(y_i(h_t))$ (the autarky value)

- How many times does $c^A(h_{17})$ appear in this problem? Once in the objective function, once in the feasibility constraint, and it appears in the participation constraint from period 0 to period 16.
- We know that the feasibility constraint is always binding so that $\gamma(h_t) > 0 \ \ \forall h_t$. On the other hand the same is not true for the participation constraint.
- Both participations cannot be binding but both can be nonbinding.
- Define $M_i(h_{-1}) = \lambda^i$
  and $M_i(h_t) = \mu_i(h_t) + M_i(h_{t-1})$
  (We will use these definitions for the recursive representation of the problem in the next class)

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25.1 Recursive Representation of the Constrained SPP

We want to transform this problem into the recursive, because it would be easier to solve the optimal allocation with a computer. Now we will show how to transform the sequential problem with the participation constraints into its recursive representation.

Before we do this transformation, first recall the Lagrangian associated with the sequential representation of the social planner's problem:

$$\lambda^A \sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t) u(c^A(h_t)) + \lambda^B \sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t) u(c^B(h_t))$$
$$+ \sum_{t=0}^{\infty} \beta^t \sum_{h_t \in H_t} \Pi(h_t) \sum_{i=1}^{2} \mu_i(h_t) \left[\sum_{r=t}^{\infty} \beta^{r-t} \sum_{h_r} \Pi(h_r|h_t) u(c^i(h_r)) - \Omega_i(h_t) \right]$$
$$+ \sum_{t=0}^{\infty} \sum_{h_t \in H_t} \gamma(h_t) \left[\sum_{i=1}^{2} c_i(h_t) - \sum_{i=1}^{2} y_i(h_t) \right]$$

Note that here the Lagrangian multiplier associated witih the participation constraint for brother i after history h_t is $\beta^t \Pi(h_t) \mu_i(h_t)$.

Now we will use the definitions from the previous class (for $M_i(h_t)$) to rewrite the above Lagrangian in a more simple form,

Collect terms and rewrite,

$$\sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t) \sum_{i} \left\{ \lambda^i u(c^i(h_t) + \mu_i(h_t) \left[\sum_{r=t}^{\infty} \beta^{r-t} \sum_{h_r} \Pi(h_r|h_t) u(c^i(h_r)) - \Omega_i(h_t) \right] \right\}$$
$$+ \sum_{t=0}^{\infty} \sum_{h_t \in H_t} \gamma(h_t) \left[\sum_{i=1}^{2} c_i(h_t) - \sum_{i=1}^{2} y_i(h_t) \right]$$

Note that, $\sum_{r=t}^{\infty} \beta^{r-t} \sum_{h_r} \Pi(h_r|h_t) u(c^i(h_r)) - \Omega_i(h_t) = u(c^i(h_t)) + \sum_{r=t+1}^{\infty} \beta^{r-t} \sum_{h_r} \Pi(h_r|h_t) u(c^i(h_r)) - \Omega_i(h_t)$,

and that $\Pi(h_r|h_t)\Pi(h_t) = \Pi(h_r)$ so using these, rewrite as,

$$\sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t) \sum_{i} \left\{ \lambda^i u(c^i(h_t) + \mu_i(h_t) u(c^i(h_t)) \right\}$$
$$+ \sum_{t=0}^{\infty} \sum_{h_r} \sum_{i} \mu_i(h_t) \left[\sum_{r=t+1}^{\infty} \beta^r \sum_{h_r} \Pi(h_r) u(c^i(h_r)) - \Omega_i(h_t) \right]$$
$$+ \sum_{t=0}^{\infty} \sum_{h_t \in H_t} \gamma(h_t) \left[\sum_{i=1}^{2} c_i(h_t) - \sum_{i=1}^{2} y_i(h_t) \right]$$

Collect the terms of $u(c^i(h_r)$,

$$\sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t) \sum_i \left\{ \left[\lambda^i + \sum_{r=0}^{t-1} \mu_i(h_r) \right] u(c^i(h_t) + \mu_i(h_t) \left[u(c^i(h_t) - \Omega_i(h_t) \right] \right\}$$
$$+ \sum_{t=0}^{\infty} \sum_{h_t \in H_t} \gamma(h_t) \left[\sum_{i=1}^{2} c_i(h_t) - \sum_{i=1}^{2} y_i(h_t) \right]$$

Introduce the variable $\mathrm{M}_i(h_t)$ and define it recursively as,

$$\begin{aligned} M_i(h_t) &= M_i(h_{t-1}) + \mu_i(h_t) \\ M_i(h_{-1}) &= \lambda^i \end{aligned}$$

where $\mathrm{M}_i(h_t)$ denotes the Pareto weight plus the cumulative sum of the Lagrange multipliers on the participation constraints at all periods from 1 to t.

So rewrite the Lagrangian once again as,

$$\sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t) \sum_i \left\{ M_i(h_{t-1}) u(c^i(h_t) + \mu_i(h_t) \left[u(c^i(h_t) - \Omega_i(h_t) \right] \right\}$$
$$+ \sum_{t=0}^{\infty} \sum_{h_t \in H_t} \gamma(h_t) \left[\sum_{i=1}^{2} c_i(h_t) - \sum_{i=1}^{2} y_i(h_t) \right]$$

Now we are ready to take the First Order Conditions:

$$\frac{u'(c^A(h_t))}{u'(c^B(h_t))} = \frac{M_A(h_{t-1}) + \mu_A(h_t)}{M_B(h_{t-1}) + \mu_B(h_t)}$$

$$\left[\sum_{r=t}^{\infty} \beta^{r-t} \sum_{h_r} \frac{\Pi(h_r)}{\Pi(h_t)} u(c^i(h_r)) - \Omega_i(h_t) \right] \mu_i(h_t) = 0$$

$$\sum_{i=1}^{2} c_i(h_t) - \sum_{i=1}^{2} y_i(h_t) = 0$$

Now let's characterize the solution using log utility:

Here x=$\frac{M_B}{M_A}$ so that x denotes the weight on person B. The above graph summarizes what happens to the participation constraints of each brother according to how lucky they get. You can see that, what happens to one brother affects the other, this is because if one brother gets lucky the planner needs to give him more and thus increasing his relative weight with respect to the other brother which translates into a worse deal for the other brother.

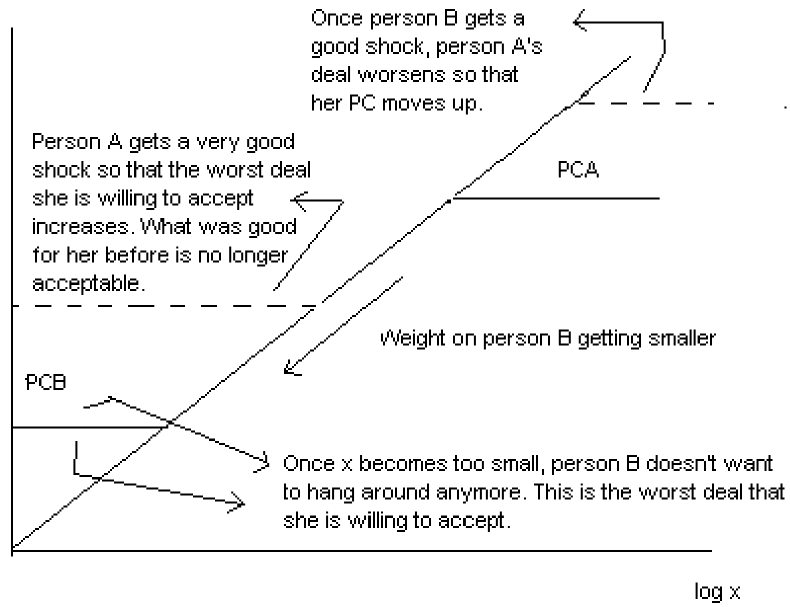

Figure 7:

There are two kinds of periods here: Periods where nothing happens so that the ratio of the consumption of the brothers stay the same, and periods where one of them gets a good shock and his deal gets bettter whereas for the other it worsens.

Now let's look at the Pareto frontier and analyze where the solution of this problem lies:

25.2 Recursive Formulation

Our goal is make the problem recursive, which is very nice when we work with computer. To do this, we need to find a set of state variables which is sufficient to describe the state of the world. We are going to use x as a state variable.So the state variables are the endowment: $y = (y^A, y^B)$ and weight to brother 2: x. Define the value function as follows:

$$V = \{(V_0, V_A, V_B) \text{ such that } V_i : X \times Y \to \mathcal{R},\ i = 1, 2, V_0(x, y) = V_A(x, y) + xV_B(x, y)\}$$

What we are going find is the fixed point of the following operator (operation is defined later):

$$T(V) = \{T_0(V), T_1(V), T_2(V)\}$$

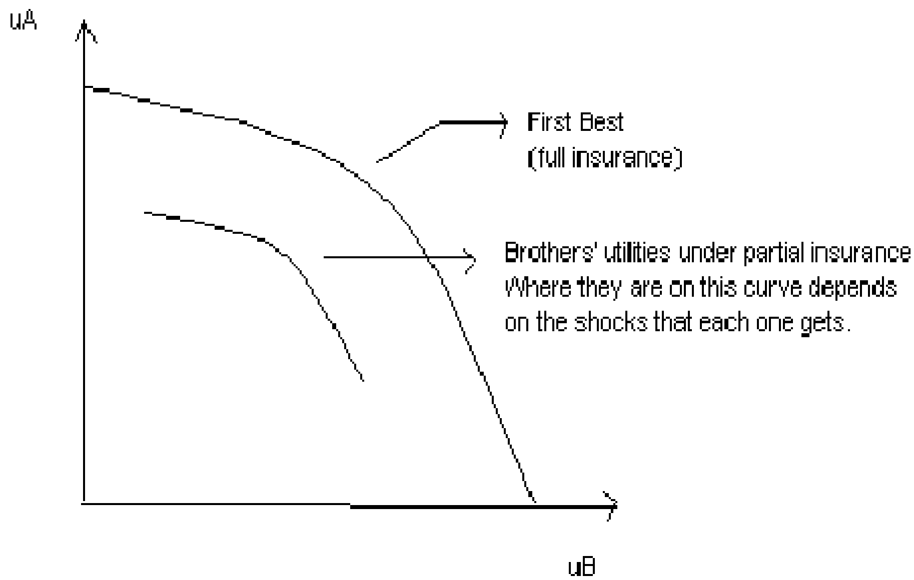

Figure 8:

Firstly, we will ignore the participation constraints and solve the problem:

$$\max_{c_A, c_B} u(c^A(y,x)) + xu(c^B(y,x)) + \beta \sum_{y'} \Gamma_{yy'} V_0(y', x)$$

subject to

$$c^A + c^B = y^A + y^B$$

First Order Conditions yield:

$$\frac{u'(c_A)}{u'(c_B)} = x$$

Second, we will check the participation constraints. There are two possibilities here:

1. Participation constraint is not binding for either 1 or 2. Then set $x(h_t) = x(h_{t-1})$. In

addition,

$$\begin{aligned} V_0^N(y,x) &= V_0(y,x) \\ V_i^N(y,x) &= u(c^i(y,x)) + \beta \sum_{y'} \Gamma_{yy'} V_i(y',x) \end{aligned}$$

2. Participation constraint is not satisfied for one of the brothers (say A).

This means that agent A is getting too little. Therefore, in order for the planner to match the outside opportunity that A has, he needs to change x so that he guarantees person A the utility from going away. We need to solve the following system of equations in this case:

$$\begin{aligned} c^A + c^B &= y^A + y^B \\ u(c^A) + \beta \sum_{y'} \Gamma_{yy'} V_A(y',x) &= u(y_A) + \beta \sum_{y'} \Gamma_{yy'} \Omega_A(y') \\ x' &= \frac{u'(c_A)}{u'(c_B)} \end{aligned}$$

This is a system of three equations and three unknowns. Denote the solution to this problem by,

$$\begin{aligned} &c^A(y,x) \\ &c^B(y,x) \\ &x'(y,x) \end{aligned}$$

So that,

$$\begin{aligned} V_0^N(y,x) &= V_A^N(y,x) + x V_B^N(y,x) \\ V_i^N(y,x) &= u(c^i(y,x)) + \beta \sum_{y'} \Gamma_{yy'} V_i(y',x'(y,x)) \end{aligned}$$

Thus we have obtained T(V)=V^N. And the next thing we need to do is find V* such that T(V*) $= V^*$.

Final question with this model is "how to implement this allocation?" or "Is there any equilibrium that supports this allocation?". The answer is yes. How? Think of this model as a repeated game. And define the strategy as follows: keep accepting the contract characterized here until the other guy walks away. If the other guy walks away, go to autarky forever. We can construct a Nash equilibrium by assigning this strategy to both of the brothers.

25.3 Some words on Economics

We use a model for answering two types of questions:

1. What accounts for........ (For example, why do we have GDP fluctuations, what accounts for these fluctuations? Possible explanations are that people are moody which is modelled with including shocks to preferences, or that nature is moody, or that the government is moody, etc.)
2. What if.........

What economists do is compute the model and compare that with data. But the important thing is to notice that whether the data generated from the model matches perfectly with the real data is not relevant. We are interested in which dimensions the data and the model generated data match and don't match with each other.

What is econometrics?

1. Descriptive Statistics
2. Estimation, $\widehat{\theta}$

   What econometricians do is look for parameters that will generate data that is close to real. But they assume that such parameters actually exist!
3. Truth

   Hypothesis Testing. The downside of the concept of hypothesis testing is that, it tests whether the model should be rejected or not. But all models are false. Whether they are the perfect match to the real or not is not relevant. What is relevant is whether they are useful or not and whether we are able to answer questions such as what is the impact of increasing social security benefits is good or not, etc.

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26.1 Time Inconsistency

The problem of time inconsistency of an optimal policy is the following: The best thing to do at period t+1 when decided at period t is not the same as the best thing to do in period t+1 when decided at t+1. Recall the smoking example. You decide that your next cigarette will be your last one and that you will never smoke again. You are making this decision not anticipating the fact that after that supposedly last next cigarette, you'll want another one. Quitting after the next cigarette might be the optimal decision for you, but once that next cigarette is finished and gone and it's time to forget about smoking once and for all, you no longer want to go ahead with what you initially planned. An example that is more relevant why we economists care about time inconsistency is the following: Think of a government that has access to a commitment technology. Due to the Chamley & Judd result, we know that the optimal tax policy is to tax capital very high in the beginning and then never to tax it again. This is in order to minimize the distortionary effects that cpaital taxation has. Now think about what would happen if this government no longer had access to a commitment technology. The optimal plan which is to tax capital high in the beginning and never again after that is not time consistent in this case. This is because at the first period, the government will indeed tax capital high as prescribed by the optimal plan but then once we are in the later periods, the government will want to tax capital instead of following what its plan was. In other words, if the government is allowed to change its optimal behaviour and optimize at any history, it won't follow the initial plan. This is a problem because when we are looking for optimal fiscal policies, we need to make sure that they are time consistent, in other words that the optimal fiscal plan constitutes a subgame perfect equilibrium, so that it's the optimal thing to do after any history.

Most maximization problems are time consistent. For example, in the Solow growth model we have time consistency.

There are two different types of time inconsistency:

1. Time inconsistent due to inconsistent perferences:

   The agent's discount factor might change over time.The smoking example is a problem of time inconsistency dye to inconsistent preferences.

2. Time inconsistent due to changing constraints (consistent preferences)

   Here there is no problem with preferences, they are consistent. On the other hand, over time the constraint that the agent is subject to changes. For example, think of the two brothers 'economy. Increasing the consumption of the brothers at period 17 is good for the planner right now (at period 0), because not only does it increase the utility of the brothers at period 17 but also it relaxes the participation constraints between periods 0 to 16. But once period 16 actually comes, the planner does not care about all those benefits anymore, because it does not impute any value on relaxing teh participation constraints from the previous periods, they are already gone, relaxing them does him no good at that point in time. Thus, giving more consumption at period 17 is not good anymore for the planner, although it was good at period 0.

   The time inconsistency problem associated with the optimal fiscal policy is in this category. Although the government is benevolent, his constraints which are the first order conditions of the private sector, are changing. Therefore, the optimal thing to do for the government is changing also. So in order to find a time consistent optimal plan, we need to find such a plan that the government never wants to deviate from its prescribed strategies.

How does the literature deal with this problem? They either ignore it, focus on trigger type of equilibria or think of the players in the next period as different guys and look for a fixed point of the policy rule (so that given that the future governments are going to follow this particular policy rule, the current government ends up actually choosing that same particular policy rule as its optimal plan).

26.2 Overlapping Generations Model

The basic differences between a pure exchange economy with infinitely lived agents and the OLG model is that in the OLG model, competitive equilibria may not be Pareto optimal and that money may have positive value.

Consider an economy where agents live for two periods. Each time period a new generation is born. Let (e^t, e^{t+1}) denote a generation's endowment and (c^t, c^{t+1}) denote their consumption in the first and second periods of their lives. Suppose $\mathrm{e}^t = 3$ and e^{t+1}=1.

Homework. *Show that each agent chooses to consume his endowment at each period of their lives, i.e. that $c^t = 3$ and $c^{t+1} = 1$.*

The problem that we run into with the Pareto optimality of the equilibrium allocations in OLG is that the prices are growing too fast in so that the value of the endowments end up being not bounded.

Now consider the space $\mathcal{C}$ with only a finite number of elements that are not 0. And take $\mathcal{X} \subset \mathcal{C}$ and let the prices that support the autarky in OG be defined on this commodity space. Then we can say that $\mathcal{C} \subset AD$. This is an Arrow-Debreu equilibrium and it is Pareto optimal because the only better allocation is the young giving to the old and that is not feasible.

Now let's consider a particular OG model with different elements like marital status, gender, health, etc. in it.

Index agents by age and gender.

g: gender

z: marital status

h: health

e: education

η : love

κ : effort

The agent's value function is the following:

$$V_{i,g}(z,h,e,\eta,a,\kappa) = \max_{z' \in B^*(z),\kappa'} \text{u(c,z)}+\beta_i \sum_{\eta',h'} \Gamma_{\eta' h',\eta h} V_{i+1,g}(z',h',e',\eta',a',\kappa')$$

subject to

$$ra + w(e,l) + w^*(.) = c + a'$$

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27.1 OLG model with some detail

- We will use this model to understand what changing social security benefits does to skills. In other words, we are interested in seeing the effects of policy on human capital accumulation.
- We will abstract from endogenous mortality (people cannot affect their survival rate)
- γ_i : probability of surviving for an agent of age i.
- We will also abstract from cross sectional differences, i.e. we will not allow people of the same age group differ from each other.
- We will let agents work even if they are retired.
- l: hours worked

The individual state variables are: i (age), h (human capital), b (accumulated level of benefits) and a

The problem of the agent if retired is as follows:

$$V_i(h,a,b) = \max_{l,c,y} \text{u(c,1-l)}+\beta\gamma_i V_{i+1}(h',a',b')$$

subject to

$$\begin{aligned}
a' &= \frac{y}{\gamma_i} \quad \text{(death insurance, return of which is } \frac{1}{\gamma_i}\text{)} \\
b' &= b \quad \text{(benefits stay the same once retired)} \\
h' &= (1-\delta)h \quad \text{(Nothing to learn after retired, h decreases)} \\
c + y' &= hw(l)l + (1+r)a + l(b,i)
\end{aligned}$$

The problem of the agent if not retired is as follows:

$$V_i(h,a,b) = \max_{l,c,y} \text{u(c,1-l)} + \beta\gamma_i V_{i+1}(h',a',b')$$

subject to

$$\begin{aligned}
a' &= \frac{y}{\gamma_i} \\
b' &= \psi(b,i,h,w(l),l) \quad \text{(benefits accumulating)} \\
h' &= \chi(h,i,n) \\
c + y' &= hw(l)l(1-\tau) + (1+r)a + l(b,i)
\end{aligned}$$

Homework. *Suppose that after \$81,000 , you don't pay social security. Prove that in this model nobody will ever make exactly \$81,000.*