Lecture Notes

Econ 702

Spring 2004

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- What is an equilibrium?
  An equilibrium is a statement about what the outcome of an economy is. Tells us **what happens in an economy**.
  An equilibrium is a mapping from environment (preference, technology, information, market structure) to allocations where,

  1. Agents maximize
  2. Agents' actions are compatible.

- One of the important questions is, given the environment what type of equilibria we should look at. The economist doesn't have the right to choose what happens. For example, for an economy with two people it might not be appropriate to think of Arrow-Debreu (maybe they are not price takers, they might not behave competitively).

- In order something to be predictable by theory it needs to exist and be unique. For this reason, existence and uniqueness are properties that we want the equilibrium to have.

- Optimality is a property of an allocation. An equilibrium allocation might or might not have this property.

- In this class, we will go over some of the 'popular' notions of equilibrium in macroeconomics that might be different from Walrasian equilibrium.

1.1 Valuation Equilibrium

- In macroeconomics, we are interested in infinite- dimensional commodity spaces. We want to look at the relationship between competitive equilibrium and Pareto optimality in models with infinite-dimensional spaces. You looked at competitive equilibrium and Pareto optimality in 701, but the proofs of the FBWT and SBWT were done in the context of finite-dimensional commodity spaces. Here we want to show that the welfare theorems hold for economies with infinite dimensional spaces. To do this, we introduce the equilibrium concept 'valuation equilibrium'.

- Before defining valuation equilibrium, we first need to define the environment:

1. $\mathcal{L}$, Commodity space:
   $\mathcal{L}$ is a topological vector space.

   Definition 1 (Vector Space) *A vector space is a space where the operations addition and scalar multiplication are defined, and where the space is closed under these two operations. i.e. If we take two sequences $a = \{a_i\} \in \mathcal{L}$ and $b = \{b_i\} \in \mathcal{L}$, it must be that $a+b \in \mathcal{L}$. And if we take $k \in \mathcal{R}^+, k > 0$, it must be that $a \in \mathcal{L} \Rightarrow d = ka \in \mathcal{L} \ \ \forall \ k > 0$.*

   Definition 2 (Topological Vector Space) *A topological vector space is a vector space which is endowed with a topology such that the maps $(x, y) \to x + y$ and $(\lambda, x) \to \lambda x$ are continuous. So we have to show the continuity of the vector operations addition and scalar multiplication.*

2. $X \subset \mathcal{L}$, Consumption Possibility Set:
   Specification of the 'things' that people could do (that are feasible to them). X contains every (individually) technologically feasible consumption point.

3. $U : X \to \mathcal{R}$, Specifies the preference ordering.

4. Y, Production possibility set.

What is an allocation in this environment? An allocation is a pair (x, y). On the other hand, a <u>feasible</u> allocation is (x, y) such that x=y (agents' actions need to be compatible).

1.1.1 Prices

Before we go ahead with defining the valuation equilibrium, we need to define one more object: prices. How we initially deal with prices will be different from what you saw before. In the first couple of classes, we treat prices more generally, in other words continuous linear functions that don't necessarily have a dot-product representation as you've always seen before.

The space of linear functionals on a space A is called the dual space of A. A price system will be an element of the dual space of our commodity space, $\mathcal{L}$.

$$p : \mathcal{L} \rightarrow \mathcal{R}$$

where $p(.)$ is continuous and linear. i.e.

$$\begin{aligned} x_n \rightarrow x \Rightarrow p(x_n) \rightarrow p(x) \;\; (\mathit{continuous}) \\ p(x_1) + p(x_2) = p(x_1 + x_2) \;\; (\mathit{linear}) \end{aligned}$$

1.1.2 Agents' Problem

- Consumer's problem is:

$$\max_{x \in X} U(x) = \tag{1}$$

subject to

$$p(x) \leq 0 \tag{2}$$

- Firm's problem is:

$$\max_{y \in Y} p(y) \tag{3}$$

1.1.3 Definition of valuation equilibrium

Definition 3 (Arrow-Debreu/Valuation Equilibrium) *Valuation equilibrium is a feasible allocation (x^*, y^*) and price p^* such that,*

1. *x^* solves the consumer's problem.*

2. *y^* solves the firm's problem.*

3. *$x^* = y^*$*

1.1.4 Pareto Optimality

Definition 4 *(x, y) is a Pareto optimum if it is feasible and if there exists no other feasible allocation (x', y') such that,*

$$\begin{aligned} U(x_i') &\geq U(x_i) \;\; \forall\, i \\ U(x_i') &\geq U(x_i) \;\; \mathit{for\ at\ least\ one}\ i \end{aligned}$$

How do we prove that a PO exists? What assumptions do we need to the existence of a PO allocation?

- If $X \cap Y$ is compact and if u is continuous, then $\bar{x} \in \arg\max_{x \in X \cap Y} u(x)$ exists and is PO.

Homework 1 *Prove this.*

- $\exists$ a unique PO if $u(.)$ is strictly concave and X, Y are convex.

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2.1 Welfare Theorems

Theorem 1 (First Basic Welfare Theorem) *Suppose that for all $x \in X$ there exists a sequence $\{x_n\}_{n=0}^{\infty}$ in$X$ converging to x with $u(x_n) \geq u(x)$ for all n (local nonsatiation). If an allocation (x^*, y^*) and a continous linear functional ν constitute a competitive equilibrium, then the allocation (x^*, y^*) is Pareto optimal.*

Theorem 2 (Second Basic Welfare Theorem) *If (i) X is convex, (ii) preference is convex (for $\forall x, x' \in X$, if $x' < x$, then $x' < (1-\theta)x' + \theta x$ for any $\theta \in (0,1)$), (iii) $U(x)$ is continuous, (iv) Y is convex, (v)Y has an interior point, then with any PO allocation (x^*, y^*) such that x^* is not a saturation point, there exists a continuous linear functional ν^* such that (x^*, y^*, ν^*) is a Quasi-Equilibrium ((a) for $x \in X$ which $U(x) \geq U(x^*)$ implies $\nu^*(x) \geq \nu^*(\nu^*)$ and (b) $y \in Y$ implies $\nu^*(y) \leq \nu^*(y^*)$)*

Note that an additional assumption we are making for SBWT to go through in infinitely dimensional spaces is that Y has an interior point i.e.

$$\exists \bar{y} \in Y,\ B \subset Y,\ B \text{ open and } \bar{y} \in B$$

Also that the SBWT states that under certain conditions listed above, we can find prices to support any Pareto optimal allocation as a quasi equilibrium. But the statement doesn't say that we can support it as an Arrow-Debreu equilibrium. The following lemma takes care of this.

Lemma 1 *If, for (x^*, y^*, ν^*) in the theorem above, the budget set has cheaper point than x^* ($\exists x \in X$ such that $\nu(x) < \nu(x^*)$), then (x^*, y^*, ν^*) is a ADE.*

Homework 2 *Show that without the assumption of continuity of the price function, the proof of First Basic Welfare Theorem doesn't go through.*

Homework 3 *Prove the following:*

$$x^* \in PO(\varepsilon) \Leftrightarrow x^* \in \arg\max_{x \in X \cap Y} \ u(x)$$

With the SBWT, we established that there exists a ν that will support our PO allocation as a competitive equilibrium. What's the problem with this approach? It is that SBWT only tells us that such a ν exists, it doesn't tell us what it is. Note that we are still in the very general case here, ν can be a very messy object, it's any continuous linear functional defined on our commodity space $\mathcal{L}$ ($\nu \in Dual(\mathcal{L})$). We will need to make some assumptions to make the price system more tractable.

What's next? Now we will write the social planner's problem and write it into an environment in which we can apply the two welfare theorems. And then we will get the solution to the social planner's problem and then we will get the prices.

Big picture: Our main purpose is to be able to apply the welfare theorems to the most commonly used models in macroeconomics where we have an infinite-dimensional commodity space. Until now, we set up an environment (Arrow-Debreu economy) (which consisted of the commodity space, consumption possibility set, production possibility set, and preferences) with infinite-dimensional commodity space and we stated that under certain conditions the Welfare Theorems hold in this environment. Now we will map the growth model into the environment that we talked about until here, and show that in the context of the growth model the assumptions we need for the Welfare Theorems are satisfied. Then we can conclude that any competitive equilibrium allocation is Pareto optimal and moreover we can support a PO allocation with some prices as a competitive equilibrium. This result is very important in macroeconomics. It helps us in solving for the equilibria. With the FBWT and SBWT, we can just solve for the PO allocations and then get the prices. This makes life much easier.

2.2 The Growth Model

Social Planner's Problem:

$$\max_{\{c_t, l_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{4}$$

subject to

$$c_t + k_{t+1} = f(k_t, l_t) \tag{5}$$

$$c_t, k_{t+1} \geq 0 \tag{6}$$

$$l_t \in [0, 1] \tag{7}$$

$$k_0 \ \ given \tag{8}$$

- Note that we want the INADA conditions everywhere so that we have interior solutions. INADA conditions make sure that the nonnegativity constraints are irrelevant. This is so that we can use the First Order Conditions and not deal with the KT conditions.

- Since leisure is not in the utility function, we don't have to worry about it. Agent doesn't care about it so he will work as much as he can, therefore it must be that $l_t = 1$.

Homework 4 *Show that the solution the above social planner's problem is unique.*

Rewrite the above problem:

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(f(k_t) - k_{t+1}) \tag{9}$$

$$k_0 \;\; given \tag{10}$$

Because of the INADA conditions we know that the solution is interior. So if $\{k_{t+1}^*\}$ is a solution then it satisfies the first order conditions,

$$-\beta^t u_c(f(k_t) - k_{t+1}) + \beta^{t+1} f_k(k_t + 1) u_c(f(k_{t+1}) - k_{t+2}) = 0 \tag{11}$$

But notice that these conditions are not sufficient. The above is a second order difference equation. It has two degrees of freedom. Therefore this equation is not enough to find the solution. There can be many sequences that satisfy it. We need to more conditions to pin down the right solution: The initial condition k_0 and the transversality condition.

2.2.1 How to solve the SPP?

In Randy's class you learned how to solve this problem through dynamic programming.
Denote $R(k', k) = u(f(k) - k')$.
The Bellman equation,

$$V(k) = \max_{k'} R(k', k) + \beta V(k') \tag{12}$$

Define the T operator,

$$(TV)(k) = \max_{k'} R(k', k) + \beta V(k') \tag{13}$$

- Randy showed that V exists and is unique and can be found by successive iterations of the T operator.

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Today we will write the growth model in AD/valuation equilibrium language. Our main purpose is to write the growth model in a suitable way to apply the welfare theorems.

3.1 Growth Model in AD language

Remember the objects that we needed to define the valuation equilibrium from the previous classes. Now we will go through the same objects and define them for the growth model.

1. $\mathcal{L}$, Commodity Space: Set of objects that are traded. The commodity space needs to be a space of sequences. What are traded in the growth model within a period? Capital services, labor services and the final output. Then $\mathcal{L}$ will consist of all three

dimensional infinite sequences (that are bounded in the sup-norm). It will represent everything the agents can do in this economy.

$$\mathcal{L} = \{\{x_{1t}, x_{2t}, x_{3t}\}_{t=0}^{\infty} : \sup_{i,t} \ x_{it} < \infty\} \tag{14}$$

Homework 5 *Prove that $\mathcal{L}$ with the supnorm topology is a topological vector space.*

Note 1 *We are making our lives easier by making firms' problems static. We are now writing everything so that we don't have to deal with dynamics. This buys us the luxury of not having to deal with dynamic objects such as capital stock in our commodity space. We have flows in the commodity space.*

2. Y, Production Possibility Set:

$$\begin{aligned} Y &= \cup_t Y_t \\ \textit{where} & \\ Y_t &= \{y \in Y : \ \{y_{ij}\} = 0 \ \forall \ j \neq t, \ y_{1t} \leq f(-y_{2t}, -y_{3t}), \ y_{1t} \geq 0, \ y_{2t}, y_{3t} \leq 0\} \end{aligned}$$

Note that the aggregate production set has no dynamic property.

3. X, Consumption Possibility Set:

$$X = \{x \in \mathcal{L} : \exists \{k_{t+1}\}_{t=0}^{\infty} \geq 0 \ \textit{such that} \tag{15}$$

$$\begin{aligned} x_{1t} + (1-\delta)k_t - k_{t+1} &\geq 0 \quad \forall t \\ x_{2t} &\in [-k_t, 0] \quad \forall t \\ x_{3t} &\in [-1, 0] \quad \forall t \\ k_0 &= \quad \textit{given}\} \end{aligned} \tag{16}$$

Note 2 *Remember the feasibility constraint*

$$k_{t+1} + c_t = F(k_t, 1) + (1-\delta)k_t \tag{17}$$

$$c_t \geq 0 \tag{18}$$

(17) and (18) imply (16).

Homework 6 *Define the consumption possiblity set with consumption.*

Homework 7 *Show that X and Y are convex.*

Homework 8 *Given the preferences,*

$$U(x) = \sum_{t=0}^{\infty} \beta_t u[x_{1t} + (1-\delta)k_t - k_{t+1}] \tag{19}$$

Verify the following,

1. *Given u is continuous, U is continuous also.*

2. *Given u is strictly concave, U is strictly concave.*

3. *Given u is locally nonsatiated, U is locally nonsatiated.*

Homework 9 *Show that the set of feasible allocations is compact ($X \cap Y$)*

Why do we need these properties? We get existence of the solution to the social planner's problem with the continuity of U and compactness of the constraint set. And we get uniqueness by the strict concavity of U and convexity of the constraint set. And First Basic Welfare Theorem holds by local nonsatiation. Then it must be that the allocation that we get from the social planner's problem is the only candidate for the AD equilibrium allocation. Notice we say 'candidate' because in order to be able to call something ADE, we need allocations AND prices. We will use Second Basic Welfare Theorem to establish the existence of prices that support the pareto optimal allocation as an ADE.

Theorem 3 (SBWT) *After verifying all assumptions (look at the previous statement of SBWT for these assumptions); with any PO allocation (x^*, y^*) such that x^* is not a saturation point, there exists a continuous linear functional ν^* such that (x^*, y^*, ν^*) is a Quasi-Equilibrium ((a) for $x \in X$ which $U(x) \geq U(x^*)$ implies $\nu^*(x) \geq \nu^*(\nu^*)$ and (b) $y \in Y$ implies $\nu^*(y) \leq \nu^*(y^*)$).*

We can use the following lemma to get from quasi equilibrium to ADE,

Lemma 2 *If $\exists \hat{x} \in X$ s.t. $v^*(\hat{x}) < v^*(x^*)$, then a quasi-equilibrium is an Arrow-Debreu equilibrium.*

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4.1 Inner Product Representation of Prices

We want the price of a good at time t in terms of good at time 0. But that's not what v^* tells us. v^* is the cost or value of a commodity point. It is an arbitrary continuous linear function that is defined on our commodity space; it may not always be possible to find a sequence of prices to represent it with. Now we will go over the conditions under which this can be done. This is the Prescott-Lucas Theorem. This theorem tells us under which conditions the valuation function, or the pricing scheme, can be represented as an inner product for the infinite time horizon case.

Theorem 4 *(based on Prescott and Lucas 1972) If, in addition to the conditions to SBWT, $\beta < 1$ and u is bounded, then $\exists \hat{p}$ such that $(x^*, y^*, \hat{p})$ is a QE and*

$$\hat{p}(x) = \sum_{t=0}^{\infty} \sum_{i=1}^{3} \hat{p}_{it} x_{it} \tag{20}$$

i.e. price system has an inner product representation.

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$$\hat{p}(x) = \sum_{t=0}^{\infty} \sum_{i=1}^{3} \hat{p}_{it} x_{it} \tag{21}$$

i.e. price system has an inner product representations.

The result above is a special case of the more general theorem proved by Prescott and Lucas (1972). Before stating the theorem, let's define some notations. Let L^n be the subspace of L such that, for $x \in L^n$, $x = ((x_{11,}x_{21,x31}), (x_{12}, x_{22}, x_{32}), (x_{13}, x_{23}, x_{33}), ..., (x_{1n-1}, x_{2n-1}, x_{3n-1}), (0,0,0), (0,0,0), ...)$, i.e. $x_{it} = 0$ for $t \geq n$. Also Let x^n denote the projection of $x \in L$ on L^n.

Now we are ready to state the theorem in a more general form.

Theorem 6 *(Prescott and Lucas 1972) If (i) X is convex, (ii) preference is convex (these two conditions are same as those in the SBWT), (iii) for every n, $x^n \in X$ and $y^n \in Y$, (iv) if $x, x' \in X$ and $U(x) > U(x')$, then there exists and integer N such that, for $\forall n \geq N$, $U(x^n) > U(x')$, then, for a QE (x^*, y^*, p^*) with non-satiation point x^*, there exists $\hat{p}$ such that (1) $\hat{p}(x) = \lim_{n\to\infty} p(x^n)$ for a $p \in Dual(L)$, and (2) $(x^*, y^*, \hat{p})$ is a QE.*

Remark 1 *The results of the theorem allows us to consider the price system of a QE as the limit of a price system of the finite commodity space and thus represent price system of a QE by inner product representations. Intuitively, the additional two conditions of the theorem ((iii) and (iv)) tell that (iii) truncated consumption or production allocation is also feasible, and (iv) truncation of the sufficiently "future" consumption does not change the preference relationship.*

With the Prescott-Lucas Theorem, from now on we can use the inner product representation of prices (assuming the conditions for this theorem hold). So the SBWT told us that for the PO allocation, we can get prices that will support it as an ADE. The Prescott-Lucas Theorem told us that these prices can be written as an inner product. Now let's find what these prices are.

Initially we solved the social planner's problem to get the allocations of ADE, and now we want to construct prices using the first order conditions of the household and firm's problem. The household's problem is,

$$\max_{x \in X} \quad U(x) \tag{22}$$

subject to

$$\sum_t (p_{1t}x_{1t} + p_{2t}x_{2t} + p_{3t}x_{3t}) = 0 \tag{23}$$

Look at the First Order Conditions with respect to x_{1t} and x_{1t+1},

$$\lambda p_{1t}^* = \beta^t u_c(x_{1t}^* - k_{t+1}^* + (1-\delta)k_t^*) \tag{24}$$

$$\lambda p_{1t+1}^* = \beta^{t+1} u_c(x_{1t+1}^* - k_{t+2}^* + (1-\delta)k_{t+1}^*) \tag{25}$$

How many unknowns do we have? λ and p. Notice that we already know x and k, we took them from our social planner's problem, and now we are trying to find p^* such that it will give the allocations that we have from the social planner's problem as a solution. To do this, we are using the FOC. For example, combine the above two FOC to get,

$$\frac{p_{1t}^*}{p_{1t+1}^*} = \frac{u_c(x_{1t}^* - k_{t+1}^* + (1-\delta)k_t^*)}{\beta u_c(x_{1t+1}^* - k_{t+2}^* + (1-\delta)k_{t+1}^*)} \tag{26}$$

The sequence of prices that we want, need to satisfy this.

In this way,i.e. through using the characterization to the solution to consumer and firm's problems you can derive also the following equations that our prices need to satisfy,

$$\frac{\hat{p}_{2t}}{\hat{p}_{1t}} = F_k(k_t^*, n_t^*) \tag{27}$$

$$\frac{\hat{p}_{3t}}{\hat{p}_{1t}} = F_n(k_t^*, n_t^*) = \frac{u_l(c_t^*, 1-n_t^*)}{u_c(c_t^*, 1-n_t^*)} \tag{28}$$

$$\frac{\hat{p}_{1t}}{\hat{p}_{1t+1}} = \frac{u_c(c_t^*, 1-n_t^*)}{\beta u_c(c_{t+1}^*, 1-n_{t+1}^*)} = 1 - \delta + \frac{\hat{p}_{2t+1}}{\hat{p}_{1t+1}} \tag{29}$$

A digression on prices:

What do these prices stand for?

p_{1t} : Amount of consumption good one must pay at t=0 in return of consumption good at time t. p_{11} is the number of units of good 10 that exchange for good 11.

p_{2t} : Amount of consumption good one must pay at t=0 in return of labor services at time t.

p_{3t} : Amount of consumption good one must pay at t=0 in return of capital services at time t.

In other words, p is the price paid now (at time 0) for the delivery of one unit of either the consumption goods, labor services or capital services at time t. In the Arrow-Debreu world, the agent is paying for promises to have certain goods delivered at specified dates in return for certain goods delivered at time 0. Thus the prices in the AD world are all in terms of units of consumption goods at time 0.

Remark 2 *Whatever discounting there is, it's already embedded in the p. Everything is in terms of time 0 goods already. The fact that people discount the future is reflected in the prices in the following sense: For something to be equilibrium, the future has to be cheaper, in other words, to induce the agent to buy a good in the distant future, we have to make that distant future good cheaper. This is reflected in p.*

Homework 10 *Derive the formula that links* p_{2t+1}^*, p_{1t}^* *and* $p_{1,t+1}^*$ *i.e. (29)*

Remark 3 (No Arbitrage) *The interpretation of the fact that the prices need to satisfy (29) is the No Arbitrage argument. No Arbitrage says that two identical ways of transferring resources across periods have to be priced at the same level. Think about the two ways our agent can transfer resources across* t *and* $t+1$ *in the growth model,*

1. *The agent can sell one unit of x_{1t} at time t and get p^*_{1t}. In this case, at time $t+1$ he gets $\frac{p^*_{1t}}{p^*_{1t+1}}$ units of time $t+1$ good, x_{1t+1}.*

2. *The agent can save one more unit of capital k_{1t+1} in his backyard at time t, and at time $t+1$ rent it to the firm and get the rental price plus the non-depreciated capital in return, which is $(1-\delta)+\frac{p^*_{2t}}{p^*_{2t+1}}$.*

4.2 Sequence of Markets Equilibrium

The market arrangement in the ADE is not realistic. The problem with the AD market structure is that everything happens at time 0, all trade are decided for at time 0. After time 0, the agents just execute the decisions that they made at time 0. We want a more reasonable market arrangement than this. Here is where we stand right now:

- We established the equivalence between SPP allocation and ADE allocation using Welfare Theorems.
- But it is not sufficient, because the market arrangement in the ADE is not realistic.
- We want to get another result which connects SPP to an equilibrium with a more reasonable market arrangement (SME). Our strategy will be to show the equivalence between ADE and SME; and then use the equivalence between the SPP allocation and ADE allocation that we showed previously to conclude that the SPP allocation also happens to be the SME allocation. This will enable us to get the SME allocation by just solving the social planner's problem.
- Later, we will see that we can use Dynamic Programming to solve SME (an associated equilibrium concept is Recursive Competitive Equilibrium, RCE).

Now we will review the consumer's problem in AD and then go on to write the consumer's problem in the sequence of markets.

4.2.1 Consumer's Problem in ADE

$$\max_{x\in X}\sum_{t=0}^{\infty}\beta^t u(c_t) \tag{30}$$

subject to

$$\sum_{t=0}^{\infty}\sum_{i=1}^{3}\hat{p}_{it}x_{it}\leq 0 \tag{31}$$

(Note that we are using the result of Lucas and Prescott (1972) and representing the price system by inner product.) How many constraints do we have? Two. One is feasibility constraint ($x\in X$) and the other is budget constraint (31). But forget the feasibility constraint now. Often we can either (i) forget the condition, or (ii) show that it is not going to be binding. Let's concentrate on the budget constraint. There is only 1 budget constraint. Why? Because we make a choice only ONCE in AD world: all the trades are made at period 0, and after the history starts, all that agents can do is to follow what was promised (full commitment is assumed). But this is a weird market arrangement. To see the point this more clearly, imagine the decision of an agent who is going to be born in period t. At period 0, although the agents is not born yet, the agent also joins the market at period 0! At period 0, she trades (by solving the consumer's problem above), and she goes to limbo from period 0 (after trade) until period t-1, and she is born in period t. As we want the market arrangement of the model to be comparable to the one in the real world, this unrealistic assumption on market arrangement is not desirable. That is the motivation to consider Sequential Market Equilibrium (SME), where markets are open every period.

4.2.2 Sequence of Markets

People have capital k_t and they rent it to the firm at rental rate $(1+r_t)$. People are endowed with one unit of time and they choose to allocate it between labor and leisure.They work for wage w. They consumer c_t and save k_{t+1}. The agents can also borrow and lend.

Two things are important here: (i) there are infinitely many markets in SME (because markets are open every period), which means that there are infinitely many budget constraints to be considered, (ii) an allocation in SME has to give as much utility as in ADE to agents in order to be PO. Otherwise, agents will choose to trade in AD markets, meaning SME doesn't work. Remember that we cannot force agents to do certain things.

Also note that there are many ways of arranging markets so that the equilibrium allocation is equivalent to that in ADE. If the number of markets open is TOO FEW, we cannot achieve the allocation in the ADE (incomplete market). To the contrary, if the number of markets are TOO MANY, we can close some of the markets and still achieve the ADE allocation in this market arrangement. Also it means that there are many ways to achieve ADE allocation because some of the market instruments are redundant and can be substituted by others. If the number of markets are not TOO FEW nor TOO MANY, we call it JUST RIGHT.

We will look at one possible market arrangement now. Our agents can lend and borrow (with bonds) and they can rent their capital to the firm. The budget constraint of the agent is:

$$(1-\delta)k_t + r_t k_t + w_t + (1+i_t)b_t = c_t + k_{t+1} + b_{t+1} \tag{32}$$

where b_t is bonds, and i_t is the interest rate on the bonds.

Next thing we notice is, we can close the market of loans without changing the resulting allocation. This is because we need someone to lend you loans in order that you borrow loans, but there is only one agents in the economy. But surprisingly, we will see that even though there is no trade in certain markets in equilibrium, we can solve for prices in those markets, because prices are determined even though there is no trade in equilibrium, and agents do not care if actually trade occurs or not because they just look at prices in the market (having market means agents do not care about the rest of the world but the prices in the market). Using this technique, we can determine prices of all market instruments even though they are redundant in equilibrium. This is the virtue of Lucas Tree Model and this is the fundamental for all finance literature (actually, we can price any kinds of financial instruments in this way. we will see this soon.)

So... close the markets for loans. We have the following budget constraint for each period:

$$[(1-\delta)+r_t]k_t + w_t = c_t + k_{t+1} \tag{33}$$

Now we're ready to write the consumer's Problem in SME

$$\max_{\{c_t,k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{34}$$

subject to

$$c_t + k_{t+1} = w_t + [(1-\delta)+r_t]k_t \ \ \forall t \tag{35}$$

$$k_0 \ \ \textit{is given} \tag{36}$$

And the firms' problem,

$$\max_{\{y_t,n_t,k_t\}} \{y_t - w_t n_t - r_t k_t\} \tag{37}$$

subject to

$$y_t \leq F(k_t, n_t) \tag{38}$$

Definition 5 (Sequential Markets Equilibrium) *A Sequential Market Equilibrium (SME) is an allocation* $\{c_t^*, k_{t+1}^*\}_{t=0}^{\infty}$ *and a sequence of prices* $\{R_t^*, w_t^*, i_t^*\}_{t=0}^{\infty}$ *such that,*

1. *Given* $\{R_t^*, w_t^*, i_t^*\}$, $\{0, c_t^*, k_{t+1}^*\}$ *solves the consumer's problem.*

2. *Given* $\{R_t^*, w_t^*, i_t^*\}$, $\{c_t^* + k_{t+1}^* - (1-\delta)k_t^*, k_t^*, 1\}$ *solves the firm's problem.*

Remark 4 *Note that it seems like we don't have market clearing conditions in the above definition of SME. But we implicitly do. For example, he fact that we required* $b_t = 0$ *to be the optimal solution to the consumer's problem clears the bonds market. And the markets for capital, labor and goods are cleared just from the fact that we use the same letters for them in both consumer's and producer's problems.*

5 Feb 10

5.1 From ADE to SME

So far we have worked with two different form of market arrangements, AD and SME, and defined equilibrium in both of these worlds. We have also seen that the problem defined in the sequential market setup is at least as hard, if not more, as the AD setup to solve. The way we went around this in the AD setup was the use of welfare theorems by making sure a unique PO solution to SPP existed, which coincided with the solution to AD problem, furthermore we were able to support this allocation with appropriate prices as the equilibrium outcome of the AD problem. If we can establish the equivalence between these two equilibrium concepts than we can figure out the euqilibrium outcome in the SME world without having to solve the sequential problem. To show this equivalence we

Proposition 7 *Let the allocation* $\{x_{it}^*, y_{it}^*\}_{t=0,i=1,2,3}^{\infty}$ *and a valuation function (assuming it has an inner product representation)* $\widehat{p}(x) = \sum_{t=0}^{\infty}\sum_{i=1}^{3} p_{it}^* x_{it}$ *form an AD equilibrium. Then there exists an allocation* $\{c_t^*,\ k_{t+1}^*, n_t^*, b_{t+1}^*\}_{t=0}^{\infty}$ *, a sequence of prices* $\{R_t^*,\ w_t^*, i_t^*\}_{t=0}^{\infty}$ *and an* $\{A_T\}_{t=0}^{\infty}$ *such that* $\forall t$*:*

$$\begin{aligned} c_t^* &= x_{1t}^* + (1-\delta)x_{2t}^* + x_{2t+1}^* && (39)\\ n_t^* &= x_{3t}^*, && (40)\\ k_t^* &= x_{2t}^* && (41)\\ b_t^* &\le -A_t \le -\infty && (42)\\ R_{t+1}^* &= \frac{p_{1t}^*}{p_{1t+1}^*} \qquad w_t^* = \frac{p_{3t}^*}{p_{1t}^*} \qquad i_t = R_t^* - 1 && (43) \end{aligned}$$

is a Sequential Market Equilibrium.

The way to prove this proposition is first to establish an implication from AD budget set and technology constraints to SME budget set and technology constraints. Once we do so, we can show that given our constructed prices, our constructed allocations satisfy the sufficient FOCs of the SM problem and thus form an SME.

Homework Prove this proposition (See solutions to Homework 3).

One important point to note here is the importance of the bond market in proving this implication. We know that in AD world our agent faces a single budget constraint whereas SM setup imposes countably infinite number of budget constraints to our agent. So if we were to argue, as we do, we the implication of SME given an ADE, we need some market that links the period by period budget constraints over time and lets us agent to behave as if she was facing a single constraint. This crucial linkage is provided by one period bonds in the sequential market setup and as the following exercise asks you to do, without a bond market the above proposition is no longer true.

Homework Show that without the existence of a bond market above proposition is no longer true(See solutions to Homework 3).

Once we prove $(AD \Rightarrow SME)$ this allows us to use First and Second Basic Welfare Theorems to obtain the SME. Now we know that, given a unique solution to the SPP exist we can solve for it and we can support it as an AD equilibrium. Given this AD equilibrium, we can construct prices and allocation that will satisfy the conditions of SME. Now we can show that opposite of the above implication is also true and thus $(AD \Leftrightarrow SME)$

Proposition 8 *Let the allocation* $\{c_t^*,\ k_{t+1}^*, n_t^*, b_{t+1}^*\}_{t=0}^{\infty}$ *and a sequence of prices* $\{R_t^*,\ w_t^*, i_t^*\}_{t=0}^{\infty}$ *form an SME equilibrium. Then there exists an allocation* $\{x_{it}^*, y_{it}^*\}_{t=0,i=1,2,3}^{\infty}$*, and a valuation function (assuming it has an inner product representation)* $\widehat{p}(x) = \sum_{t=0}^{\infty}\sum_{i=1}^{3} p_{it}^* x_{it}$ $\{A_T\}_{t=0}^{\infty}$ *such* *that* $\forall t$*:*

$$\begin{aligned}
x_{1t}^* &= c_t^* - (1-\delta)k_t^* + k_{t+1}^* && (44)\\
x_{2t}^* &= k_t^* && (45)\\
x_{3t}^* &= n_t^* && (46)\\
\left(\prod_{s=1}^{t} R_s^*\right)^{-1} &= p_{1t}^* \ , \ \ p_{3t}^* = p_{1t}^* w_t^* \quad , \quad i_t = R_t^* - 1 && (47)
\end{aligned}$$

is an ADE.

A similar line of logic can be followed to prove this proposition as well with one additional condition. We know by the definition of the AD equilibrium, the valuation function must be linear and continuous. Linearity is trivial to show in this case with our implied valuation function. The other important property continiuty seems to be harder to show but we know that in the infinite dimensional spaces, boundedness is a sufficient condition for continuity and we can exploit this fact. Once we do so, we have our AD equilibrium constructed from SME objects.

Homework Prove the proposition above. (See the solutions to Homework 3)

Homework Derive a sufficient condition for the implied valuation function given by the constructed AD prices (See the solutions to Homework 3).

Our next step will be to define the problem of the household in a recursive manner, which we know how to deal with using DP methods, and define a new form of equilibrium concept in this new environment. So why bother to even to go back to the recursive formulation when we could solve SPP exploting recursivity to begin with. We have already shown that we can support this allocation as an equilibrium outcome by way of welfare theorems. The answer is because we have to if want to be able to deal with environmets that are interesting in the economic sense. The important linkage between these different problems as we point in the last sentence, is the applicability of the welfare theorems. So the question is what if (i) assumptions of Welfare Theorems do not hold or (ii) we have more than one agent, thus we have many equilibrium depending on the choice of the Pareto weight in the Social Planner's Problem. Then obviously we no longer can follow the same argument, and need to solve the equilibrium directly. Since (i) solving ADE is "almost impossible", (ii) solving SME is "very hard", but (iii) solving RCE is "possible", RCE is important for analyzing this class of economies, where Welfare Theorems fail to hold and we have to solve for equilibrium outcome directly.

5.2 Recursive representation of the HH problem

As you have seen in 704, the beauty of the recursive representation lies in the fact that, in a stationary environment, the nature of the problem do not change with passage of time. Unlike the sequential formulation, in which the solution to the problem depends on at what point in time you solve it, the solution to the recursive problem do not depend on time and we do not have to keep track of time.

Homework Show that the solution to the recursive problem is not history dependent

So what do we keep track of? Everything that matters to the structure of our problem. These are the variables that our agents respond to either directly or indirectly and we call them the STATE VARIABLES. State variables need to satisfy the following criteria:

1. PREDETERMINED: when decisions are made, the state variables are taken as given and cannot be effected by the agent.

2. It must MATTER for decisions of agents: there is no sense of adding irrelevant variables as state variable.

3. It must VARY across time and state: otherwise, we can just take it as a parameter.

One important thing is to be able to distinguish the aggregate and individual state variables. Aggregate state is not affected by individual choice. But aggregate state should be consistent with the individual choice (we will consider the meaning of "consistency" more formally later), because aggregate state represents the aggregated state of individuals. In particular, in our RA-NGM aggregate state turns out to be the same as individual state in equilibrium, but this does not mean that the agent decide the aggregate state or the agent is forced to follow the average behavior, but rather the behavior of the agent turns out to be the aggregate behavior, in equilibrium.

Also note that prices (wages, and rental rates of capital) is determined by aggregate capital, rather than individual capital, and since individual takes aggregate state as given, she also takes prices as given (because they are determined by aggregate state). Again, the aggregate capital turns out to coincide with the individual choice, but it is not because of the agent's choice, rather it is the result of consistency requirement.

One notational note. Victor is going to use a for individual capital and K for aggregate capital, in order to avoid the confusion between K and k. But the problem with aggregate and individual capital is often called as "big-K, small-k" problem, because the difference of aggregate capital and individual capital is crucial. So for our case, the counterpart is "big-K, small-a" problem.

6 Feb 11

6.1 Recursive representation of the HH problem (continued)

What are the possible candidates for state variables? $\{a, w, R\}$. Why? Because these are the variables our agent responds to. Our agent has to know how wealthy she is to solve her problem. She also needs to know the prices but we do not need $\{R, w\}$ directly. Why? Because they are redundant: K is a sufficient statistic to calculate $\{R, w\}$ as they must be equal to marginal products in the firm problem at equilibrium.If we put K as a state variable instead of these prices, we do not need $\{R, w\}$ [1] and we reduce the number of state variables our agent has to follow and make her life easier. So are we done? Not yet. As the problem

[1] Of course, you can construct an equilibrium using R or w, but these turns out to be the redundant.

of the HH is formulated below, our agent not only needs to know $\{R, w\}$ but $\{R', w'\}$ as well thus K'. But this is a variable that our agent has no control over and the best she can do is to have a 'belief' about it. These beliefs in our model are parametrized by the G^e function which maps todays state to a unique belief about next period's value. As it is formulated, it is an exogenous parameter, i.e. we can solve this problem for any sort of beliefs under which the problem is well defined. But from the beginning of this course we want to be able to 'predict' the outcome, once we setup our environment as precise as possible. To continue to be able to do so, as we will see later, we will impose an additional constraint on the beliefs and make them an equilibrium object as well, i.e. endogenize them.

Now let's write the representative consumer's problem in the recursive way.

$$V(K, a; G^e) = \max_{c,a'}[u(c) + \beta V(K', a'; G^e)] \tag{48}$$

subject to

$$c + a' = w + Ra \tag{49}$$

$$w(K) = F_L(K, 1) \tag{50}$$

$$R(K) = 1 + F_K(K, 1) - \delta \tag{51}$$

$$K' = G^e(K) \tag{52}$$

Couple of comments:

- For the above problem to be a well-defined one, all the variables in the maximand (in the problem above: $[u(c) + \beta V(K', a'; G^e)]$) have to be either (i) a state variable (so an argument of V(.)), (ii) a choice variable (so appear below max operator), (iii) or defined by a constraint, in order for the problem to be well defined. In the case above, note (i) c is a choice variable, (ii) K' is defined by (52) (which we will discuss below), (iii) a' is defined by (49), (iv) the variables in (49) (especially r and w) are also defined by constraints, which only contains state variables (K), thus we know that the problem is well defined.

- Again, prices $\{R, w\}$ are functions of aggregate variables, so agents have to take them as given. Note that this is because individual is measure zero, by assumption (so, although we are dealing with representative agent, at the same time we assume that agents are measure zero and have no power to affect aggregate state of the world, hence prices).

- (52) might look strange, but without it the problem is not well defined. In other words, we have to allow agents to make "belief" or "forecast" or "expectations" about the future state of the world, to solve the problem, because agents need to make expectations about the return to capital in the next period to make consumption - saving choice.

- What can be the arguments of G^e function? individual variables $\{c, a, a'\}$ cannot be, because by assumptions, individual agents have no power to affect the aggregate state of world. $\{R, w\}$ cannot be if K is an argument, because K is a sufficient statistics for prices. Thus, we know that K is the only argument of G^e function.

- As we argue above, we index the value function with G^e because the solution of the problem above depends on the choice of G^e. But what is "appropriate" G^e? This is revealed when we see the definition of an equilibrium below.

- Notice that $\{K, w(K), R(K), G^e(K)\}$ are enough to generate all future prices if today's aggregate capital is K.

The solution to the above HH problem is a pair of functions$\{V^*(.), g^*(.)\}$,a value function and an optimal decision rule. But as macroeconomists we are interested in the aggragate implications of individual outcome so we want to go beyond solving the problem of the household.

Homework Prove that the value function that solves the HH problem V(.) exists, continious, bounded, strictly concave and strictly increasing given enough structure is assumed about the primitives and state these sufficient assumptions.

How would a RCE look like once we solve the above HH problem? That is to say what happens in this economy given all the households solve the above problem and firms solve their one period problem? We observe that the solution to the Firm's problem is implicitly taken care of by the definition of prices. One would expect from an equilibrium definition, a prediction about the evolution of this economy and this is exactly what we get as we see next. Now, Let's define the Recursive Competitive Equilibrium:

Definition 6 *A Recursive Competitive Equilibrium with arbitrary expectations* $G^e(.)$ *is* $\{V^*(.), g^*(.), G^*(.),$ *such that*

1. *Given* $\{G^e(.), R(.), w(.)\}$, $\{V^*(.), g^*(.)\}$ *solves the household problem.:*

$$V^*(K, a, G^*) = \max_{c,a'}[u(c) + \beta V(K', a'; G^*)]$$

subject to (49), (50), (51), and $K' = G^*(K)$, *and*

$$a' = g^*(K, a; G^e) \in \arg\max(\textit{the same problem})$$

2. $\{R(.), w(.)\}$ *are characterized by the optimal decisions of firms.*

3. $G^*(K; G^e) = g^*(K, K; G^e)$

Some comments on the third condition. This condition is called 'Representative agent condition' and is a specific case of the compatibility condition that any equilibrium must satisfy. It basically means that if a consumer turns out to be average this period (her individual capital stock is K, which is aggregate capital stock), the consumer will choose to be average in the next period (she chooses $G^*(K)$, which is a belief on the aggregate capital stock in the next period if today's aggregate capital stock is K). This condition guarantees that in an equilibrium, individual choice turns out to be consistent with the aggregate law of motion. This is true not because our agent is constrained to do so but because the prices are such that she choses to do so.

As argued previously, our model is not fully closed yet in the sense it still cannot give us a sharp prediction about the outcome given the environment. Depending on the nature of beliefs there are infinitely many possible outcomes our model can generate. To take care of this and make our model fully operational we impose the following constraint to beliefs and define the Rational Expectations RCE.

Definition 7 *A RE RCE is a RCE with the additional requirement, given* G^e*,*

$$G^e(K) = G(K; G^e)$$

This condition states that the beliefs of our agent about the evolution of the economy is exactly identical to outcome of our model about the evoulution of the economy, which takes our agent's beliefs as given. Although it is a very strong restriction to impose, it makes our model fully operational and lets it to pin down a unique outcome out of infinitely many.

7 Feb 12

7.1 Relating RCE to SME

As we define the RE RCE it is important to note that the SME is also a RE equilibrium. In the sequential problem, when our agent makes her decisions, she bases them on a infinite sequence of prices that she thinks as the prices that will be realized in future. The equilibrium prices that our solution to sequential problem generates must be identical to the prices that our agent took as given for consistency of our equilibrium. So it is a RE equilibrium as well in the sense the agent's expectations are indeed the ones generated by the model A natural way to proceed at this point would be to show the equivalence of the two equilibrium definitions as we did before. It is natural to be inclined to argue that given an SME we should be able to construct an RCE since we have derived the recursive problem directly from the sequential one. Although for most of the cases this argument this line of logic holds there can be some 'abnormal' SME's that this is not true and we cannot claim $SME \Rightarrow RCE$ .What can go wrong? The answer to this question lies in the mathematical definitions of the solutions to these two problems. We know that the solution to a sequential problem is the mathematical object 'sequence'. Mathematically, a sequence is a mapping from the integers to reals whereas the solution to the recursive problem is he mathematical object 'function' which is a mapping from the state space we defined onto itself.. If we were to be able to show this implication, we would have to be able to define a function that would generate the exact sequence that is the solution to the sequential problem. Now suppose that we have the solution as a sequence that passes through the same point in its range but follows a different path afterward for each of them. This is an equilibrium outcome that cannot be ruled out. By definition if our function maps a single point in the domain to a single point in its range and there does not exist a function that can generate the solution to the described sequential problem. That said as the following proposition states we can show $RCE \Rightarrow SME$ by construction.

Proposition 9 *For any RE RCE* $\{G^*, V^*, g^*\}$ *, we can construct an allocation* $\{c_t^*,\, k_{t+1}^*\}_{t=0}^{\infty}$ *and prices* $\{R_t^*, w_t^*, i_t^*\}_{t=0}^{\infty}$ *that satisfies the definition of SME.*

Homework Prove the proposition both by, given the recursive competitive equilibrium objects, constructing the necessary objects for SME and showing that they satisfy the sufficient FOCs of SME and by contradiction.(see the solutions to Homework 3)

What do we know about the competitive equilibrium? Unlike the finite dimensional spaces where the existence of competitive equilibrium is based on a fixed point argument of a function, in the infinite dimensional case, we are working on the space of functions and we need a fixed point theorem on a functional space. Note that, directly solving for equilibrium by solving the RCE involves finding out the functions $\{V, G\}$ When we look at SPP in recursive form, we find a single contraction mapping to work on The SPP is solved as the fixed point of the following contraction mapping since SPP do not have to worry about prices and thus G. We define

$$V_1(K) = T(V_0)(K) = \max_{K' \in X} R(K, K') + \beta V_0(K')$$

where T maps a continuous, strictly concave function to a continuous and strictly concave function given enough structure on the primitives. And we can show T is a contraction mapping. To find the fixed point of this contraction V^*, we can use iteration: for any continuous and strictly concave function V_0,

$$V^* = \lim_{n \to \infty} T^n(V_0)$$

such that

$$V^* = \max_{K' \in X} R(K, K') + \beta V^*(K')$$

But to solve a RE RCE, we cannot use such fixed point theorem because we need find (V, G, g) jointly. Let V, G, g denote the spaces of functions on the compact set $\left[0, \overline{K}\right]^2$ and let $V \in V$ be continuous, strictly concave and differentiable in a.Let $G \in G$ be continuous and define the following mapping $T(V, G) = (T_1, T_2)(V, G) : V \times G \to V \times G$.

$$\begin{aligned} V_1(K, a) &= T_1(V_0, G_0) = \max_{c, a'} u(c) + \beta V_0(G_0(K), a') \\ s.t. \quad c + a' &= w(K) + R(K)a \end{aligned}$$

and the decision rule is

$$a' = g(K, a; G_0)$$

and

$$G_1(K) = T_2(V_0, G_0) = g(K, K; G_0)$$

We can see the first component of T mapping gives V, and the second part gives G. If it exists, the fixed point of this mapping $T$is RE RCE. But in theory you cannot establish contraction property of T because of its second component and thus its not possible to argue the successive iteration approcach for the existence and solution of the equilibrium. It is more difficult to find RE RA RCE in theory but in practice usually this problem is managable numerically.

Homework Show that T_2 cannot be a contraction.

7.2 RCE with Leisure

Let's try to write down the problem of consumer.

$$V(K,a;G)=\max_{c,n,a'}\{u(c,1-n)+\beta V(K',a';G)\} \tag{53}$$

subject to

$$c+a'=R(K)a+w(K)n \tag{54}$$

$$K'=G(K) \tag{55}$$

This is an ill-defined problem. Why? Something is missing $R(K)$ and $w(K)$ are wrong function of price because now K is not sufficient determinant of w and R.From firm's problem, we know

$$\begin{aligned} w &= F_2(K,N) \\ R &= 1+F_1(K,N)-\delta \end{aligned}$$

We do now N is not a state variable because it is not predetermined. Agents' behaviour determine aggragete employment and unknown to the agent at the beginning of each period. Our agents are smart enough to be able to solve the problem the firms are solving and determine the aggragate employment as a function aggragate capital.

From now on, we will only look at RCE with rational expectation and the following formulation of the consumer's problem is complete;

$$V(K,a;G,H)=\max_{c,n,a'}\{u(c,n)+\beta V(K',a';G,H)\} \tag{56}$$

subject to

$$c+a'=Ra+wn \tag{57}$$

$$K'=G(K) \tag{58}$$

$$N=H(K)$$

$$\begin{aligned} w(K). &= F_2(K,N) \\ R(K) &= 1+F_1(K,N)-\delta \end{aligned}$$

And the solutions are:

$$a'=g(K,a;G,H) \tag{59}$$

$$n=h(K,a;G,H) \tag{60}$$

Definition 8 *A representative agent RE RCE is a set of functions* $\{V(.),\ G(.),H(.),\ g(.),\ h(.)\}$ *such that*

1. *Given* $\{G(.), H(.)\}$, $\{V(.), g(.), h(.)\}$ *solves the consumer's problem.*

2.
$$G(K) = g(K, K; G, H) \tag{61}$$
$$H(K) = h(K, K; G, H) \tag{62}$$

Although compared with SPP, RCE is hard to solve, it can be used to characterize more kinds of economies, including those environments when welfare theorem does not hold and next we analyze one such economy.

8 Feb 13

8.1 RCE for non-PO economies

What we did with RCE so far can be claimed to be irrelevant. Why? Because, since the Welfare Theorems hold for these economies, equilibrium allocation, which we would like to investigate, can be solved by just solving SPP allocation. But RCE can be useful for analyzing much broader class of economies, many of them is not PO (where Welfare Theorems do not hold). That's what we are going to do from now. Let's define economies whose equilibria are not PO.

8.2 Economy with Externality

Suppose agents in this economy care about other's leisure. We would like to have beer with friends and share time with them. So other people's leisure enters my utility function. That is, the preference is given by

$$U(c, 1-n, N)$$

where L=1-N is the aggregate leisure.

One example may be
$$U = c^{\theta}(1-n)^{1-\theta}N^{\gamma} \qquad , \ \gamma < 0$$

With externality in the economy, competitive equilibrium cannot be solved from SPP.

The problem of consumer is as follows:

$$V(K, a) = \max_{c,n,a'}\{u(c, n, N) + \beta V(K', a')\} \tag{63}$$

subject to

$$\begin{aligned} c + a' &= Ra + wn && (64)\\ R &= F_K(K, N) && (65)\\ w &= F_N(K, N) && (66)\\ K' &= G(K) && (67)\\ N &= H(K) && (68) \end{aligned}$$

And the solutions are:

$$a' = g(K, a)$$

$$n = h(K, a)$$

So a RCE for this economy is a set of functions $\{V(.),\, G(.)\,, H(.),\, g(.)\,,\, h(.)\}$ satisfying the HH problem solution and RA condition. To get a better feel of the problem lets look at the sufficient FOC and Euler Equation characterizing this equilibrium;

FOC(n)

$$u_c\,(c, 1-n, N) \cdot F_2\,(K, N) - u_2\,(c, 1-n, N) = 0 \tag{69}$$

Euler equation:

$$u_c\,(c, 1-n, N) - \beta u_c\,(c', 1-n', N')\, R(K) = 0 \tag{70}$$

Homework Derive the FOC for asset holdings and envelope condition explicitly to get the Euler equation above.

We can also derive the sufficient FOC and Euler Equation characterizing the solution to SPP using the fact that we are in RA framework and our population is normalized to one thus $n = N$;

FOC(N)

$$u_c(c, 1-N, N) \cdot F_2\,(K, N) - u_2(c, 1-N, N) + u_3(c, 1-N, N) = 0 \tag{71}$$

FOC(a')

$$-u_c(c, 1-N, N) + \beta V_1'\,(K') = 0 \tag{72}$$

Envelope condition

$$V_1\,(K) = F_1\,(K, N)\, u_c(c, 1-N, N) \tag{73}$$

Thus we get Euler equation:

$$u_c(c, 1-N, N) = \beta u_c(c', 1-N', N') F_1\,(K', N') \tag{74}$$

The solution to the SP problem is a optimal policy function $G^{spp}(K)$.A useful digression at this point is couple of comments about the nature of the problem the SP solves in a RA setting. A SP solves a weighted utility maximization problem subject to feasibility constraints where the weights are to be determined as exogenous parameters. So far we always wrote down a single individual's problem to be solved by the SP and by doing so as the next homework will demonstrate, implicitly solved an equal weighted total utility maximization problem Since we are in a RA framework, we did not have to worry about the issue of transfers and we have been able to argue the solution to the weighted utility maximization under total feasibility constraint is equivalent to our individual problem in the sense, the FOC that characterizes the SP solution to individual problem is also sufficient to characterize the SP solution to the equal weighted total utility maximization.

Homework Consider the following equal weight SPP problem with identical agents,

$$\max_{\substack{\{n_j\} \\ n_j \in [0,1]}} \sum_{j=1}^{T} u(1-n_j, \sum_{i=1}^{T} n_i)$$

Derive the FOC(s) that characterizes the solution to this problem and show that the characterization do not depend on T.

Another important point is about characterizing the solution to the HH problem in RA RCE framework. Once we take the FOCs that characterize HH behavior we can use the RA equlibrium condition and exploit the fact that $a = K$ and $n = N$ by substituting for individual decision rules and variables with aggragate ones. By doing so we will be able to write down the two functional equations that characterize the equilibrium and solution to SPP only as functions of $G(K), G^{spp}(K), H(K), H^{spp}(K)$ and see that the solutions to them are indeed different.

After substitution of the budget constraint and using the CRTS property of technology, the two functional equations that characterize the RCE becomes;

$$\begin{aligned} & u_c(F(K, H(K)) - G(K), 1 - H(K), H(K)) \\ = \; & \beta u_c(F(G(K), H(G(K))) \\ & -G(G(K)), 1 - H(G(K)), H(G(K))) F_1\left(G(K), H(G(K))\right) \end{aligned} \tag{75}$$

$$\begin{aligned} & u_c(F(K, H(K)) - G(K), 1 - H(K), H(K)) \cdot F_2\left(K, H(K)\right) \\ & = u_2(F(K, H(K)) - G(K), 1 - H(K), H(K)) \end{aligned} \tag{76}$$

with $G(K), H(K)$ as the only arguments. Solving for equilibrium is to solve for these two functions that makes the two conditions hold for any K. Writing down the corresponding two equations for the SPP we can see that they are different and thus the solutions to two are different and equilibrium is no longer PO.

Homework Derive the two functional equations that characterizes the solution to SP version of this problem and show that they imply a different solution than a competitive solution.

So how do we solve these equations without exploiting any contraction property? Although there is no theory about the existence of such functions, the method of 'Euler Equation approximation' usually works well numerically in practice to obtain the equilibrium.

Lets stop and see where we are for a moment. In 704 we learned to solve the relatively easy recursive formulation of SPP. Then in 702 we see that under 'appropriate conditions' we can use the solution to SPP to construct competitive solutions. The problem arises when these 'appropriate conditions' are not there and what do we do then? We go back to what we did with the sequential SPP and represent the hard to solve sequential problem of RA recursively and try to solve that directly. We saw, although not trivial and easy as solving SP's recursive problem, that poblem is managable. Once we are able to solve these kind of problems, we have the necessary tools to analyze all kind of interesting economies.

9 February 17-19

9.1 RCE with Government

When we are considering RCE with government, there are several issues that we need to consider before we begin writing down the problem of the agent and the government's budget constraint. We need to make some choices about the economy that we are modelling:

Can the government issue debt? If no: The government is restricted by his period by period budget constraint. He cannot run a budget deficit or a surplus. Whatever he gets as revenues from taxes, he spends that and no more or no less. His budget constraint needs to hold at each period, the government cannot borrow from the public. Here the government expenditures are exactly equal to the tax revenues. If yes: The government can issue bonds at each period When it turns out that the government's expenditures are higher than its revenues (the tax revenues) he can issue debt or when it turns out that the expenditures are less than his revenues, he can retire debt that was issued before. If we model the economy so that the government is allowed to issue debt, then we need to deal with the issue of restricting it to accumulate public debt indefinitely. This is where the No-Ponzi Scheme comes into play. But we'll have more on that later.

We will do the second case today. So the economy is as follows:

- Government issues debt and raises tax revenues to pay for a constant stream of expenditures.
- Government debt is issued at face value with a strem of interest rate $\{\mathrm{r}_{b,t}\}$
- No shocks.
- No labor/leisure choice.

Let's write down the problem of the consumer. Notice that the consumer can transfer resources across time in two ways here: He can either save in the form of capital or he can buy bonds. In equilibrium, the rates of return on both ways of saving should be the same by no arbitrage so that $\mathrm{r}_b = r_k$. Since the rates of return on both is the same, the agent shouldn't care in what form he saves, i.e. the composition of the asset portfolio doesn't matter. So let a denote the agent's asset which consists of physical capital holding k and financial asset b. We don't need to make the distinction between the two. And let r denote the rate of return on a (which is in turn the rate of return on capital and rate of return on bonds).

Aggregate state variables are K and B where B is the government debt. Notice that G, the government expenditures, is not a state variable here since it's constant across time. The individual state variable is a, the consumer's asset holdings.

The consumer's problem:

$$V(K,B,a) = \max_{c,a'} \ \text{u(c)} + \beta V(K',B',a')$$

subject to

$$\begin{aligned}
c + a' &= a + [ra + w](1-\tau) \\
r &= r(K) = f_k(K,1) - \delta \\
w &= w(K) = f_n(K,1) \\
K' &= H(K,B) \\
B' &= \Psi(K,B) \\
\tau &= \tau(K,B)
\end{aligned}$$

Solution is: a'=$\psi(K,B,a)$

Definition 9 *Given $\tau(K,B)$, a RCE is a set of functions $\{V(.),\psi(.),H(.),\Psi(.)\}$ such that*

1. *(Household's optimization) Given $\{H(.),\Psi(.)\},\{V(.),\psi(.)\}$ solve the household's problem.*

2. *(Consistency) $H(K,B)+\Psi(K,B)=\psi(K,B,K+B)$*

3. *(No Arbitrage Condition)*

$$r_b(K,B) = 1 + F_K(H(K,B),1) - \delta$$

(The rate of return on bond is equal to the rate of return on capital; notice we already used this fact when we were writing down the problem of the consumer by letting r denote the rate of return on both and not distinguishing between them)

4. *(Government Budget Constraint)*

$$\Psi(K,B) + [f(K,1) - \delta K + (f_k(K,1) - \delta)B](1 - \tau(K,B)) = \bar{G} + B[1 + f_k(K,1) - \delta]$$

(So that the government's resources each period are the bonds that it issues ($\Psi(K,B)$), plus its revenues from tax on rental income, wage income, and income on the interest on bonds. Its uses are the government expenditures (G), and the debt that it pays back.)

5. *(No Ponzi Scheme Condition) $\exists \underline{B}$, $\overline{B}, \underline{K}$, $\overline{K}$ such that $\forall K,B \in [\underline{K},\overline{K}]x[\underline{B},\overline{B}]$*

$$\Psi(K,B) \in [\underline{B},\bar{B}]$$

10 Feb 24-26

10.1 Extensions to our standard economy

10.1.1 Neo-classical firm with a dynamic problem

Our analysis so far have left the firm's static problem lingering in the background and primarily focused on the HH behaviour. This is merely a matter of choice and as we will show below firm's problem can be formulated in a dynamic manner without resulting any substansive changes in our main results. The firm is defined as an entity with a unit of land. The land is not used in the production process. The firm makes the dynamic investment decision and owns the capital and households owns the shares of the firm. Then the problem of our household and firm is,

$$\begin{aligned} V(K,a) &= \max_{a',c}\{U(c)+\beta V[G(K),a']\} \qquad (77) \\ s.t. \quad c+a' &= R(K)a \end{aligned}$$

$$\Omega(K,k) = \max_{k'}\{F(k)-k'+q(G(K))\Omega(G(K),k')\} \qquad (78)$$

with solutions;

$$\begin{aligned} a' &= h(K,a) \\ k' &= g(K,k) \end{aligned}$$

The way to proceed in defining the recursive equilibrium and characterizing it is similar only a bit more tedious and left as an homework.

Homework Define RCE and derive the relevant Euler Equation for both the firm and the household problem using First Order and Envelope conditions. Simplify them as one functional equation in one unknown using the relevant RA conditions.

10.1.2 Two country economy

We have two countries (A,B), i.e. two growth models. We can think of these economies as different islands. We will assume capital and labor cannot move but output can, so that agents cannot increase the capital stock of the other country at any given period. The remaining structure of the HH problem is identical to our previous economies What is RCE? What are the state variables?

The state variables are aggregate wealth and capital stock of the agents in both countries What are the ways of holding wealth in this economy? Our agents have the option to either invest in their home country or in the other country.as K^A or K^B. Note that we no longer have the equality of aggregate wealth and capital stock.in a particular country. But individual $K's$ are not a sufficient as aggregate state variable because we also need the aggregate wealth for the evolution of aggregate capital stock in each country. Since we know $W^A+W^B=K^A+K^B$ has to be true, keeping track of only one country's wealth share S^A, and K^A and K^B will be sufficient. Individual state variable is asset holdings, but as we will see by arbitrage, we do not have to specify which sort of asset the agent is holding in her budget constraint. So our problem becomes,

$$\begin{aligned} V^j(S,a) &= \max_{c,a'} \left\{ U(c) + \beta V^j(G_A(S), G_B(S), S_A(S), a' \right\} && (79) \\ s.t. \quad c + a' &= w_j(S) + R(S)a \\ j &\in \{A,B\} \quad \text{and } S = \{K_A, K_B, S_A\} && (80) \\ \text{with solution} \qquad a'_j &= g_j(S,a) && (81) \end{aligned}$$

What is RCE?

Definition 10 *RCE is a set of functions* $\{V^j(.), g_j(.)\}$ *and* $\{G_j(.), S_j(.), w_j(S), R(S)\}$, $j \in \{A,B\}$*, such that;*

1. Given $\{G_j(.), S_j(.), w_j(S), R(S)\}, \{V^j(.), g_j(.)\}$ solves the households problem.
2. $\{w_j(S), R(S)\}$ solves the firms problem
3. (RA condition)

$$g_j(S, S_j * (K^A + K^B)) = S_j(S) * (G_A(S) + G_B(S))$$

4. No arbitrage

$$R(G_A(S), G_B(S), S_A(S)) = F_1^A(G_A(S)) = F_1^B(G_B(S))$$

Homework Derive the FOC for an individual living in country j and using the Envelope condition and RA conditions write the Euler equation as a functional equation.

10.2 Stochastic Growth Model

10.2.1 Shock and History

We will now look at the stochastic RA-NGM.

What is a shock? Unanticipated change? Not really:

In a stochastic environment, we don't know exactly what will happen but we know where it's coming from (we know something about the stochastic process, i.e. the process that the shocks are following)

Markov Chains In this course, we will concentrate on Markov productivity shock. Markov shock is a stochastic process with the following properties.

1. There are finite number of possible states for each time. More intuitively, no matter what happened before, tomorrow will be represented by one of a finite set.

2. The only thing that matters for the realization tomorrow is today's state. More intuitively, no matter what kind of history we have, the only thing you need to predict realization of shock tomorrow is today's realization.

   More formally, for each period, suppose either z^1 or z^2 happens[2] [3]. Denote z_t is the state of today and Z_t is a set of possible state today, i.e. $z_t \in Z_t = \{z^1, z^2\}$ for all t. Since the shock follow Markov process, the state of tomorrow will only depend on today's state. So let's write the probability that z^j will happen tomorrow, conditional on today's state being z^i as $\Gamma_{ij} = prob[z_{t+1} = z^j | z_t = z^i]$. Since Γ_{ij} is a probability, we know that

$$\sum_j \Gamma_{ij} = 1 \qquad for\ \forall i \tag{82}$$

   Notice that 2-state Markov process is summarized by 6 numbers: z^1, z^2, Γ_{11}, Γ_{12}, Γ_{21}, Γ_{22}.

The great beauty of using Markov process is we can use the explicit expression of probability of future events, instead of using weird operator called expectation, which very often people don't know what it means when they use.

Representation of History

- Let's concentrate on 2-state Markov process. In each period, state of the economy is $z_t \in Z_t = \{z^1, z^2\}$.
- Denote the history of events up to t (which of $\{z^1, z^2\}$ happened from period 0 to t, respectively) by

  $h_t = \{z_0, z_1, z_2, ..., z_t\} \in H_t = Z_0 \times Z_1 \times ... \times Z_t$.
- In particular, $H_0 = \emptyset$, $H_1 = \{z^1, z^2\}$, $H_2 = \{(z^1, z^1), (z^1, z^2), (z^2, z^1), (z^2, z^2)\}$.
- Note that even if the state today is the same, past history might be different. By recording history of event, we can distinguish the two histories with the same realization today but different realizations in the past (think that the current situation might be "you do not have a girl friend", but we will distinguish the history where "you had a girl friend 10 years ago" and the one where you didn't
- Let $\Pi(h_t)$ be the unconditional probability that the particular history h_t does occur. By using the Markov transition probability defined in the previous subsection, it's easy to show that (i) $\Pi(h_0) = 1$, (ii) for $h_t = (z^1, z^1)$, $\Pi(h_t) = \Gamma_{11}$ (iii) for $h_t = (z^1, z^2, z^1, z^2)$, $\Pi(h_t) = \Gamma_{12}\Gamma_{21}\Gamma_{12}$.
- $\Pr\{z_{t+1} = z^i \mid z_t = z^j, z_{t-1}, z_{t-2,..........}\} = \Gamma_{ji}$
- Having finite support of the distribution is very convenient.

[2]In this class, superscript denotes the state, and subscript denotes the time.

[3]Here we restrict our attention to the 2-state Markov process, but increasing the number of states to any finite number does not change anything fundamentally.

Social Planner's Problem with Shocks

- Social Planner's Problem (the benevolent God's choice) in this world is a state-contingent plan, i.e. optimal consumption and saving (let's forget about labor-leisure choice in this section for simplicity) choice for all possible nodes (imagine the nodes of a game tree. we need to solve optimal consumption and saving for each node in the tree).
- Notice that the number of nodes for which we have to solve for optimal consumption and saving is countable. This feature allows us to use the same argument as the deterministic case to deal with the problem. The only difference is that for deterministic case, the number of nodes is equal to number of periods (which is infinite but countable), but here the number of nodes is equal to the number of date-events (which is also infinite but countable).
- More mathematically, the solution of the problem is the mapping from the set of date-events (which is specified by history) to the set of feasible consumption and saving.

$$\max_{\{c_t(h_t), k_{t+1}(h_t)\}} \sum_{t=0}^{\infty} \beta^t \sum_{h_t \in H_t} \Pi(h_t) u(c_t(h_t))$$

subject to

$$c_t(h_t) + k_{t+1}(h_t) = (1-\delta)k_t(h_{t-1}) + z_t F[k_t(h_{t-1}), 1] \quad \forall t \ \forall h_t$$

$$k_0, z_0 \text{ given}$$

What is the dynamic programming version of this problem?

$$V(z,k) = \max_{c,k\prime} u(c) + \beta \sum_{z\prime \in Z} \Gamma_{zz\prime} V(z\prime, k\prime)$$

subject to

$$c + k\prime = (1-\delta)k + zF(k,1)$$

11 March 16-18

11.1 Lucas Tree Model

We will study Lucas Tree Model (Lucas 1978[4]) and look at the things that Finance people talk about. Lucas tree model is a simple but powerful model.

[4]Lucas, R. (1978). "Asset prices in an exchange economy." Econometrica 46: 1429-1445

11.1.1 The Model

Suppose there is a tree which produces random amount of fruits every period. We can think of these fruits as dividends and use z_t to denote the stochastic process of fruits production. $z_t \in \{z^1, ...z^{nz}\}$. Further, assume z_t follows Markov process. Formally:

$$z_t \sim \Gamma(z_{t+1} = z_i \mid z_t = z_j) = \Gamma_{ji} \tag{83}$$

Let h_t be the history of realization of shocks, i.e., $h_t = (z_0,\ z_1,\ ...,\ z_t)$. Probability that certain history h_t occurs is $\pi(h_t)$.

Household in the economy consumes the only good, which is fruit. We assume representative agent in the economy, and there is no storage technology. In an equilibrium, the first optimal allocation is that the representative household eats all the dividends every period. We will look at what the price has to be when agents use markets and start to trade. First, we study the Arrow-Debreu world. And then, we use sequential markets to price all kinds of derivatives, where assets are entitlement to consumption upon certain date-event.

11.1.2 Arrow-Debreu World

Consumers;s problem

$$\max_{\{c(h_t)\}_{t=0}^{\infty}} \sum_t \beta^t \sum_{h_t \in H_t} \pi(h_t) u(c_t(h_t)) \tag{84}$$

subject to

$$\sum_t \sum_{h_t \in H_t} p(h_t) c_t(h_t) = \bar{a} = \sum_t \sum_{h_t \in H_t} p(h_t) z_t(h_t) \tag{85}$$

Equilibrium allocation is autarky

$$c_t(h_t) = z_t(h_t) \tag{86}$$

Now the key thing is to find the price which can support such equilibrium allocation.

Normalize

$$p\left(h_0\right) = 1$$

Take first order condition of the above maximization problem and also substitute (86)

FOC

$$\beta^t \pi(h_t) u_c(z_t(h_t)) = p_t(h_t)\lambda \tag{87}$$

$$u_c\left(z_0\right) = \lambda \tag{88}$$

We get the expression for the price of the state contingent claim in the Arrow-Debreu market arrangement.

$$p_t(h_t) = \frac{\beta^t \pi(h_t) u_c(z_t(h_t))}{u_c(z_0)} \tag{89}$$

Note that the price $p_t(h_t)$ is in terms of time 0 consumption.

What is the value of the tree? It is the value of its fruits. Value of an asset is the value of the things it yields. This is key in calculating the price of an asset. Then the value of the tree is,

$$\text{Value of the tree} = \sum_t \sum_{h_t} p(h_t) z(h_t)$$

$$q(z_0) = \sum_t \sum_{h_t} \frac{\beta^t \pi(h_t) u_c[z(h_t)]}{u_c[z(z_0)]} z(h_t)$$

11.1.3 Sequence of Markets

How will the agent transfer resources across periods in the sequence of markets? By buying and selling shares of the tree. And in equilibrium, he can hold only 1 share of the tree, so we want to find the prices that will make him do exactly that.

In sequential market, we can think of stock market where the tree is the asset. Household can buy and sell the asset. Let s_t be share of asset and q_t be the asset price at period t. The budget constraint at every time-event is then:

$$qs' + c = s(q + z) \tag{90}$$

First, we can think of any financial instruments and use the A-D prices $p_t(h_t)$ to price them.

1. The value of the tree in terms of time 0 consumption is indeed

$$\sum_t \sum_{h_t \in H_t} p(h_t) z_t(h_t)$$

2. A contract that gives agent the tree in period 3 and get it back in period 4: This contract is worth the same as price of harvests in period 3:

$$\sum_{h_3 \in H_3} p(h_3) z_3(h_3)$$

3. Price of 3-year bond: 3 year bond gives agents 1 unit of good at period 3 with any kinds of history. The price is thus

$$\sum_{h_3 \in H_3} p(h_3)$$

11.1.4 Market Equilibrium

We will write it in a recursive form. Then, first can we get rid of h_t and write it in a recursive form? Or are prices stationary? The answer depends on whether the stochastic process is stationary.

Look at Problem Set 7 solutions for 2004, question 1 for the recursive formulation of the problem and the characterization of q.

11.1.5 Constructing price of a bond

Remember that the budget constraint of the agent is:

$$c(h_t) + s(h_t)q(h_t) = s(h_{t-1})\left[z_t + q_t(h_t)\right]$$

Now let's add one period ahead securities,

$$c(h_t)+s(h_t)q(h_t)+\sum_{z_{t+1}} a(h_t, z_{t+1})p^A(h_t, z_{t+1})+p^b(h_t)b(h_t) = s(h_{t-1})\left[z_t + q_t(h_t)\right]+a(h_{t-1}, z_t)+b(h_{t-1})$$

where b is the bond. Do we have enough securities or do we have too much? We have too much, because one of the assets here is redundant. A bond can be replicated by the one period ahead state contingent securities. A one period ahead state contingent security that gives one unit of the good at each state tomorrow would be equivalent to a bond. By no arbitrage, the following holds (the price of the two ways of getting one unit of good for sure tomorrow has to equal to each other):

$$\sum_{z_{t+1}} p^A(h_t, z_{t+1}) = p^b(h_t)$$

Homework 11 *Pricing two period American and European options at a node h_t.*

1. *Price an option to sell shares at price $\overline{q} = 3$, only two periods ahead (i.e. you can only exercise the option two periods ahead). This is called a European option.*

2. *Price an option to sell shares at price $\overline{q} = 3$, at any time before its maturity (i.e. you can exercise the option either tomorrow or the day after). This is called an American option.*

12 March 23-25

12.1 Measure Theory

- We will use measure theory as a tool to describe a society with heterogenous agents. Most of the previous models that we dealt with, there was a continuum of identical agents, so we saw the economy as consisting of only one type of agent.But from now on in the models that we deal with there will be heterogenous agents in the economy. The agents will differ in various ways: in their preferences, in the shocks they get, etc. Therefore, the decisions they make will differ also. In order to describe such a society, we need to be able to keep track of each type of agent. We use measure theory to do that.

- What is measure?

  Measure is a way to describe society without having to keep track of names. But before we define measure, there are several definitions we need to learn.

Definition 11 *For a set A, $\mathcal{A}$ is a set of subsets of A.*

Definition 12 $\sigma - a\lg ebra$ *$\mathcal{A}$ is a set of subsets of A, such that,*

1. *$A, \emptyset \in \mathcal{A}$*
2. *$B \in \mathcal{A} \Rightarrow B^c/A \in \mathcal{A}$ (closed in complementarity)*
   where $B^c/A = \{a \in A : a \notin B\}$
3. *for $\{B_i\}_{i=1,2...}$, $B_i \in \mathcal{A} \Rightarrow [\cap_i B_i] \in \mathcal{A}$ (closed in countable intersections)*

- An example for the third property: Think of a σ-algebra defined on the set of people in a classroom. The property of being closed in countable intersections says that if the set of tall people is in the σ-algebra, and the set of women is in the σ-algebra, then the set of tall women should be in the σ-algebra also.
- Consider the set A={1,2,3,4} Here are some examples of σ-algebras defined on the set A:

$\mathcal{A}^1$= $\{\emptyset, A\}$
$\mathcal{A}^2 = \{\emptyset, A, \{1\}, \{2, 3, 4\}\}$
$\mathcal{A}^3 = \{\emptyset, A, \{1\}, \{2\}, \{2, 3, 4\}, \{1, 3, 4\}, \{3, 4\}, \{1, 2\}\}$
$\mathcal{A}^4 =$ The set of all subsets of A.

Remark 5 *A topology is a set of subsets of a set also, just like a σ-algebra. But the elements of a topology are open intervals and it does not satisfy the property of closedness in complementality (since a complement of an element is not an element of the topology). Therefore topology is not a σ-algebra.*

Remark 6 *Topologies and Borel sets are also family of sets but we use them to deal with continuity, and σ-algebra we use to deal with weight.*

- Think of the σ-algebras defined on the set A above. Which one provides us with the least amount of information? It is $\mathcal{A}^1$. Why? Because from $\mathcal{A}^1$, we only know whether an element is in the set A or not. Think of the example of the classroom. From $\mathcal{A}^1$,all we get to is whether a person is in that classroom or not, we learn nothing about the tall people, short people, males, females, etc. The more sets there are in a σ-algebra, the more information we have.This is where the Borel sets are useful. A Borel set is a σ-algebra which is generated by a family of open sets. Since Borel σ-algebra contains all the subsets generated by intervals, you can recognize any subset of set, using Borel σ-algebra. In other words, Borel σ-algebra corresponds to complete information.
- Now we are ready to define measure:

Definition 13 *A measure is a function $x : \mathcal{A} \to \mathcal{R}_+$ such that*

1. *$x(\emptyset) = 0$*

2. *if* $B_1, B_2 \in \mathcal{A}$ *and* $B_1 \cap B_2 = \emptyset \Rightarrow x(B_1 \cup B_2) = x(B_1) + x(B_2)$ *(finite additivity)*

3. *if* $\{B_i\}_{i=1}^{\infty} \in \mathcal{A}$ *and* $B_i \cap B_j = \emptyset$ *for all* $i \neq j \Rightarrow x(\cup_i B_i) = \sum_i x(B_i)$ *(countable additivity)*

Definition 14 *Probability (measure) is a measure such that* $x(A) = 1$

- Consider the set A=[0,1] where a $\in A$ denotes wealth. So A is the set of wealth levels normalized to 1.

- We will define x: $\mathcal{A} \to \mathcal{R}_+$ as a probability measure so that the total population is normalized to one. Using measure, we can represent various statistics in a simple form:

1. The total population:
$$\int_A dx = x(A) = 1$$

2. Average wealth:
$$\int_A adx$$
We go through each levels of wealth in the economy and multiply the wealth level by the proportion of people that have that wealth level, and because the size of the society is normalized to 1, this gives us average wealth.

3. Variance of wealth:
$$\int_A [a - \int_A adx]^2 dx$$

4. Coefficient of variation:
$$\frac{\{\int_A [a - \int_A adx]^2 dx\}^{1/2}}{\int_A adx}$$

5. Wealth level that seperates the 1% richest and the poorest 99% is $\widetilde{a}$ that solves the following equation:
$$0.99 = \int_A 1_{[a \geq \widetilde{a}]} dx$$

Homework *Write the expression for the Gini index.*

12.2 Transition Function

Definition 15 *A transition function $Q : A \times \mathcal{A} \rightarrow \mathcal{R}$ such that:*

1. *$\forall \bar{B} \in \mathcal{A}$, $Q(., \bar{B}) : A \rightarrow \mathcal{R}$ is measurable,*
2. *$\forall \bar{a} \in A$, $Q(\bar{a}, .) : \mathcal{A} \rightarrow \mathcal{R}$ is a probability measure.*

Q function is a probability that a type a agent ends up in the type which belongs to B. What the first condition above says is the following: Whatever we need to know for today has to be sufficient to specify what the probability tomorrow is.

Consider the set A the elements of which are the possiblel states of the world, say good and bad.

A={good, bad}

And let Γ be the transition matrix associated with A.

$$\Gamma_{ii'} = \begin{bmatrix} \Gamma_{gg} & \Gamma_{gb} \\ \Gamma_{bg} & \Gamma_{bb} \end{bmatrix}$$

The $\sigma-$algebra we would want to use is $\mathcal{A} = \{\emptyset, A, \{good\}, \{bad\}\}$

Notice that what we need to know to compute the probability of a certain state tomorrow is today's state and the $\sigma-$algebra lets us know that.

- The measure x defined on a sigma algebra over the set of intervals of wealth is complete description of the society today.
- x'(B): Measure of people who have characteristics in B$\in \mathcal{A}$ tomorrow.
- The pair x,Q will tell us about tomorrow.

  Pick one measure that's defined over the $\sigma-$algebra, $\mathcal{A}$, on the set of wealth levels A=[0,1]

  $$x'(B) = \int_A Q((s,a), B)dx$$

  So with x and Q, we get x'.i.e. with the measure of people today and the transition function Q, we get the measure of people with wealth level in a certain interval B=[a,b] tomorrow.

  What people are now (x) + What people do (Q) $\rightarrow x'$

12.3 Introduction to the Economy with Heterogenous Agents

Imagine a Archipelago that has a continuum of islands. There is a fisherman on each island. The fishermen get an endowment s each period. s follows a Markov process with transition $\Gamma_{ss'}$ and,

s$\in \{s^1, ..., s^{n_s}\}$

The fishermen cannot swim. There is a storage technology such that, if the fishermen store q units of fish today, they get 1 unit of fish tomorrow. (s,a) is the type of a fisherman and the set consisting of all possible such pairs is,

$$S \times A = \{s^1, s^2, ..., s^n\} \times [0, \overline{a}]$$

Let $\mathcal{A}$ be the set of Borel sets on SxA. And define a probability measure x on $\mathcal{A}$,

$$x : \mathcal{A} \rightarrow [0, 1]$$

The fisherman's problem is:

$$V(s, a) = \max_{c,a'} \text{ u(c) } + \beta \sum_{s'} \Gamma_{ss'} V(s', a')$$

subject to

$$c + qa\prime = s + a$$

$$c \succeq 0 \text{ and a' } \in [0, \overline{a}]$$

With the decision rule a'=g(s,a) and the transition matrix for the endowment process $\Gamma_{ss'}$, we can construct the transition matrix. The transition function Q(s,a,B) tells us the probability that a fisherman with (s,a) today ends up in some $B_s \times B_a \in \mathcal{A}$ tomorrow (where B_s and B_a are the projections of B over the spaces S and A).

The transition function is constructed as follows

$$Q(s, a, B) = 1_{[g(s,a) \in B_a]} \sum_{s' \in B_s} \Gamma_{ss'}$$

The First Order Conditions are,

$$u_c(\text{s+a-}qa\prime)\text{=}\frac{\beta}{q}\sum_{s'}\Gamma_{ss'}u_c(\text{s'+a'-qa''})$$

This is a second order difference equation and there are many solutions that satisfy it. But only one of those solutions does not violate compactness. So if we can find some natural bounds for a', or impose conditions so that a' has bounds, the above first order conditions along with the bounds that we have for savings, will characterize the optimal solution.

You'll notice that $a' \in [0, \overline{a}]$ is already one of the constraints of the above maximization problem. But now rather than just imposing such a constraint, we will find a natural reason that savings should have a lower bound and we will consider a condition that ensures an upper bound for savings.

For the lower bound, we assume that there is no technology which allows negative amount of saving and this sounds natural since storing a negative amount of fish does not make much sense. So savings has a lower bound because Mother Nature says so.

For the time being, consider an economy where there is no uncertainty. In such an economy, the following theorem holds,

Theorem 10 *If $\beta < q$, then $\exists \bar{a}$ such that, if $a_0 < \bar{a}$, $a_t < \bar{a} \quad \forall t$.*

You can see this formally through the usual Euler equations,

$$u_c(c_t) = \frac{\beta}{q} u_c(c_{t+1}) \quad \forall t$$

From these equations, it's clear that $\beta < q \Rightarrow u_c(c_{t+1}) > u_c(c_t) \Rightarrow c_{t+1} < c_t \quad \forall t$

If you are impatient enough compared with the returns from technology, you will consume today rather than tomorrow. Gains from saving will disappear eventually and you will stop saving more.

Now think about the economy with uncertainty. Here, the fisherman has the risk of getting a very bad shock tomorrow. So the fisherman would save just in case he has this bad shock; he would want to store some fish today in order to insure himself against getting very small number of fish tomorrow so he is not hungry in case that happens. In this case we need to think more about how to put an upper bound on savings, because with uncertainty even if $\beta < q$, the fisherman is willing to save due to gains from insurance. The kind of savings to protect oneself from risk in the future in the absence of state contingent commodities, we call precautionary savings. In order to ensure an upper bound for savings, we need to bound the gains from insurance somehow. The way to do this is to impose the condition on the utility function that its negative curvature (keeping in mind that the utility function is concave) is diminishing as wealth increases. This means that wealthier agents are less risk-averse. Formally, that u' is convex. The wealthier the agent is, the smaller the variance of his endowment next period proportional to his wealth so he doesn't want to save if he is very wealthy

So in the economy with uncertainty, in order to have an upper bound on savings, we need the first derivative of the utility function to be convex so that the following Jensen's Inequality holds:

$$\frac{\beta}{q} \int \Gamma_{ss'} u_c(c') > \frac{\beta}{q} u_c(\int \Gamma_{ss'} c')$$

Theorem 11 *If $\beta < q$ and u' is convex then $\exists \bar{a}$ such that $a_0 < \bar{a}$, g(s,a)$< \bar{a} \quad \forall s$.*

13 March 30: Industry Equilibrium

We will go over a version of Hopenhayn's 1992 paper "Entry, Exit, and Firm Dynamics in Long Run Equilibrium". We will look at stationary industry equilibria. Before proceeding with defining the environment and considering several versions of Hopenhayn's model, we will first look at the firm's problem in more detail.

Note that we are ignoring the problem of the household here. There is a representative household and the wages and interest rates are constant.

Consider a firm who uses labor to produce the output good and has productivity $s \in S$. The production technology of the firm is given by $y = sf(n)$. The firm chooses how much labor to employ given the wage, w, and given its productivity,s. The firm's problem is,

$$\max_n psf(n) - wn$$

The solution n^* solves,

$$psf'(n) = w$$

Then the two period profit of the firm is,

$$\pi_2 = [p^* sf(n^*) - wn^*]\left[1 + \frac{1}{1+r}\right]$$

Now suppose that the firm will only operate next period with probability $(1-\delta)$. With probability δ it will die. In that case, the two period profit of the firm is,

$$\pi_2 = [p^* sf(n^*) - wn^*]\left[1 + \frac{1-\delta}{1+r}\right]$$

Now consider the infinite periods profit of the firm,

$$\begin{aligned}\pi_\infty &= [p^* sf(n^*) - wn^*] \sum_{t=0}^{\infty}\left(\frac{1-\delta}{1+r}\right)^t \\ &= [sf(n^*) - wn^*]\left(\frac{1+r}{r+\delta}\right)\end{aligned}$$

The zero profit condition is that the profit from entry is equal to the cost of entry, denoted by c^0. This condition says that there are no further incentives to enter the industry:

$$c^0 = \pi_\infty$$

What is a firm here? A firm is technology s and size n. This is how we define a firm.

14 April 1: Industry Equilibrium (contd.)

14.1 Basic Model

Consider an industry that is composed of only one firm. Let's first describe the environment:

1. Aggregate demand is given by $y^D(p)$ and the input price by w. There is a single input, labor, and the production technology is $y = sf(n)$, where $s \in$ S is the productivity of firm and is constant, and n is the labor employed by the firm. The production technology exhibits decreasing returns to scale. S is compact.

2. r and w are given.

3. δ is the exogenous death rate of the firm.

4. c^0 is the fixed cost of entry.

Define $x : \mathcal{S} \rightarrow R$ as the measure of firms, where $\mathcal{S}$ is the $\sigma - algebra$ defined on the set S).

Definition 16 *A stationary industry equilibrium is* (p^*, x^*, n^*, e^*) *such that,*

1. *(Firm optimizes)* $n^*(s)$ *solves the problem of the firm:*

$$n^* \in \arg\max_n \quad p^* s f(n) - w^* n$$

2. *(Markets clear)*

$$y^D(p^*) \quad = \quad sf(n^*(s)) \;\; (\textit{Mkts clr for the output good}) \tag{91}$$

where y^D *denotes the demand for the output good.*

3. *Mass of new entrants is equal to the mass of incumbents who die:*

$$e^* = \delta x^*(S) \tag{92}$$

where e^* *denotes the mass of new entrants.*

4. *The stationary distribution* x^* *satisfies,*

$$x^*(s) = (1-\delta)x^*(s) + \delta x^*(s) \tag{93}$$

5. *Zero profit condition,*

$$c^0 = \frac{1+r}{r+\delta} \;\; [p^* s f(n^*) - w^* n^*] \tag{94}$$

14.2 Tweak 1: Many Firms

Now we add the following elements to the basic model described above: Instead of only one type of firm, now consider an industry that is composed of a continuum of firms which produce a homogenous product. Each firm has productivity $s \in S$, S is compact. The incumbent firms have constant productivities, and the new entrants draw their productivity shock from a distribution with cdf $F(s)$. Again, firms die at exogenous rate δ.

Note that here, the distribution of firms completely reflects the distribution from which they draw their productivity shocks, $F(s)$. This is because what types of firms remain or what types of firms exit is not an issue since there is exogenous entry and exit. For example, if exit was endogenous we would expect the 'bad' firms to exit and the better ones to stay, and therefore the type distribution of incumbent firms would be different than the initial distribution F(s). But in our case, the distribution of incumbents and the initial type distribution are identical.

Homework 12 *Define stationary equilibrium in this economy.*

14.3 Tweak 2: Changing Productivity

Now we are dropping the assumption that the productivity of the incumbent firms stays constant; s follows a Markov process independent across firms with conditional probability $\Gamma(s, s')$, which has the stochastic dominance property:

$$For\ s_1, s_2 \in S, \quad s_1 < s_2 \Rightarrow \int_{\tilde{s}}^{\bar{s}} \Gamma(s_1, s)\ ds \leq \int_{\tilde{s}}^{\bar{s}} \Gamma(s_2, s)\ ds \ \ \forall \tilde{s} \in S$$

The entry and exit decisions are still exogenous. Incumbent firms die at rate δ. The following are the equilibrium conditions:

1. (Market clearing condition)

$$y^D(p^*) \ \ = \ \ \int_S sf(n^*(s))$$

2. (Stationarity)

$$x^*(B) = \int_S \int_S (1-\delta)\Gamma(s, s')1_{s' \in B}\ dx^*(s)\ ds' + \int_S 1_{s' \in B} F(ds') \tag{95}$$

where B is a Borel set over S.

The first term in (95) counts the 'survivors' (incumbents who don't die) who get productvity shock $s' \in B$ given s. The second term is the new entrants who have productivity shocks $s' \in B$.

Notice that until this point, whatever results we get out of this model, they will be a mere reflection of our choice of F and Γ. For example, we can get the result that the incumbent firms' sizes are bigger compared to the new entrants simply by making assumption on the transition matrix that enables them to move up. But this is not very interesting: What we put in this model is what we get. Now we will start looking at a more interesting 'tweak' and we will endogenize the entry and exit decisions.

14.4 Tweak 3: Endogenous Entry and Exit

Now at each period firms make entry and exit decisions. What is a sufficient mechanism to get firms to quit? Having fixed costs. A fixed cost c_f must be paid every period by incumbent firms. Each period, incumbent firms decide to stay or exit.

The value of a firm with productivity s is the following:

$$\pi(s) = \max \ \ \left[\max_n \ p^* sf(n) - wn - c^f + \frac{1}{1+r} \int_S \Gamma(s, s')\pi(s') ds', \ \ 0 \right] \tag{96}$$

Homework 13 *Consider the decision of the firm to exit or stay. Show that the stochastic dominance property of* Γ *is a sufficient condition to characterize the firms decision with a threshold* s^* *such that, if* $s \in [\underline{s}, s^*)$ *the firm quits and if* $s \in [s^*, \overline{s}]$ *the firm stays.*

Suppose s^* is the threshold productivity shock, below which the firm quits. In this industry the distribution of firms evolves according to:

$$x'(B) = \int_S \int_{s^*}^{\bar{s}} \Gamma(s,s') 1_{s' \in B}\, dx(s)\, ds' + \left(\int_{\underline{s}}^{s^*} dx \right) \int_S 1_{s' \in B} dF(s) \tag{97}$$

The first term in (97) counts all the firms who decide the stay this period (i.e. all firms such that $s \in [s^*, \bar{s}]$) and who end up having some productivity s' in B. The second term counts the new entrants who get $s' \in B$. Now let's write down the steady state conditions:

$$x^*(B) = \int_S \int_{s^*}^{\bar{s}} \Gamma(s,s') 1_{s' \in B}\, dx^*(s)\, ds' + \left(\int_{\underline{s}}^{s^*} dx^* \right) \int_S 1_{s' \in B} dF(s) \tag{98}$$

The market clearing condition is,

$$y^D(p^*) = \int_S sf(n^*) dx^*(s) + \left(\int_{\underline{s}}^{s^*} dx^* \right) \int_S sf(n^*(s)) dF(s) \tag{99}$$

And the zero profit condition:

$$c^0 = \frac{1}{1+r} \int \pi(s) dF(s) \tag{100}$$

15 April 6-8

15.1 Growth

In our analysis so far, we have used Neo-classical Growth Model as our benchmark model and built on it for the analysis of more interesting economic questions. One peculiar characteristic of our benchmark model, unlike its name suggested, was lack of growth. Many interesting questions in economics are related to the cross-country differences of growth rates and we will cover some models that will allow for growth so that we will be able to attempt to answer such questions.

15.1.1 Exogenous growth

What does it take for an economy to grow? Before answering that question, we know in our standard NGM there is basically two ways of growth, one in which everything grows, which is not necessarily a per-capita growth, and the other is per-capita growth. We will be focusing on per-capita growth. The title exogenous growth refers to the structure of models in which growth rate is determined exogenously, and is not an outcome of the model. First and the simplest one of these is one in which the determinant of the growth rate is population growth.

Growth with population Suppose the population of our economy grows at rate γ and we have the classical CRTS technology in capital and labor inputs.

$$\begin{aligned} Y_t &= AF(K_t, N_t) \\ N_t &= N_0 * \gamma^t \end{aligned} \tag{101}$$

Note that our economy is no longer stationary but as we will see, within the exogenous growth framework we can make these economies look like stationary ones by re-normalizing the variables. Thus, at the end of the day it will only be a mathematical twist on our standard growth model. Once we do that, we will be looking for the counterpart of a steady state that we have in our stationary economies, the Balanced Growth Path, in which all the variables will be growing at constant rates but not necessarily equal. Back to our population growth model, we know

$$AF(K, N) = A[KF_k(K, N) + NF_N(K, N)] \tag{102}$$

Question is, if N is growing at rate γ, can this economy have a balanced growth path. Can we construct one? We know by CRTS property F_K and F_N are homogenous of degree zero. If we assume capital stock grows at rate γ as well, then prices stay constant and per-capita variables are constant and output grow at the same rate. So we get growth on a balanced growth path without per-capita growth. One question is how do we model population growth in our representative agent model. One way is to assume there is a constant proportion of immigration to our economy from outside but this has to assume the immigrants are identical to our existing agents in our economy, which is a bid problematic. The other way could be to assume growing dynasties which preserves the representative agent nature of our economy. If we do so, the problem of the social planner becomes,

$$\begin{aligned} & \max \sum_{t=0}^{\infty} \beta^t N_t U(\frac{C_t}{N_t}) \\ st \quad C_t + K_{t+1} &= AF(K_t, N_t) + (1-\delta)K_t \end{aligned} \tag{103}$$

To transform the budget set to per capita terms, divide all terms by N_t and to make the environment stationary by dividing all the variables by γ^t and assume $N_0 = 1$, we get,

$$\begin{aligned} & \max \sum_{t=0}^{\infty} (\beta\gamma)^t U(\widehat{c}_t) \\ st \quad \widehat{c}_t + \gamma\widehat{k}_{t+1} &= AF(\widehat{k}_t, 1) + (1-\delta)\widehat{k}_t \end{aligned} \tag{104}$$

So how is this transformed model any different than our NGM? By the discount factor, the agents in this economy with growth discounts the future less but everything else is identical to NGM of course with the exception of this economy growing at a constant rate.

Now suppose we have a 'labor augmenting' productivity growth with constant population normalized to one, i.e. have the following CRTS technology,

$$Y_t = AF(K_t, \gamma^t N_t) \tag{105}$$
$$AF(K_t, \gamma^t N_t) = A[K_t F_k(K_t, \gamma^t N_0) + \gamma^t N_0 F_N(K_t, \gamma^t N_0)] \tag{106}$$

Can we have an BGP? The problem is,

$$\begin{aligned} & \max \sum_{t=0}^{\infty} \beta^t U(C_t) && (107) \\ st \quad C_t + K_{t+1} &= AF(K_t, \gamma^t N_0) + (1-\delta)K_t \end{aligned}$$

and since we have a population of one, these variables are already per-capita terms. For stationarity, we have to normalize the variables to 'per productivity' units, by dividing all by γ^t. Then the problem becomes,

$$\begin{aligned} & \max \sum_{t=0}^{\infty} \beta^t U(\gamma^t \widehat{c}_t) && (108) \\ st \quad \widehat{c}_t + \gamma \widehat{k}_{t+1} &= AF(\widehat{k}_t, 1) + (1-\delta)\widehat{k}_t \end{aligned}$$

Suppose we have a CRRA preferences, then the question is how can we represent the preferences as a function of $\widehat{c}_t$ only. Writing the CRRA,

$$\sum_{t=0}^{\infty} \beta^t \frac{(\gamma^t \widehat{c}_t)^{1-\sigma} - 1}{1-\sigma} = \sum_{t=0}^{\infty} (\beta(\gamma^{1-\sigma}))^t \frac{\widehat{c}_t^{\,1-\sigma} - 1}{1-\sigma} \tag{109}$$

and the problem becomes,

$$\begin{aligned} & \max \sum_{t=0}^{\infty} (\beta(\gamma^{1-\sigma}))^t \frac{\widehat{c}_t^{\,1-\sigma} - 1}{1-\sigma} && (110) \\ st \quad \widehat{c}_t + \gamma \widehat{k}_{t+1} &= AF(\widehat{k}_t, 1) + (1-\delta)\widehat{k}_t \end{aligned}$$

and once again it is exact same problem as the NGM with a different discount factor. Note that the existence of a solution to this problem depends on $\beta(\gamma^{1-\sigma})$.In this set-up we get per-capita growth at rate γ. Also note that CRRA is the only functional form for preferences that is compatible with BGP. This is because as per-capita output grows, for consumption to grow at a constant rate, our agent has to face the same trade-off at each period.

Now suppose we have the TFP growing at rate γ with a CRTS Cobb-Douglas technology

$$\begin{aligned} Y_t &= A_t F(k_t, 1) \\ \frac{A_{t+1}}{A_t} &= \gamma \end{aligned}$$

What would be the growth rate of this economy? We can show that like the previous cases the growth rate of the economy is the growth rate for the productivity of labor, which is $\gamma^{\frac{1}{1-\alpha}}$ in this case.

15.1.2 Endogenous Growth

So far in the models we covered growth rate has been determined exogenously. Next we will look to models in which the growth rate is chosen by the model itself. We do know for a fixed amount of labor, the curvature of our technology limits the growth due to diminishing marginal return on capital and with depreciation there is an upper limit on capital accumulation. So if our economy is to experience sustainable growth for a long period of time, we either give up the curvature assumption on our technology or we have to be able to shift our production function up. Given a fixed amount of labor, this shift is possible either by an increasing TFP parameter or increasing labor productivity, . The simplest of such models where we can see t is the AK model, where the technology is linear in capital stock so that diminishing marginal return on capital does not set in.

AK Model We have the usual social planner's problem with linear technology and full depreciation,

$$\max \sum_{t=0}^{\infty} \beta^t U(C_t) \tag{111}$$
$$st \;\; C_t + K_{t+1} = AK_t$$

and the FOCs

$$(c_t) \;:\; \beta^t U_c(.) = \lambda_t \tag{112}$$
$$(k_{t+1}) \;:\; \lambda_t = \lambda_{t+1} \tag{113}$$

together implies the Euler equation,

$$U_c(c_t) = A\beta U_c(c_{t+1}) \tag{114}$$

and on the BGP with consumption growing at rate γ with CRRA utility we get,

$$\gamma = (A\beta)^{1/\sigma} \tag{115}$$

and the growth rate is determined by the model parameters endogenously. Note that capital also grows at rate γ and the fate of the economy is determined by pre-determined parameters of the model. The capital stock will diverge to infinity if $(A\beta)^{1/\sigma} > 1$ or the economy is destined to vanish if $(A\beta)^{1/\sigma} < 1$. Also note that there is no transitional dynamics in this model (we loose the state variables in the euler equation after substituting for the balanced growth rate relation) and conditional on γ,asymptotically all economies are same regardless of the initial capital level. If we de-centralize this economy we know wages will be zero since labor has no use and gross rental rate of capital will be fixed at the A. This is at odds with what we observe in real world. We would rather like to have a model that allows for both transitional dynamics, labor and growth at the same time. Allowing for labor implies that we need a variable that proxies the increasing productivity of labor endogenously and be reproducible in terms of output, such that we are able to shift our production function continually in the output-capital space without hitting a natural bound.

Human Capital and Growth One way of doing this is, introducing the variable 'Human Capital' as an input of production, to proxy continuously and endogenously increasing labor efficiency. We have two ways of modelling the human capital, one way is to see it very much like physical capital, in the sense output has to be invested to increase the existing stock of human capital. That is the Lucas' approach, in which you can think of investing in education by building more schools as a way to increase the existing human capital stock. The alternative way would be to reserve a part of the leisure time for increasing the human capital stock, which can be thought of studying harder to get better in a fraction of the leisure time. Unfortunately, the second approach puts limit on the rate human capital can grow and might fail to generate sustainable endogenous growth. Next, we look at the Lucas' human capital model.

Lucas' Human Capital Model We have an Cobb-Douglas technology with CRTS and human capital (H) as an input of production instead of labor and the laws of motion for the inputs,

$$\begin{aligned} F(H,K) &= AK^{\alpha}H^{1-\alpha} && (116)\\ K' &= i_k + (1-\delta_k)K && (117)\\ H' &= i_h + (1-\delta_h)H && (118) \end{aligned}$$

Now that there is no limit to the accumulation of human capital and sustainable growth on a BGP is feasible. Furthermore, an analysis of the characterization of the balanced growth path will indicate that this model indeed has transitional dynamics, so unlike the AK model if economy starts out of this optimal growth path economy can adjust and converge to it by responding to prices in a de-centralized setting. If we model the law of motion for human capital as,

$$H' = (1-N) + (1-\delta_h)H \tag{119}$$

where $(1-N)$ is the time devoted to accumulating human capital, say by studying harder, we see there is a natural limit to the growth of human capital and such an economy might not have a BGP. The key ingredient of endogenous growth with labor is then the reproducibility of the human capital without such a limit.

Growth through Externalities (Romer) We have seen in the AK model the growth rate is determined solely by model primitives and endogenized but still it is not a directly or indirectly determined by the agents' choices in our model. In Lucas' human capital model, the growth rate is determined by the choice of agents, specifically by the optimal ratio of human and physical capital. The source of growth in Lucas' model is reproducibility of human capital. In the next model, Romer introduces the notion of externality generated by the aggregate capital stock to go through the problem of diminishing marginal returns to aggregate capital. In this model, the source of growth will be the aggregate capital accumulation, which is possible with a linear aggregate technology in capital as we saw in the AK model. The firms in our model will not be aware of this externality and will have the usual CRTS technology and observe the source of growth coming from the TFP parameter.

As usual with externalities, the equilibrium outcome will not be optimal. Each firm has the following technology,

$$y_t = AK_t^{1-\alpha}k_t^{\alpha}n_t^{1-\alpha} \tag{120}$$

but since the firms are not aware of the positive externality they are facing they are solving the problem with the following technology.

$$\begin{aligned} y_t &= \overline{A}_t k_t^{\alpha} n_t^{1-\alpha} && (121) \\ where\ \overline{A}_t &= A_t K_t^{1-\alpha} && (122) \end{aligned}$$

We can see the social planner in fact is solving an AK model in per-capita terms. So does the de-centralized version of this economy have a BGP and if it does how would it look like? Assuming CRRA preferences without leisure and , we can derive the BGP condition and pin down the growth rate from the euler equation of a typical household,

$$1 = \beta\gamma^{-\sigma}(1+r) \tag{123}$$

where $\gamma = \dfrac{c_{t+1}}{c_t}$is the growth rate at the balanced path as usual and r=MP$_k$. So to find out the marginal product of capital for the firm we differentiate the technology w.r.t. k_t,

$$1 + r_t = \alpha AK_t^{1-\alpha}k_t^{\alpha-1}n_t^{1-\alpha} + (1-\delta) \tag{124}$$

and since the prices are determined by aggregate state variables $K_t = k_t$ gives,

$$A\alpha - \delta = r \tag{125}$$

and substituting this into the euler equation we get the growth rate of consumption.

$$[(A\alpha - \delta + 1)\beta]^{\frac{1}{\sigma}} = \gamma \tag{126}$$

Solving the AK problem the SP faces we can verify the optimal growth rate for consumption is,

$$[(A - \delta + 1)\beta]^{\frac{1}{\sigma}} = \gamma^{sp}. \tag{127}$$

The important properties of the decentralized model are,

1. It is sub-optimal due to firms' unawareness of the externality they are facing and thus have lower growth rate.

2. Once again, the rental rate do not depend on the capital stock (due to the linear technology in aggregate the state variable capital stock drops out of the euler equation) and there is no transitional dynamics generated by the model.

To sum up what we have done so far, we have started with models that had exogenous growth and saw that we can make these models look an behave like our NGM after appropriate transformation. Then we went on to look at the models that generate growth endogenously and saw that a prerequisite for growth in these models is linearity of the technology in reproducible factors. We looked at the simple AK model, where the technology is linear in capital stock and analyzed the BGP of such an economy. Then we looked at Lucas' human capital model, in which we had two forms of capital, human and physical, both of which are reproducible in terms of output. Then we analyzed the model by Romer, which again has linearity in the reproducible factor at the aggregate level (capital stock), but firms were facing the CRTS technology with diminishing marginal return on capital and not aware of the positive externality they face. Next we will see another model by Romer with monopolistic competition and a R&D sector which can generate endogenous growth.

Monopolistic Competition, Endogenous Growth and R&D Romer's monopolistic competition model has three production sectors, the final goods production, intermediate goods production and R&D i.e. variety production. Our usual TFP parameter in production function will represent the 'variety' in production inputs and as we will see the growth of varieties through research and development firms will make sure a balanced growth path is sustainable. The production function in this economy is,

$$Y_t = L_{1t}^{\alpha} \int_0^{A_t} x_t(i)^{1-\alpha} di \tag{128}$$

where $x_t(i)$ is the type i intermediate good and there is a measure A_t of different intermediate goods and L_{1t} is the amount of labor allocated to the final good production. The production function exhibits CRTS. The intermediate goods are produced with the following linear technology,

$$\int_0^{A_t} \eta x_t(i) di = K_t \tag{129}$$

Now suppose the variety of goods grows at rate γ, $A_{t+1} = \gamma A_t$, is long run sustainable growth possible? The answer to this question will depend whether our final goods production technology is linear in growing terms. We do know by the curvature of the technology, optimality implies equal amount of each variety will be used in production, $x_t(i) = x_t$,then we have,

$$A_t \eta x_t = K_t \tag{130}$$

and our output at this equal variety becomes,

$$Y_t = L_{1t}^{\alpha} A_t {x_t}^{1-\alpha} \tag{131}$$

then substituting for x_t we have,

$$Y_t = \frac{L_{1t}^{\alpha}}{\eta^{1-\alpha}} A_t^{\alpha} K_t^{1-\alpha} \tag{132}$$

thus if both A_t and K_t are growing at rate γ, then production function is linear in growing terms and long run balanced growth is feasible. Note that this model becomes very similar to our previous exogenous labor productivity growth under these assumptions. The purpose of this model is to determine γ endogenously. What will be the source of growth, where does γ come from? As we will see, there will be incentives for R&D firms to produce new 'varieties' because there will be a demand for it. These new varieties will be patented to intermediate good production firms, where a patent will mean exclusive rights to produce that intermediate good. So we will have monopolistic competition in the intermediate goods production. Now suppose the law of motion for 'varieties', which is the technology in R&D sector is given by,

$$A_{t+1} = (1 + L_{2t}\zeta)A_t \tag{133}$$

where L_{2t} is the labor employed in R&D sector. Note that this is not a regular law of motion in the sense every new variety produced helps the production of further new varieties.such that there is a positive externality to variety production. Also assume leisure is not valued and we have aggragate feasibility condition for labor as,

$$L_{2t} + L_{1t} = 1 \tag{134}$$

As a homework, we have calculated the BG rate of SP version of this economy, now we will de-centralize this economy and characterize the equlibrium growth rate.and see that it is sub-optimal. The period t problem of a firm in the competitive final good production sector is,

$$\max_{x_t(i),L_{1t}} \{L_{1t}^{\alpha} \int_0^{A_t} x_t(i)^{1-\alpha} di - w_t L_{1t} - \int_0^{A_t} q_t(i)x_t(i)di\} \tag{135}$$

and since we have CRTS with perfect competition we have zero profit with following FOCs,

$$\begin{aligned} w_t &= \alpha L_{1t}^{\alpha-1} \int_0^{A_t} x_t(i)^{1-\alpha} di & (136) \\ q_t(i) &= (1-\alpha)L_{1t}^{\alpha} x_t(i)^{-\alpha} & (137) \end{aligned}$$

notice that the inverse demand function for good of variety i is,

$$\left(\frac{q_t(i)}{(1-\alpha)L_{1t}^{\alpha}}\right)^{\frac{-1}{\alpha}} = x_t(i) \tag{138}$$

The intermediate goods industry will be monopolistic competition, in which there is only one firm, that is one patent holder, producing each variety. Each firm takes the demand of its variety and prices as given, and solves the following problem each period,

$$\begin{aligned} \Pi_t(i) &= \max_{x_t(i),K_t(i)} \{q_t(i)x_t(i) - R_t K_t(i)\} & (139) \\ s.t. \quad x_t(i) &= \frac{K_t(i)}{\eta} \end{aligned}$$

plugging in the inverse demand function and the technology constraint, the FOC is,

$$(1-\alpha)^2 x_t(i)^{-\alpha} L_{1t} = R_t \eta \tag{140}$$

and because of the symmetry we mentioned $x_t(i) = x_t = \frac{K_t}{\eta A_t}$ we can write this FOC as,

$$(1-\alpha)^2 (\frac{K_t}{\eta A_t})^{-\alpha} L_{1t} = R_t \eta \tag{141}$$

i.e. the rental price of capital is not equal to it's marginal product and there is opportunuties for positive profit. But also remember there is a fixed cost of entering to this industry, namely the price paid for the patent. Then as we will see the relation between the two will be one of our equilibrium conditions. Now lets look at the problem of R&D firms,

$$\max_{A_{t+1},L_{2t}} \{p_t^P(A_{t+1} - A_t) - w_t L_{2t}\} \tag{142}$$

$$s.t. A_{t+1} = (1 + L_{2t}\zeta)A_t$$

where p_t^P is the patent of the price. Free entry is assumed thus there will be zero profit in equilibrium. Notice also the R&D firm is solving a static problem without realizing the positive externality this period's decision creates on next periods production. As we will see, this and the monopoly power of the patent owners will be the sources of sub-optimality in decentralized solution.The FOC is,

$$p_t^P = \frac{w_t}{\zeta A_t} \tag{143}$$

where wage is determined in the final goods market and given this price equilibrium quantity will come from the deman function. As we mentioned before, one equilibrium condition will be that at any point in time, total profit a patent generates will be equal to price of it such that there will also be zero profit in the intermediate goods market.

$$p_t^P = \sum_{\tau=t}^{\infty} \frac{\Pi_t(i)}{(1+r)^{\tau-t}} \tag{144}$$

These conditions with constant growht equations for the growing variables is sufficient to characterize the equilibrium growth rate of this economy.

Homework Solve this economy for the balanced growth rate and show that equilibrium allocation and growth rate is sub-optimal.

16 April 13-15

16.1 Transitions and Non-Stationary Equilibria

Take the representative agent economy with production technology $Y = z_0 K^\alpha$. The steady state condition is,

$$\alpha z_0 {K^*}^{\alpha-1} + (1-\delta) = \frac{1}{\beta}$$

What happens if z changes? Is this a legitimate question? No. This is because we have not allowed our consumers to anticipate z changing. They think z is constant and this is how they solve their problem. Instead, we can ask the following question: What happens in an economy where $\bar{z} = 2z_0$ and $K_0 = K^*$? What would happen in a world where we start from K^* (the steady state of the economy with $z = z_0$) but with the new technology parameter $\bar{z} = 2z_0$. Then the problem would be computing the transition path of the economy until it reaches the new steady state K^{**}.

How much does K^* change? Is the new steady state capital, K^{**}, twice as much as K^*?

No. This would be true if it was the case that we doubled z^α, but in our case we are just doubling z.

And how should do welfare comparison? Here it is important to see why it would be wrong to compare $\sum \beta^t u(c_t^{**})$ and $\sum \beta^t u(c_t^*)$ (where c_t^{**} is the steady state consumption level in the economy with with $\bar{z} = 2z_0$). This comparison would be wrong because:

1. The two numbers are in utils and utils don't mean anything!

2. Changing z_0 to $\bar{z}$ doesn't change K^* to K^{**} immediately. Starting from K^*, once we change the technology parameter, we know that it will take the economy a while to reach the new steady state K^{**}. Therefore we have to compute the transition path and get the path of consumption along this transition path, let's denote it by $\{\bar{c}_t\}_t$. And then we need to compare this to the path that the economy would be on if we hadn't changed the technology parameter (basically just K^* along the whole path, let's call the consumption path here $\{c_t^*\}_t$).

After we solve for the transition path of consumption $\{\bar{c}_t\}_t$, in order to do the welfare comparison, we need to calculate the proportional increase in consumption that makes the agent indifferent between the two paths. This is λ.

$$\sum \beta^t u((1+\lambda)c_t^*) = \sum \beta^t u(\bar{c}_t)$$

About solving for the transition path: We assume that the economy converges to the new steady state in, say, 200 periods. Then we have a second order difference equation with initial and last conditions known.

$$\begin{aligned} \varphi(K_t, K_{t+1}, K_{t+2}) &= 0 \\ K_0 &= K^* \\ K_{200} &= K^{**} \end{aligned}$$

16.2 Business Cycle Economy

Now consider the standard neoclassical growth model with technological shocks but this time with an 'extravagant' technological constraint:

$$\begin{aligned} &\max \quad E \sum \beta^t u(c_t) \qquad (145) \\ \text{s.t.} \quad & z_t K_t^\alpha = c_t + K_{t+1} \\ & K_{t+1} \in \{K^1, K^2\} \\ & z_t \in \{z^1, ...z^{n_z}\} \text{and } z_t \sim \Gamma_{zz'} \end{aligned}$$

How do we characterize the solution to this problem? First Order Conditions are not good because we have a discrete choice set here. How do we mathematically describe what happens in this economy?

The state space is $Z \times K = \{z^1, ...z^{n_z}\} \times \{K^1, K^2\}$. We describe what happens by specifying what will be chosen (what level of capital for the next period) for each point in the state space (for each pair of (z,K) in the state space). This is given by $K' = G_{z,K}$. Then we need to specify how often this economy is at each state and given where it is at, what state will it end up at. For this kind of description, we need to use probability measure. The stationary distribution defined below, x^* tells us how much time the economy stays at each node. It is the time series interpretation of the same object we defined previously for the cross-section distribution.

$$x^* = T(x^*, \{G, \mu\})$$

In macroeconomics, we usually don't derive an analytical solution for this stationary distribution. Instead, we use simulation in order to learn the properties of our economy's long run distribution.

Given $K' = G_{z,K}$ and $\Gamma_{zz'}$, we can generate the time series of the artificial economy and then compare the time series to the data. In order to do this we first HP filter the time series that we generate and the data. This is in order to make both data stationary. We want to transform the data in order to be able to concentrate on the properties that we are interested in. We compare the statistics from our model economy to the ones from the data. The ideal thing would be to solve for x^* and compare the properties of x^* to the statistics from the data. But for no reason other than 'laziness' we simulate the model instead. For example, we simulate an economy of 150 periods and compute the ratio of standard deviation of consumption to standard deviation of output, $\frac{\sigma_c}{\sigma_y}$. Then we simulate this economy of 150 periods, 100 times. As a result we have 100 different paths of key variables for 150 periods (100 different samples). And we get the mean of this statistic $\frac{\sigma_c}{\sigma_y}$ across these 100 samples and this is the statistic that we compare to $\frac{\sigma_c}{\sigma_y}$ from the data. Usually business cycle people report the standard deviation of this statistic (across simulations). It is important to note that this is not the standard error in the econometric sense. This is more like a modelling error. If the samples are not very different from each other, then the model has the right predictions.

16.2.1 Transition in the Huggett Economy

Recall the problem of the agent in the Huggett economy:

$$\max \quad E \sum \beta^t u(c_t) \tag{146}$$

$$\text{s.t.} \qquad q_t a_{t+1} + c_t = e_t + a_t$$

Now suppose that suddenly the stochastic process for e changes from Γ to Γ'. How do we solve for the transition path? We guess that in 200 years the economy will go to its new stationary equilibrium. And we guess the q's along this transition path, $\{q_t\}_{t=1}^{200}$ that will take the economy from the initial stationary distribution x_0^* to x_1^*.

16.3 Economy with Aggregate Shock

Consider an economy with heterogeneous agents with production and without labor-leisure choice (exogenous labor supply).

$$
\begin{aligned}
Y_t &= z_t K_t^{\alpha} N_t^{\alpha} \\
K_t &= \int a \; dx_t \\
N_t &= \int e \; dx_t \\
&\Gamma_{zz'}, \pi_{e'|zz'e}
\end{aligned}
$$

Are (z_t, K_t) sufficient to know the prices? They are for the prices of today but they are not sufficient statistics for the prices of tomorrow. We need the distribution, x, for tomorrow's prices. Who has what, matters, unless the decision rules are linear. We need to know how much each type of agent saves in this economy, because the poor save and consume differently than the rich. If the decision rule was linear we wouldn't have this problem (in that case K_t would be a sufficient statistic for K_{t+1}) (For more on this, look at Problem Set 5 of 2003, question 6).

17 April 19-20

17.1 Endogenous incomplete market models

So far we modelled our economy as an incomplete market one by assumption. From now, we do not do this. Instead, we will define the fundamental environment and assume more on what information agents have and what agents can see or commit to, which might result in non- first best outcomes within the model . We will look at two big classes for models. One is the economy with private information. In other words, there is asymmetric information or incomplete information in the mod. The second class is the models with lack of commitment. In the world without commitment, the contract among agents need to be self-enforceable. Otherwise, agents will just quit the contract and walk away.

17.1.1 Optimal unemployment insurance[5]

Consider an economy where the probability of finding a job $p(a)$ is a function of effort $a \in [0, 1]$. And we assume that once the agent gets a job, she will have wage w for ever. Thus, the individual problem is

$$\max_{a_t} E \sum_t \beta^t \left[u(c_t) - a_t \right]$$

[5] The source of this part is the revised chapter of Tom Sargent's Recursive Economic Theory on the web.

There are two cases: when the agent has got a job, she will pay no effort and enjoy w for ever. The life long utility is

$$V^E = \sum_t \beta^t u(w) = \frac{u(w)}{1-\beta} \tag{147}$$

When the agent is still unemployed, she will have nothing to consumer. Her problem is

$$V^u = \max_a \left\{ u(0) - a + \beta \left[p(a) V^E + (1 - p(a) V^{u\prime}) \right] \right\} \tag{148}$$

If the optimal solution of a is interior, $a \in (0,1)$, then the first order condition gives

$$-1 + \beta p'(a) \left(V^E - V^u \right) = 0 \tag{149}$$

And since the V^u is stationary,

$$V^u = \max_a \left\{ u(0) - a + \beta \left[p(a) V^E + (1 - p(a) V^u) \right] \right\} \tag{150}$$

Solving (149)(150) gives the optimal a and V^u. Another way is to successively substitute a and obtain solution because (**??**) defines a contraction mapping operator. We can fix V_0^u, then solve (150) to get $a(V_0^u)$ and obtain V_1^u. Keeping going until $V_n^u = V_{n+1}^u$. In a word, optimal effort level a^* solves (150) with $V^u = V^{u\prime}$.

The probability of finding a job $p(a)$ is called hazard rate. If agents did not find a job with effort level a^*, next period, she will still execute the same effort level a^*. Why? Because the duration of unemployment is not state variable in agent's problem. (If agents do not have enough realization about the difficulty of getting a job. With learning, their effort a will increase as they revise their assessment of the difficulty. But such revision of belief is not in this model.)

Now suppose resource is given to people who is unemployed to relive her suffering by a benevolent planner. This planner has to decide the minimal cost of warranting agent a utility level V: $c(V)$. To warrant utility level V, the planner tells the agent how much to consume, how much effort to exert and how much utility she will get if she stay unemployed next period. Obviously, the cost function $c(V)$ is increasing in V.

Problem 12 *Show that $c(V)$ is strictly convex.*

The cost minimization problem of the planner can be written in the following recursive problem:

$$c(V) = \min_{c,a,V^u} c + [1 - p(a)] \frac{1}{1+r} c(V^u) \tag{151}$$

subject to

$$V = u(c) - a + \beta \left[p(a) V^E + (1 - p(a)) V^u \right] \tag{152}$$

To solve the problem, construct Lagragian function

$$\mathcal{L} = c + [1 - p(a)] \beta c(V^u) + \theta \left[V - u(c) + a - \beta \left[p(a) V^E + (1 - p(a)) V^u \right] \right]$$

FOC: (c)

$$\theta = \frac{1}{u_c} \tag{153}$$

(a)

$$c(V^u) = \theta\left[\frac{1}{\beta p'(a)} - \left(V^E - V^u\right)\right] \tag{154}$$

(V^u)

$$c'(V^u) = \theta \tag{155}$$

Envelope condition

$$c'(V) = \theta \tag{156}$$

We will work on some implication of these conditions:

1. Compare (154) and (149), we can see that the substitution between consumption and effort is different from the one in agent's problem without unemployment insurance. This is because the cost of effort is higher for work that it is from the viewpoint of planner.

2. (155) tells us that the marginal cost of warranting an extra unit of utility tomorrow is θ.,provided that tomorrow V^u is optimally chosen when today's promise is V. And (156) tells us that the marginal cost of warranting an extra unit of V today is θ.

3. Given that c is strictly concave, $V = V^u$.

4. Regardless of unemployment duration, $V = V^u$. So, effort required the the planner is the same over time. Hazard rate is still constant.

Next , we will study the case when effort is not observable. Planner can only choose consumption and V^u. Effort level is chosen optimally by worker and it is unobservable.

Now suppose there is a social planner who will warrant utility level V for unemployed agent, where V summarize all the past information. The cost minimization problem is

$$c(V) = \min_{c,a,V^u} c + [1 - p(a)]\beta c(V^u) \tag{157}$$

subject to

$$V = u(c) - a + \beta\left[p(a)V^E + (1 - p(a))V^u\right] \tag{158}$$

FOC: (c)

$$\theta = \frac{1}{u_c} \tag{159}$$

(a)

$$c(V^u) = \theta\left[\frac{1}{\beta p'(a)} - \left(V^E - V^u\right)\right] \tag{160}$$

(V^u)

$$c'(V^u) = \theta \tag{161}$$

Envelope condition

$$c'(V) = \theta \tag{162}$$

Problem 13 *Write out envelope condition for the above problem explicitly*

The optimal promise for tomorrow is $V^u(V)$. Now, let's work out the property of $V^u(.)$.

Lemma 3 *If V ¿ V^A, then $c(V) > 0$ where V^A is the utility for unemployed agent when they are in autarky..*

The intuition for the lemma is that if the planner promises the agent something more than what agent can achieve by herself, it will cost the planner something because the planner cannot do anything more than what people can do on their own.

Lemma 4 *Lagrangian multiplier $\theta > 0$.*

The second lemma tells us that if the planner promise more, she has to pay more.

In the problem without unemployment insurance, (**??**) implies that

$$\frac{1}{\beta p'(a)} = V^E - V^u$$

In the planner's problem, since $c(V) > 0$, (160) implies that the effort level chosen by the planner are different from agent's choice in autarky. The reason is that effort does not cost that much in planner's thought.

Unobservable effort When a is not observable, planner can only choose c and V^u. And households choose a optimally. Now it becomes a principle-agent problem. We will solve the problem backward.

If given c and V^u, the agent will solve

$$\max_a u(c) - a + \beta \left[p(a) V^E + (1 - p(a)) V^u \right] \tag{163}$$

FOC is

$$[p'(a)\beta]^{-1} = V^E - V^u \tag{164}$$

This FOC gives an implicit function of a as a function of V^u: $a = g(V^u)$. (Because c and a are separate in the utility function, a is not a function of c).

Then, the planner solve her cost minimization problem, in which the optimality condition is also one constraint.

$$c(V) = \min_{c,a,V^u} c + [1 - p(a)]\beta c(V^u)$$

subject to

$$\begin{aligned} V &= u(c) - a + \beta \left[p(a) V^E + (1 - p(a)) V^u \right] && (165) \\ 1 &= [p'(a)\beta] \left[V^E - V^u \right] && (166) \end{aligned}$$

Lagragian is

$$\begin{aligned} & c + [1 - p(a)]\beta c(V^u) + \theta \left[V - u(c) + a - \beta \left[p(a) V^E + (1 - p(a)) V^u \right] \right] \\ & + \eta \left[1 - [p'(a)\beta] \left[V^E - V^u \right] \right] \end{aligned}$$

FOC: (c)

$$\theta^{-1} = u_c$$

(a)

$$c\left(V^u\right) = \theta\left[\frac{1}{\beta p'\left(a\right)} - \left(V^E - V^u\right)\right] - \eta\frac{p''\left(a\right)}{p'\left(a\right)}\left(V^E - V^u\right) \tag{167}$$

(V^u)

$$c'\left(V^u\right) = \theta - \eta\frac{p'\left(a\right)}{1-p\left(a\right)} \tag{168}$$

Envelope condition

$$c'\left(V\right) = \theta \tag{169}$$

Again, (168) tells the marginal cost to warrant additional amount of delayed promise. (169) gives the marginal cost to increase today's utility. The Lagrangian multiplier associated with constraint (166) is positive, $\eta > 0$, which means that the constraint is binding. So,

$$\eta\frac{p'\left(a\right)}{1-p\left(a\right)} > 0$$

Therefore, we have

$$c'\left(V^u\right) < c'\left(V\right) \Rightarrow V^u < V$$

from the strict concavity of $c\left(.\right)$. The delayed promised utility decreases over time.

Let $\theta^u = \theta - \eta\frac{p'(a)}{1-p(a)}$, then $\theta^u < \theta$,which tells us about the consumption path. Consumption decreases over time because $\theta^{-1} = u_c$.

Problem 14 *Prove that c_t is a decreasing when people are unemployed.*

How about effort level?

Problem 15 *Show that a_t is increasing over time when people are unemployed.*

Overall, we get the following model implications: optimal unemployment insurance says that longer unemployment period the agent stays, the less insurance she will be insured for. In this way, the planner induces the higher effort level. Although you cannot let people do what is optimal, such behavior can be achieved by giving out less consumption and promised utility over time. This model implies that time-varying unemployment insurance plan is optimal, under which the replacement rate θ goes down over time.

17.1.2 One sided lack of commitment[6]

We will study a model with one-said lack of commitment. This is an endowment economy (no production). There is no storage technology. Consider the village of fisherladies, where

[6]The source of this part is the online revised second edition of Tom Sargent's Recursive Macroeconomic Theory.

young granddaughters receive $y_s \in \{y_1, y_2, ..., y_S\}$ every period. y is iid. The probability that certain y_s realizes is Π_s. h_t is a history of shocks up to period t, i.e. $h_t = \{y_0,\, y_1,\, y_2,\, ...,\, y_t\}$.

First, if the granddaughter stays autarky, she will solve the problem

$$V_{AUT} = \sum_{t=0}^{\infty} \beta^t \sum_s \Pi_s u(y_s) = \frac{\sum_s \Pi_s u(y_s)}{1-\beta}$$

Note that here V^A is the utility of the young lady before endowment shock realizes.

Now we assume that the grandmother offers a contract to the granddaughter, which transfer resources and provide insurance to her. Grandmother is subject to commitment. But the young granddaughter may leave grandmother and break her word. Thus, this model is one-sided commitment model: an agent can walk away from a contract but the other cannot. Therefore, the contract should be always in the interest of granddaughter for her to stay.

We define a contract $f_t : H_t \to c \in [0, \tau]$. We will see next class that incentives compatibility constraint requires that at each node of history H_t, the contract should guarantee a utility which is higher than that in autarky.

Notice that the problem is different from Lucas tree model because of the shock realization timing. In Lucas tree model, shock is state variable because action takes place after shock is realized. Thus, action is indexed by shock. Here action is chosen before shock realization. Therefore, shock is not a state variable and action is state contingent.

In Lucas tree model, $V(s) = \max_c u(c) + \beta \sum_{s'} \Pi_{ss'} V(s')$. Here, if we write the problem recursively, it is $V = \max_{c_s} \sum_s \Pi_s u(c_s) + \beta V$.

Remember, the grandmother will make a deal with her granddaughter. They sign a contract to specify what to do in each state. $h_t \in H_t$. Contract is thus a mapping $f_t(h_t) \to c(h_t)$. With this contract, granddaughter gives y_t to the grandmother and receives $c_t = f_t(h_{t-1}, y_t)$. But if the granddaughter decided not to observe the contract, she consumes y_t this period and cannot enter a contract in the future, i.e. she has to live in autarky in the future.

For grandmother to keep granddaughter around her, the contract has to be of interest to granddaughter because although grandmother keeps her promise, granddaughter does not. There are two possible outcome if this contract is broken. One is that granddaughter goes away with current and future endowment. The other is that they renegotiate. We ignore the second possibility as no renegotiation is allowed. But we need deal with the possibility that the granddaughter says no to the contract and steps away.

The first best outcome is to warrant a constant consumption c_t to granddaughter who is risk averse. But because of the one-side lack of commitment, the first best is not achievable. The contract should always be attractive to granddaughter, otherwise, when she gets lucky with high endowment y_s, she will feel like to leave. So, this is a dynamic contract problem which the grandmother will solve in order to induce good behavior from granddaughter. The contract is dynamic because the nature keeps moving.

We say the contract $f_t(h_t)$ is incentive compatible or satisfies participation constraint if for all h_t,

$$u(f_t(h_t)) + \sum_{\tau=1}^{\infty} \beta^\tau \sum_s \Pi_s u(f_{t+\tau}(h_{t+\tau})) \geq u(y_s(h_t)) + \beta V^A \tag{170}$$

The left hand side is utility guaranteed in the contract. And the right hand side is the utility that granddaughter can get by herself. The participation constraint is not binding if y_s is low. And when y_s is high, PC is binding.

Problem of the grandmother In this model, problem of the grandmother is to find an optimal contract that maximizes the value of such a contract of warranting V to her. We define the problem using recursive formula. Firstly, let's define the value of contract to grandmother if she promised V to her granddaughter by $P(V)$. $P(V)$ can be defined recursively as the following:

$$P(V) = \max_{\{c_s,\omega_s\}_{s=1}^S} \sum_s \Pi_s[(y_s - c_s) + \beta P(\omega_s)] \tag{171}$$

subject to

$$u(c_s) + \beta\omega_s \geq u(y_s) + \beta V^A \quad \forall s \tag{172}$$

$$\sum_s \Pi_s[u(c_s) + \beta\omega_s] \geq V \tag{173}$$

Notice that there are $1 + S$ constraints. The choice variables c_s, ω_s are state-contingent where ω_s is the promised utility committed to granddaughter in each state. In the objective function, $\sum_s \Pi_s(y_s - c_s)$ is the expected value of net transfer.

There are two sets of constraints. (172) is PC and (173) is promise keeping constraint.

18 April 21-22

18.1 One sided lack of commitment (continued)

Recall the problem of the grandmother. We defined the problem just before the realization of the shock.

$$P(V) = \max_{c_s,w_s} \sum_s \pi_s[(y_s - c_s) + P(w_s)]$$

$$\text{s.t.} \qquad u(c_s) + \beta w_s \geq u(y_s) + \beta V^A \ \ \forall s$$

$$\sum_s \pi_s[u(c_s) + \beta w_s] \geq V$$

The First Order Conditions to the grandmother's problem are:

(c_s)

$$\Pi_s = (\lambda_s + \mu\Pi_s)u'(c_s) \tag{174}$$

(ω_s)

$$-\Pi_s P'(\omega_s) = \mu\Pi_s + \lambda_s \tag{175}$$

(μ)

$$\sum_s \Pi_s[u(c_s) + \beta\omega_s] = V \tag{176}$$

(λ)

$$u(c_s) + \beta\omega_s \geq u(y_s) + \beta V^A \tag{177}$$

In addition, Envelope Theorem tells that:

$$P'(v) = -\mu \tag{178}$$

Interpret the first order conditions:

1. (174) tells that in an optimal choice of c_s, the benefit of increasing one unit of c equals the cost of doing so. The benefit comes from two parts: first is $\mu\Pi_s u'(c_s)$ as increasing consumption helps grandmother to fulfill her promise and the second part is $\lambda_s u'(c_s)$ since increase in consumption helps alleviated the participation constraint. And the cost is the probability of state s occurs.

2. (175) equates the cost of increasing one unit of promised utility and the benefit. The cost to grandmother is $-\Pi_s P'(\omega_s)$ and the benefit is $\mu\Pi_s + \lambda_s$ which helps grandmother deliver promise and alleviate participation constraint.

How about the contract value $P(V)$. First, $P(V)$ can be positive or negative.

Claim 16 *(1) There exits V such that $P(V) > 0$*[7]*. (2) There exits V such that $P(V) > 0$*

What's the largest V we will be concerned with? When PC will be binding for sure. If PC binds for the best endowment shock y_S, then PC holds for all the shock y_s. When granddaughter gets the best shock y_S, the best autarky value is then

$$V_{AM} = u(y_S) + \beta V_A$$

And the cheapest way to guarantee V_{AM} is to give constant consumption $\overline{c_S}$, such that

$$V_{AM} = \frac{u(c_S)}{1-\beta}$$

From this case, we can see that because of lack of commitment, the grandmother will have to give more consumption in some states. While when there is no lack of commitment, strict concavity of $u(.)$ implies that constant stream of consumption beats any $\{c_t\}$ that have the same present value, as there is no PC.

18.1.1 Characterizing the Optimal Contract

We will characterize the optimal contract by considering the two cases: (i) $\lambda_s > 0$ and (ii) $\lambda_s = 0$.

[7] When $P(V)$ is positive, it shows that there is gain from trade.

Firstly, if $\lambda_s = 0$, we have the following equations from FOC and EC:

$$P'(\omega_s) = -\mu \tag{179}$$

$$P'(V) = -\mu \tag{180}$$

Therefore, for s where PC is not binding,

$$V = \omega_s$$

c_s is the same for all s. For all s such that the Participation Constraint is not binding, the grandmother offers the same consumption and promised future value.

Let's consider the second case, where $\lambda_s > 0$. In this case, the equations that characterize the optimal contract are:

$$u'(c_s) = \frac{-1}{P'(\omega_s)} \tag{181}$$

$$u(c_s) + \beta\omega_s = u(y_s) + \beta V^A \tag{182}$$

Note that this is a system of two equations with two unknowns (c_s and ω_s). So these two equations characterize the optimal contract in case $\lambda_s > 0$. In addition, we can find the following properties by carefully observing the equations:

1. The equations don't depend on V. Therefore, if a Participation Constraint is binding, promised value does not matter for the optimal contract.

2. From the first order condition with respect to ω_s, $P'(\omega_s) = P'(v) - \frac{\lambda_s}{\Pi_s}$, where $\frac{\lambda_s}{\Pi_s}$ is positive. Besides, we know that P is concave. This means that $v < \omega_s$. In words, if a Participation Constraint is binding, the moneylender promises more than before for future.

Combining all the results we have got, we can characterize the optimal contract as follows:

1. Let's fix V_0. We can find a $y_s(V_0)$, where for $\forall y_s \leq y_s(V_0)$, the participation constraint is not binding. And vice versa.

2. The optimal contract that the moneylender offers to an agent is the following:

   If $y_t \leq y_s(v_0)$, the moneylender gives $(v_0, c(v_0))$. Both of them are the same as in the previous period. In other words, the moneylender offers the agent the same insurance scheme as before.

   If $y_t > y_s(v_0)$, the moneylender gives $(v_1, c(y_s))$, where $v_1 > v_0$ and c doesn't depend on v_0. In other words, the moneylender promises larger value to the agent to keep her around.

   So the path of consumption and promised value for an agent is increasing with steps.

18.2 Two sided lack of commitment

18.2.1 The Model

- Two brothers, A and B, and neither of them has access to a commitment technology. In other words, the two can sign a contract, but either of them can walk away if he does not feel like observing it.

- This is an endowment economy (no production) and there is no storage technology. Endowment is represented by $(y_s^A, y_s^B) \in Y \times Y$, where y_s^i is the endowment of brother i. s=(y_s^A, y_s^B) follows a Markov process with transition matrix $\Gamma_{ss'}$.

18.2.2 First Best Allocation

We will derive the first best allocation by solving the social planner's problem:

$$\max_{\{c_i(h_t)\}_{\forall h_t, \forall i}} \lambda^A \sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t) u(c^A(h_t)) + \lambda^B \sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t) u(c^B(h_t))$$

subject to the resource constraint:

$$\sum_i c^i(h_t) - y^i(h_t) = 0 \qquad \forall h_t \quad \text{w/ multiplier } \gamma(h_t)$$

The First Order Conditions are:

$$\begin{aligned} FOC(c^A(h_t)) &: \lambda^A \beta^t \Pi(h_t) u'(c^A(h_t)) - \gamma(h_t) = 0 \\ FOC(c^B(h_t)) &: \lambda^B \beta^t \Pi(h_t) u'(c^B(h_t)) - \gamma(h_t) = 0 \end{aligned}$$

Combining these two yields:

$$\frac{\lambda^A}{\lambda^B} = \frac{u'(c^A(h_t))}{u'(c^B(h_t))}$$

The first best allocation will not be achieved if there is no access to a commitment technology. Therefore, the next thing we should do is look at the problem the planner is faced with in the case of lack of commitment. Due to lack of commitment, the planner needs to make sure that at each point in time and in every state of the world, h_t, both brothers prefer what they receive to autarky. Now we will construct the problem of the planner adding these participation constraints to his problem.

18.2.3 Constrained Optimal Allocation

The planner's problem is:

$$\max_{c^A(h_t), c^B(h_t)} \lambda^A \sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t) u(c^A(h_t)) + \lambda^B \sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t) u(c^B(h_t))$$

$$\sum_i c^i(h_t) - y^i(h_t) = 0 \qquad \forall h_t \quad \text{w/ multiplier } \gamma(h_t)$$

$$\sum_{r=t}^{\infty} \beta^{r-t} \sum_{h_r} \Pi(h_r|h_t) u(c^i(h_r)) \geq \Omega_i(h_t) \qquad \forall h_t, \forall i \quad \text{w/ multiplier } \mu_i(h_t)$$

where $\Omega_i(h_t) = \sum_{r=0}^{\infty} \beta^{r-t} \sum_{h_r} \Pi(h_r|h_t) u(y_i(h_t))$ (the autarky value)

- How many times does $c^A(h_{17})$ appear in this problem? Once in the objective function, once in the feasibility constraint, and it appears in the participation constraint from period 0 to period 16.
- We know that the feasibility constraint is always binding so that $\gamma(h_t) > 0 \ \ \forall h_t$. On the other hand the same is not true for the participation constraint.
- Both participations cannot be binding but both can be nonbinding.
- Define $M_i(h_{-1}) = \lambda^i$

  and $M_i(h_t) = \mu_i(h_t) + M_i(h_{t-1})$

  (We will use these definitions for the recursive representation of the problem in the next class)

18.2.4 Recursive Representation of the Constrained SPP

We want to transform this problem into the recursive, because it would be easier to solve the optimal allocation with a computer. Now we will show how to transform the sequential problem with the participation constraints into its recursive representation.

Before we do this transformation, first recall the Lagrangian associated with the sequential representation of the social planner's problem:

$$\begin{aligned}
&\lambda^A \sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t) u(c^A(h_t)) + \lambda^B \sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t) u(c^B(h_t)) \\
&+ \sum_{t=0}^{\infty} \beta^t \sum_{h_t \in H_t} \Pi(h_t) \sum_{i=1}^{2} \mu_i(h_t) \left[\sum_{r=t}^{\infty} \beta^{r-t} \sum_{h_r} \Pi(h_r|h_t) u(c^i(h_r)) - \Omega_i(h_t) \right] \\
&+ \sum_{t=0}^{\infty} \sum_{h_t \in H_t} \gamma(h_t) \left[\sum_{i=1}^{2} c_i(h_t) - \sum_{i=1}^{2} y_i(h_t) \right]
\end{aligned}$$

Note that here the Lagrangian multiplier associated witih the participation constraint for brother i after history h_t is $\beta^t \Pi(h_t) \mu_i(h_t)$.

Now we will use the definitions from the previous class (for $M_i(h_t)$) to rewrite the above Lagrangian in a more simple form,

Collect terms and rewrite,

$$\begin{aligned}
&\sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t) \sum_{i} \left\{ \lambda^i u(c^i(h_t) + \mu_i(h_t) \left[\sum_{r=t}^{\infty} \beta^{r-t} \sum_{h_r} \Pi(h_r|h_t) u(c^i(h_r)) - \Omega_i(h_t) \right] \right\} \\
&+ \sum_{t=0}^{\infty} \sum_{h_t \in H_t} \gamma(h_t) \left[\sum_{i=1}^{2} c_i(h_t) - \sum_{i=1}^{2} y_i(h_t) \right]
\end{aligned}$$

Note that, $\sum_{r=t}^{\infty} \beta^{r-t} \sum_{h_r} \Pi(h_r|h_t) u(c^i(h_r)) - \Omega_i(h_t) = u(c^i(h_t)) + \sum_{r=t+1}^{\infty} \beta^{r-t} \sum_{h_r} \Pi(h_r|h_t) u(c^i(h_r)) - \Omega_i(h_t)$, and that $\Pi(h_r|h_t)\Pi(h_t) = \Pi(h_r)$ so using these, rewrite as,

$$\sum_{t=0}^{\infty}\beta^t\sum_{h_t}\Pi(h_t)\sum_i\left\{\lambda^i u(c^i(h_t) + \mu_i(h_t)u(c^i(h_t)\right\}$$
$$+\sum_{t=0}^{\infty}\sum_{h_r}\sum_i\mu_i(h_t)\left[\sum_{r=t+1}^{\infty}\beta^r\sum_{h_r}\Pi(h_r)u(c^i(h_r)) - \Omega_i(h_t)\right]$$
$$+\sum_{t=0}^{\infty}\sum_{h_t\in H_t}\gamma(h_t)\left[\sum_{i=1}^{2}c_i(h_t) - \sum_{i=1}^{2}y_i(h_t)\right]$$

Collect the terms of $u(c^i(h_r)$,

$$\sum_{t=0}^{\infty}\beta^t\sum_{h_t}\Pi(h_t)\sum_i\left\{\left[\lambda^i + \sum_{r=0}^{t-1}\mu_i(h_r)\right]u(c^i(h_t) + \mu_i(h_t)\left[u(c^i(h_t) - \Omega_i(h_t)\right]\right\}$$
$$+\sum_{t=0}^{\infty}\sum_{h_t\in H_t}\gamma(h_t)\left[\sum_{i=1}^{2}c_i(h_t) - \sum_{i=1}^{2}y_i(h_t)\right]$$

Introduce the variable $\mathrm{M}_i(h_t)$ and define it recursively as,

$$\begin{aligned} M_i(h_t) &= M_i(h_{t-1}) + \mu_i(h_t) \\ M_i(h_{-1}) &= \lambda^i \end{aligned}$$

where $\mathrm{M}_i(h_t)$ denotes the Pareto weight plus the cumulative sum of the Lagrange multipliers on the participation constraints at all periods from 1 to t.

So rewrite the Lagrangian once again as,

$$\sum_{t=0}^{\infty}\beta^t\sum_{h_t}\Pi(h_t)\sum_i\left\{M_i(h_{t-1})u(c^i(h_t) + \mu_i(h_t)\left[u(c^i(h_t) - \Omega_i(h_t)\right]\right\}$$
$$+\sum_{t=0}^{\infty}\sum_{h_t\in H_t}\gamma(h_t)\left[\sum_{i=1}^{2}c_i(h_t) - \sum_{i=1}^{2}y_i(h_t)\right]$$

Now we are ready to take the First Order Conditions:

$$\frac{u'(c^A(h_t))}{u'(c^B(h_t))} = \frac{M_A(h_{t-1}) + \mu_A(h_t)}{M_B(h_{t-1}) + \mu_B(h_t)}$$

$$\left[\sum_{r=t}^{\infty}\beta^{r-t}\sum_{h_r}\frac{\Pi(h_r)}{\Pi(h_t)}u(c^i(h_r)) - \Omega_i(h_t)\right]\mu_i(h_t) = 0$$

$$\sum_{i=1}^{2}c_i(h_t) - \sum_{i=1}^{2}y_i(h_t) = 0$$

18.2.5 Recursive Formulation

Our goal is make the problem recursive, which is very nice when we work with computer. To do this, we need to find a set of state variables which is sufficient to describe the state of the world. We are going to use x as a state variable.So the state variables are the endowment: $y = (y^A, y^B)$ and weight to brother 2: x. Define the value function as follows:

$$V = \{(V_0, V_A, V_B) \text{ such that } V_i : X \times Y \rightarrow \mathcal{R},\ i = 1, 2, V_0(x, y) = V_A(x, y) + xV_B(x, y)\}$$

What we are going find is the fixed point of the following operator (operation is defined later):

$$T(V) = \{T_0(V), T_1(V), T_2(V)\}$$

Firstly, we will ignore the participation constraints and solve the problem:

$$\max_{c_A, c_B} u(c^A(y, x)) + xu(c^B(y, x)) + \beta \sum_{y'} \Gamma_{yy'} V_0(y', x)$$

subject to

$$c^A + c^B = y^A + y^B$$

First Order Conditions yield:

$$\frac{u'(c_A)}{u'(c_B)} = x$$

Second, we will check the participation constraints. There are two possibilities here:

1. Participation constraint is not binding for either 1 or 2. Then set $x(h_t) = x(h_{t-1})$. In addition,

$$\begin{aligned} V_0^N(y, x) &= V_0(y, x) \\ V_i^N(y, x) &= u(c^i(y, x)) + \beta \sum_{y'} \Gamma_{yy'} V_i(y', x) \end{aligned}$$

2. Participation constraint is not satisfied for one of the brothers (say A).

This means that agent A is getting too little. Therefore, in order for the planner to match the outside opportunity that A has, he needs to change x so that he guarantees person A the utility from going away. We need to solve the following system of equations in this case:

$$\begin{aligned} c^A + c^B &= y^A + y^B \\ u(c^A) + \beta \sum_{y'} \Gamma_{yy'} V_A(y', x) &= u(y_A) + \beta \sum_{y'} \Gamma_{yy'} \Omega_A(y') \\ x' &= \frac{u'(c_A)}{u'(c_B)} \end{aligned}$$

This is a system of three equations and three unknowns. Denote the solution to this problem by,

$$\begin{aligned} &c^A(y,x) \\ &c^B(y,x) \\ &x'(y,x) \end{aligned}$$

So that,

$$\begin{aligned} V_0^N(y,x) &= V_A^N(y,x) + xV_B^N(y,x) \\ V_i^N(y,x) &= u(c^i(y,x)) + \beta \sum_{y'} \Gamma_{yy'} V_i(y', x'(y,x)) \end{aligned}$$

Thus we have obtained T(V)=V^N. And the next thing we need to do is find V* such that T(V*) $= V^*$.

Final question with this model is "how to implement this allocation?" or "Is there any equilibrium that supports this allocation?". The answer is yes. How? Think of this model as a repeated game. And define the strategy as follows: keep accepting the contract characterized here until the other guy walks away. If the other guy walks away, go to autarky forever. We can construct a Nash equilibrium by assigning this strategy to both of the brothers.

18.3 A Brief Introduction to OLG

Now let's consider a particular OG model. The agents will age in this model. We can either have that age increases by one year each period or we can have it such that it increases by one with probability π and stays the same with probability $(1-\pi)$. Here we will have the first case. People of the same age are identical so all we need to care about is the wealth level at each level of age, i. The problem of the agent of age i is:

$$V_i(z,A,a) = \max_{c,a'} u(c) + \beta E[V_{i+1}(z', A', a')|z]$$

$$\text{s.t.} \qquad c + a' = w(z,A)\epsilon_i + [1 + r(z,A)]a$$
$$A' = G(z,A)$$

where A is the aggregate wealth level and ϵ is number of efficiency units of labor. We can also have the agents die at each period with a probability γ_i. In this case, what could we do with the assets of the agents who die? There are two options. One is the 'Pharaoh model' where the agents are just buried with their wealth. The other one is to have the assets of the dead shared among others (accidental bequests).