Macro 702, Spring 2005, First Midterm
Suggested Solutions

Growth Model

Solution 1

Our commodity space should include everything that is traded by the household. Let $z_t \in Z$ and $H = Z \times Z \times ....$ An element of H, $h_t = (z_0, z_1, ..., z_t) \in H_t$ is a history of shocks up to period t. The commodity space is[1]:

$$L = \{s \mid s_t(h_t) = (s_{1t}(h_t), s_{2t}(h_t), s_{3t}(h_t)) \in R^3 \ \forall t, h_t \text{ and } \left\|x^i\right\|_\infty < \infty\}$$

The consumption possibility set is:

$$\begin{aligned} X &= \{x \in L \mid \exists\{c_t(h_t), k_{t+1}(h_t)\}_{t=0}^\infty \geq 0 \text{ such that} \\ x_{1t}(h_t) &= (1-\delta)k_t(h_{t-1}) + k_{t+1}(h_t) + c_t(h_t), \\ x_{2t}(h_t) &\leq k_t(h_{t-1}),\ x_{3t}(h_t) \in [0,1] \quad \forall t, h_t \text{ and } k_0, h_0 \text{ given}\} \end{aligned}$$

Note that here x_{1t} is the produced final output, x_{3t} is the labor services, x_{2t} is the capital services.

Production possibility set is:

$$L \supset Y = \cup_{t=0}^\infty Y_t : Y_t = \{y_{1t}(h_t) \geq 0, y_{2t}(h_t), y_{3t}(h_t) \leq 0 \text{ and} \tag{1}$$
$$0 \leq y_{1t}(h_t) \leq z_t F(y_{3t}(h_t), y_{2t}(h_t)) \quad \forall t, h_t)\} \tag{2}$$

For a given price vector $p(x)$, assuming prices have an inner product representation, the household's problem is:

$$\max \sum_{t=0}^\infty \beta^t \sum_{h_t \in H_t} \pi(h_t) u(c_t(h_t), n_t(h_t), N_{t-1}(h_{t-1}))$$

$$s.t. \qquad \sum_t \sum_{h_t \in H_t} \sum_{i=1}^3 p_{it}(h_t) x_{it}(h_t) \leq 0$$

where $\pi(h_t)$ is unconditional probability of history h_t.

Producers' problem:

$$\max_{y_t \in Y_t} \sum_{i=1}^3 p_{it}(h_t) y_{it}(h_t)$$

An Arrow-Debreu Competitive Equilibrium is (p^*, x^*, y^*) such that

1. x^* solves the consumer's problem.

[1]Note that there is no mention of measurability issues here since we always assume it is satisfied.

2. y^* solves the firm's problem.

3. Markets clear,

$$
\begin{aligned}
x^*_{1t}(h_t) &= y^*_{1t}(h_t) \text{ for all } h_t, t \\
x^*_{2t}(h_t) &= -y^*_{2t}(h_t) \text{ for all } h_t, t \\
x^*_{3t}(h_t) &= -y^*_{3t}(h_t) \text{ for all } h_t, t
\end{aligned}
$$

Solution 2

(FBWT) If the preferences of consumers are nonsatiated ($\exists\{x_n\} \in X$ that converges to $x \in X$ such that $U(x_n) > U(x)$), an allocation (x^*, y^*) of an ADE (p^*, x^*, y^*) is PO. (Note the implicit assumtion of no externality)

(SBWT) If (i) X is convex, (ii) preference is convex (for $\forall x, x' \in X$, if $x' < x$, then $x' < (1-\theta)x' + \theta x$ for any $\theta \in (0,1)$), (iii) $U(x)$ is continuous, (iv) Y is convex, (v)Y has an interior point, then with any PO allocation (x^*, y^*) such that x^* is not a satiation point, there exists a continuous linear functional p^* such that (x^*, y^*, p^*) is a Quasi-Equilibrium ((a) for $x \in X$ which $U(x) \geq U(x^*)$ implies $p^*(x) \geq p^*(x^*)$ and (b) $y \in Y$ implies $p^*(y) \leq p^*(y^*)$)

Solution 3

Note that there is a term N_{t-1} which creates an externality in the utility function. The agent is not aware of, thus does not internalize, the fact that the work decision she makes this period will effect her utility next period, whereas the social planner will solve this problem being aware of the fact $n_{t-1} = N_{t-1}$. The SPP solution will not coincide with the competitive allocation and thus competitive allocation is not PO. The first welfare theorem fails due to existence of externality.

Solution 4

Lets define the state variables first, i.e what matters when the household makes the her decisions and also varies over time. The wealth of the household, a ,one individual state variable that concerns only the household, the aggregate shock to production z , the aggregate capital stock K to pin down prices and depending on the functional form of the utility function the houshold also has to know N_{t-1}(and since there is no specific assumption about the utility function we will define RCE in the most general form). Note that since this is a representative agent framework the state contingent claims market is closed and complete markets and incomplete markets are equivalent. Lets write down the household problem with a little different timing convention than we usually do to incorporate lagged aggragate hours, where prime variables denote this period, double prime next period and no prime last period,

$$
\begin{aligned}
V(a', z', N, K') &= \max_{a'', n', c'} \{U(c', n', N) + \beta \sum \Gamma_{z'z''} V(a'', z'', N', K'')\} \\
s.t.\ c' + a'' &= R(z', N, K')a' + w(z', N, K')n'
\end{aligned}
$$

given the aggragate law of motions,

$$\begin{aligned} K'' &= G(z', N, K') \\ N' &= H(z', N, K') \end{aligned}$$

where the prices come from the firm's problem and has the solution,

$$\begin{aligned} a'' &= g(a', z', N, K') \\ n' &= h(a', z', N, K') \end{aligned}$$

This problem is well defined and we can define the RCE.

Definition (RCE): RCE is a set of functions $\{V, g, h, G, H, R, w\}$ such that,

1. Given $\{R, w, G, H\}$, $\{V, g, h\}$ solves the household problem.
2. Aggragate consistency

$$\begin{aligned} G(N, K') &= g(K', z', N, K') \\ H(N, K') &= h(K', z', N, K') \end{aligned}$$

Apart from the existence of the externatlity, the non-standart part of this problem is the existence of a lagged aggragate variable as a state which we have to carry on through periods. This makes the information set the household needs larger than usual but other than that the definition is as usual.

Solution 5

The government budget constraint is the formula that links the equilibrium subsidy to tax rate and state variables without debt,

$$T(z', N, K') = \tau w(z', N, K') H(N, K')$$

Solution 6

Note that the only reason we need the lagged aggragate hours as a state variable is because it is assumed to effect the optimal decision of the household. If the utility function is separable in all its arguments then the FOC with respect to n and a' and the envelope condition (thus the euler equation) will not depend on N_{t-1} and the household no longer need this information to determine its optimal behavior. Thus we can drop N_{t-1} as a state variable and simplfy our definition of RCE.

Lucas Trees

Solution 7

The state variables are s, d. The problem of the represenatative agent can be written recursively as

$$V\left(s,d^i\right)=\max_{s',c,n}\left\{u\left(c,n\right)+\beta\sum_j\Gamma_{i\ j}\ V\left(s',d^{\ j}\right)\right\}$$

$$s.t:\ \ c+s'p^i=s\left(\ p^i+d^i\right)+n\quad and\quad n\in[0,1].$$

or equivalently

$$V\left(s,d^i\right)=\max_{s',c,n}\left\{u\left[s\left(\ p^i+d^i\right)-s'p^i+n,n\right]+\beta\sum_j\Gamma_{i\ j}\ V\left(s',d^{\ j}\right)\right\}$$

or letting $V\left(s,d^i\right)=V_i\left(s\right)$

$$V_i\left(s\right)=\max_{s',c,n}\left\{u\left[s\left(\ p^i+d^i\right)-s'p^i+n,n\right]+\beta\sum_j\Gamma_{i\ j}\ V_j\ \left(s'\right)\right\}$$

Suppose that the solutions to this problem are of the form $s'=g\left(s,d\right),\ n=h\left(s,d\right)$. Let's take the FOCs:

$$\{s'\}:\ -p_i\ u_c\left(c,n\right)+\beta\sum_j\Gamma_{i\ j}\ \frac{\partial\ V_j\ \left(s'\right)}{\partial s'}=0\qquad(1)$$

$$\{n\}:\ u_c\left(c,n\right)+u_n\left(c,n\right)=0\qquad(2)$$

and to obtain an expression for the partial derivative in (1), use the envelope condition

$$\frac{\partial\ V_i\ (s)}{\partial s}=\ u_c\left(c,n\right)\left(p_i+d^i\right),\ \text{ so (1) becomes}$$

$$p_i\ u_c\left(c,n\right)=\beta\sum_j\Gamma_{i\ j}\ u_c\left(c',n'\right)\left(p^{\ j}+d^{\ j}\right)=0\qquad(1')$$

Of course, we have to use the specific functional forms given in the exercise. The necessary FOCs become

$$p_i\ c^{-\ \sigma}=\beta\sum_j\Gamma_{i\ j}\ \left(c'\right)^{-\ \sigma}\left(p^{\ j}+d^{\ j}\right)=0\qquad(3)$$

$$c^{-\ \sigma}=a\qquad(4)$$

Definition 8 *A Recursive Equilibrium is a list of functions* $\{V(s,d),\ g(s,d),\ h(s,d),\ \phi(d)\}$, *such that*

1) Agents maximize, i.e,

$$\phi\left(d^{\,i}\right)\ \left[\ \ s\left(\ \phi\left(d^{\,i}\right)+d^{i}\right)-\phi\left(d^{\,i}\right)g\left(s,d^{\,i}\right)+h\left(s,d^{\,i}\right)\right]^{-\ \sigma}=$$

$$=$$

$$\beta\sum_{j}\Gamma_{i\ j}\ \left(\phi\left(d^{\,j}\right)+d^{\,j}\right)\left[g\left(s,d^{\,i}\right)\left(\phi\left(d^{\,j}\right)+d^{\,j}\right)-\phi\left(d^{\,j}\right)g\left(g\left(s,d^{\,i}\right)\right)+h\left(s,d^{\,j}\right)\right]^{-\ \sigma}$$

and

$$\left(\ s\left(\ \phi\left(d^{\,i}\right)+d^{i}\right)-\phi\left(d^{\,i}\right)g\left(s,d^{\,i}\right)+h\left(s,d^{\,i}\right)\ \ \right)^{-\ \sigma}=a$$

2) Agent is representative, i.e,

$$g(1,d)=1\ \Leftrightarrow\ c=d^{\,i}\ +n$$

or, more formally

$$\left(\ \phi\left(d^{\,i}\right)+d^{i}\right)-\phi\left(d^{\,i}\right)g\left(1,d^{\,i}\right)+h\left(1,d^{\,i}\right)=d^{\,i}\ +h\left(1,d^{\,i}\right)$$

Solution 9

First, we have to find the price of an option to sell at price p_2 one period ahead. Let the price of this option be $p_j^1(p_2)$. We have

$$p_j^1(p_2)=\ \sum_m q_{j\ m}\ \max\{0,\ p_2-p_m^s\}$$

where $q_{j\ m}$ is the price of a state m contigent claim given that the current state is j,

and p_m^s is the price of a share of the tree in state m.

NOTE: The interpretation here is the following: The agent can buy in period $t+1$ and then sell or not sell in period $t+2$, but if she doesn't buy in period $t+1$ then she cannot sell in period $t+2$ (because she never had the share in her hands).

Suppose now that the state in period t is i and in $t+1$ the agent finds herself in state j. What's the return of the option she is holding?

$$\max\left\{0,\ \max\left\{p_j^s,\ p_j^1\left(p_2\right)\right\}-p_1\right\}$$

where the second max operator indicates precisely the fact that she can sell or not sell in period $t+2$.

Hence, the price of the option under consideration (let it be denoted as $p_i^{1,2}\ (p_1,\ p_2)$) is given by

$$p_i^{1,2}\ (p_1,\ p_2)=\ \sum_j q_{i\ j}\ \max\left\{0,\ \max\left\{p_j^s,\ p_j^1\left(p_2\right)\right\}-p_1\right\}=$$

$$=\sum_j q_{i\ j}\ \max\left\{0,\ \max\left\{p_j^s,\sum_m q_{j\ m}\ \max\{0,\ p_2-p_m^s\}\right\}-p_1\right\}$$

Solution 10

We saw in Question 7 that the FOCs imply a constant consumption level. We also know that in equilibrium $c=d^{\ i}+n$. This means that we can keep consumption constant by letting the labor adjust for the variation of the fruit yield. For example, in bad periods the agent will have to work more, and whenever $d=d^M$ (the maximum possible value), she will choose $n=0$. But since $n\in[0,1]$, we have to restrict the range of the values of the yield. The less restrictive assumption that works here is

$$d_t\in[k,\ k+1],\ \text{ where } k\in\mathbb{R}_+, \text{ for every } t\,.$$

Solution 11

The price of a tree is given by

$$P=\ \sum_{t=0}^{\infty}\ \sum_{h_t} p_t\left(h_t\right)\ d_t\left(h_t\right),$$

where $p_t\left(h_t\right)$ is the (Arrow-Debreu) price of the fruit in period t after the realization of history h_t.

The way to obtain a formula for this price is pretty standard. Writting down the AD specification of the problem, and obtaining the FOCs with respect to $c_t(h_t)$ and c_0, we can find that

$$p_t(h_t) = \beta^t \pi(h_t) \frac{u_c(c_t(h_t), n_t(h_t))}{u_c(c_0, n_0)}$$

But under the assumptions of the previous part, we know that the consumption is constant. Moreover, with the particular functional form for preferences (additively separable) we have that $u_c(c, n)$ will be a function of c only. So the formula for the price of the tree becomes

$$P = \sum_{t=0}^{\infty} \sum_{h_t} \beta^t \pi(h_t) \frac{u_c(c_t(h_t), n_t(h_t))}{u_c(c_0, n_0)} d_t(h_t), =$$

$$\sum_{t=0}^{\infty} \beta^t \sum_{h_t} \pi(h_t) \frac{(c_t(h_t))^{-\sigma}}{c_0^{-\sigma}} d_t(h_t) =$$

$$= \sum_{t=0}^{\infty} \beta^t \sum_{h_t} \pi(h_t) \quad d_t(h_t)$$