Suggested Solutions to Problem Set 1

Econ 702, Spring 2005

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Problem 1 *For a representative agent economy prove the following:*

$$x^* \in PO(\varepsilon) \Leftrightarrow x^* \in \arg\max_{x\in X} u(x) \tag{1}$$

Solution:

1. First showing $\Rightarrow$
   Assume this is an economy with a large (finite) number of identical agents. Suppose $x^* \in PO(\varepsilon)$ but $x_i^* \notin \arg\max_{x_i\in X} u(x_i)$ for some i, where i is the index for identical agents.
   $$\Rightarrow \exists \tilde{x}_i \; s.t. \; u(\tilde{x}_i) > u(x_i^*) \; and \; \tilde{x}_i \in X$$
   Construct the following allocation,
   $$\tilde{x} = (x_1^*, x_2^*, ..., \tilde{x}_j, ...)$$
   The allocation $\tilde{x}$ is feasible and it gives more utility to agent j while keeping all the others with the same utility.This contradicts with the fact that x^* is Pareto optimal.

2. Showing $\Leftarrow$
   Suppose $x_i^* \in \arg\max_{x_i\in X} u(x_i)\forall i$ but $x^* \notin PO(\varepsilon)$
   Since $x^* \notin PO(\varepsilon)$, $\exists$ a feasible allocation $\tilde{x}$ such that,
   $$\begin{aligned} u(\tilde{x}_i) &\geq u(x_i^*) \; for \; all \; i \\ u(\tilde{x}_i) &> u(x_i^*) \; for \; some \; i \end{aligned} \tag{2}$$
   But this means that for some i $x_i^* \notin \arg\max_{x_i\in X} u(x_i)$. Contradiction.

Problem 2 *Consider the following social planner's problem:*

$$\max_{\{c_t,l_t,n_t,k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t, \ell_t) \tag{3}$$

$$
\begin{aligned}
s.t. \qquad & c_t + k_{t+1} \leq f(k_t, n_t) \\
& k_0 \quad given, \\
& c_t, k_{t+1}, n_t, \ell_t \geq 0 \\
& \ell_t + n_t = 1
\end{aligned}
$$

Show that the set of feasible allocations is convex.

Solution:

1. Showing that the constraint set is convex. Define, $\mathcal{B} = \{\{c_t, l_t, k_{t+1}\}_{t=0}^{\infty} \in \ell_\infty : c_t + k_{t+1} \leq f(k_t, 1 - l_t), c_t \geq 0 \text{ and } k_{t+1} \geq 0 \, \forall t\}$

   Let $\{c_t^1, l_t^1, k_{t+1}^1\} \in \mathcal{B}, \{c_t^2, l_t^2, k_{t+1}^2\} \in \mathcal{B}$ and $\theta \in [0, 1]$.
   The following holds,

$$
c_t^1 + k_{t+1}^1 = f(k_t^1, 1 - l_t^1) \tag{4}
$$

$$
c_t^2 + k_{t+1}^2 = f(k_t^2, 1 - l_t^2) \tag{5}
$$

   Then for $0 < \theta < 1$,

$$
\begin{aligned}
& [\theta c_t^1 + (1-\theta) c_t^2] + [\theta k_{t+1}^1 + (1-\theta) k_{t+1}^2] \\
& = \theta f(k_t^1, 1 - l_t^1) + (1-\theta) f(k_t^2, 1 - l_t^2)
\end{aligned} \tag{6}
$$

   Assuming f is concave as usual,

$$
\begin{aligned}
& \theta f(k_t^1, 1 - l_t^1) + (1-\theta) f(k_t^2, 1 - l_t^2) \\
& \leq f(\theta k_t^1 + (1-\theta) k_t^2, \theta(1 - l_t^1) + (1-\theta)(1 - l_t^2))
\end{aligned} \tag{7}
$$

   (6) and (7) imply,

$$
\begin{aligned}
& [\theta c_t^1 + (1-\theta) c_t^2] + [\theta k_{t+1}^1 + (1-\theta) k_{t+1}^2] \\
& \leq f(\theta k_t^1 + (1-\theta) k_t^2, \theta(1 - l_t^1) + (1-\theta)(1 - l_t^2))
\end{aligned} \tag{8}
$$

   $\Rightarrow \{(1-\theta)c_t^1 + \theta c_t^2, (1-\theta) l_t^1 + \theta l_t^2, (1-\theta) k_{t+1}^1 + \theta k_{t+1}^2\} \in \mathcal{B}$
   $\Rightarrow \mathcal{B}$, the set of sequences $\{c_t, l_t, k_{t+1}\}_{t=0}^{\infty}$ that satisfy the constraints is a convex subset of $\mathcal{R}^\infty$.

Problem 3 *Consider the social planner's problem (SPP)above with Cobb-Douglas technology, partial depreciation and CRRA preferences. Derive the Euler equation.*

Solution:

The particular functional forms are;

$$f(k_t, n_t) = k_t^{\theta} n_t^{(1-\theta)}$$

$$u(c_t, l_t) = (\frac{c_t^{1-\sigma}}{1-\sigma}) + (\frac{l_t^{1-\gamma}}{1-\gamma})$$

Letting;

$$V(k_0) = \max_{\{c_t, l_t, n_t, k_{t+1}\} \in C} \sum_{t=0}^{\infty} \beta^t u(c_t, \ell_t)$$

where C is the constraint set then after substituting for the consumption from the feasibility constraint,the generic Euler equation is;

$$\frac{dV(k_0)}{dk_{t+1}} = u_c(c_t, l_t) - \beta u_c(c_{t+1}, l_{t+1})[1 - \delta + f_k(k_{t+1}, n_{t+1})] = 0$$

where the derivatives are;

$$u_c(c_t, l_t) = c_t^{-\sigma}$$
$$f_k(k_t, n_t) = \theta k_t^{\theta-1} n_t^{1-\theta}$$

substituting

Problem 4 *Defining the commodity space as a space of bounded real sequences,*

$$\mathcal{L} = \{ \ \{\ell_{it}\}_{t=0}^{\infty}, \ \sup_{i,t} |\ell_{it}| < \infty \quad \forall \ell\} \tag{9}$$

Prove that $\mathcal{L}$ endowed with the supnorm is a topological vector space (TVS). Also prove that R^n endowed with the usual Euclidian norm is a TVS.

Solution:

1. Showing that $\mathcal{L}$ endowed with the supnorm topology is a topological vector space.
   A topological vector space is a vector space which is endowed with a topology such that the maps $(x, y) \rightarrow x + y$ and $(\lambda, x) \rightarrow \lambda x$ are continuous. So we have to show the continuity of the vector operations addition and scalar multiplication.

   Take $(x, y) \in \mathcal{L}$ s.t. $x_i \rightarrow x$ and $y_i \rightarrow y$, then

   $$sup_i|(x_i - x) + (y_i - y)| = sup_i|(x_i + y_i) - (x + y)| \rightarrow 0$$

   so $(x_i, y_i) \rightarrow (x, y)$.
   Also,

   $$sup_i|\lambda x_i - \lambda x| \leq |\lambda| sup_i|(x_i - x)| \rightarrow 0$$

   so $(\lambda, x_i) \rightarrow (\lambda, x)$.

Problem 5 *Show that the consumption possibility set X is convex, and the production possibility set, Y are convex and has an interior point (endowed with supnorm).*

Solution:

1. Showing X is convex, Let $x^1 \in X$ and $x^2 \in X$ and $\theta \in [0,1]$

$$\begin{aligned} k^1_{t+1} + c^1_t &= x^1_{1t} \\ k^2_{t+1} + c^2_t &= x^2_{1t} \\ \Rightarrow [\theta k^1_{t+1} + (1-\theta)k^2_{t+1}] + [\theta c^1_t + (1-\theta)c^2_t] &= [\theta x^1_{1t} + (1-\theta)x^2_{1t}] \end{aligned} \tag{10}$$

   Also, $-x^1_{2t} \in [0, k^1_t]$ and $-x^2_{2t} \in [0, k^2_t]$

$$\Rightarrow (\theta x^1_{2t} + (1-\theta)x^2_{2t}) \in [\theta k^1_t + (1-\theta)k^2_t, 0] \tag{11}$$

   And finally, $-x^1_{3t} \in [0,1]$ and $-x^2_{3t} \in [0,1]$

$$\Rightarrow (\theta x^1_{3t} + (1-\theta)x^2_{3t}) \in [0,1] \tag{12}$$

   (10),(11),and (12) $\Rightarrow [\theta x^1 + (1-\theta)x^2] \in X$ so that X is convex.

2. Showing Y is convex,

$$Y = \{y \in \mathcal{L} : y_{1t} \le f(-y_{2t}, -y_{3t}), y_{1t} \ge 0,\ y_{2t}, y_{3t} \le 0\} \tag{13}$$

   Let $y^1 \in Y$ and $y^2 \in Y$ and $\theta \in [0,1]$

$$\begin{aligned} y^1_{1t} &\le f(-y^1_{2t}, -y^1_{3t}) \\ y^2_{1t} &\le f(-y^2_{2t}, -y^2_{3t}) \\ \Rightarrow (\theta y^1_{1t} + (1-\theta)y^2_{1t}) &\le \theta f(-y^1_{2t}, -y^1_{3t}) + (1-\theta)f(-y^2_{2t}, -y^2_{3t}) \end{aligned} \tag{14}$$

   Assuming f is concave,

$$\begin{aligned} &\theta f(-y^1_{2t}, -y^1_{3t}) + (1-\theta)f(-y^2_{2t}, -y^2_{3t}) \\ &\le f(\theta(-y^1_{2t}) + (1-\theta)(-y^2_{2t}), \theta(-y^1_{3t}) + (1-\theta)(-y^2_{3t})) \end{aligned} \tag{15}$$

   (14) and (15) imply,

$$(\theta y^1_{1t} + (1-\theta)y^2_{1t}) \le f(\theta(-y^1_{2t}) + (1-\theta)(-y^2_{2t}), \theta(-y^1_{3t}) + (1-\theta)(-y^2_{3t})) \tag{16}$$

Also,

$$\begin{aligned}
(\theta y_{1t}^1 + (1-\theta) y_{1t}^2) &\geq 0 \\
(\theta y_{2t}^1 + (1-\theta) y_{2t}^2) &\leq 0 \\
(\theta y_{3t}^1 + (1-\theta) y_{3t}^2) &\leq 0
\end{aligned} \tag{17}$$

(16) and (17) $\Rightarrow [\theta y^1 + (1-\theta) y^2] \in Y$ so that Y is convex.

Next we want to show that the PPS has an interior point (which is necessary for SBWT to hold on infinite dimensional spaces). To do so we will construct one, take $x = ((1,-1,-1),(1,-1,-1)......,(1,-1,-1),....) \in$Y$\in \ell_\infty$. To check it is an interior point take an $\epsilon = 1/2$ ball around x and we want to show $B_\epsilon(x) \subseteq \ell_\infty$. Take any $\tilde{x}_t \in B_\epsilon(x)$ than we know

$$sup_t |\tilde{x}_t| \leq (3/2) < \infty \ , \forall \tilde{x}_t \in B_\epsilon(x)$$

and x is an interior point.

Problem 6 *Show that the set of feasible allocations is compact ($X \cap Y$)*

Solution: To be completed.

Problem 7 *Let (p^*, x^*, y^*) be an AD equilibrium. Setup the household and firm problem in AD language and derive the prices from the given equilibrium allocations and FOCs. Show that the following mapping constitutes a SME (by verifying the FOCs of SME problem is satisfied)*

$$\begin{aligned}
c_t^* &= x_{1t}^* - x_{2t+1}^* && \forall t && (18) \\
n_t^* &= x_{3t}^*, \quad \ell_t^* = 0 && \forall t && (19) \\
k_t^* &= x_{2t}^* && \forall t && (20) \\
R_t^* &= \frac{p_{2t}^*}{p_{1t}^*} && \forall t && (21) \\
w_t^* &= \frac{p_{3t}^*}{p_{1t}^*} && \forall t && (22)
\end{aligned}$$

Solution: Please see the extra notes for an extended discussion.